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AGGREGATION OF THE RANDOM COEFFICIENT GLARCH(1,1) PROCESS

LIUDAS GIRAITIS
Queen Mary, University of London

REMIGIJUS LEIPUS AND DONATAS SURGAILIS
Vilnius University and Institute of Mathematics and Informatics

The paper discusses contemporaneous aggregation of the Linear ARCH (LARCH) model as defined in (1), which was introduced in Robinson (1991) and studied in Giraitis, Robinson, and Surgailis (2000) and other works. We show that the limiting aggregate of the (G)eneralized LARCH(1,1) process in (3)–(4) with random Beta distributed coefficient β exhibits long memory. In particular, we prove that squares of the limiting aggregated process have slowly decaying correlations and their partial sums converge to a self-similar process of a new type.

1. INTRODUCTION

Modeling of asset returns has attracted considerable interest in financial econometrics. Particular attention is paid to modeling the long memory phenomenon, the leverage effect, and other well-known empirical stylized facts of asset returns. However, the widely used class of ARCH/GARCH models, specified in such a way that the conditional variance is a weighted linear combination of *squares* of past returns and the conditional variances itself, is not able to reproduce the covariance long memory of squared returns (see Giraitis, Kokoszka, and Leipus, 2000, Giraitis, Leipus, and Surgailis, 2009). Recall that a covariance stationary series X_t is said to have *covariance long memory* whenever $\sum_{t=0}^{\infty} |\mathrm{cov}(X_0, X_t)| = \infty$; see, e.g., Leipus, Paulauskas, and Surgailis (2005); Zaffaroni (2007a).

On the other hand, since economic reasons for the above-mentioned long memory phenomenon are unclear, attempts have been made to explain it by simple heterogeneous short memory models involving regime shifts or aggregation (Ding and Granger, 1996; Mikosch, 2003; Davidson and Sibbertsen, 2005; and Leipus et al., 2005). The classical Granger (1980) contemporaneous aggregation scheme (summing and averaging across observations), based on the simplest AR(1)

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regression schemes near the nonstationarity regime, provides an explanation of the long memory effect in homoskedastic time series, and a simulation method of such series as well. Further justification of the aggregation procedure in linear models was provided by Gonçalves and Gouriéroux (1988), Oppenheim and Viano (2004), and Zaffaroni (2004). Ding and Granger (1996), Leipus and Viano (2002), Kazakevičius, Leipus, and Viano (2004), and Zaffaroni (2007a, 2007b) considered contemporaneous aggregation of conditionally heteroskedastic models, with mostly negative outcome, in the sense that the limit aggregate does not have covariance long memory. The only case where the limiting aggregated conditionally heteroskedastic process was previously shown to have this property is the stochastic volatility model in Zaffaroni (2007a), with volatility given by the AR(1) process as in Granger's (1980) original scheme. In some other cases (Ding and Granger, 1996; Kazakevičius et al., 2004) the limiting aggregate was shown to have hyperbolically decaying, albeit summable, correlations and therefore not to have covariance long memory in the above sense, too.

The present paper studies contemporaneous aggregation of the linear ARCH (LARCH) model. This model was introduced by Robinson (1991) and studied in Giraitis, Robinson, and Surgailis (2000). See also Giraitis and Surgailis (2002), Giraitis, Leipus, Robinson, and Surgailis (2004), Berkes and Horváth (2003), and Beran (2006). In the LARCH model, the observed series is described by equations

$$X_t = \zeta_t V_t, \qquad V_t = a + \sum_{j=1}^{\infty} b_j X_{t-j}, \tag{1}$$

where $a \neq 0$ is a constant, $\zeta_t, t \in \mathbb{Z}$ are i.i.d.$(0, 1)$, and $b_j, j \geq 1$ are (nonrandom) weights satisfying

$$\|b\| := \left\{ \sum_{j=1}^{\infty} b_j^2 \right\}^{1/2} < 1. \tag{2}$$

In the financial context, X_t can be interpreted as asset return and V_t as volatility. Condition (2) allows a broad range of dependence behavior including long memory in volatility and its powers (see Giraitis, Robinson, and Surgailis, 2000). The parametric class of (G)eneralized LARCH(p, d, q) processes (defined in analogy with the well-known parametric classes GARCH(p, q) and FARIMA(p, d, q)) corresponds to V_t in (1), solving FARIMA(p, d, q) equation $(1 - \phi(L))(1 - L)^d (V_t - a) = \psi(L) X_t$ (see Giraitis et al., 2004). This class is different from the well-known FIGARCH class where volatility depends on a linear combination of the squares of returns. Let us note that, contrary to the LARCH case, the FIGARCH equation has no stationary solution with finite variance (see Giraitis et al., 2009; and Douc, Roueff, and Soulier, 2008), and its long memory properties have not been rigorously established.

The present paper shows that aggregation of the short memory GLARCH(1,1) process with random Beta distributed coefficient can generate long memory. In particular, we prove that the squared limiting aggregated process has

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nonsummable correlations decaying as t^{-2q} for $0 < q < 1/2$ and its partial sums converge to a non-Gaussian self-similar process of a new type.

2. THE AGGREGATED GLARCH(1,1) PROCESS

Consider the GLARCH(1,1) model with

$$X_t = \zeta_t V_t, \quad V_t = a + \beta V_{t-1} + c\beta\sqrt{1-\beta^2}X_{t-1} \tag{3}$$

$$= a' + c\sqrt{1-\beta^2}\sum_{j=1}^{\infty}\beta^j X_{t-j}, \tag{4}$$

with parameters $a \in \mathbb{R}$, $0 < \beta < 1$, $a' = a/(1-\beta)$, $0 < c < 1$. Note that (4) is a special case of (1) corresponding to

$$b_j = c\sqrt{1-\beta^2}\beta^j. \tag{5}$$

Then condition (2) becomes

$$\|b\| = c\beta < 1. \tag{6}$$

The stationary solution of (4) under condition (6) can be obtained by iteration (see Giraitis, Robinson, and Surgailis, 2000):

$$X_t = \sum_{k=0}^{\infty} X_t^{(k)}, \quad X_t^{(k)} = \zeta_t V_t^{(k)}, \tag{7}$$

$$V_t^{(k)} = a'\big(c\sqrt{1-\beta^2}\big)^k \sum_{s_k < \cdots < s_1 < t} \beta^{t-s_k}\zeta_{s_1}\cdots\zeta_{s_k} \quad (k \geq 1), \quad V_t^{(0)} = a'. \tag{8}$$

It is easy to check that all series in (8) converge in mean square; indeed, since ζ_t are i.i.d.$(0,1)$ and $\mathrm{E}X_t^{(k)}X_t^{(\ell)} = 0 (k \neq \ell)$, so

$$\begin{aligned}\mathrm{E}X_t^2 &= \sum_{k=0}^{\infty}\mathrm{E}(X_t^{(k)})^2 = (a')^2\sum_{k=0}^{\infty}c^{2k}(1-\beta^2)^k \sum_{s_k<\cdots<s_1<t}\beta^{2(t-s_k)} \\ &= (a')^2\sum_{k=0}^{\infty}c^{2k}(1-\beta^2)^k\left(\sum_{i=1}^{\infty}\beta^{2i}\right)^k \\ &= (a')^2\sum_{k=0}^{\infty}c^{2k}\beta^{2k} = \frac{(a')^2}{1-c^2\beta^2} < \infty\end{aligned} \tag{9}$$

in view of (6).

Remark 2.1. Note that $X_t^{(0)} = a'\zeta_t$ and $X_t^{(1)} = \zeta_t a' c(1-\beta^2)^{1/2}\sum_{j=1}^{\infty}\beta^j\zeta_{t-j}$ is a particular case of the nonlinear MA model of Robinson and Zaffaroni (1998).

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Aggregation of nonlinear MA was discussed in Zaffaroni (2007a). (The parameter δ in Zaffaroni, 2007a, Assump. A(ϑ) corresponds to $q - 1/2$, with q defined through (17).)

