ANALYTICAL SOLUTION FOR THREE-DIMENSIONAL HYPERBOLIC HEAT CONDUCTION EQUATION WITH TIME-DEPENDENT AND DISTRIBUTED HEAT SOURCE

M. R. Talaee * **V. Sarafrazi**

Rolling Stock
School of Railway Engineering
Iran University of Science and Technology
Tehran, Iran

ABSTRACT

This paper is devoted to the analytical solution of three-dimensional hyperbolic heat conduction equation in a finite solid medium with rectangular cross-section under time dependent and non-uniform internal heat source. The closed form solution of both Fourier and non-Fourier profiles are introduced with Eigen function expansion method. The solution is applied for simple simulation of absorption of a continues laser in biological tissue. The results show the depth of laser absorption in tissue and considerable difference between the Fourier and Non-Fourier temperature profiles. In addition the solution can be applied as a verification branch for other numerical solutions.

Keywords: Hyperbolic heat conduction, Source term, Laser interaction of biological tissue, Eigen function expansion method.

1. INTRODUCTION

In recent years Non-Fourier or hyperbolic heat conduction was in attention especially in cases of high temperature gradients, very large heat fluxes or very short time duration such as laser interaction of biological tissue. Despite of the Fourier theory, the non-Fourier or hyperbolic theory predicts that heat propagate with wave behavior and finite speed. The hyperbolic equation is based on the constitutive relation of heat flux vector $\mathbf{q}$ to temperature gradient in solids which was firstly introduced by Cattaneo and Vernotte [1,2]:

$$\mathbf{q}+\tau_q \frac{\partial \mathbf{q}}{\partial t}=-\lambda \nabla \theta \tag{1}$$

where θ is temperature, λ is the thermal conductivity of material and the constant τ_q is called the thermal relaxation time which physically represents the time lag between heat flux and imposed temperature gradient on material points. It is known that $\sqrt{\alpha/\tau_q}$ denotes the propagation speed of temperature wave which $\alpha = \lambda/\rho c$ is the thermal diffusivity and ρc is the material heating capacity. The resulted three-dimensional hyperbolic heat conduction equation from Eq. (1) in Cartesian coordinate with internal generation term is:

$$\tau_q \frac{\partial^2 \theta}{\partial t^2}+\frac{\partial \theta}{\partial t}=\alpha \nabla^2 \theta+\frac{1}{\rho c}\left(\tau_q \frac{\partial Q}{\partial t}+Q\right) \tag{2}$$

where Q is the source term. We derived pure analytical solutions for this equation in three-dimensional coordinate considering no generation term with separation of variables method [3]. The analytical solution for this problem with separation of variables method is difficult due to the presence of the generation term, so the procedure most be improved. Due to the importance of analytical solutions in verification of other numerical ones, solving Eq. (2) in Cartesian coordinates is under focus in the past few years. Lewandowska and Malinowski [4] solved the hyperbolic heat conduction in the semi-infinite body with the heat source which capacity linearly depends on temperature in an analytical way using Laplace transform. They [5] also found an analytical solution of the hyperbolic heat conduction equation for the case of a finite medium symmetrically heated on both sides using an internal heat source with capacity dependent on coordinate and time. Al-Khairy and Al-Ofey [6] presents an analytical solution of the hyperbolic heat conduction equation for moving semi-infinite medium under the effect of time dependent laser heat source using Laplace Method. Most of the solutions are focused in laser interaction simulation. Jaegwon Yoo *et al*. [7] studied the physical processes in ultra short laser irradiation of a material,

* Corresponding author (mrtalaee@iust.ac.ir)

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and developed a thermal wave solution of the hyperbolic heat conduction equation in a finite medium on the basis of the Green's function formalism. Zhang *et al.* [8] processed the analytical one-dimensional non-Fourier conduction model with heat source term for modeling high power ultra short laser pulse heating by the method of Laplace transform. Torii and Yang [9] numerically solved the case of a thin film subjected to a symmetrical heating on both sides. Ahmadikia *et al.* [10] solved one dimensional thermal wave and panes bioheat transfer models using Laplace transform under laser irradiation. Trujillo *et al*. [11] applied Laplace scheme to obtain a general solution of hyperbolic temperature profile in a one-dimensional model of non-perfused homogeneous biological tissue under laser heating. Zobier and Chaudhry [12] derive an analytical model for computation of temperature and heat flux distribution in a semi-infinite solid under spatially decaying, time-dependent laser source, using Laplace transform.

By reviewing the literature it can be seen that there is not any analytical three-dimensional solution for hyperbolic heat conduction problem under time-dependent heat source term and in addition most of the solutions are derived with Laplace transform method, which usually needs numerical procedure for inverse Laplace. All the analytical solutions are said to be exact but the analytical solution here is the pure exact solution, which do not use numerical procedure such as inverse Laplace for simple separable functions. In addition here there are introduced closed form solutions, for both hyperbolic and parabolic problem which can be used as a verification branch of other numerical ones.

In the presented paper here, following our past work [3], the hyperbolic heat conduction Eq. (2) is solved in three-dimensional Cartesian coordinates under the arbitrary time-dependent and non-uniform distributed heat generation term. The method of solution is Eigen function expansion procedure. For demonstration, following the works [11], the solution, is used for simulation of a continues laser interaction in a cubical tumor tissue. The innovations of the solution are:

- Using any numerical procedure such as inverse procedure in Laplace method for separable simple function.
- Ability to introduce closed form answers for parabolic and hyperbolic problems.
- Introduce a good foundation in order to verify the other numerical solutions.
- Ability to estimate exact temperature in some sensitive phenomena such as tumor thermal ablation under laser in simple situation *etc*.

