*Probability in the Engineering and Informational Sciences*, **14**, 2000, 385–392. Printed in the U.S.A.

# A NOTE ON PERTURBATION ANALYSIS ESTIMATORS FOR AMERICAN-STYLE OPTIONS

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In this note, we correct an error in the paper by Fu and Hu [1] for the perturbation analysis estimator given for the gradient of an American call option payoff on an underlying asset paying multiple dividends. We then introduce a different asset

*Supported in part by the National Science Foundation under grant DMI-9713720 and by the Semiconductor Research Corporation under grant 97-FJ-491.

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https://doi.org/10.1017/S0269964800143086 Published online by Cambridge University Press

price model that is more straightforward than the previous model, and derive the corresponding gradient estimators. We conclude with a brief discussion of extensions of the estimator to other American-style options.

## 1. INTRODUCTION

Section 3 of Fu and Hu [1] considers an American call option defined on a stock paying dividends at discrete-time points, and gradient estimators are derived for the expected payoff of the option with respect to various parameters (see also Fu and Hu [2], and Gürkan et al. [3]). The one-dividend case is analyzed in detail and a rigorous proof is provided for the correctness of the estimator. The estimator is then extended to an arbitrary number of dividends without derivation. It turns out that the estimator is, in fact, incorrect, and the main purpose of this note is to provide the correct estimator. We provide a detailed derivation for the two-dividend case and then show how it generalizes to any number of dividends.

A secondary purpose of this note is to consider an alternative model for the dynamics of the underlying stock price process. The model considered in Fu and Hu [1] follows Stoll and Whaley [4] and others by discounting dividends back to time 0 and assuming that the process modeling the stock price minus the discounted dividends follows a geometric Brownian motion. This makes the model more analytically tractable in many cases, but it implicitly assumes that all future dividends are known at time 0, which may not be realistic, for example, in the case of uncertain dividend payouts. Here, we consider a model in which the stock price process and dividend process are generated separately (but not necessarily independently) and provide the corresponding gradient estimators. We show that this estimator can also be used with slight modifications for another American-style option: a Bermudan call option defined on an underlying asset that pays dividends at a continuous rate.

## 2. THE CORRECTED ESTIMATOR

We first briefly recall the model with the associated notation. An American call option is defined on a stock that distributes dividend $D_j$ at time $t_j = \sum_{i=1}^{j} \tau_i (\tau_j > 0)$, $j = 1, \ldots, \eta(T)$, where $\eta(T)$ is the number of dividends distributed during the lifetime of the call contract, taking $\tau_{\eta(T)+1} = T - \sum_{i=1}^{\eta(T)} \tau_i$, $t_0 = 0$, $t_{\eta(T)+1} = T$ for notational convenience. Following standard models, we assume that the stock price drops after ex-dividend by the amount of the dividend (i.e., $S_{t_j^+} = S_{t_j^-} - D_j$). The American feature allows exercise of the option at any time up to and including the expiration date, but under the assumption of a frictionless market, the call option should only be exercised—if at all—right before an ex-dividend date or at the expiration date, resulting in the following threshold exercise policy: exercise at $t_j$ if $S_{t_j^-} > s_j \ (\geq K)$.

https://doi.org/10.1017/S0269964800143086 Published online by Cambridge University Press

The sample performance is the net present (at time 0) value of the option payoff (not the option value) and can be written as

$$L = \sum_{i=1}^{\eta(T)+1} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j, S_{t_i^-} > s_i\right\}(S_{t_i^-} - K)e^{-rt_i}, \tag{1}$$

where we define $s_{\eta(T)+1} = K$ for notational convenience, and we are interested in estimating $\partial E[L]/\partial\theta$. The indicator functions simply indicate whether or not the option has been exercised at each potential exercise point.