Let $\beta_i \in (0,1)$ and $a_i' \in \mathbb{R}$ be random variables such that the random vectors $(\beta_i, a_i'), i = 1,2,\ldots$ are independent and have a common distribution (β, a') and $\mathrm{E}(a')^2 < \infty$. Let $X_{i,t} = \zeta_t V_{i,t}$, $V_{i,t}, t \in \mathbb{Z}$ be a stationary solution to random coefficient GLARCH(1,1) equations (4) with random parameters $a' = a_i', \beta = \beta_i$ satisfying $0 < \beta_i < 1$ a.s., with a common (nonrandom) $c \in (0,1)$ and a common noise $\zeta_t, t \in \mathbb{Z}$, which is assumed to be independent of the sequence $(\beta_i, a_i'), i = 1,2,\ldots$.

For $N = 1,2,\ldots$, define the *aggregated process* as

$$\bar{X}_{N,t} = \frac{1}{N}\sum_{i=1}^{N} X_{i,t}, \qquad \bar{V}_{N,t} = \frac{1}{N}\sum_{i=1}^{N} V_{i,t}. \tag{10}$$

Similarly, define the aggregate components in (7)–(8):

$$\bar{X}_{N,t}^{(k)} = \frac{1}{N}\sum_{i=1}^{N} X_{i,t}^{(k)}, \qquad \bar{V}_{N,t}^{(k)} = \frac{1}{N}\sum_{i=1}^{N} V_{i,t}^{(k)}. \tag{11}$$

Note that $\bar{X}_{N,t} = \sum_{k=0}^{\infty} \bar{X}_{N,t}^{(k)}$, $\bar{V}_{N,t} = \sum_{k=0}^{\infty} \bar{V}_{N,t}^{(k)}$.

The following theorem describes the limiting behavior of the aggregated process $X_t^{(N)}$. Notation $a_t \sim b_t$ means that $a_t/b_t \to 1$ as $t \to \infty$.

THEOREM 1. *Let* $\bar{X}_{N,t}, \bar{V}_{N,t}, \bar{X}_{N,t}^{(k)}, \bar{V}_{N,t}^{(k)}$ *be given by* (10)–(11).

(i) For any $t \in \mathbb{Z}, k = 0,1,\ldots,$

$$\bar{X}_{N,t} \to \bar{X}_t, \qquad \bar{V}_{N,t} \to \bar{V}_t, \qquad \bar{X}_{N,t}^{(k)} \to \bar{X}_t^{(k)}, \quad \textit{and} \quad \bar{V}_{N,t}^{(k)} \to \bar{V}_t^{(k)}, \tag{12}$$

in mean square, as $N \to \infty$, *where* $\bar{X}_t = \mathrm{E}[X_t | \zeta_s, s \in \mathbb{Z}]$, $\bar{V}_t = \mathrm{E}[V_t | \zeta_s, s \in \mathbb{Z}]$,

$$\bar{X}_t^{(k)} = \mathrm{E}\big[X_t^{(k)} | \zeta_s, s \in \mathbb{Z}\big], \qquad \bar{V}_t^{(k)} = \mathrm{E}\big[V_t^{(k)} | \zeta_s, s \in \mathbb{Z}\big], \tag{13}$$

and where $X_t = \sum_{k=0}^{\infty} X_t^{(k)}$, $V_t = \sum_{k=0}^{\infty} V_t^{(k)}$ *is the stationary solution of the GLARCH*(1,1) *equations in* (3)–(4) *with random* a', β. *More explicitly,*

$$\bar{X}_t = \sum_{k=0}^{\infty} \bar{X}_t^{(k)}, \qquad \bar{V}_t = \sum_{k=0}^{\infty} \bar{V}_t^{(k)}, \tag{14}$$

$$\bar{X}_t^{(k)} = \zeta_t \bar{V}_t^{(k)}, \qquad \bar{V}_t^{(k)} = \sum_{s_k < \ldots < s_1 < t} g_{t-s_k}^{(k)} \zeta_{s_1} \cdots \zeta_{s_k}, \tag{15}$$

where

$$g^{(0)} = \mathrm{E}a', \qquad g_t^{(k)} = c^k \mathrm{E}\big[a'(1-\beta^2)^{k/2}\beta^t\big] \quad (k \le t, k, t = 1,2,\ldots). \tag{16}$$

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The Volterra series in (14)–(15) *converge in mean square and define strictly stationary, covariance stationary and ergodic processes.*

(ii) Assume in the definition of $\bar{X}_{N,t}$, *that* $a' = 1$ *and* β *in* (4) *has Beta distribution with probability density*

$$f(x) = B(p,q)^{-1} x^{p-1}(1-x)^{q-1}, \qquad x \in (0,1), \qquad p,q > 0. \tag{17}$$

Then for any $k = 1, 2, \ldots$, *as* $t \to \infty$,

$$g_t^{(k)} \sim \frac{c^k 2^{k/2} \Gamma(k/2+q)}{B(p,q)}\, t^{-(k/2)-q}. \tag{18}$$

The function $g_t^{(k)}$ *is explicitly written in terms of the hypergeometric function in* (34). *Moreover, if* $0 < q < 1/2$, *then*

$$\operatorname{cov}(\bar{V}_0, \bar{V}_t) \sim c_{\bar{V}}^2 t^{-2q}, \qquad \operatorname{cov}(\bar{V}_0^{(k)}, \bar{V}_t^{(k)}) \sim c_{\bar{V}^{(k)}}^2 t^{-2q} \quad (k \geq 1), \tag{19}$$

where the constants $c_{\bar{V}}^2 = \sum_{k=1}^{\infty} c_{\bar{V}^{(k)}}^2$ *are given in* (29).

Remark 2.2. It can be shown that for $q > 1/2$, the covariances in Theorem 1(ii), (19) are absolutely summable: $\sum_{t\in\mathbb{Z}} |\operatorname{cov}(\bar{V}_0, \bar{V}_t)| < \infty$, $\sum_{t\in\mathbb{Z}} \left|\operatorname{cov}(\bar{V}_0^{(k)}, \bar{V}_t^{(k)})\right| < \infty$.

Remark 2.3. Zaffaroni (2004, 2007a, 2007b) studied the possibility of *nonstationary* limit aggregate arising from AR(1) and stochastic volatility models. In the case of the stationary GLARCH(1,1) equation in (4) with $a' = 1$ and Beta distributed β, the existence of a nonstationary limit aggregate is unlikely, since the variance of X_t, as well as the variance of aggregated process $X_{N,t}$, is bounded by $1/(1-c^2)$; see (9). On the other hand, it is possible that a nonstationary limit regime could arise from aggregation of *nonstationary* LARCH models. The study of nonstationary LARCH models and their asymptotic properties is an open and interesting problem for future work.

Theorem 1, (19) shows that aggregation of short memory LARCH processes can lead to long memory (nonsummable covariance) in volatility $\bar{V}_t$ of the limiting aggregated process. Note from (19) that the same decay rate $O(t^{-2q})$ of covariances applies to *all* terms $\bar{V}_t^{(k)} (k \geq 1)$ of the Volterra expansion (14) of $\bar{V}_t$. For $k = 1$, this result follows easily, since $\bar{V}_t^{(1)} = \sum_{s<t} g_{t-s}^{(1)} \zeta_s$ is a linear process with coefficient $g_t^{(1)} \sim C t^{-(1/2)-q}$, $C \neq 0$, and so the covariance of this term decays as t^{-2q}, too.