2. MATHEMATICAL MODEL

Let's take into account an isotropic finite cubic tissue with length **L**, which is under the time-dependent and distributed in one dimension heat source term. The governing hyperbolic heat conduction equation considering the homogeneous properties is:

$$\tau_q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{1}{\rho c}\left[Q(x, y, z, t) + \tau_q \frac{\partial Q(x, y, z, t)}{\partial t}\right] \quad (3)$$

where $Q(x, y, z, t)$ is the source term and it is considered that $Q(x, y, z, t)$ is separable function of space and time.

$$Q(x, y, z, t) = g(x, y, z) f(t) \quad (4)$$

And g is a separable function of x, y, z, so the Eq. (3) cab be written as:

$$\tau_q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{1}{\rho c} G(x, y, z, t) \quad (5)$$

where $G(x, y, z, t)$ is separable function of space and time as:

$$G(x, y, z, t) = g(x, y, z) \cdot \left(f(t) + \tau_q \, \partial f(t) / \partial t \right) \quad (6)$$

The boundary conditions of tissue are considered as:

$$\begin{cases} T(0, y, z, t) = T_0 \\ T(a, y, z, t) = T_0 \end{cases}$$
$$\begin{cases} T(x, 0, z, t) = T_0 \\ T(x, b, z, t) = T_0 \end{cases}$$
$$\begin{cases} T(x, y, 0, t) = T_0 \\ T(x, y, c, t) = T_0 \end{cases} \quad (7)$$

Furthermore the initial conditions of the problem are considered to be:

$$\begin{cases} T(x, y, z, 0) = T_0 \\ \left.\dfrac{\partial T}{\partial t}\right|_{x,y,z,0} = 0 \end{cases} \quad (8)$$

where, T_0 is the initial temperature of the tissue. For normalization of the problem, it is introduced the following dimensionless parameters [13]:

$$\theta = \lambda \frac{T - T_0}{L q_0}, \ \xi = \frac{x}{L}, \ \varepsilon = \frac{y}{L}, \ \omega = \frac{z}{L}, \ Fo = \frac{\alpha t}{L^2},$$
$$\Pi = \frac{\alpha t_p}{L^2}, \ Ve = \sqrt{\frac{\alpha \tau}{L^2}} \quad (9)$$

where θ is dimensionless temperature and ξ, ε, ω are dimensionless coordinates. Fo is the dimensionless number of time, and Ve is the dimensionless number of relaxation time. Introducing the dimensionless parameters, Eq. (5) can be written as:

$$Ve^2 \frac{\partial^2 \theta}{\partial Fo^2} + \frac{\partial \theta}{\partial Fo} = \left[\frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \varepsilon^2} + \frac{\partial^2 \theta}{\partial \omega^2}\right] + G(\xi, \varepsilon, \omega, Fo) \quad (10)$$

where $G(\xi, \varepsilon, \omega, Fo)$ is a separable function:

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$$G(\xi, \varepsilon, \omega, Fo) = g(\xi, \varepsilon, \omega)\left[f(Fo) + Ve^2 \frac{\partial f(Fo)}{\partial Fo} \right] \quad (11)$$

Also, the boundary conditions can be shown as:

$$\begin{cases} \theta(0, \varepsilon, \omega, Fo) = 0 \\ \theta(1, \varepsilon, \omega, Fo) = 0 \end{cases}$$
$$\begin{cases} \theta(\xi, 0, \omega, Fo) = 0 \\ \theta(\xi, 1, \omega, Fo) = 0 \end{cases}$$
$$\begin{cases} \theta(\xi, \varepsilon, 0, Fo) = 0 \\ \theta(\xi, \varepsilon, 1, Fo) = 0 \end{cases} \quad (12)$$

and the initial conditions will be:

$$\begin{cases} \theta(\xi, \varepsilon, \omega, 0) = 0 \\ \left.\dfrac{\partial \theta}{\partial t}\right|_{\xi, \varepsilon, \omega, 0} = 0 \end{cases} \quad (13)$$

Solving the problem (10) ~ (13) in this state is done analytically using the Eigen function expansion method. Consequently, for the first step, the solution of the problem will be considered as:

$$\theta(\xi, \varepsilon, \omega, Fo) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A_{mnk}(Fo)\sin(m\pi\xi)\sin(n\pi\varepsilon)\sin(k\pi\omega) \quad (14)$$

where, $A_{mnk}(Fo)$ is time dependent coefficient of the solution and can be achieved by solving the second-order differential equations as will be shown later. It can be seen that solution (14) satisfies all the boundary conditions of Eq. (12). The property of this solution is that, it satisfy the below Eigenvalue Helmhotz problem too [14]:

$$\nabla^2\theta(\xi, \varepsilon, \omega, Fo) = -\lambda^2{}_{mnk}\,\theta(\xi, \varepsilon, \omega, Fo) \quad (15)$$

where $\lambda_{mnk} = \pi\sqrt{m^2 + n^2 + k^2}$ are the eigenvalues.