The stock price model in Fu and Hu [1] followed that of Stoll and Whaley [4] and others, in considering the dynamics of the stock price net of the present value of escrowed dividends, given by

$$\tilde{S}_t = h(Z; \tilde{S}_0, t, \mu, \sigma),$$

for some random vector $Z$ independent of the parameters. In particular, they took

$$\tilde{S}_{t_j} = h(Z_j; \tilde{S}_{t_{j-1}}, \tau_j, \mu, \sigma), \quad j = 1, \ldots, \eta(T) + 1,$$

with independent $Z_j \sim f_j$, probability density functions. The actual stock price process was then given by

$$S_t = \tilde{S}_t + \sum_{i=j+1}^{\eta(T)} D_i e^{-r(t_i - t)} \quad \text{for } t_j \le t < t_{j+1}, j = 0,1,\ldots,\eta(T).$$

In particular, at exercise decision points,

$$S_{t_j^-} = \tilde{S}_{t_j} + \sum_{i=j}^{\eta(T)} D_i \exp\left(-r \sum_{k=j+1}^{i} \tau_k\right), \quad j = 1, \ldots, \eta(T).$$

The estimator for $\partial E[L]/\partial\theta$ given in Fu and Hu [1, Eqs. (13)–(16)] was

$$\begin{aligned}
&\sum_{i=1}^{\eta(T)} \frac{\partial h^{-1}(y_i^*)}{\partial\theta} f_i(h^{-1}(y_i^*))\{E[L|S_{t_i^-} = s_i^-] - E[L|S_{t_i^-} = s_i^+]\} \\
&\quad + \sum_{i=1}^{\eta(T)} \left[\prod_{j=1}^{i-1} \mathbf{1}\{S_{t_j^-} \le s_j\}\right] \mathbf{1}\{S_{t_i^-} > s_i\} \frac{\partial}{\partial\theta}[(S_{t_i^-} - K)e^{-rt_i}] \\
&\quad + \prod_{j=1}^{\eta(T)} \mathbf{1}\{S_{t_j^-} \le s_j\} \frac{\partial}{\partial\theta}[(S_T - K)^+ e^{-rT}],
\end{aligned} \tag{2}$$

where

$$E[L|S_{t_i^-} = s_i^-] = E[(S_T - K)^+ e^{-rT}|S_{t_i^-} = s_i^-], \tag{3}$$

$$E[L|S_{t_i^-} = s_i^+] = (s_i - K)e^{-rt_i}, \tag{4}$$

$$y_i^* = (s_i - D_i; \tilde{S}_{t_{i-1}^-}, \tau_i, \mu, \sigma), \quad i = 1, \ldots, \eta(T), \tag{5}$$

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(we have corrected a typographical error of a missing tilde over $S_{t_{i-1}^-}$ in the definition of $y_i^*$).

The correct estimator is

$$\sum_{i=1}^{\eta(T)} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j\right\} \frac{\partial h^{-1}(y_i^*)}{\partial \theta} f_i(h^{-1}(y_i^*))$$

$$\times \left\{E\left[L \middle| \bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j, S_{t_i^-} = s_i^-\right] - (s_i - K)e^{-rt_i}\right\}$$

$$+ \sum_{i=1}^{\eta(T)} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j, S_{t_i^-} > s_i\right\} \frac{\partial}{\partial \theta}\left[(S_{t_i^-} - K)e^{-rt_i}\right]$$

$$+ \mathbf{1}\{S_{t_1^-} \le s_1, \ldots, S_{t_{\eta(T)}^-} \le s_{\eta(T)}\} \frac{\partial}{\partial \theta}\left[(S_T - K)^+ e^{-rT}\right], \qquad \textbf{(6)}$$

where

$$y_i^* = \left(s_i - \sum_{j=i}^{\eta(T)} D_j \exp\left(-r \sum_{k=i+1}^{j} \tau_k\right); \tilde{S}_{t_{i-1}}, \tau_i, \mu, \sigma\right), \quad i = 1, \ldots, \eta(T). \qquad \textbf{(7)}$$

Comparison of the two expressions reveals three errors in the previous estimator (2)–(5):

1. An incorrect expression for $y_i^*$, Eq. (5), was given.
2. An incorrect expression for $E[L|S_{t_i^-} = s_i^-]$, Eq. (3), was given.
3. Missing conditions involving $\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j$ in the first summation of Eq. (2).