3. LONG MEMORY PROPERTIES OF THE LIMIT AGGREGATES

One of the main tasks of the present paper is to study the covariance structure of the squared limit aggregates $\bar{X}_t^2$ and $\bar{V}_t^2$ and the limiting behavior of their partial sums. According to Lemma 1 below, this problem reduces to the study of the limit

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behavior of the partial sums process of $\bar{V}_t$ in (14)–(15). To that end, we introduce a new class of processes, as follows. Let $W(s) = W_+(s)\mathbf{1}(s \geq 0) + W_-(-s)\mathbf{1}(s \leq 0), s \in \mathbb{R}$ be a Brownian motion on the real line $\mathbb{R}$, where $W_+(s), W_-(s), s \geq 0, W_\pm(0) = 0$ are two independent standard Brownian motions. Let $G = G(x), x \in \mathbb{R}$ be a real-valued function such that $\mathrm{E}G^2(Z) < \infty$, $Z \sim N(0,1)$. The process

$$Y_G(t) = \int_{-\infty}^{t} \left\{ \int_{0\vee s}^{t} G\left(\frac{W(\tau) - W(s)}{\sqrt{\tau - s}} \right) \frac{d\tau}{(\tau - s)^{q+(1/2)}} \right\} dW(s), \quad t \geq 0 \tag{20}$$

is well defined for any $q \in (0, 1/2)$, as (backward) Itô stochastic integral, and satisfies $\mathrm{E}Y_G(t) = 0$ and

$$\begin{aligned} \mathrm{var}(Y_G(t)) &= \int_{-\infty}^{t} \mathrm{E}\left[\int_{0\vee s}^{t} G\left(\frac{W(\tau) - W(s)}{\sqrt{\tau - s}} \right) \frac{d\tau}{(\tau - s)^{q+(1/2)}} \right]^2 ds \\ &\leq \mathrm{E}G^2(Z) \int_{-\infty}^{t} \left[\int_{0\vee s}^{t} \frac{d\tau}{(\tau - s)^{q+(1/2)}} \right]^2 ds \\ &= c_1^2 t^{2-2q} \mathrm{E}G^2(Z) < \infty. \end{aligned} \tag{21}$$

Moreover, the process in (20) has stationary increments and is self-similar with index $H = 1 - q \in (1/2, 1)$; in other words, for any $\lambda > 0$,

$$\{Y_G(\lambda t), t \geq 0\} =_{\mathrm{law}} \{\lambda^H Y_G(t), t \geq 0\}. \tag{22}$$

The self-similarity property in (22) follows from the self-similarity of the Brownian motion: $\{W(\lambda t)\} =_{\mathrm{law}} \{\lambda^{1/2} W(t)\}$, by change of variables $s \to \lambda s, \tau \to \lambda \tau$. For constant function $G \equiv 1$, the process in $Y_G(t)$ in (20) becomes

$$Y_1(t) = \int_{-\infty}^{t} \left\{ \int_{0\vee s}^{t} \frac{d\tau}{(\tau - s)^{q+(1/2)}} \right\} dW(s),$$

which is nothing but a fractional Brownian motion with variance $c_1^2 t^{2-2q}$ and Hurst parameter $H = 1 - q$ (see Taqqu, 2003, (9.1)). Let

$$G(x) = \sum_{k=0}^{\infty} J_k H_k(x)$$

be the Hermite expansion of G in Hermite polynomials $H_k(x) = (-1)^k d^k \mathrm{e}^{-x^2/2}/dx^k$, $J_k = \mathrm{E}G(Z)H_k(Z)/k!$ (see, e.g., Taqqu, 1979). Then

$$Y_G(t) = \sum_{k=0}^{\infty} J_k Y_{H_k}(t), \tag{23}$$

where $Y_{H_k}(t)$ is the process in (20) corresponding to $G = H_k$. Note that the series in (23) is orthogonal and converges in mean square. The process $Y_{H_k}(t)$ can be

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written as a $(k+1)$-tuple Itô–Wiener integral (see the proof of Theorem 2, below), and is related to the class of Gaussian subordinated self-similar processes discussed in Dobrushin (1979, Thm. 6.2). However, to our knowledge, stochastic integrals of the type (20) have not been previously encountered in limit problems of time series analysis.

In the sequel, we shall use the process $Y_G(t)$ with a special function $G(x)$ defined by the Hermite expansion

$$G(x) = \frac{2^{1/2}c}{B(p,q)} \sum_{k=0}^{\infty} \frac{2^{k/2}}{k!} c^k \Gamma\left(\frac{k+1}{2}+q\right) H_k(x) \tag{24}$$

$$= \frac{2^{1/2}c}{B(p,q)} \left[\frac{\Gamma(0.5+q)}{(1+c^2)^{0.5+q}} \Phi\left(0.5+q, 0.5, \frac{c^2x^2}{2(1+c^2)}\right) \right.$$

$$\left. + \frac{2^{1/2}xc\Gamma(1+q)}{(1+c^2)^{1+q}} \Phi\left(1+q, 1.5, \frac{c^2x^2}{2(1+c^2)}\right) \right].$$

Here, $\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \mathrm{e}^{xu} u^{a-1}(1-u)^{b-a-1}du$, $b > a > 0$ is the degenerated hypergeometric function; see Bateman and Erdélyi, 1953, Sect. 6.5, eq. (1). Write $\Rightarrow$ for weak convergence of finite dimensional distributions.

THEOREM 2. *Let the conditions of Theorem 1(ii) be satisfied, with* $0 < q < 1/2$. *Then*

$$n^{q-1} \sum_{s=1}^{[nt]} (\bar{V}_s - 1) \Rightarrow Y_G(t), \tag{25}$$

$$n^{q-1} \sum_{s=1}^{[nt]} \bar{V}_t^{(k)} \Rightarrow J_{k-1} Y_{H_{k-1}}(t) \quad (k \geq 1), \tag{26}$$

where $J_{k-1} = 2^{k/2} c^k \Gamma\left(\frac{k}{2}+q\right) / B(p,q)(k-1)!$ *and where the limit processes in* (25), *and* (26) *are the self-similar processes as defined in* (20) *and corresponding to G in* (24) *and the Hermite polynomial* H_{k-1}, *respectively. Moreover, the limit processes are related as in* (23), *and the process* $J_0 Y_{H_0}(t)$ *corresponding to* $k = 1$ *in* (26) *is a fractional Brownian motion.*

To study the memory properties of the squared limit aggregates, we impose the additional moment condition

$$c\sqrt{11}(\mathrm{E}\zeta^4)^{1/4} < 1, \tag{27}$$

which guarantees finiteness of the 4th moments $\mathrm{E}\bar{X}_t^4$ and $\mathrm{E}\bar{V}_t^4$ (see Giraitis, Robinson, and Surgailis, 2000). Theorem 3 below shows that the squared limit aggregates $\bar{X}_t^2$ and $\bar{V}_t^2$ also exhibit long memory.

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THEOREM 3. *Assume conditions of Theorem 1(ii), with* $0 < q < 1/2$, *and condition* (27). *Then, as* $t \to \infty$,

$$\left.\begin{matrix}\operatorname{cov}(\bar{V}_t^2, \bar{V}_0^2)\\ \operatorname{cov}(\bar{X}_t^2, \bar{X}_0^2)\end{matrix}\right\} \sim 4c_{\bar{V}}^2 t^{-2q}, \tag{28}$$

where

$$c_{\bar{V}}^2 = \sum_{k=1}^{\infty} c_{\bar{V}(k)}^2, \qquad c_{\bar{V}(k)}^2 = \frac{1}{(k-1)!}\left(\frac{c^k \Gamma\left(\frac{k}{2}+q\right)}{B(p,q)}\right)^2 B(2q, (k/2)-q). \tag{29}$$

Moreover,

$$\left.\begin{matrix} n^{q-1}\sum_{s=1}^{[nt]}(\bar{V}_s^2 - \mathrm{E}\bar{V}_s^2)\\ n^{q-1}\sum_{s=1}^{[nt]}(\bar{X}_s^2 - \mathrm{E}\bar{X}_s^2)\end{matrix}\right\} \Rightarrow 2Y_G(t), \tag{30}$$

where $Y_G(t)$ *is the same as in Theorem 2.*

Remark 3.1. The nonstandard form of the limit distribution in (30) can be explained by the presence of higher-order terms in the Volterra expansion of the GLARCH(1,1) model. Indeed, for the stochastic volatility model involving only the first one or two terms of this expansion, viz. $X_t = \zeta_t V_t^{(1)}$ or $X_t = \zeta_t\left(1 + V_t^{(1)}\right)$, where $V_t^{(1)}$ is a random coefficient AR(1) process as in Theorem 1(ii), the limit distribution of partial sums of the squared aggregate is a Gaussian or the Rosenblatt process (see Robinson and Zaffaroni, 1998; Surgailis, 2003).

