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A note on blocks of modular representations

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1. The notation used will be that of (2).

G is a finite group,

$\mathfrak{O}$ is a ring which is either

(1·1) a field of characteristic p, or

(1·2) the ring of valuation integers of a field of characteristic 0 with a complete discrete valuation. In this case, $\mathfrak{P}$ is the maximal ideal of $\mathfrak{O}$.

If H is a subgroup of G, $\Gamma(\mathfrak{O}, H)$ is the group algebra of H over $\mathfrak{O}$. $Z(\mathfrak{O}, H)$ is its centre and $R(\mathfrak{O}, H)$ is its radical. Let $\Gamma(\mathfrak{O}, H) = \Gamma_H$, $Z(\mathfrak{O}, H) = Z_H$, and $R(\mathfrak{O}, H) = R_H$. If $\mathfrak{O}$ is of type (1·2), then $R_H \geqslant \mathfrak{P}\Gamma_H$, and R_H maps onto $R(\mathfrak{O}/\mathfrak{P}, H)$ in the natural homomorphism

$$\Gamma(\mathfrak{O}, H) \to \Gamma(\mathfrak{O}/\mathfrak{P}, H).$$

Also Γ_H is complete in the topology defined by the powers of R_H. K_α ($\alpha = 1, 2, \ldots, k$) are the *class sums* of G, i.e. each K_α is the sum of all the elements in a conjugacy class of G. If $K_\alpha = x + \ldots$, the *defect group* of K_α is a Sylow p-subgroup of the centralizer $C_G(x)$ of x in G and is defined uniquely up to conjugacy in G. Here p is the characteristic of $\mathfrak{O}$ if $\mathfrak{O}$ is of type (1·1), or the characteristic of $\mathfrak{O}/\mathfrak{P}$ if $\mathfrak{O}$ is of type (1·2).

2. In (2), Green considered the primitive idempotents of Z_G. If E is such an idempotent, he defined the *defect group* of E, which is a p-subgroup D of G defined uniquely up to conjugacy in G. If $\mathfrak{O}$ is of type (1·1) this reduces to the definition of Rosenberg (4). If, further, $\mathfrak{O}$ is a splitting field for G, then this is the definition given by Brauer (1) and Osima (3) for the defect group D of the *block* of G associated with E. In these special cases, if H is a subgroup of G such that

$$DC(D) \leqslant H \leqslant N(D), \tag{2·1}$$

there is an algebra homomorphism of Z_G into Z_H which maps E on an idempotent e of Z_H ($C(D)$ is the centralizer of D, $N(D)$ is the normalizer of D). We say e is associated with E under the *Brauer correspondence*. If $H = N(D)$ this correspondence is (1–1) from the set of primitive idempotents of defect group D of Z_G onto the set of primitive idempotents of defect group D of Z_H ((1), (3), (4)).

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If $\mathfrak{O}$ is of type (1·2), there is a (1–1) correspondence between the primitive idempotents of $Z(\mathfrak{O}, H)$ and those of $Z(\mathfrak{O}/\mathfrak{P}, H)$ for any subgroup H of G. Hence, if H satisfies the conditions of (2·1), we can still talk of the idempotent e of Z_H corresponding to an idempotent E of defect group D of Z_G ((2), 1·5).

One of Green's results ((2), Lemma 4·2C) shows how the Brauer correspondence can be viewed in a different light. We may regard Γ_G as a $G \times G$-module by the rule

$$a(x, y) = x^{-1}ay \quad (a \in \Gamma_G;\ x, y \in G).$$

From now on we shall suppose that Γ_G is regarded as a $G \times G$-module in this way. Then $\Gamma_G E$ is an indecomposable submodule of Γ_G. If $H = N(D)$ and e corresponds to E, Green showed that $\Gamma_H e$ is isomorphic to a direct summand of the restricted module $(\Gamma_G E)_{H \times H}$. His argument, however, is rather involved. It is the object of this note to prove the following theorem.

THEOREM. *Let E be a primitive idempotent of Z_G of defect group D, and H a subgroup of G such that*

$$DC(D) \leqslant H \leqslant N(D).$$

Let e be the idempotent of Z_H associated with E under the Brauer correspondence. Then

(i) *$\Gamma_H e$, as $H \times H$-module, is isomorphic to a direct summand of $(\Gamma_G E)_{H \times H}$, and*

(ii) *if $H = N(D)$, $\Gamma_G E$ is isomorphic to a direct summand of the induced $G \times G$-module $(\Gamma_H e)^{G \times G}$.*

Thus the Brauer correspondence is exhibited as a restriction-induction correspondence between $\Gamma_H e$ and $\Gamma_G E$.

3. We prove the theorem when $\mathfrak{O}$ is of type (1·2). The proof is similar when $\mathfrak{O}$ is of type (1·1).

Proof of theorem. (i) We can write

$$E \equiv \Sigma \lambda_\alpha K_\alpha \pmod{\mathfrak{P}\Gamma_G},$$

where

(3·1) each K_α has defect group conjugate to a subgroup of D ((4), Proposition 3·2).

Let $K_\alpha = K_\alpha^1 + K_\alpha^2 + K_\alpha^3$, where K_α^1, K_α^2, K_α^3 are the sums of all the elements of K_α lying in $C(D)$, $H - C(D)$, and $G - H$, respectively. It is known that

$$\Sigma \lambda_\alpha K_\alpha^1 \equiv e \pmod{\mathfrak{P}\Gamma_H}$$

((3), p. 181). Further, by (3·1) K_α^2 is the sum of class sums of H which have defect group properly contained in a conjugate (in H) of D. Hence

$$K_\alpha^2 e \equiv 0 \pmod{R_H e}$$

((4), Proposition 3·2). So we can write

$$E = e + n + f,$$

where $n \in R_H$ and f is a $\mathfrak{O}$-linear combination of elements of G which do not lie in H.

