Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Beth Woollacott, 20 Rectory Close, Sutton Bonington, Loughborough, Nottinghamshire, LE12 5PJ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Entries should arrive by 20th May 2021 and solutions will be published in the July 2021 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the MA website

https://www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others****.*

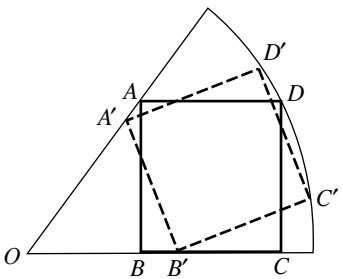
Problem 2021.1 **(Stan Dolan)**

Each vertex of a triangle is reflected in its opposite side. For what triangles will at least two of the new points be equal?

Problem 2021.2 **(Paul Stephenson)**

The diagram shows an acute-angled sector of a circle. The sector circumscribes a unit square $ABCD$ which shares a side with the sector and circumscribes a unit square $A'B'C'D'$ which shares an axis of symmetry with the sector.

Find the angle of the sector.



Solutions to 2020.5 and 2020.6

Both problems were solved by Gaurav Chaurasia (R.P.S. Academy, Bharauli, Uttar Pradesh), Christopher Mattocks (Bishop Wordsworth's School), James Mitchell (Sir John Leman High School, Beccles), Anujan Sribavananthan (William Perkin C. of E. High School, Greenford) and Student A (Indian Institute of Technology, Varanasi). Problem 2020.5 was solved by Student B (St. Michael's School, Durgapur, West Bengal). Both James and Student B submitted several answers to 2020.5 and these have been noted in the solutions below.

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Problem 2020.5 **(Gaurav Chaurasia)**

Let $O(n) = 1 + 9 + \ \ldots\ + (2n-1)^2$ and $E(n) = 4 + 16 + \ \ldots\ + (2n)^2$.

Find and prove expressions for $O(n)$ and $E(n)$ in terms of T, where

$$T(n) = 1 + 3 + \ \ldots\ + \frac{n(n+1)}{2},$$

is the sum of the first n triangular numbers.

Solution

A natural and direct method of solution was used by the setter, Gaurav, and the other solvers. Following James's solution, one can use standard results as follows.

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1), \qquad \sum_{i=1}^{n} i^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

and

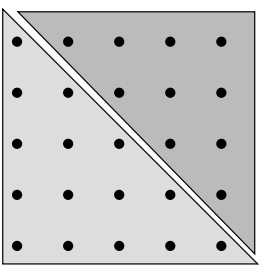
$$T(n) = \sum_{i=1}^{n} \frac{i(i+1)}{2} = \tfrac{1}{6}n(n+1)(n+2).$$

Then $$E(n) = \sum_{i=1}^{n}(2i)^2 = 4\sum_{i=1}^{n} i^2 = \tfrac{2}{3}n(n+1)(2n+1) = T(2n)$$

and
$$\begin{aligned} O(n) &= \sum_{i=1}^{n}(2i-1)^2 = 4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= \tfrac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n \\ &= \tfrac{1}{3}n\left(4n^2-1\right) = T(2n-1). \end{aligned}$$

Christopher, James and Student B also solved the problem very neatly by noticing and using a simple result for triangular numbers.

Let $t_i = \frac{1}{2}i(i+1)$ be the i th triangular number. Then $t_{i-1} + t_i = i^2$.



The diagram illustrates the result

$$t_4 + t_5 = 5^2.$$

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Then

$$E(n) = (t_1 + t_2) + (t_3 + t_4) + \ \ldots \ + (t_{2n-1} + t_{2n}) = T(2n)$$

$$O(n) = t_1 + (t_2 + t_3) + (t_4 + t_5) + \ \ldots \ + (t_{2n-2} + t_{2n-1}) = T(2n-1).$$

Problem 2020.6 (**Paul Stephenson**)

Let n be any positive integer. Define b to be the number of 1s in the binary expansion of n and define f to be the power of 2 in a prime power factorisation of $n!$.

Find and prove a simple connection between b and f.

Solution

Let the binary expansion of n have b 1s. Then, following James's solution,

$$n = 2^{a(1)} + 2^{a(2)} + \ \ldots \ + 2^{a(b)},$$

where $2^{a(1)} > 2^{a(2)} > \ \ldots \ > 2^{a(b)}$.

Then, from Legendre's formula,

$$\begin{aligned} f &= \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \ \ldots \\ &= \left(1 + 2 + \ \ldots \ + 2^{a(1)-1}\right) + \ \ldots \ + \left(1 + 2 + \ \ldots \ + 2^{a(b)-1}\right) \\ &= \left(2^{a(1)} - 1\right) + \left(2^{a(2)} - 1\right) + \ \ldots \ + \left(2^{a(b)} - 1\right) \\ &= n - b. \end{aligned}$$

The first prize of £25 is awarded to James Mitchell. The second prize of £20 is awarded to Christopher Mattocks.

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Note from the Editor

See *www.m-a.org.uk/the-mathematical-gazette* for the SPC questions.

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