4. PROOFS OF THEOREMS 1–3

Proof of Theorem 1. (i) Recall (14) and (15). From (7) and (8) we obtain a similar convergent Volterra expansion of the aggregated process in (10):

$$\bar{X}_{N,t} = \sum_{k=0}^{\infty} \bar{X}_{N,t}^{(k)}, \qquad \bar{V}_{N,t} = \sum_{k=0}^{\infty} \bar{V}_{N,t}^{(k)}, \tag{31}$$

$$\bar{X}_{N,t}^{(k)} = \zeta_t \bar{V}_{N,t}^{(k)}, \qquad \bar{V}_{N,t}^{(k)} = \sum_{s_k < \cdots < s_1 < t} \bar{g}_{N,t-s_k}^{(k)} \zeta_{s_1} \cdots \zeta_{s_k}, \tag{32}$$

where

$$\bar{g}_N^{(0)} = N^{-1}\sum_{i=1}^{N} a_i', \qquad \bar{g}_{N,t}^{(k)} = c^k N^{-1}\sum_{i=1}^{N} a_i'(1-\beta_i^2)^{k/2}\beta_i^t.$$

By the law of large numbers, for any $k, t \geq 1$,

$$\bar{g}_N^{(0)} \to \mathrm{E}a' = g^{(0)}, \qquad \bar{g}_{N,t}^{(k)} \to c^k \mathrm{E}\left[a'(1-\beta^2)^{k/2}\beta^t\right] = g_t^{(k)} \tag{33}$$

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a.s. as $N \to \infty$, so that, formally, all quantities in (31) and (32) converge to the corresponding quantities in (14) and (15) as $N \to \infty$. Thus in order to prove (12), it suffices to show that the above-mentioned convergences take place in mean square.

To that end, write

$$\mathrm{E}(\bar{V}_t - \bar{V}_{N,t})^2 = \mathrm{E}(\bar{g}_N^{(0)} - g^{(0)})^2 + \sum_{k=1}^{\infty} \sum_{s_k < \ldots < s_1 < t} \mathrm{E}(\bar{g}_{N,t-s_k}^{(k)} - g_{t-s_k}^{(k)})^2.$$

By independence of $(a_i', b_i), i \geq 1$, we have that

$$\begin{aligned}\mathrm{E}(\bar{g}_{N,t-s_k}^{(k)} - g_{t-s_k}^{(k)})^2 &= N^{-1} c^{2k} \mathrm{var}\big(a'(1-\beta^2)^{k/2}\beta^{t-s_k}\big) \\ &\leq N^{-1} c^{2k} \mathrm{E}\big[(a')^2 (1-\beta^2)^k \beta^{2(t-s_k)}\big]\end{aligned}$$

and therefore, as in (9),

$$\begin{aligned}&\mathrm{E}(\bar{V}_t - \bar{V}_{N,t})^2 \\ &\quad \leq N^{-1} \mathrm{E} \sum_{k=0}^{\infty} \sum_{s_k < \cdots < s_1 < t} c^{2k} (a')^2 (1-\beta^2)^k \beta^{2(t-s_k)} = N^{-1} \mathrm{E} \frac{(a')^2}{1-c^2\beta^2} \to 0\end{aligned}$$

and the convergence in mean square follows. The above computation obviously entails all four relations in (12) as well as the convergence of all Volterra series in (14)–(15). Strict stationarity of these series stem from the fact that the innovations and the coefficients of the Volterra series are invariant with respect to time shifts. Ergodicity of these processes follows from Stout (1974, Thm. 3.5.8).

(ii) According to the definitions (16) and (17),

$$\begin{aligned}g_t^{(k)} &= \frac{c^k}{B(p,q)} \int_0^1 (1-x^2)^{k/2} x^{t+p-1} (1-x)^{q-1} dx \\ &= \frac{c^k \Gamma(p+t)\Gamma\left(\frac{k}{2}+q\right)}{B(p,q)\left(p+t+\frac{k}{2}+q\right)} F\left(-\frac{k}{2}, p+t; p+t+\frac{k}{2}+q; -1\right), \qquad \textbf{(34)}\end{aligned}$$

where F is the hypergeometric function. Relation (18) follows from asymptotics of the hypergeometric function (see Abramowitz and Stegun, 1972, (15.7.1)).

Let us prove (19). From (18) one can easily deduce the bound

$$\begin{aligned}g_t^{(k)} &\leq 2^{k/2} c^k \mathrm{E}\big[(1-\beta)^{k/2}\beta^t\big] \\ &= \frac{2^{k/2}c^k}{B(p,q)} \int_0^1 (1-x)^{(k/2)+q-1} x^{t+p-1} dx \\ &\leq \frac{2^{k/2}c^k}{B(p,q)} \int_0^1 y^{(k/2)+q-1} \mathrm{e}^{-(t+p-1)y} dy \\ &\leq C 2^{k/2} c^k \Gamma\left(\frac{k}{2}+q\right) t^{-(k/2)-q}, \qquad \textbf{(35)}\end{aligned}$$

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for any $1 \le k \le t, t = 2, 3, \ldots$, where $C = 1/B(p,q)$ does not depend on k, t. We have

$$\operatorname{cov}(\bar{V}_t, \bar{V}_0) = \sum_{k=1}^{\infty} \sum_{s_k < \ldots < s_1 < 0} g_{t-s_k}^{(k)} g_{-s_k}^{(k)} = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} g_{t+s}^{(k)} g_s^{(k)} p_s^{(k)}, \tag{36}$$

where

$$p_s^{(k)} := \sum_{0 < u_1 < \ldots < u_{k-1} < s} 1 = \frac{(s-1)\cdots(s-k+1)}{(k-1)!} \sim \frac{s^{k-1}}{(k-1)!} \qquad (s \to \infty) \tag{37}$$

and

$$p_s^{(k)} \le s^{k-1}/(k-1)! \quad (\forall k, s \ge 1). \tag{38}$$

Asymptotic relations in (18) and (37) together with dominating bounds in (35) and (38) enable the use of the Lebesgue dominated convergence theorem in (36), resulting in

$$\begin{aligned} \operatorname{cov}(\bar{V}_t, \bar{V}_0) &\sim \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left(\frac{c^k 2^{k/2} \Gamma\left(\frac{k}{2} + q\right)}{B(p,q)} \right)^2 \sum_{s=1}^{\infty} \frac{s^{k-1}}{(t+s)^{k/2+q} s^{k/2+q}} \\ &\sim \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left(\frac{c^k 2^{k/2} \Gamma\left(\frac{k}{2} + q\right)}{B(p,q)} \right)^2 \int_0^{\infty} \frac{s^{k-1} ds}{(t+s)^{(k/2)+q} s^{(k/2)+q}} \\ &= c_{\bar{V}}^2 t^{-2q} \qquad (t \to \infty), \end{aligned} \tag{39}$$

see the constant $c_{\bar{V}}^2$ in (29). Since $\operatorname{cov}(\bar{V}_t, \bar{V}_0) = \sum_{k=1}^{\infty} \operatorname{cov}(\bar{V}_t^{(k)}, \bar{V}_0^{(k)})$, the computation leading to (39) also entails the proof of the second relation in (19). Theorem 1 is proved. ■

Proof of Theorem 2. Let us first rewrite the limiting process as an infinite series of multiple Itô–Wiener integrals. By definition,

$$Y_{H_{k-1}}(t) = \int_{-\infty}^{t} \left\{ \int_{0 \vee s}^{t} H_{k-1}\left(\frac{W(\tau) - W(s)}{\sqrt{\tau - s}} \right) \frac{d\tau}{(\tau - s)^{(1/2)+q}} \right\} dW(s).$$

By using

$$H_{k-1}\left(\frac{W(\tau) - W(s)}{\sqrt{\tau - s}} \right) = \frac{(k-1)!}{(\tau - s)^{(k-1)/2}} \int_{\{s < u_{k-1} < \ldots < u_1 < \tau\}} dW(u_{k-1}) \ldots dW(u_1),$$

(see, e.g., Kwapień and Woyczyński, 1992, (10.3.3), Thms. 10.3.1 and 10.3.2), we obtain the expression for $J_{k-1} Y_{H_{k-1}}(t)$ as a k-tuple Itô–Wiener integral:

$$J_{k-1} Y_{H_{k-1}}(t) = \int_{\mathbb{R}^k} f_k(t; u_1, \ldots, u_k) \, dW(u_k) \ldots dW(u_1),$$

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with the kernel f_k given by

$$f_k(t;u_1,\dots,u_k)=\frac{2^{k/2}c^k\Gamma\left(\frac{k}{2}+q\right)}{B(p,q)}\int_{u_1\vee 0}^{t}\frac{d\tau}{(\tau-u_k)^{(k/2)+q}}, \tag{40}$$

for $u_k<\cdots<u_1<t$, $f_k(t;u_1,\dots,u_k)=0$ elsewhere.

