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Application of Δ and E operators to evaluate certain integrals

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1. *Introduction.* In a recent paper Ragab and Simary (6) have deduced some integrals involving the products of generalized Whittaker functions. We feel that some of the integrals are not correctly evaluated. Indeed the integral (2) should be symmetrical in α and β, as we see by changing the variables by $s = -t$ in the integral, while it is not symmetrical. A similar remark will follow in the case of the integrals (3) and (6).

Toscano (7) used the operator Δ only in deducing some integrals. In this paper a number of integrals (including the correct forms of the above-mentioned integrals) have been evaluated by the application of the different combinations of the finite difference operators Δ and E.

The proofs given are only formal, except in the case of integral (4) where the result obtained is justified by a valid term by term integration. Similar justifications can probably be provided for the other finite integrals, namely (5) and (6) although they would be more complicated.

2. *Results to be used in this paper*

$$\Delta f(\alpha) = f(\alpha+1) - f(\alpha), \quad \underset{\alpha}{\Delta^\nu} f(\alpha) = \underset{\alpha}{\Delta}(\underset{\alpha}{\Delta^{\nu-1}} f(\alpha))$$

and

$$\underset{\alpha}{E} f(\alpha) = f(\alpha+1).$$

We have ((1), pp. 297, 300, 301)

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\,\Gamma(\beta-x)}\,dx = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}, \quad \Re(\alpha+\beta) > 1, \quad n \text{ integer}; \tag{2·1}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\beta+\gamma-1)\,\Gamma(\gamma+\delta-1)} \times \frac{1}{\Gamma(\delta+\alpha-1)}, \quad \Re(\alpha+\beta+\gamma+\delta) > 3 \tag{2·2}$$

and

$$\int_{-\infty}^{\infty} \frac{\begin{matrix}\sin\\ \cos\end{matrix}(\pi x)\,dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)} = \frac{\begin{matrix}\sin\\ \cos\end{matrix}[\tfrac{1}{2}\pi(\beta-\alpha)]}{2\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\gamma+\delta}{2}\right)\Gamma(\alpha+\delta-1)},$$

$$\alpha+\delta = \beta+\gamma, \quad \Re(\alpha+\beta+\gamma+\delta) > 2. \tag{2·3}$$

We have also ((2), p. 9)

$$\int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\,\Gamma(\beta)}\,dx = \frac{1}{\Gamma(\alpha+\beta)}, \quad \Re(\alpha) > 0,\ \Re(\beta) > 0; \tag{2·4}$$

$$\int_0^1 \frac{x^{\alpha+\sigma}(1-x)^{\beta}}{\Gamma(1+\beta)}\,dx = \frac{\Gamma(\alpha+\sigma+1)}{\Gamma(\alpha+\beta+\sigma+2)}, \quad \Re(\alpha+\sigma+1) > 0, \quad \Re(\beta+1) > 0; \tag{2·5}$$

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and

$$\int_x^1 \frac{(1-t)^{-\mu+\alpha}(t-x)^{\lambda-1}}{\Gamma(\lambda)\,\Gamma(1-\mu+\alpha)}\,dt = \frac{(1-x)^{-\mu+\alpha+\lambda}}{\Gamma(1-\mu+\lambda+\alpha)}, \quad x<1, \quad \Re(1-\mu+\alpha)>0, \quad \Re(\lambda)>0. \tag{2·6}$$

3. *Integral* 1.

$$\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\,\Gamma(\beta-x)}\,{}_1F_1\left[\begin{matrix}\alpha'; & v\\ \alpha+x & \end{matrix}\right]{}_1F_1\left[\begin{matrix}\beta'; & v\\ \beta-x & \end{matrix}\right]dx$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\,{}_1F_1\left[\begin{matrix}\alpha'+\beta'; & 2v\\ \alpha+\beta-1 & \end{matrix}\right], \quad \Re(\alpha+\beta)>1, \quad n \text{ integer}. \tag{3·1}$$

Proof. Consider the integral (2·1), multiply by the factor $\Gamma(\alpha')\,\Gamma(\beta')\,v^{\alpha+\beta}$ and apply the operator $\exp[E_\alpha E_{\alpha'}+E_\beta E_{\beta'}]$ on the both sides to obtain:

Left-hand side

$$\int_{-\infty}^{\infty}\left[\sum_{r=0}^{\infty}\frac{\Gamma(\alpha'+r)\,v^{\alpha+r}}{\Gamma(\alpha+x+r)\,r!}\right]\left[\sum_{n=0}^{\infty}\frac{\Gamma(\beta'+n)\,v^{\beta+n}}{\Gamma(\beta-x+n)\,n!}\right]\frac{\sin[(2n+1)\pi x]}{\sin\pi x}\,dx$$

$$= \Gamma(\alpha')\,\Gamma(\beta')\,v^{\alpha+\beta}\int_{-\infty}^{\infty}\frac{\sin[(2n+1)\pi x]}{\sin(\pi x)\,\Gamma(\alpha+x)\,\Gamma(\beta-x)}\,{}_1F_1\left[\begin{matrix}\alpha'; & v\\ \alpha+x & \end{matrix}\right]{}_1F_1\left[\begin{matrix}\beta'; & v\\ \beta-x & \end{matrix}\right]dx.$$

Right-hand side

$$\sum_{n,r=0}^{\infty}\frac{2^{\alpha+\beta+n+r-2}}{\Gamma(\alpha+\beta+n+r-1)}\,\frac{\Gamma(\alpha'+r)\,\Gamma(\beta'+n)}{r!\;\;n!}\,v^{\alpha+\beta+n+r}$$

$$= \frac{2^{\alpha+\beta-2}\,v^{\alpha+\beta}\,\Gamma(\alpha')\,\Gamma(\beta')}{\Gamma(\alpha+\beta-1)}\sum_{n=0}^{\infty}\frac{(2v)^n}{(\alpha+\beta-1)_n\,n!}\sum_{r=0}^{n}\binom{n}{r}(\alpha')_r\,(\beta')_{n-r}$$

$$= \frac{2^{\alpha+\beta-2}\,v^{\alpha+\beta}\,\Gamma(\alpha')\,\Gamma(\beta')}{\Gamma(\alpha+\beta-1)}\,{}_1F_1\left[\begin{matrix}\alpha'+\beta'; & 2v\\ \alpha+\beta-1 & \end{matrix}\right],$$

(using Vandermonde's theorem). On equating the left-hand side with the right-hand side, we get the required result.

Integral 2.

$$\int_{-\infty}^{\infty}\frac{1}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)}\,{}_1F_1\left[\begin{matrix}\alpha+\beta-1; & v\\ \alpha+x & \end{matrix}\right]{}_1F_1\left[\begin{matrix}\gamma+\delta-1; & v\\ \gamma+x & \end{matrix}\right]dx$$

$$= \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\delta+\alpha-1)\,\Gamma(\beta+\gamma-1)}$$

$$\times\,{}_2F_2\left[\begin{matrix}\dfrac{\alpha+\beta+\gamma+\delta-3}{2}, & \dfrac{\alpha+\beta+\gamma+\delta-2}{2}; & 4v\\ \beta+\gamma-1,\ \delta+\alpha-1 & & \end{matrix}\right],$$

$$\Re(\alpha+\beta+\gamma+\delta)>3. \tag{3·2}$$

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Proof. Multiply by the factor $\Gamma(\alpha+\beta-1)\,\Gamma(\gamma+\delta-1)\,v^{\alpha+\gamma}$ and apply the operator $\exp[E_\alpha+E_\gamma]$ on the both sides of the integral (2·2) to obtain,

$$\int_{-\infty}^{\infty}\frac{1}{\Gamma(\beta-x)\,\Gamma(\delta-x)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+\beta-1+n)v^{\alpha+n}}{\Gamma(\alpha+x+n)n!}\right]\left[\sum_{r=0}^{\infty}\frac{\Gamma(\gamma+\delta-1+r)v^{\gamma+r}}{\Gamma(\gamma+x+r)r!}\right]dx$$
$$=\sum_{n,r=0}^{\infty}\frac{\Gamma(\alpha+\beta+\gamma+\delta-3+n+r)v^{\alpha+\gamma+n+r}}{n!r!\,\Gamma(\alpha+\delta+n-1)\,\Gamma(\beta+\gamma-1+r)},$$