At the next step, the Fourier expansion of generation function $G(\xi, \varepsilon, \omega, Fo)$ according to the Eigenfunction used in Eq. (14) is needed as:

$$G(\xi, \varepsilon, \omega, Fo) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} G_{mnk}(Fo)\sin(m\pi\xi)\sin(n\pi\varepsilon)\sin(k\pi\omega) \quad (16)$$

where the coefficients $G_{mnk}(Fo)$ can be found from the concept of orthogonality of Eigen functions as [14]:

$$G_{mnk}(Fo) = 8\int_0^1\int_0^1\int_0^1 G(\xi, \varepsilon, \omega, Fo)\sin(m\pi\xi)\sin(n\pi\varepsilon)\sin(k\pi\omega)\, d\xi \cdot d\varepsilon \cdot d\omega \quad (17)$$

Implying the solution (14) and (16) into the Eq. (10), lead to a liner second order non-homogeneous ordinary differential equation, which can be shown as below, considering the initial conditions (13):

$$\begin{cases} Ve^2 \dfrac{\partial^2 A_{mnk}(Fo)}{\partial Fo^2} + \dfrac{\partial A_{mnk}(Fo)}{\partial Fo} + (\lambda^2{}_{mnk})A_{mnk}(Fo) = G_{mnk}(Fo) \\ A_{mnk}(0) = 0\,, \quad \dfrac{\partial A_{mnk}(0)}{\partial Fo} = 0 \end{cases} \quad (18)$$

Solution of this equation is obtained in two stages. First the homogeneous problem is solved to get the general solution $A_h(Fo)$. Then the non-homogeneous problem is solved with the aid of the set of general solution to get the particular solution $A_p(Fo)$. The final solution is sum of the general and particular solution. This method is the method of variation of parameters for determining the particular solution of homogenous equation [15]. Coefficient of the solution is determined using initial conditions. Thus the final solution for $Ve > 1/2\lambda_{mnk}$ can be shown as:

$$A_{mnk}(Fo) = e^{\frac{-Fo}{2Ve^2}}\left[u(Fo)\sin\left(\frac{\Gamma Fo}{2Ve^2}\right) + v(Fo)\cos\left(\frac{\Gamma Fo}{2Ve^2}\right)\right] \quad (19)$$

where $\Gamma = \sqrt{4Ve^2\lambda_{mnk}^2 - 1}$ and $u(Fo)$ and $v(Fo)$ are determined as below:

$$u(Fo) = \frac{2Ve^2}{\Gamma}\int e^{\frac{Fo}{2Ve^2}} G_{mnk}(Fo)\cos\left(\frac{\Gamma Fo}{2Ve^2}\right) dFo \quad (20)$$

$$v(Fo) = \frac{-2Ve^2}{\Gamma}\int e^{\frac{Fo}{2Ve^2}} G_{mnk}(Fo)\sin\left(\frac{\Gamma Fo}{2Ve^2}\right) dFo \quad (21)$$

So the solution of hyperbolic Eq. (10) is Eq. (14), where $A_{mnk}(Fo)$ can be found from Eqs. (19) ~ (21). Accordingly, undergoing this procedure for parabolic problem (Fourier conduction) the solution is Eq. (14), where $A_{mnk}(Fo)$ can be found as:

$$A_{mnk}(Fo) = e^{-\lambda_{mnk}^2 Fo}\left[\int e^{\lambda_{mnk}^2 Fo} G_{mnk}(Fo)\, dFo\right] \quad (22)$$

3. DEMONSTRATION AND RESULTS

For the case study a cubic tumor tissue has been considered with equal length of L which enclosed in human body and all the surfaces remain at the same initial temperature. The tissue is under a step laser pulsed function from the top (z direction) as shown in Fig. (1). The laser heating function can be considered in simple situation as a heat source term with exponential variation in absorption depth and step function of time pulse [16]:

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$$Q(x,\,y,\,z,\,t)=(1-R)I_0\,\mu.e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}e^{(-\mu z)}\,f(t) \tag{23}$$

where R, μ and I_0 are light reflectance of surface, absorption coefficient of the material and laser intensity respectively. A step time pulse function $f(t) = H(t) - H(t - t_p)$ with the duration $t_p = 10$ is adopted here according to the Ooi & Ang, [17]. Due to the position of origin of coordinates in cubic tissue as shown in Fig. 1, for application of the Gaussian laser function in the central point of cube, coordinates x and y must be transformed to the centre. The solution of this problem is Eq. (14) where $A_{mnk}(Fo)$ is determined from the below procedure. The non-dimensional term $G(\xi, \varepsilon, \omega, Fo)$ of Eq. (11) becomes:

$$G(\xi,\,\varepsilon,\,\omega,\,Fo)=\frac{L(1-R)I_0\,\mu}{q_0}\cdot e^{-\left(\frac{(\varepsilon-1/2)^2+(\xi-1/2)^2}{2\sigma^2}\right)}e^{(-\mu\omega)}\left[\left(H(Fo)-H(Fo-\Pi)\right)+Ve^2\left(\delta(Fo)-\delta(Fo-\Pi)\right)\right] \tag{24}$$

Using the above source term, Eq (17) for this problem reduced to:

$$G_{mnk}(Fo)=W_{mnk}\left[\left(H(Fo)-H(Fo-\Pi)\right)+Ve^2\left(\delta(Fo)-\delta(Fo-\Pi)\right)\right] \tag{25}$$

where W_{mnk} is separable function as:

$$W_{mnk}=\frac{8k\pi L(1-R)I_0\mu\left(1-e^{-\mu L}(-1)^k\right)}{q_0\left[(\mu L)^2+(k\pi)^2\right]}\int_0^1\int_0^1 e^{-\left(\frac{(\varepsilon-1/2)^2+(\xi-1/2)^2}{2\sigma^2}\right)}\sin(m\pi\xi)\sin(n\pi\varepsilon)\,d\xi d\varepsilon \tag{26}$$