Next, we rewrite in a similar way the left-hand sides of (25)–(26). We have

$$\mathcal{Y}_n(t)=n^{q-1}\sum_{\tau=1}^{[nt]}(\bar V_\tau-\mathrm{E}\bar V_\tau)=\sum_{k=1}^{\infty}\mathcal{Y}_n^{(k)}(t), \tag{41}$$

where

$$\mathcal{Y}_n^{(k)}(t):=n^{q-1}\sum_{-\infty<s_k<\dots<s_1<\infty}\left\{\sum_{\tau=1\vee(s_1+1)}^{[nt]}g_{\tau-s_k}^{(k)}\right\}\zeta_{s_1}\cdots\zeta_{s_k}. \tag{42}$$

Clearly, the statement of the theorem follows from the following two facts: For any finite $K<\infty$,

$$\left(\mathcal{Y}_n^{(1)}(t),\dots,\mathcal{Y}_n^{(K)}(t)\right)\Rightarrow(J_0Y_{H_0}(t),\dots,J_{K-1}Y_{H_{K-1}}(t)) \tag{43}$$

and

$$\mathrm{E}\left(\sum_{k>K}\mathcal{Y}_n^{(k)}(t)\right)^2+\mathrm{E}\left(\sum_{k>K}J_{k-1}Y_{H_{k-1}}(t)\right)^2\le\delta_K, \tag{44}$$

where δ_K does not depend on n and vanishes as $K\to\infty$.

To prove (43) we use the so-called scheme of discrete multiple integrals (see Surgailis, 1982, 2003). To simplify the notation, we restrict below to the proof of "one-dimensional" convergence

$$\mathcal{Y}_n^{(k)}\Rightarrow J_{k-1}Y_{H_{k-1}}, \tag{45}$$

where $k\ge 1$ is fixed and $\mathcal{Y}_n^{(k)}=\mathcal{Y}_n^{(k)}(1)$, $Y_{H_{k-1}}=Y_{H_{k-1}}(1)$, since the general case in (43) can be treated by a similar argument.

To that end, introduce a discrete stochastic measure $Z_n(du)$ with independent increments defined on finite intervals $(u',u'']$, $u'<u''$ by

$$Z_n(u',u'']:=n^{-1/2}\sum_{u'<s/n\le u''}\zeta_s.$$

A discrete multiple integral $\int_{\mathbb{R}^k}h(u_1,\dots,u_k)Z_n(du_1)\dots Z_n(du_k)\equiv\int h\,d^kZ_n$ is defined for *simple* functions h taking finite number of constant values on "cubes" $(i_1/n,(i_1+1)/n]\times\cdots\times(i_k/n,(i_k+1)/n]\subset\mathbb{R}^k$, $i_p\in\mathbb{Z}$, $p=1,\dots,k$, and has

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similar properties as the usual Itô–Wiener integral (see Surgailis, 2003). According to Surgailis (2003, Prop. 3.2), if $(h_n)_{n\geq 1}$ is a sequence of simple functions convergent to $h \in L^2(\mathbb{R}^k)$ in $L^2(\mathbb{R}^k)$, then

$$\int h_n \, d^k Z_n \Rightarrow \int h \, d^k W \equiv \int_{\mathbb{R}^k} h(u_1, \ldots, u_k) \, dW(u_1) \ldots dW(u_k), \tag{46}$$

where on the right-hand side is the k-tuple Itô–Wiener integral of h. Note $\mathcal{Y}_n^{(k)} = \mathcal{Y}_n^{(k)}(1)$ in (42) and (45) can be written as a discrete multiple integral:

$$\mathcal{Y}_n^{(k)} = \int h_n \, d^k Z_n,$$

where

$$h_n(u_1, \ldots, u_k) = n^{(k/2)+q-1} \sum_{t=(s_1+1)\vee 1}^{n} g_{t-s_k}^{(k)} \tag{47}$$

for

$$(u_1, \ldots, u_k) \in \left(\frac{s_1}{n}, \frac{s_1+1}{n} \right] \times \cdots \times \left(\frac{s_k}{n}, \frac{s_k+1}{n}, \right], \qquad -\infty < s_k < \cdots < s_1 < n \tag{48}$$

and integers $s_1, \ldots, s_k \in \mathbb{Z}$, $h_n(u_1, \ldots, u_k) = 0$ elsewhere. Then, since h_n is simple, relation (45) follows from (46) and

$$\lim_{n\to\infty} \|h_n - h\|_{L^2} = 0, \tag{49}$$

where $h(u_1, \ldots, u_k) = f^{(k)}(1; u_1, \ldots, u_k)$ is defined in (40) and $\|\cdot\|_{L^2}$ is the norm in $L^2(\mathbb{R}^k)$.

To show (49), one needs to verify the two following steps, namely to

(i) show the pointwise convergence $h_n(u_1, \ldots, u_k) \to h(u_1, \ldots, u_k)$ for a.e. $(u_1, \ldots, u_k) \in \mathbb{R}^k$; and

(ii) show a dominating bound $|h_n(u_1, \ldots, u_k)| + |h(u_1, \ldots, u_k)| \leq \bar{h}(u_1, \ldots, u_k)$, for a.e. $(u_1, \ldots, u_k) \in \mathbb{R}^k$ and any $n \geq n_0$ large enough, with $\bar{h} \in L^2(\mathbb{R}^k)$ independent of n.

Step (i) follows by rewriting h_n of (47)–(48) in the form

$$h_n(u_1, \ldots, u_k) = \frac{C_k}{n} \sum_{\left(\frac{s_1}{n}\vee 0\right) < \frac{t}{n} \leq 1} \frac{\Delta_k(t, s_k)}{\left(\frac{t}{n} - \frac{s_k}{n}\right)^{(k/2)+q}} \tag{50}$$

with the arguments in (50) defined as in (48), where

$$C_k := 2^{k/2} c^k \Gamma\left(\frac{k}{2} + q\right) / B(p, q), \tag{51}$$

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and where

$$\Delta_k(t,s_k)=\frac{g^{(k)}_{t-s_k}}{C_k(t-s_k)^{-(k/2)-q}}\to 1 \qquad (t-s_k\to\infty), \tag{52}$$

see (18). The right-hand side of (52) can be easily seen to converge to the right-hand side of (40), thus proving step (i). Step (ii) follows from the bound in (35). Indeed, according to this bound, $|\Delta_k(t,s_k)|\le 2$ for any $t>s_k, k\ge 1$, so that $|h_n(u_1,\dots,u_k)|\le 2h(u_1,\dots,u_k)\equiv \bar h(u_1,\dots,u_k)$, where, for $k\ge 2$,

$$\|\bar h\|^2_{L^2}\le 4C_k^2\int_{\{-\infty<u_k<\dots<u_1<1\}}\left(\int_{u_1\vee 0}^1\frac{d\tau}{(\tau-u_k)^{(k/2)+q}}\right)^2 du_1\dots du_k<\infty.$$

This proves $\bar h\in L^2(\mathbb{R}^k)$ for $k\ge 2$; for $k=1$, the proof is simpler and we omit it. The above argument also proves relations (49), (45), and (43).