or

$$\int_{-\infty}^{\infty}\frac{1}{\Gamma(\beta-x)\,\Gamma(\delta-x)\,\Gamma(\alpha+x)\,\Gamma(\gamma+x)}\,{}_1F_1\left[\begin{matrix}\alpha+\beta-1;\ v\\ \alpha+x\end{matrix}\right]{}_1F_1\left[\begin{matrix}\gamma+\delta-1;\ v\\ \gamma+x\end{matrix}\right]dx$$
$$=\frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\alpha+\delta-1)\,\Gamma(\beta+\gamma-1)}$$
$$\times\sum_{n=0}^{\infty}\frac{(\alpha+\beta+\gamma+\delta-3)_n v^n}{n!\,(\alpha+\delta-1)_n}\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^r(2-\alpha-\delta-n)_r}{(\beta+\gamma-1)_r}$$
$$=\frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\alpha+\delta-1)\,\Gamma(\beta+\gamma-1)}$$
$$\times\sum_{n=0}^{\infty}\frac{(\alpha+\beta+\gamma+\delta-3)_n v^n}{n!\,(\alpha+\delta-1)}\,{}_2F_1\left[\begin{matrix}-n;\ 2-\alpha-\delta-n\\ \beta+\gamma-1\end{matrix}\right]$$
$$=\frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\alpha+\delta-1)\,\Gamma(\beta+\gamma-1)}$$
$$\times\sum_{n=0}^{\infty}\frac{(\alpha+\beta+\gamma+\delta-3)_{2n}}{(\beta+\gamma-1)_n\,(\alpha+\delta-1)_n\,n!}\frac{v^n}{n!}\quad\text{(using Gauss's Theorem)}$$
$$=\frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\,\Gamma(\gamma+\delta-1)\,\Gamma(\alpha+\delta-1)\,\Gamma(\beta+\gamma-1)}$$
$$\times\,{}_2F_2\left[\begin{matrix}\dfrac{\alpha+\beta+\gamma+\delta-3}{2},\ \dfrac{\alpha+\beta+\gamma+\delta-2}{2};\\ \beta+\gamma-1,\ \alpha+\delta-1\end{matrix}\ 4v\right].$$

This completes the proof.

Integral 3.

$$\int_{-\infty}^{\infty}\frac{\begin{matrix}\cos\\ \sin\end{matrix}(\pi x)}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)}\,{}_2F_2\left[\begin{matrix}\tfrac{1}{2}(\alpha+\beta),\ \alpha';\ v\\ \alpha+x,\ \beta-x\end{matrix}\right]{}_2F_2\left[\begin{matrix}\tfrac{1}{2}(\gamma+\delta),\ \delta';\ v\\ \gamma+x,\ \delta-x\end{matrix}\right]dx$$
$$=\frac{\begin{matrix}\cos\\ \sin\end{matrix}\tfrac{1}{2}\pi(\beta-\alpha)}{2\Gamma(\alpha+\delta-1)\,\Gamma[\tfrac{1}{2}(\alpha+\beta)]\,\Gamma[\tfrac{1}{2}(\gamma+\delta)]}\,{}_1F_1\left[\begin{matrix}\delta'+\alpha';\ v\\ \alpha+\delta-1\end{matrix}\right]$$
$$\Re(\alpha+\beta+\gamma+\delta)>2,\ \alpha+\delta=\beta+\gamma.\quad(3\cdot3)$$