Thus the time dependent coefficient of $A_{mnk}(Fo)$ in Eqs. (18) ~ (21) is determined, as:

$$A_{mnk}(Fo)=\frac{2Ve^4W_{mnk}e^{\frac{-Fo}{2Ve^2}}}{\Gamma(\Gamma^2+1)}\left[\begin{array}{l}2\Gamma e^{\frac{Fo}{2Ve^2}}\left(H(Fo)-H(Fo-\Pi)\right)+H(Fo)\left((\Gamma^2-1)\sin\left(\frac{\Gamma Fo}{2Ve^2}\right)-2\Gamma\cos\left(\frac{\Gamma Fo}{2Ve^2}\right)\right)+\\ e^{\frac{\Pi}{2Ve^2}}H(\Pi-Fo)\left((\Gamma^2-1)\sin\left(\frac{\Gamma(Fo-\Pi)}{2Ve^2}\right)-2\Gamma\cos\left(\frac{\Gamma(Fo-\Pi)}{2Ve^2}\right)\right)\end{array}\right] \tag{27}$$

For parabolic problem (Fourier conduction) the solution is Eq. (14), where $A_{mnk}(Fo)$ can be found as:

$$A_{mnk}(Fo)=e^{-\lambda_{mnk}^2Fo}\left[\frac{W_{mnk}}{\lambda_{mnk}^2}\left(e^{\lambda_{mnk}^2Fo}-H(\Pi-Fo)(e^{\lambda_{mnk}^2Fo}+e^{\lambda_{mnk}^2\Pi})-1+H(-\Pi)+H(\Pi)\,e^{\lambda_{mnk}^2\Pi}\right)\right] \tag{28}$$

The resulted temperature profiles are calculated using below thermal properties for tissue and laser function:

Table 1 value of applied parameters [17].

Parameter	k ($Wm^{-1}k^{-1}$)	c ($Jkg^{-1}k^{-1}$)	ρ (kgm^{-3})	t_p (sec)	τ (sec)	μ (m^{-1})	I_0 (Wm^{-2})
Value	0.58	4178	1050	10	16	1900	4.42×10^5

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The contours of non-Fourier temperature profiles of vertical plain at the center of tissue are shown in Fig. 2. It can be seen that there is a time lag for penetration of temperature profiles to the body tissue compared with Fourier one in Fig. 3. The penetration depth of the laser pulse due to the time can be estimated with the exact analytical solution as shown in Fig. 2. The results show that the hyperbolic profiles predict slower penetration rate of laser compared with Fourier one, which can be important for some tumor ablation operation.

The contours of non-Fourier temperature profiles in horizontal plain at the center of the tissue are shown in Fig. 4. It can be seen that the laser concentration (intensity) on the center of cubic tissue may lead to increase of temperature there.

The temperature history of central point of tissue due to time is plotted in Fig. 5 for various *Ve* number. The results show the oscillatory behavior of hyperbolic profiles compared with Fourier one. The results show that non-Fourier temperature profiles predict higher maximum temperature compared with Fourier one and the maximum amplitude is increased with increasing *Ve* number which show the higher inertial effect of hyperbolic profiles. In addition, the wave behavior of hyperbolic profiles, leads to oscillation in temperature distribution and postpone reaching to stability.

The hyperbolic and parabolic temperature distribution in vertical line in centre of the tissue is shown in Figs. 6 and 7 respectively for initial time of laser implication. The penetration depth of temperature profile due to laser absorption and it's time lag can be deduced from the figures. It can be seen that the hyperbolic conduction predicts a time lag and or slower penetration of temperature profiles in tissue depth compared with Fourier one. The effect of laser intensity is I_0 on the produced temperature profiles is shown in Fig. 8. It is shown that the maximum temperature is increased by increasing the laser intensity.

The three-dimensional temperature plots of the middle horizontal and vertical plain in the cubic tissue are shown in Figs. 9 and 10 respectively. The ability of the procedure for driving the solution in all dimensions can be seen from these figures. It can be seen that the temperature distribution is concentrated around the vertical axis and near to the top surface of the tissue (laser impact plane) such as an elliptic (nap form) shape.

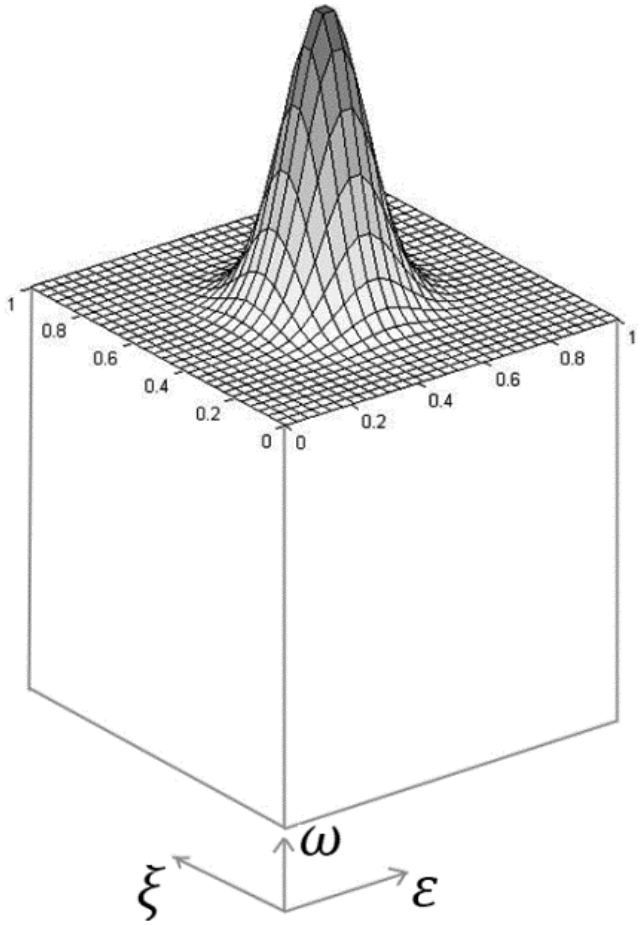


Fig. 1 Schematic of laser incidence to the top of the cube.