It remains to prove (44). We shall verify this bound for the first sum on the right-hand side in (44) only since the second sum can be estimated in a similar way. We have

$$\begin{aligned}
&\mathrm{E}\left(\sum_{k>K}\mathcal{Y}^{(k)}_n(t)\right)^2\\
&\quad=n^{2q-2}\sum_{k>K}\sum_{t,t'=1}^n\sum_{s_k<\dots<s_1<t\wedge t'}g^{(k)}_{t-s_k}g^{(k)}_{t'-s_k}\\
&\quad\le Cn^{2q-1}\sum_{k>K}\sum_{t=1}^n\sum_{s=1}^\infty g^{(k)}_s g^{(k)}_{t+s}p^{(k)}_s\\
&\quad\le Cn^{2q-1}\left(\sum_{t=1}^n t^{-2q}\right)\sum_{k>K}\frac{1}{(k-1)!}\left(c^k2^{k/2}\Gamma\left(\frac{k}{2}+q\right)\right)^2 B\left(2q,\frac{k}{2}-q\right)
\end{aligned}$$

exactly as in the proof of (39), where the last sum is independent of n. It remains to show that this sum, denoted by Σ_K, converges. Since $B\left(2q,\frac{k}{2}+q\right)=\Gamma(2q)\Gamma\left(\frac{k}{2}-q\right)/\Gamma\left(\frac{k}{2}+q\right)$ and using Stirling's formula, we obtain

$$\begin{aligned}
\Sigma_K&=\Gamma(2q)\sum_{k>K}\frac{1}{(k-1)!}c^{2k}2^k\Gamma\left(\frac{k}{2}+q\right)\Gamma\left(\frac{k}{2}-q\right)\\
&\le C\sum_{k>K}\mathrm{e}^{-k+(1/2)}c^{2k}2^k\mathrm{e}^{-k}\left(\frac{k}{2}+q\right)^{(k/2)+q-1/2}\left(\frac{k}{2}-q\right)^{(k/2)-q-1/2}\\
&\le C\sum_{k>K}k^{-k+(1/2)}c^{2k}2^k(k/2)^k\left(\frac{k}{2}+q\right)^{q-1/2}\left(\frac{k}{2}-q\right)^{-q-1/2}\\
&\le C\sum_{k>K}k^{-1/2}c^{2k}\to 0\quad(K\to\infty),
\end{aligned}$$

since $|c|<1$. This proves (44) and Theorem 2 as well. ∎

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Proof of Theorem 3. Denote $U_t = \bar{V}_t - 1 = \sum_{k=1}^{\infty} \bar{V}_t^{(k)}$. Note $\mathrm{E}U_t = 0$ and $\mathrm{cov}(U_t, U_s) = \mathrm{cov}(\bar{V}_t, \bar{V}_s)$. Then

$$\bar{V}_t^2 = 1 + 2U_t + U_t^2. \tag{53}$$

The proof of the theorem easily follows from decomposition (53), Theorems 1 and 2, (19), and Lemma 1 below, whose proof is postponed to Section 5. ■

LEMMA 1. *Under the assumptions of Theorem 2, there exists $\delta > 0$ such that, as $t \to \infty$,*

$$\mathrm{cov}\big(U_t^2, U_0^2\big) = O(t^{-2q-\delta}), \tag{54}$$

$$\mathrm{cov}\big(U_t, U_0^2\big) = O(t^{-2q-\delta}), \quad \text{and} \tag{55}$$

$$\mathrm{cov}\big(U_t^2, U_0\big) = O(t^{-2q-\delta}). \tag{56}$$

5. PROOF OF LEMMA 1

First we shall prove some auxiliary bounds. Recall the definition $b_j = c\sqrt{1-\beta^2}\,\beta^j\,(j = 1, 2, \ldots), b_j = 0 (j = 0, -1, \ldots)$ in (5). For ordered subset $S = \{s_k, \ldots, s_1\} \subset \mathbb{Z}, s_k < \cdots < s_1, k = 1, 2, \ldots$ denote

$$b_t^S = b_{t-s_1} \cdots b_{s_{k-1}-s_k}, \qquad \zeta^S = \zeta_{s_1} \cdots \zeta_{s_k}. \tag{57}$$

LEMMA 2. *There exists a constant $C > 0$ such that for any integers $\tau, t, k \geq 1$, the following estimates hold:*

$$\sum_{|S|=k:0<\wedge(S),\vee(S)<\tau} \mathrm{E}\big[b_{\tau+t}^S\big]\mathrm{E}\big[b_\tau^{S\cup\{0\}}\big] \leq Cc^{2k}k^{2q} \begin{cases} \tau^{-q}t^{-.5-q}, & 0 < \tau \leq t, \\ \tau^{-.5-2q}, & \tau > t \end{cases} \tag{58}$$

and

$$\sum_{|S|=k:0<\wedge(S),\vee(S)<\tau} \mathrm{E}\big[b_{\tau+t}^{S\cup\{0\}}\big]\mathrm{E}\big[b_\tau^S\big] \leq Cc^{2k}k^{2q}(\tau+t)^{-1-q}\tau^{.5-q}. \tag{59}$$

In (58)–(59)*, the sums are taken over all ordered subsets $S = \{s_k, \ldots, s_1\} \subset \mathbb{Z}, s_k < \cdots < s_1, 0 < s_k = \wedge(S) < \vee(S) = s_1 < t$.*

Proof. Denote Σ_1, Σ_2 the left-hand sides of (58) and (59), respectively. Then (with $u = \wedge(S)$) and using the Stirling formula,

$$\Sigma_1 = \sum_{0<u<\tau} \sum_{u<s_{k-1}<\ldots<s_1<\tau} g_{k,\tau+t-u}\, g_{k+1,\tau}$$

$$= \sum_{0<u<\tau} \sum_{u<s_{k-1}<\ldots<s_1<\tau} c^{2k}\mathrm{E}\big[(1-\beta^2)^{k/2}\beta^{\tau+t-u}\big]\mathrm{E}\big[(1-\beta^2)^{(k+1)/2}\beta^\tau\big]$$

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$$\le C\frac{2^{k+(1/2)}c^{2k+1}\Gamma\left(\frac{k}{2}+q\right)\Gamma\left(\frac{k+1}{2}+q\right)}{(k-1)!}\frac{1}{\tau^{\frac{k+1}{2}+q}}\sum_{0<u<\tau}\frac{(\tau-u)^{k-1}}{(\tau+t-u)^{\frac{k}{2}+q}}$$

$$\le Cc^{2k}k^{2q}\frac{1}{\tau^{1+q}}\sum_{0<u<\tau}\frac{1}{(\tau+t-u)^{.5+q}},$$

where the last sum can be easily estimated to yield the bound in (58). In a similar way,

$$\Sigma_2=\sum_{0<u<\tau}\sum_{u<s_{k-1}<\cdots<s_1<\tau}g_{k+1,\tau+t}g_{k,\tau-u}$$

$$=\sum_{0<u<\tau}\sum_{u<s_{k-1}<\cdots<s_1<\tau}c^{2k}\mathrm{E}\left[(1-\beta^2)^{(k+1)/2}\beta^{\tau+t}\right]\mathrm{E}\left[(1-\beta^2)^{k/2}\beta^{\tau-u}\right]$$

$$\le C\frac{2^{k+(1/2)}c^{2k+1}\Gamma\left(\frac{k}{2}+q\right)\Gamma\left(\frac{k+1}{2}+q\right)}{(k-1)!}\frac{1}{(\tau+t)^{\frac{k+1}{2}+q}}\sum_{0<u<\tau}\frac{(\tau-u)^{k-1}}{(\tau-u)^{\frac{k}{2}+q}}$$

$$\le Cc^{2k}k^{2q}\frac{1}{(\tau+t)^{1+q}}\sum_{0<u<\tau}\frac{1}{(\tau-u)^{.5+q}},$$

proving (59) and the lemma, too. ■

Next we describe a diagram formalism for computation of (joint) moments up to order 4 of Volterra series (see Giraitis, Robinson, and Surgailis, 2000, for more details). Let $(k)_4=(k_1,\dots,k_4)\in\mathbb{Z}_+^4$ be a collection of integers; $\mathbb{Z}_+=\{s\in\mathbb{Z}:s\ge 1\}$. Let $I=I(k)_4$ be a table having 4 rows $I_j=\{(k_j,j),\dots,(1,j)\}$ of length k_j, $j=1,\dots,4$. A *diagram* is an ordered partition $\gamma=(G_1,\dots,G_r)$ of the table I by subsets (*edges*) G_q, $q=1,\dots,r$, $r\ge 1$, having at least 2 elements and containing at most one element of any row, i.e., such that $|G_q|\ge 2$, $|G_q\cap I_j|\le 1$, $q=1,\dots,r$, $j=1,\dots,4$. The class of all diagrams $\gamma=(G_1,\dots,G_r)$ over $I(k)_4$ is denoted by $\Gamma_{I(k)_4}$.