Proof. Multiply by

$$\Gamma[\tfrac{1}{2}(\alpha+\beta)]\,\Gamma[\tfrac{1}{2}(\gamma+\delta)]\,\Gamma(\alpha')\,\Gamma(\delta')\,v^{\alpha+\delta}$$

apply the operator $\exp(E_\alpha E_{\alpha'}E_\beta+E_\delta E_{\delta'}E_\gamma)$ on both sides of the integral (2·3) we get,

$$\int_{-\infty}^{\infty}\begin{matrix}\cos\\ \sin\end{matrix}(\pi x)\left[\sum_{n=0}^{\infty}\frac{\Gamma[\frac{1}{2}(\alpha+\beta+2n)]\,\Gamma(\alpha'+n)\,v^{\alpha+n}}{\Gamma(\alpha+x+n)\,\Gamma(\beta-x+n)\,n!}\right]\left[\sum_{r=0}^{\infty}\frac{\Gamma[\frac{1}{2}(\gamma+\delta+2r)]\,\Gamma(\delta'+r)\,v^{\delta+r}}{\Gamma(\gamma+x+r)\,\Gamma(\delta-x+r)\,r!}\right]dx$$
$$=\frac{\begin{matrix}\sin\\ \cos\end{matrix}\tfrac{1}{2}(\beta-\alpha)}{2}\sum_{n,r=0}^{\infty}\frac{\Gamma(\alpha'+n)\,\Gamma(\delta'+r)\,v^{\alpha+\delta+n+r}}{\Gamma(\alpha+\delta+n+r-1)\,n!\,r!},$$

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or

$$\int_{-\infty}^{\infty} \frac{\sin \atop \cos (\pi x)}{\Gamma(\alpha+x)\,\Gamma(\beta-x)\,\Gamma(\gamma+x)\,\Gamma(\delta-x)}\, {}_2F_2\left[\begin{matrix}\frac{1}{2}(\alpha+\beta),\ \alpha';\ v\\ \alpha+x,\ \beta-x\end{matrix}\right] {}_2F_2\left[\begin{matrix}\frac{1}{2}(\gamma+\delta),\ \delta';\ v\\ \gamma+x,\ \delta-x\end{matrix}\right] dx$$

$$= \frac{\sin \atop \cos \frac{1}{2}\pi(\beta-\alpha)}{2\,\Gamma(\alpha+\delta-1)\,\Gamma[\frac{1}{2}(\alpha+\beta)]\,\Gamma[\frac{1}{2}(\gamma+\delta)]} \sum_{n=0}^{\infty} \frac{v^n}{(\alpha+\delta-1)_n\, n!} \sum_{r=0}^{n} \binom{n}{r} (\alpha')_{n-r} (\delta')_r.$$

Using Vandermonde's Theorem we get the required result.

Integral 4.

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, {}_1F_1\left[\begin{matrix}\alpha';\ vx\\ \alpha\end{matrix}\right] {}_1F_1\left[\begin{matrix}\beta';\ v(1-x)\\ \beta\end{matrix}\right] dx$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, {}_1F_1\left[\begin{matrix}\alpha'+\beta';\ v\\ \alpha+\beta\end{matrix}\right]; \quad \Re(\alpha) > 0, \quad \Re(\beta) > 0. \qquad (3{\cdot}4)$$

Proof. Multiply by $\Gamma(\alpha')\,\Gamma(\beta')\,v^{\alpha+\beta}$ and apply the operator $\exp(E_\alpha E_{\alpha'} + E_\beta E_{\beta'})$ on both sides of the integral (2·4) we get,

$$\int_0^1 \left[\sum_{n=0}^{\infty} \frac{\Gamma(\alpha'+n)\, v^{\alpha+n} x^{\alpha+n-1}}{\Gamma(\alpha+n)\, n!}\right]\left[\sum_{r=0}^{\infty} \frac{\Gamma(\beta'+r)\, v^{\beta+r}(1-x)^{\beta+r-1}}{\Gamma(\beta+r)\, r!}\right] dx$$

$$= \sum_{n,\,r=0}^{\infty} \frac{\Gamma(\alpha'+n)\,\Gamma(\beta'+r)\, v^{\alpha+\beta+n+r}}{\Gamma(\alpha+\beta+n+r)\, n!\, r!},$$

or

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, {}_1F_1\left[\begin{matrix}\alpha';\ vx\\ \alpha\end{matrix}\right] {}_1F_1\left[\begin{matrix}\beta';\ v(1-x)\\ \beta\end{matrix}\right] dx$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=0}^{\infty} \frac{v^n}{(\alpha+\beta)_n\, n!} \sum_{r=0}^{\infty} \binom{n}{r} (\alpha')_{n-r} (\beta')_r$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, {}_1F_1\left[\begin{matrix}\alpha'+\beta';\ v\\ \alpha+\beta\end{matrix}\right].$$