In addition, there is an implicit, but unstated, assumption of independence that is generally satisfied for asset models of practical interest, including the derivations to be provided in the next section. Note that, otherwise, the two estimators match for $\eta(T) = 1$, the one-dividend estimator in Fu and Hu [1] being correct.

## 3. AN ALTERNATIVE STOCK PRICE MODEL

In the alternative stock price model, we separate the stock price and dividend processes in a natural way, as follows:

$$S_{t_j^-} = h(Z_j; S_{t_{j-1}^+}, \tau_j, r, \sigma), \quad j = 1, \ldots, \eta(T), S_{t_0^+} = S_0,$$

with, as before, independent $Z_j \sim f_j$, probability density functions, where at exercise decision points

$$S_{t_j^+} = S_{t_j^-} - D_j, \quad j = 1, \ldots, \eta(T).$$

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For this model, the same estimator given by Eq. (6) applies with one minor change in the definition of $y_i^*$ given by Eq. (7):

$$y_i^* = (s_i; S_{t_{i-1}^+}, \tau_i, \mu, \sigma), \quad i = 1, \ldots, \eta(T). \tag{8}$$

One caution to heed in using this model is that by separating the dividends from the stock price process, there is the possibility of obtaining negative stock prices if the dividend process is not properly gauged (correlated) with the stock price process (e.g., if the dividends are specified a priori). Of course, in general, the probability is infinitesimal, but the corresponding binomial tree model may have the undesirable properties of nonrecombining and negative nodes.

We now give a derivation for $\eta(T) = 2$, for which the sample performance is given by

$$L = \mathbf{1}\{S_{t_1^-} > s_1\}(S_{t_1^-} - K)e^{-rt_1} + \mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\}(S_{t_2^-} - K)e^{-rt_2}$$
$$+ \mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\}(S_T - K)^+ e^{-rT}.$$

The derivative of the first term with respect to $\theta$ is given similar to the term in the one-dividend derivation of Fu and Hu [1] by

$$\frac{\partial}{\partial\theta} E[\mathbf{1}\{S_{t_1^-} > s_1\}(S_{t_1^-} - K)e^{-rt_1}]$$
$$= -E\left[\frac{\partial h^{-1}(y_1^*)}{\partial\theta}(s_1 - K)e^{-rt_1} f_1(h^{-1}(y_1^*))\right]$$
$$+ E\left[\mathbf{1}\{S_{t_1^-} > s_1\}\frac{\partial}{\partial\theta}((S_{t_1^-} - K)e^{-rt_1})\right],$$

where $y_1^* = (s_1; S_0, \tau_1, \mu, \sigma)$. Taking the expectation of the second term of $L$ and using the independence of $Z_1$ and $Z_2$, we have

$$E[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\}(S_{t_2^-} - K)e^{-rt_2}]$$
$$= E[\mathbf{1}\{h(Z_1; S_0, \tau_1) \le s_1, h(Z_2; h(Z_1; S_0, \tau_1) - D_1, \tau_2) > s_2\}$$
$$\times (h(Z_2; h(Z_1; S_0, \tau_1) - D_1, \tau_2) - K)e^{-rt_2}]$$
$$= \int_{x_1=-\infty}^{h^{-1}(y_1^*)} \int_{x_2=h^{-1}(y_2^*(x_1))}^{\infty} (h(x_2; h(x_1; S_0, \tau_1) - D_1, \tau_2) - K)e^{-rt_2}$$
$$\times f_1(x_1) f_2(x_2)\, dx_2\, dx_1,$$