Let $S_j\subset\mathbb{Z}$, $|S_j|=k_j$ be a collection of ordered integers:

$$S_j=\{s_{k_j,j},\dots,s_{1,j}\},\qquad s_{k_j,j}<\cdots<s_{1,j},\qquad j=1,\dots,4. \tag{60}$$

Put $(S)_4:=(S_1,\dots,S_4)$. Let $f=f((S)_4)=f(s_{i,j}:(i,j)\in I)$ be a function defined on the set of all such collections $(S)_4$. With any such f and any diagram $\gamma=(G_1,\dots,G_r)\in\Gamma_{I(k)_4}$, we associate the sum

$$\sum_{(S)_4\sim\gamma}f((S)_4):=\sum_{s_{i,j}=\tilde{s}_q,(i,j)\in G_q,q=1,\dots,r}f(s_{i,j}:(i,j)\in I) \tag{61}$$

over integers $s_{i,j}$, $i=1,\dots,k_j$, $j=1,\dots,4$ satisfying the inequalities in (60) and such that

$$\tilde{s}_1<\cdots<\tilde{s}_r. \tag{62}$$

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In other words, a diagram γ determines the type of the "diagonal," or the order of the coinciding elements $s_{i,j}$ of sets $S_1, \dots, S_4$. We have

$$\mathrm{E}[U_{t_1} U_{t_2}] = \sum_{S} \mathrm{E}\big[b^{S}_{t_1,1} b^{S}_{t_2,2}\big], \tag{63}$$

$$\mathrm{cov}\big(U_t^2, U_0^2\big) = \sum_{(k)_4} \sum_{\gamma \in \Gamma_{I(k)_4}} \bar{\mu}_\gamma \sum_{(S)_4 \sim \gamma} \mathrm{E}\big[b^{S_1}_{t,1}\, b^{S_2}_{t,2}\, b^{S_3}_{0,3}\, b^{S_4}_{0,4}\big], \tag{64}$$

where $\bar{\mu}_\gamma = \mathrm{cov}(\zeta^{S_1}\zeta^{S_2}, \zeta^{S_3}\zeta^{S_4})$ for $(S)_4 = (S_1, \dots, S_4)$ satisfying (60) and (62) depends only on γ and $\bar{\mu}_\gamma$ vanishes unless γ is block-connected. Following Giraitis, Robinson, and Surgailis (2000, p. 1013), we call $\gamma = (G_1, \dots, G_r) \in \Gamma_{I(k)_4}$ *block-connected* if it contains an edge $G_q (1 \le q \le r)$ which has a nonempty intersection with the blocks $I' = I_1 \cup I_2$ and $I'' = I_3 \cup I_4$.

Proof of Lemma 1. The proof below refers to (54) only, since the other two bounds, (55) and (56), are simpler and can be proved analogously. Using the representation (64) of the covariance, it suffices to show that for any $\delta > 0$ and any $A > 1$, there exists $C > 0$ such that for any $k_1, \dots, k_4 \ge 1$ and any block-connected diagram $\gamma = (G_1, \dots, G_r) \in \Gamma_{I(k)_4}$ and any $t \ge 1$, the following inequality holds:

$$\sum_{(S)_4 \sim \gamma} \mathrm{E}\big[b^{S_1}_{t,1}\, b^{S_2}_{t,2}\, b^{S_3}_{0,3}\, b^{S_4}_{0,4}\big] \le C(Ac)^k t^{-2q-\delta}, \tag{65}$$

where $k = k_1 + \cdots + k_4$. Indeed, if (65) is true then, by (64) and using the facts that $|\Gamma_{I(k)_4}| \le 11^{k/2}, |\bar{\mu}_\gamma| \le \mu_4^{k/4}$ (see Giraitis, Robinson, and Surgailis, 2000, Lems. 3.1 and 3.2 for details), we obtain

$$\begin{aligned} |\mathrm{cov}(U_t^2, U_0^2)| &\le C t^{-2q-\delta} \sum_{(k)_4} (Ac)^k \mu_4^{k/4} \sum_{\gamma \in \Gamma_{I(k)_4}} 1 \\ &\le C t^{-2q-\delta} \left(\sum_{k=1}^{\infty} (Ac\sqrt{11}\mu_4^{1/4})^k \right)^4 = O(t^{-2q-\delta}), \end{aligned}$$

where the last sum converges according to condition (27), by choosing $A > 1$ sufficiently close to 1.

Again, for notational simplicity, we restrict the proof of (65) to (block-connected) diagrams $\gamma = (G_1, \dots, G_r)$ whose edges G_i have exactly two points: $|G_i| = 2, i = 1, \dots, r$. Given a subset $J \subset \{1, 2, 3, 4\}$, $\mathcal{G}_J$ stands for the class of edges G that connect rows $I_j, j \in J$, in other words, such that $|G \cap I_j| = 1$ $(j \in J)$, $|G \cap I_j| = 0$ $(j \notin J)$. Thus, any edge G with $|G| = 2$ necessarily belongs to one of the disjoint classes $\mathcal{G}_{12}, \mathcal{G}_{13}, \mathcal{G}_{14}, \mathcal{G}_{23}, \mathcal{G}_{24}, \mathcal{G}_{34}$. Two edges G' and G'' *are of the same type* if they belong to the same class (from the above list of six classes).

Recall from (62) that edges G_i of a diagram $\gamma = (G_1, \dots, G_r)$ are *ordered from the left*. Let us examine the type of edges starting from G_r, i.e., the utmost

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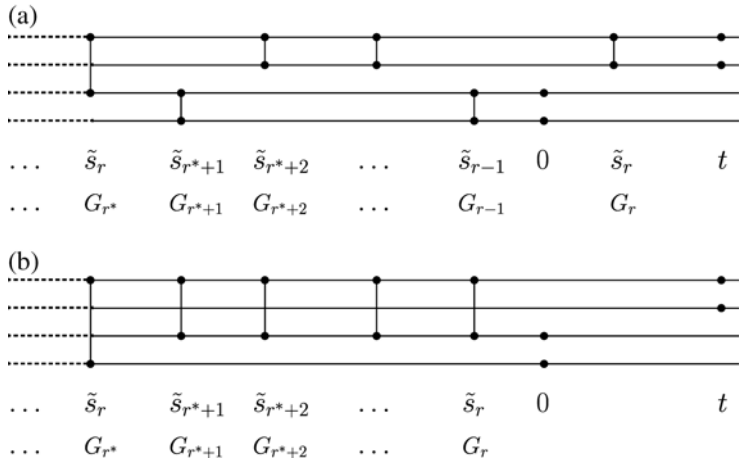


FIGURE 1. (a) Case 1 diagram; (b) Case 2 diagram.

right edge of γ. Also recall that our diagram γ is block-connected, meaning that it necessarily contains an edge that intersects both blocks $I' = I_1 \cup I_2$ and $I'' = I_3 \cup I_4$. There are two cases (see Figure 1).

Case 1. Starting from the right, edges $G_r, G_{r-1}, \ldots, G_{r_*+1}$ connect only the rows inside each block, i.e., $G_i \in \mathcal{G}_{12} \cup \mathcal{G}_{34}, i = r, r-1, \ldots, r_*$ for some $1 \le r_* < r$, and $G_{r_*} \in \mathcal{G}_{13} \cup \mathcal{G}_{14} \cup \mathcal{G}_{23} \cup \mathcal{G}_{24}$ (i.e., G_{r_*} connects the blocks I' and I'').