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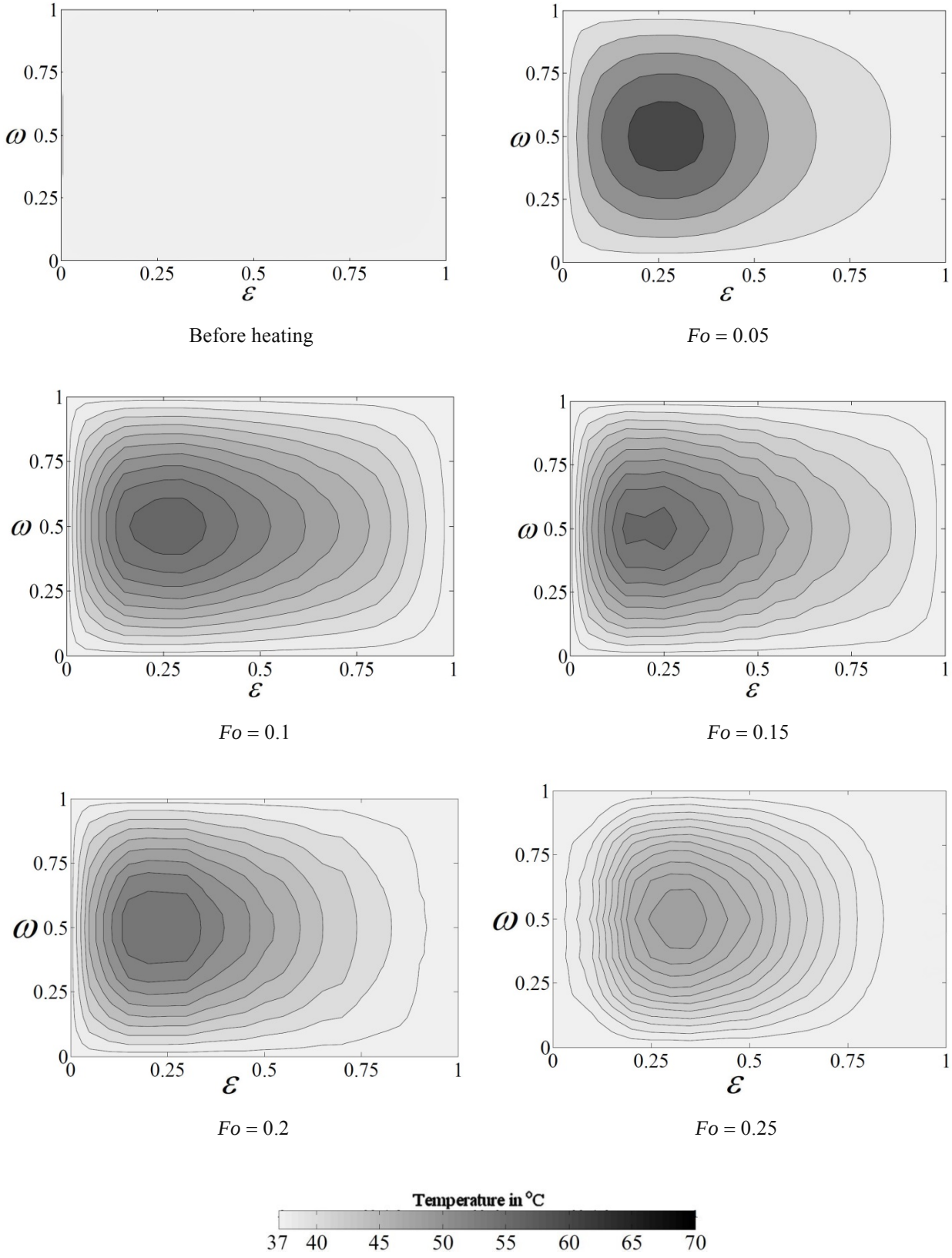


Fig. 2 Hyperbolic temperature distribution in central vertical section of tissue ($Ve = 0.3$ and $\xi = 0.5$) during continuous laser irradiation.