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where we have defined $y_2^*(x_1) = (s_2; h(x_1; S_0, \tau_1) - D_1, \tau_2)$. Differentiating and assuming an interchange of differentiation and expectation, we have

$$\begin{aligned}
&\frac{\partial}{\partial\theta} E[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\}(S_{t_2^-} - K)e^{-rt_2}] \\
&\quad = E\left[\frac{\partial h^{-1}(y_1^*)}{\partial\theta} f_1(h^{-1}(y_1^*)) E[\mathbf{1}\{S_{t_2^-} > s_2\}(S_{t_2^-} - K)e^{-rt_2} | S_{t_1^-} = s_1^-]\right] \\
&\qquad - E\left[\frac{\partial h^{-1}(y_2^*)}{\partial\theta} f_2(h^{-1}(y_2^*))\mathbf{1}\{S_{t_1^-} \le s_1\}(s_2 - K)e^{-rt_2}\right] \\
&\qquad + E\left[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\}\frac{\partial}{\partial\theta}((S_{t_2^-} - K)e^{-rt_2})\right],
\end{aligned}$$

where we have defined $y_2^* = (s_2; S_{t_1^+}, \tau_2, \mu, \sigma)$. Considering the third term, we have

$$\begin{aligned}
&E[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\}(S_T - K)^+ e^{-rT}] \\
&\quad = E[\mathbf{1}\{h(Z_1; S_0, \tau_1) \le s_1, h(Z_2; S_{t_1^-} - D_1, \tau_2) \le s_2\}e^{-rT} \\
&\qquad \times (h(Z_3; S_{t_2^-} - D_2, \tau_3) - K)^+] \\
&\quad = E\left[\int_{-\infty}^{h^{-1}(y_1^*)} \int_{-\infty}^{h^{-1}(y_2^*(x_1))} (h(Z_3; h(x_2; h(x_1; S_0, \tau_1) - D_1, \tau_2) - D_2, \tau_3) - K)^+ \right. \\
&\qquad \left. \times e^{-rT} f_1(x_1) f_2(x_2)\, dx_2\, dx_1\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{\partial}{\partial\theta} E[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\}(S_T - K)^+ e^{-rT}] \\
&\quad = E\left[\frac{\partial h^{-1}(y_1^*)}{\partial\theta} f_1(h^{-1}(y_1^*)) E[\mathbf{1}\{S_{t_2^-} \le s_2\}(S_T - K)^+ e^{-rT} | S_{t_1^-} = s_1^-]\right] \\
&\qquad + E\left[\frac{\partial h^{-1}(y_2^*)}{\partial\theta} f_2(h^{-1}(y_2^*)) E[\mathbf{1}\{S_{t_1^-} \le s_1\}(S_T - K)^+ e^{-rT} | S_{t_2^-} = s_2^-]\right] \\
&\qquad + E\left[\mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\}\frac{\partial}{\partial\theta}((S_T - K)^+ e^{-rT})\right].
\end{aligned}$$

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Combining all these results, we obtain the perturbation analysis (PA) estimator for the two dividends case:

$$\begin{aligned}
&\frac{\partial h^{-1}(y_1^*)}{\partial\theta} f_1(h^{-1}(y_1^*))(E[L|S_{t_1^-} = s_1^-] - (s_1 - K)e^{-rt_1}) \\
&\quad + \mathbf{1}\{S_{t_1^-} \le s_1\} \frac{\partial h^{-1}(y_2^*)}{\partial\theta} f_2(h^{-1}(y_2^*)) \\
&\quad \times (E[(S_T - K)^+ e^{-rT}|S_{t_1^-} \le s_1, S_{t_2^-} = s_2^-] - (s_2 - K)e^{-rt_2}) \\
&\quad + \mathbf{1}\{S_{t_1^-} > s_1\} \frac{\partial}{\partial\theta}[(S_{t_1^-} - K)e^{-rt_1}] \\
&\quad + \mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} > s_2\} \frac{\partial}{\partial\theta}[(S_{t_2^-} - K)e^{-rt_2}] \\
&\quad + \mathbf{1}\{S_{t_1^-} \le s_1, S_{t_2^-} \le s_2\} \frac{\partial}{\partial\theta}[(S_T - K)^+ e^{-rT}],
\end{aligned}$$

which matches Eq. (6) for $\eta(T) = 2$, with the modification on $y_i^*$ from Eq. (7) to Eq. (8).

