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A bivariate Poisson queueing process that is not infinitely divisible

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We are using the term 'bivariate Poisson process' to describe a bivariate point process $(N_1(.), N_2(.))$ whose components (or, marginal processes) are Poisson processes. In this we are following Milne (2) who amongst his examples cites the case where $N_1(.)$ and $N_2(.)$ refer to the input and output processes respectively of the $M/G/\infty$ queueing system. Such a bivariate point process is infinitely divisible. We shall now show that in a stationary $M/M/1$ queueing system (i.e. Poisson arrivals at rate λ, exponential service at rate $\mu > \lambda$, single-server) a similar identification of $(N_1(.), N_2(.))$ yields a bivariate Poisson process that is not infinitely divisible.

In the first place, $N_1(.)$ is Poisson by assumption and $N_2(.)$ is known to be Poisson when the system is stationary (see Reich (3) for a simple proof).

If $(N_1(.), N_2(.))$ is infinitely divisible then because its components are Poisson, the joint probability generating function for $(N_1, N_2) \equiv (N_1(0,t], N_2(0,t])$ $(t > 0)$ must be of the form

$$E(x^{N_1}y^{N_2}) = \exp(\alpha(x-1)+\beta(y-1)+\gamma(xy-1)) \tag{1}$$

for some non-negative α, β, γ (Dwass and Teicher (1)). In particular then, we should have

$$\sum_{i=0}^{\infty} x^i \operatorname{pr}\{N_1 = i, N_2 = 0\} = \exp(\alpha(x-1)-\beta-\gamma). \tag{2}$$

But with $\rho = \lambda/\mu$,

$$\begin{aligned}\sum_{i=0}^{\infty} x^i \operatorname{pr}\{N_1 = i, N_2 = 0\} &= \sum_{i=0}^{\infty} x^i \rho \operatorname{pr}\{N_1 = i, N_2 = 0 | \text{server busy at } 0\} \\ &\quad + \sum_{i=0}^{\infty} x^i (1-\rho) \operatorname{pr}\{N_1 = i, N_2 = 0 | \text{system empty at } 0\} \\ &= \rho e^{-\lambda(1-x)t} e^{-\mu t} + (1-\rho) e^{-\lambda t} + (1-\rho) \\ &\quad \times \int_0^t \lambda e^{-\lambda(t-u)} x e^{-\lambda(1-x)u} e^{-\mu u} \, du \\ &= [\lambda(1-x) e^{-(\lambda(1-x)+\mu)t} + (\mu-\lambda) e^{-\lambda t}]/(\mu - \lambda x) \\ &\qquad\qquad (\text{all } |x| \leqslant 1), \end{aligned} \tag{3}$$

which is not of the required form (2).

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