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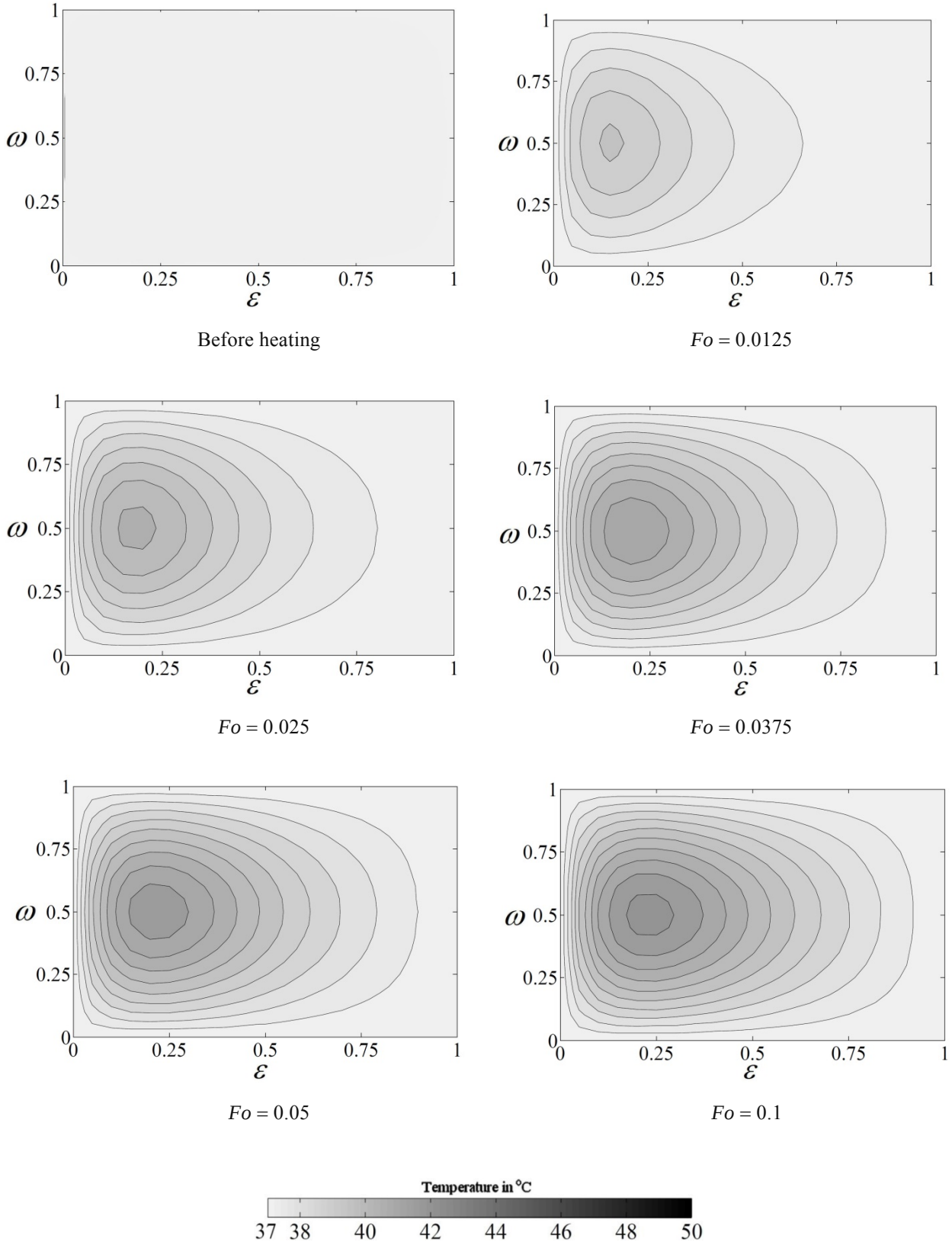


Fig. 3 Parabolic temperature distribution in central vertical section of tissue ($Ve = 0$ and $\xi = 0.5$) during continuous laser irradiation.

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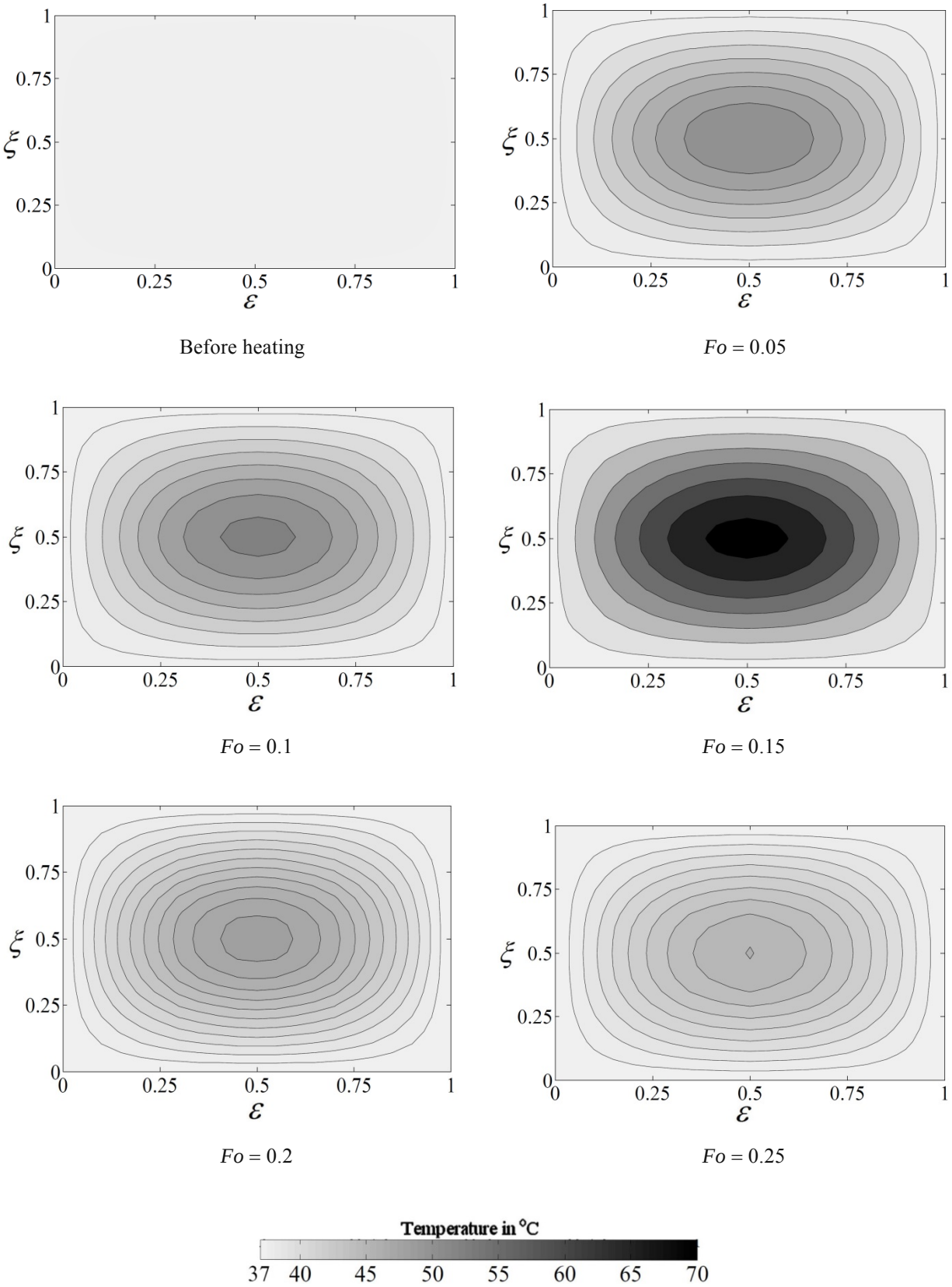