Validity. Consider the product of two absolute convergent power series,

$${}_1F_1\left[\begin{matrix}\alpha';\ vx\\ \alpha\end{matrix}\right] {}_1F_1\left[\begin{matrix}\beta';\ v(1-x)\\ \beta\end{matrix}\right] = \sum_{n=0}^{\infty} u_n(x),$$

where

$$u_n(x) = \frac{v^n}{n!} \sum_{r=0}^{n} \binom{n}{r} \frac{(\alpha')_{n-r}(\beta')_r}{(\alpha)_{n-r}(\beta)_r} (1-x)^r x^{n-r}$$

which also converges absolutely. Also let $g(x) = x^{\alpha-1}(1-x)^{\beta-1}$. Now the following observations can be made:

(1) $u_0(x), u_1(x), \ldots$ are continuous functions in $(0, 1)$ and the series $\Sigma u_n(x)$ converges uniformly in $(0, 1)$.

(2) $g(x)$ has finite number of infinite discontinuity in $(0, 1)$ and $\int_0^1 (x)^{\alpha-1}(1-x)^{\beta-1}\, dx$ is absolutely convergent when $\Re(\alpha) > 0$, $\Re(\beta) > 0$.

Hence on integrating the series term by term, which is permissible (see (4), p. 173), we get the required result.

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Integral 5.

$$\int_0^1 x^{\alpha+\sigma}(1-x)^{\beta} P_n^{\alpha,\beta}(1-2x)\, {}_pF_q\left[\begin{matrix}\alpha_p+\sigma;\ vx\\ \beta_q+\sigma\end{matrix}\right] dx$$

$$= \frac{\Gamma(\alpha+\sigma+1)\,\Gamma(1+\sigma)\,\Gamma(1+\beta+n)\,(-1)^n}{n!\,\Gamma(2+\alpha+\beta+\sigma+n)\,\Gamma(\sigma-n+1)}$$

$$\times {}_{p+2}F_{q+2}\left[\begin{matrix}\alpha_p+\sigma,\ \alpha+\sigma+1,\ 1+\sigma,\ v\\ \beta_q+\sigma,\ 2+\alpha+\beta+n+\sigma,\ \sigma-n+1\end{matrix}\right],$$

$$\Re(\alpha+\sigma+1)>0, \quad \Re(\beta+1)>0; \quad \text{if} \quad p=q+1, \quad |v|<1,$$

where $\alpha_p+\sigma$ is understood as $(\alpha_1+\sigma, \alpha_2+\sigma, \ldots, \alpha_p+\sigma)$.

Proof. Multiply by

$$\frac{\Gamma(1+\alpha+\beta+n)\,\Gamma(\alpha_p+\sigma)\,v^{\sigma}}{\Gamma(\beta_q+\sigma)}$$

and apply the operator $e^{E\sigma}\Delta_{\beta}^{\eta}$ on both sides of the integral (2·5) we get,

$$\int_0^1 \left[\sum_{m=0}^{\infty} \frac{x^{\alpha+\sigma+m}\Gamma(\alpha_p+\sigma+m)\,v^{\sigma+m}}{\Gamma(\beta_q+\sigma+m)\,m!}\right]\left[(-1)^n \sum_{r=0}^{n} \frac{(-n)_r}{r!} \frac{(1-x)^{\beta+r}\Gamma(1+\alpha+\beta+n+r)}{\Gamma(1+\beta+r)}\right] dx$$

$$= \sum_{m=0}^{\infty} \frac{v^{\sigma+m}\Gamma(\alpha+\sigma+m+1)\,\Gamma(\alpha_p+\sigma+m)}{m!\,\Gamma(\beta_q+\sigma+m)} (-1)^n \sum_{r=0}^{n} \frac{(-n)_r}{r!} \frac{\Gamma(1+\alpha+\beta+n+r)}{\Gamma(\alpha+\beta+\sigma+m+r+2)}.$$

On simplifying both sides we get the required result.