For the general $\eta(T)$ dividends case, we have

$$\begin{aligned}
&\frac{\partial}{\partial\theta} E\left[\mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j, S_{t_i^-} > s_i\right\}(S_{t_i^-} - K)e^{-rt_i}\right] \\
&\quad = \sum_{j=1}^{i-1} E\left[\mathbf{1}\left\{\bigcap_{l=1}^{j-1} S_{t_l^-} \le s_l\right\} \frac{\partial h^{-1}(y_j^*)}{\partial\theta} f_j(h^{-1}(y_j^*))\right. \\
&\qquad \left. \times E\left[\mathbf{1}\left\{\bigcap_{m=j+1}^{i-1} S_{t_m^-} \le s_m, S_{t_i^-} > s_i\right\}(S_{t_i^-} - K)e^{-rt_i}\middle| S_{t_j^-} = s_j^-\right]\right] \\
&\quad - E\left[\mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j\right\} \frac{\partial h^{-1}(y_i^*)}{\partial\theta} f_i(h^{-1}(y_i^*))(s_i - K)e^{-rt_i}\right] \\
&\quad + E\left[\mathbf{1}\left\{\bigcap_{j=1}^{i-1} S_{t_j^-} \le s_j, S_{t_i^-} > s_i\right\} \frac{\partial}{\partial\theta}((S_{t_i^-} - K)e^{-rt_i})\right], \quad i = 1,2,\ldots,\eta(T),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial\theta} E\left[\mathbf{1}\left\{\bigcap_{j=1}^{\eta(T)} S_{t_j^-} \le s_j\right\}(S_T - K)^+ e^{-rT}\right] \\
&\quad = \sum_{j=1}^{\eta(T)} E\left[\mathbf{1}\left\{\bigcap_{l=1}^{j-1} S_{t_l^-} \le s_l\right\} \frac{\partial h^{-1}(y_j^*)}{\partial\theta} f_j(h^{-1}(y_j^*))\right. \\
&\qquad \left. \times E\left[\mathbf{1}\left\{\bigcap_{m=j+1}^{\eta(T)} S_{t_m^-} \le s_m\right\}(S_T - K)^+ e^{-rT}\middle| S_{t_j^-} = s_j^-\right]\right] \\
&\quad + E\left[\mathbf{1}\left\{\bigcap_{j=1}^{\eta(T)} S_{t_j^-} \le s_j\right\} \frac{\partial}{\partial\theta}((S_T - K)^+ e^{-rT})\right].
\end{aligned}$$

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Combining these results leads to the PA estimator given by Eq. (6) for general $\eta(T)$, with the modification on $y_i^*$ from Eq. (7) to Eq. (8).

To obtain the correct estimator for the original stock price model, the above derivations can be repeated with the following slight modifications. Define for notational convenience $D^{(i)} = \sum_{j=i}^{\eta(T)} D_j \exp(-r \sum_{k=i+1}^{j} \tau_k)$, and observe the following:

$$S_{t_i^-} > s_i \Leftrightarrow \tilde{S}_{t_i} + D^{(i)} > s_i \Leftrightarrow h(Z_i; \tilde{S}_{t_i}, \tau_i) > s_i - D^{(i)} \Leftrightarrow Z_i > h^{-1}(s_i - D^{(i)}; \tilde{S}_{t_i}, \tau_i).$$

## 4. A DIFFERENT OPTION

Another extension is the case in which the stock pays continuous dividends at a rate $\delta$, but early exercise is restricted to discrete points, say $\{t_i, i = 1, \ldots, \eta(T)\}$. The latter restriction makes this a Bermudan option. In this case, assume that the stock price net of the present value of escrowed dividends changes continuously according to

$$S_{t_j} = S_{t_j^-} = S_{t_j^+} = h(Z; S_{t_{j-1}}, \tau_j, \mu - \delta, \sigma), \quad j = 1, \ldots, \eta(T).$$

For this option, again the same estimator given by Eq. (6) is applicable with the change in the definition of $y_i^*$ given by Eq. (7) to the following:

$$y_i^* = (s_i; S_{t_{i-1}}, \tau_i, \mu - \delta, \sigma), \quad i = 1, \ldots, \eta(T).$$

Note that this stock model corresponds to a generalization of the first model, because the dividends are incorporated directly into the stock price dynamics.

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