Fig. 4 Hyperbolic temperature distribution in central horizontal section of tissue ($Ve = 0.3$ and $\omega = 0.5$) during continuous laser irradiation.

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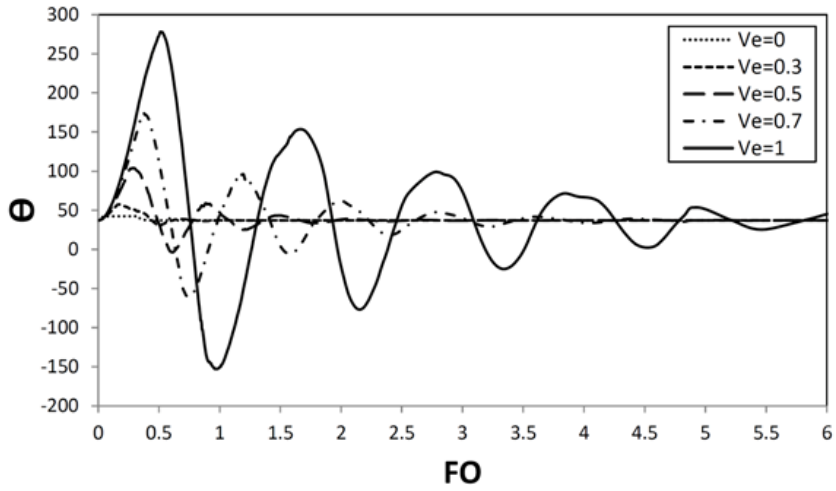


Fig. 5 Temperature variation of middle point of cube due to *Fo* with t_p = 10 and various *Ve* during continuous laser irradiation.

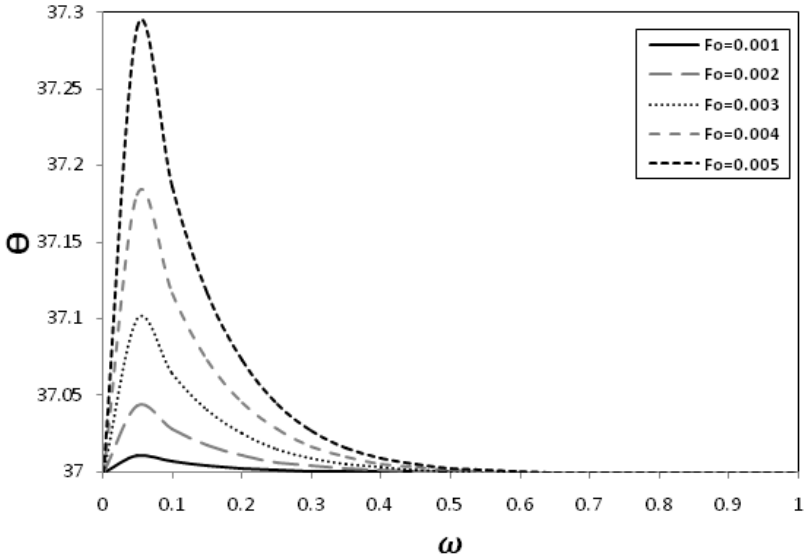


Fig. 6 Hyperbolic temperature variation in ω direction with *Ve* = 0.3 for various *Fo* during continuous laser irradiation.

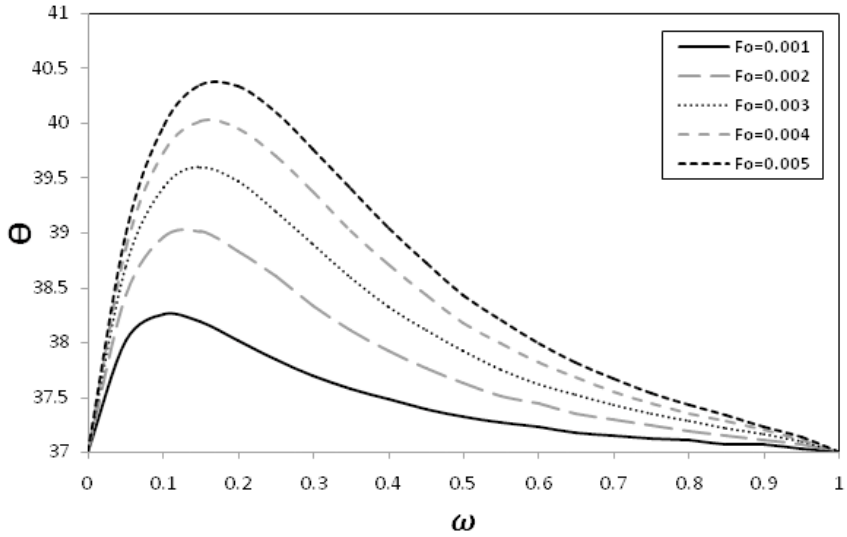


Fig. 7 Parabolic temperature variation in ω direction with *Ve* = 0 for various *Fo* during continuous laser irradiation.