Integral 6.

$$\int_x^1 (1-t)^{-\mu+\alpha}(t-x)^{\lambda-1}\, {}_{p+2}F_{q+1}\left[\begin{matrix}-\nu,\ \nu+1+\alpha+\beta,\ a_p+\alpha;\ \frac{1}{2}(1-t)\\ 1-\mu+\alpha,\ b_q+\alpha\end{matrix}\right] dt$$

$$= \frac{\Gamma(1-\mu+\alpha)\,(1-x)^{-\mu+\alpha+\lambda}\Gamma(\lambda)}{\Gamma(1-\mu+\lambda+\alpha)}\, {}_{p+2}F_{q+1}\left[\begin{matrix}-\nu,\ \nu+1+\alpha+\beta,\ a_p+\alpha;\ \frac{1}{2}(1-x)\\ 1-\mu+\lambda+\alpha,\ b_q+\alpha\end{matrix}\right]$$

$$\Re(1-\mu+\alpha)>0, \quad \Re(\lambda)>0;\ p\leqslant q, \quad \text{if} \quad p=q, \quad |1-x|<2.$$

Proof. Multiply by $\Gamma(\nu+1+\beta+\alpha)\,\Gamma(a_p+\alpha)/2^{\alpha}(b_q+\alpha)$ and apply the operator Δ_{α}^{ν} on both sides of (2·6) we get,

$$\int_x^1 \sum_{r=0}^{\infty} \frac{(-1)^{\nu}(-\nu)_r}{r!} \frac{\Gamma(\nu+1+\beta+\alpha+r)\,\Gamma(a_p+\alpha+r)}{2^{\alpha+r}\,\Gamma(b_q+\alpha+r)} \frac{(1-t)^{-\mu+\alpha+r}}{\Gamma(\lambda)} \frac{(t-x)^{\lambda-1}}{\Gamma(1-\mu+\alpha+r)}\, dt$$

$$= (-1)^{\nu} \sum_{r=0}^{\infty} \frac{(-\nu)_r}{r!} \frac{\Gamma(\nu+1+\beta+\alpha+r)\,\Gamma(a_p+\alpha+r)\,(1-x)^{-\mu+\alpha+r+\lambda}}{2^{\alpha+r}\Gamma(b_q+\alpha+r)\,\Gamma(1-\mu+\lambda+\alpha+r)}.$$

On simplifying both sides we get the required result.

Validity. If ν is a positive integer then we do not require any proof as the series is finite.

Particular cases

(i) Integral (1) is the correct form of ((5), result (3)) and reduces to the correct form of ((5), result (4)) when $\alpha'=\alpha$, $\beta'=\beta$ and $n=0$.

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(ii) When $\alpha = \alpha'$ and $\delta = \delta'$ we obtain the correct form of the ((5), result (6)) from the integral (3).

(iii) If we replace $\alpha_p + \sigma$ by a_p and $\beta_q + \sigma$ by b_q in the integral (5) we get ((3), p. 188).

(iv) On putting $\alpha = \beta = 0$, $p = q = 0$, and $\lambda = \alpha$ and on using ((1), p. 143) we get ((6), p. 447) from the integral (6).

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REFERENCES

(1) BATEMAN PROJECT. *Table of integral transforms*, vol. II (New York, 1954).
(2) BATEMAN PROJECT. *Higher transcendental functions*, vol. I (New York, 1953).
(3) BHONSLE, B. R. On some results involving Jacobi Polynomials. *J. Indian Math. Soc.* **26** (1962), 187–190.
(4) CARSLAW, H. S. *Introduction to the theory of Fourier's series and integrals*, third edition (Dover).
(5) RAGAB, F. M. and SIMARY, M. A. Integrals involving the products of generalised Wittaker functions. *Proc. Cambridge Philos. Soc.* **61** (1965), 429–432.
(6) LOVE, E. R. Franz Neumann's Integral of 1948. *Proc. Cambridge Philos. Soc.* **61** (1965), 445–456.
(7) TOSCANO, L. I polinomi ipergeometrici nel calcolo delle differenze finite. *Boll. Un. Mat. Italia.* (3) **4** (1949), 398–409.

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