Let η be the $H \times H$-homomorphism of Γ_G into Γ_H given by

$$(3\cdot2) \qquad g\eta = \begin{cases} g & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$

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Consider the $H \times H$-homomorphisms

$$\psi: \Gamma_H e \to \Gamma_G E \quad \text{and} \quad \phi_0: \Gamma_G E \to \Gamma_H e$$

given by

$$(he)\psi = heE \quad (h \in H),$$

$$(gE)\phi_0 = (gE)\eta e \quad (g \in G).$$

Then

$$(he)\psi\phi_0 = he + hne.$$

Now $hne \in R_H$, and so if χ is the $H \times H$-endomorphism of $\Gamma_H e$ given by

$$(he)\chi = hne,$$

the mapping $\iota + \chi$ has the inverse $(\iota - \chi + \chi^2 - \ldots)$ (ι is the identity automorphism of $\Gamma_H e$). Let $\phi = \phi_0(\iota - \chi + \chi^2 - \ldots)$, then ϕ maps $\Gamma_G E$ into $\Gamma_H e$, and

$$(he)\psi\phi = he \quad (h \in H).$$

This proves (i).

(ii) Let $H = N(D)$, and let $\{x_1, x_2, \ldots, x_m\}$ be a set of right coset representatives of G over H. Then $(\Gamma_H e)^{G \times G}$ has as $\mathfrak{O}$-basis the elements

$$he \otimes (x_i, x_j) \quad (h \in H; i, j = 1, 2, \ldots, m).$$

This is regarded as a $G \times G$-module by the rule

$$\begin{aligned}(he \otimes (x_i, x_j))(g_\lambda, g_\mu) &= he \otimes (x_i g_\lambda, x_j g_\mu) \\ &= h_\lambda^{-1} h e h_\mu \otimes (x_{\lambda(i)}, x_{\mu(j)}),\end{aligned}$$

where $x_i g_\lambda = h_\lambda x_{\lambda(i)}$, $x_j g_\mu = h_\mu x_{\mu(j)}$, and $1 \leqslant \lambda(i), \mu(j) \leqslant m$.

Let the $G \times G$-homomorphisms $\pi: (\Gamma_H e)^{G \times G} \to \Gamma_G E$ and $\theta_0: \Gamma_G E \to (\Gamma_H e)^{G \times G}$ be given by

$$[\sum_{i,j} h_{ij} e \otimes (x_i, x_j)]\pi = \sum_{i,j} x_i^{-1} h_{ij} e x_j E,$$

$$(gE)\theta_0 = \sum_{i,j} gE(x_i, x_j)^{-1} \eta e \otimes (x_i, x_j) \quad (g \in G; h_{ij} \in H). \tag{3·3}$$

Here η is given by (3·2). We verify that

$$(gE)\theta_0\pi = gE \sum_{j=1}^{m} x_j^{-1} e x_j \quad (g \in G).$$

We now compute $\sum_j x_j^{-1} e x_j$. We may write

$$E = \Sigma \lambda_\beta K_\beta + \Sigma \lambda_\gamma K_\gamma \pmod{\mathfrak{P}\Gamma_G},$$

where each K_β is of defect group D, and each K_γ has defect group conjugate to a proper subgroup of D. In particular, $K_\gamma E \in R_G E$ for each γ ((4), Proposition 3·2). Since $H = N(D)$ and each K_β is of defect group D, each K_β^1 is the sum of the elements of H in a single conjugacy class of defect group D ((1), 10A). Let

$$K_\beta^1 = y + \ldots \quad (y \in H),$$

so that D is a Sylow p-subgroup of $Q = C_G(y)$. Then

$$\sum_j x_j^{-1} K_\beta^1 x_j = t K_\beta,$$

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where
$$t = |Q:Q \cap H|$$
$$= |Q:Q \cap N(D)|.$$

Since D is a Sylow p-subgroup of Q, we have

$$t \equiv 1 \pmod{p}.$$

However,
$$e \equiv \Sigma \lambda_\beta K_\beta^1 \pmod{\mathfrak{P}\Gamma_H},$$

and so
$$\sum_j x_j^{-1} e x_j \equiv \Sigma \lambda_\beta K_\beta \pmod{\mathfrak{P}\Gamma_G},$$

and
$$E(\sum_j x_j^{-1} e x_j) \equiv (E - \Sigma \lambda_\gamma K_\gamma E) \pmod{\mathfrak{P}\Gamma_G}$$
$$\equiv E \pmod{R_G}.$$

From (3·3) we then get
$$(gE)\,\theta_0 \pi = gE(E+n)$$

where $n \in R_G$. The proof can now be completed in the usual way. That is, we can find a $G \times G$-homomorphism

$$\theta: \Gamma_G E \to (\Gamma_H e)^{G \times G}$$

such that $(gE)\,\theta\pi = gE$. Hence $\Gamma_G E$ is isomorphic to a direct summand of $(\Gamma_H e)^{G \times G}$ and the theorem is proved.

I am very grateful to Dr J. A. Green for allowing me to see, before publication, an earlier version of (2), which suggested this theorem.

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