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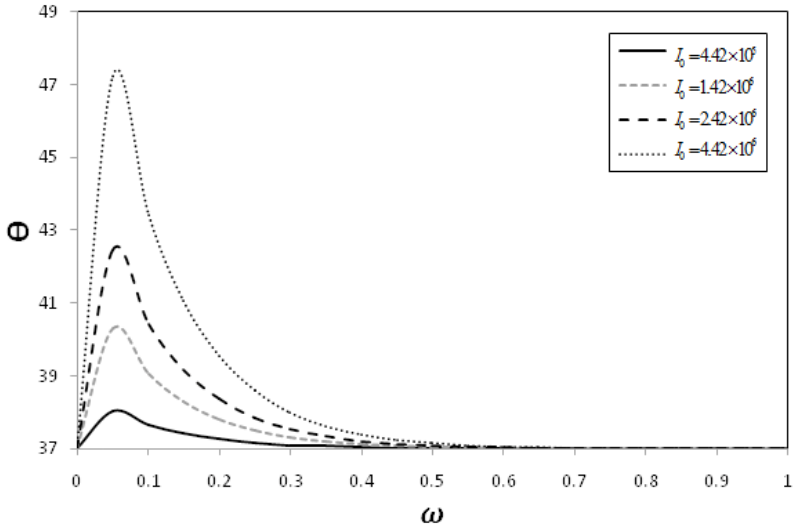


Fig. 8 Hyperbolic temperature variation in ω direction with $Ve = 0.3$ and $Fo = 0.01$ under various laser intensity (I_0) during continuous laser irradiation.

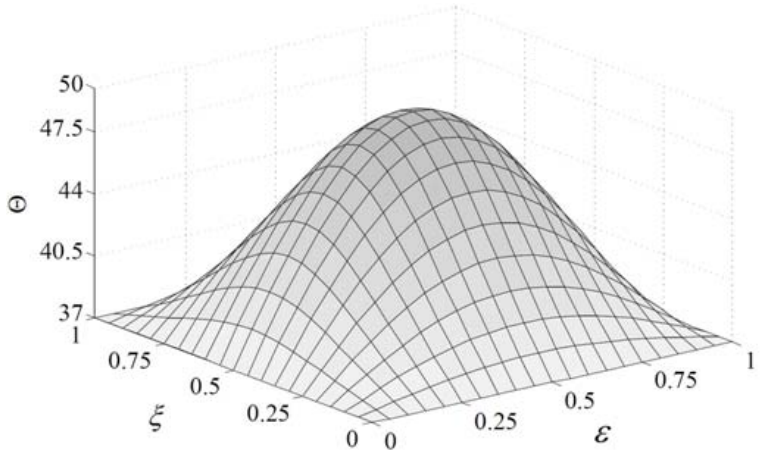


Fig. 9 Three dimensional temperature profile at $\omega = 0.5$ for $Ve = 0.3$ and $Fo = 0.1$.

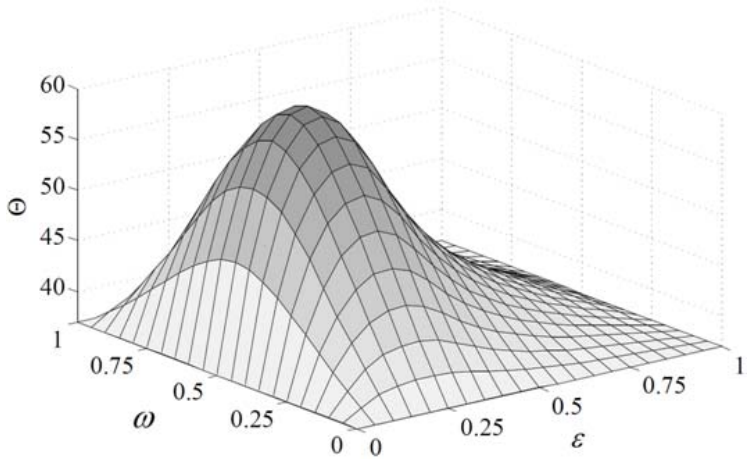


Fig. 10 Three dimensional temperature profile at $\varepsilon = 0.5$ for $Ve = 0.3$ and $Fo = 0.1$.

4. CONCLUSIONS

In this problem the three-dimensional hyperbolic and parabolic heat conduction equations with time-dependent, non-uniform distributed heat source is analytically solved in a finite solid cube. The straightforward solution is introduced for hyperbolic and parabolic conduction using Eigen Function expansion method. The solution is used for modeling of continues laser interaction of tumor tissue. The generation

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function is distributed in impacted surface and vertical direction. The hyperbolic temperature profiles predict a time lag for penetration depth of the laser pulse and higher temperature amplitude compared with parabolic one. The exact analytical solution can be used as verification branch for other numerical solutions and operations of biological tissue under laser treatment.

NOMENCLATURE

I_0	Laser Intensity
L	Length of the cube
Fo	Non-dimensional number of time
$f(t)$	Time function of heat source
H	Heaviside function
R	Surface light reflectance
μ	Material absorption coefficient
t	time
Ve	Vernotte number
x, y, z	Spatial coordinate
$\boldsymbol{q}$	Heat flux vector
m, n, k	Summation counter

Greek symbols

α	Thermal diffusivity
ρ	Mass density
Δ	Laplace's differential Operator
∇	Gradient operator
τ_q	Thermal relaxation time
τ_p	Laser pulse duration
θ	Dimensionless temperature
σ	Laser concentration
δ	Dirac Function
λ	Thermal conductivity
λ_{mnk}	Eigen Values
ξ, ε, ω	non-dimensional space parameters for x, y, z respectively

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