Case 2 contains all other cases.

Case 2. The utmost right edge G_r connects I' and I'', i.e., $G_r \in \mathcal{G}_{13} \cup \mathcal{G}_{14} \cup \mathcal{G}_{23} \cup \mathcal{G}_{24}$, edges $G_{r-1}, \ldots, G_{r_*+1}$ are of the same type as G_r, followed by G_{r_*} which is of a different type. (Note that such $G_{r_*}, 1 \le r_* < r$ necessarily exists.)

The proof of (65) in Case 1 is simpler and is postponed to the end of the proof of Lemma 1. Consider Case 2. Assume for concreteness that $G_i \in \mathcal{G}_{13}, i = r, r-1, \ldots, r_*+1$.

There are five possibilities for G_{r_*}:

$$G_{r_*} \in \mathcal{G}_{12}, \qquad G_{r_*} \in \mathcal{G}_{14}, \qquad G_{r_*} \in \mathcal{G}_{34}, \qquad G_{r_*} \in \mathcal{G}_{23}, \qquad G_{r_*} \in \mathcal{G}_{24}. \quad \textbf{(66)}$$

Actually, we need to examine only three possibilities from this list, since the remaining two follow by symmetry.

As in Giraitis, Robinson, and Surgailis (2000, (3.17)), we can apply the Cauchy-Schwarz inequality to the summation over (common points) $\tilde{s}_1 < \cdots < \tilde{s}_{r_*-1}$ of subsets $S_1, \ldots, S_4$ (see (60), (62)) in (65), and therefore reduce the proof of (65) to the case $r_* = 1$. Consider the following subcases of Case 2.

Case 2.1. $G_1 (= G_{r_*}) \in \mathcal{G}_{12}, G_i \in \mathcal{G}_{13} (i = 2, \ldots, r)$. (Actually, this means that we deal with a diagram over a table having only three rows instead of four, and

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the summation in (65) reduces to triplets $(S_1, S_2, S_2) \subset \mathbb{Z}^3$.) Using Lemma 2 (59) with $\tau = -u$, we obtain

$$\begin{aligned}\sum_{(S)_3 \sim \gamma} \mathrm{E}\big[b_{t,1}^{S_1} b_{t,2}^{S_2} b_{0,3}^{S_3}\big] &= \sum_{u<0} \mathrm{E}[b_{t-u}] \sum_{|S|=k-2: u<s_{k-2}<\cdots<s_1<0} \mathrm{E}\big[b_t^{S\cup\{u\}}\big]\mathrm{E}\big[b_0^S\big] \\ &\le Cc^{k-1}k^{2q} \sum_{u<0} (t-u)^{-.5-q}(t-u)^{-1-q}(-u)^{.5-q} \\ &\le C(Ac)^k \sum_{u<0} (t-u)^{-1-q}(-u)^{-2q} \\ &\le C(Ac)^k t^{-3q}, \end{aligned} \tag{67}$$

with some $A > 1$ arbitrarily close to 1 and some $C > 0$ independent of t and k.

Case 2.2. $G_1 (= G_{r_*}) \in \mathcal{G}_{14}, G_i \in \mathcal{G}_{13} (i = 2, \ldots, r)$. This case is symmetric to Case 2.1 and we omit the proof.

Case 2.3. $G_1 (= G_{r_*}) \in \mathcal{G}_{34}, G_i \in \mathcal{G}_{13} (i = 2, \ldots, r)$. Again, in this case we deal with a diagram over a table having only three rows. Using Lemma 2 (58) with $\tau = -u$, we obtain, with $(S)_3 = (S_1, S_3, S_4)$,

$$\begin{aligned}\sum_{(S)_3 \sim \gamma} \mathrm{E}\big[b_{t,1}^{S_1} b_{0,3}^{S_3} b_{0,4}^{S_4}\big] &= \sum_{u<0} \mathrm{E}[b_{-u}] \sum_{|S|=k-2: u<s_{k-2}<\cdots<s_1<0} \mathrm{E}\big[b_t^S\big]\mathrm{E}\big[b_0^{S\cup\{u\}}\big] \\ &\le Cc^{k-1}k^{2q} \sum_{u<0} (-u)^{-.5-q}(-u)^{-q}(t \vee (-u))^{-.5-q} \\ &\le C(Ac)^k \left\{ t^{-.5-q} \sum_{0<v\le t} v^{-.5-2q} + \sum_{v>t} v^{-1-3q} \right\} \\ &\le C(Ac)^k t^{-3q}, \end{aligned} \tag{68}$$

with some $A > 1$ arbitrarily close to 1 and some $C > 0$ independent of t and k.

Case 2.4. $G_1 (= G_{r_*}) \in \mathcal{G}_{23}, G_i \in \mathcal{G}_{13} (i = 2, \ldots, r)$. This case is symmetric to Case 2.3 and so the proof is omitted.

Case 2.5. $G_1 (= G_{r_*}) \in \mathcal{G}_{24}, G_i \in \mathcal{G}_{13} (i = 2, \ldots, r)$. (This is the only case among Cases 2.1–2.5 where the diagram $\gamma^* = (G_{r_*}, \ldots, G_r)$ has four rows.) But this case easily follows from the covariance bounds in Section 4, as the left-hand side of (65) reduces to a product of two covariances of the type (63), each of which has the order $O(t^{-2q})$. Indeed, using the bound in (39), we obtain

$$\begin{aligned}&\sum_{(S)_4 \sim \gamma} \mathrm{E}\big[b_{t,1}^{S_1} b_{t,2}^{S_2} b_{0,3}^{S_3} b_{0,4}^{S_4}\big] \\ &\quad\le \left(\sum_{|S|=(k-2)/2: \vee(S)<0} \mathrm{E}\big[b_{t,1}^S b_{0,3}^S\big] \right) \left(\sum_{|S|=1: \vee(S)<0} \mathrm{E}\big[b_{t,2}^S b_{0,4}^S\big] \right) \le C(Ac)^k t^{-4q} \end{aligned} \tag{69}$$

with some $A > 1$ arbitrarily close to 1 and some $C > 0$ independent of t and k.

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Relations (67)–(69) prove (65) in Case 2, with $\delta = q > 0$. It remains Case 1. Again, using the Cauchy-Schwarz inequality, we reduce the proof to the case $r_* = 1$. Assume for concreteness $G_1 (= G_{r_*}) \in \mathcal{G}_{13}$ (see Figure 1(a)). We have the following subcases: $G_i \in \mathcal{G}_{12} (1 < i \le r)$ (Case 1.1), $G_i \in \mathcal{G}_{34} (1 < i \le r)$ (Case 1.2), and $G_i \in \mathcal{G}_{12}, G_j \in \mathcal{G}_{34}$ for some $1 < i, j \le r, i \ne j, G_i \in \mathcal{G}_{12} \cup \mathcal{G}_{34} (1 < i \le r)$ (Case 1.3). Case 1.3 reduces to a product of two covariances as in Case 2.5 and we omit it. Cases 1.1 and 1.2 are similar. Consider Case 1.2. The corresponding diagram has only three rows and, using (18) (with $k = 1$) and (59) (with $t = 0$), we obtain, with $(S)_3 = (S_1, S_3, S_4)$,

$$\begin{aligned}\sum_{(S)_3 \sim \gamma} \mathrm{E}\big[b_{t,1}^{S_1} b_{0,3}^{S_3} b_{0,4}^{S_4}\big] &\le \sum_{u<0} \mathrm{E}[b_{t-u}] \sum_{|S|=(k-2)/2 : u<\wedge(S)<\vee(S)<0} \mathrm{E}\big[b_0^{S\cup\{u\}}\big] \mathrm{E}\big[b_0^{S}\big] \\ &\le C c^{2k} k^{2q} \sum_{u<0} (t-u)^{-q-.5} (-u)^{-.5-2q} \\ &\le C (Ac)^k t^{-3q}\end{aligned}$$

with some $A > 1$ arbitrarily close to 1 and some $C > 0$ independent of t and k.

This concludes the proof of (65) and Lemma 1. ■

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