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Self-organized field structures in electron-depleted multi-ion dusty plasma

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It is shown that there exists a strong interaction between the magnetic and kinetic aspects of a multi-ion plasma. The interaction appears as a system of simultaneous equations which show the alignment of vortices to flows and satisfy the Beltrami condition. Solving these equations lead to a non-force-free magnetic field which can be cast as a superposition of three multiscale force-free magnetic field configurations. It is the consequence of different Beltrami parameters of positive and negative ion fluids. It is also shown that self-organized paramagnetic and diamagnetic field structures could be created by varying the vorticities and flows of ion fluids.

1. Introduction

The topology of magnetic fields play a key role in the determination of plasma state, for instance, the force-free magnetic field configuration represents the self-organized state of a turbulent plasma (Taylor 1974). The force-free magnetic field reads as $\nabla \times \mathbf{B} = \alpha \mathbf{B}$, where α is a field line constant. Such a field whose vorticity becomes parallel to itself is called as Beltrami field. The Beltrami fields have been used to describe the twisted, sheared, spiral and helical structures in different plasma systems. It is thus reasonable to describe and understand the possible equilibrium states of a plasma in terms of Beltrami fields. The present work is an attempt to do so and devoted to study the Beltrami fields generated in multi-ion plasmas. The multi-ion plasmas are observed in Earth's auroral zone, magnetosphere, magnetotail, bow shock in front of the magnetopause boundary layers (Hultqvist et al. 1999; Stasiewics 2004), heliosphere (Schwenn and Marsch 1990), Saturn's magnetosphere (Shemansky and Hall 1992), Mercury's ionized exosphere and magnetosphere (Zurbuchen et al. 2008; Kim et al. 2013), magnetic clouds (Burlaga et al. 1998) and comets (Brinca and Tsurutani 1987; Chaizy et al. 1991). The multi-ion plasmas can also be generated in the laboratory (Liang et al. 1998; Oohara and Hatakeyama 2003; Oohara et al. 2005; Oohara and Hatakeyama 2007) and it significantly increases the need to study multi-ion plasmas.

The Hall magnetohydrodynamics model (HMHD) for a two-fluid plasma (ignoring electron inertia) introduces the emergence of multi Beltrami fields due to a singular perturbation (Yoshida and Mahajan 1999; Ohsaki and Yoshida 2005). The double Beltrami field (superposition of two Beltrami fields) which is the relaxed state of two fluid plasma is the well-known example of collisionless Hall effect (Mahajan

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and Yoshida 1998). When the inertial effects of both the species are taken into account, three multiscale Beltrami fields may appear in the system (Iqbal et al. 2008). The Beltrami parameters (ratios of generalized (canonical) vorticities to corresponding flows of the fluids constituting the plasma) also affect the creation of multi Beltrami fields. For instance, when the Beltrami parameters in an electron-depleted electronegative dusty plasma composed of positive ions and two-negative ions are equal, the plasma relaxes to double Beltrami field (Iqbal 2014). In the present work, it is shown that when the Beltrami parameters of two-negative ions are equal whereas the Beltrami parameter for positive ions is different, the electron-depleted dusty plasma self-organizes to triple Beltrami field (TCB). The TCB is the linear sum of three Beltrami fields.

The layout of the paper is as follows: In Sec. 2, beginning with the normalized momentum balance equations for the species, the steady-state solutions are obtained. The equilibrium magnetic field satisfies the triple curl Beltrami equation. In Sec. 3, it is shown that a linear sum of three force-free magnetic field configurations yield the triple curl Beltrami equation. Equation of state is described in Sec. 4 and a brief summary is given in Sec. 5.

2. Triple curl Beltrami field

We consider a magnetized and collisionless electronegative dusty plasma. The constituents of plasma are immobile charged dust grains, positive ions and two types of negative ions. The dust grain only play the role to keep the plasma neutral. All the ions are singly ionized and possess different inertia. We assume that $z_d n_d \gg n_e$, where z_d are the electrons attached to the surface of dust grains, n_d and n_e are the number density of dust grains and electrons respectively. As a result of the attachment of electrons to dust particles, the electrons are highly depleted (Mamun et al. 1996; Tribeche et al. 2008). The magnetic field affects the dust charging (Tsytovich et al. 2003). For simplicity, we have not taken this effect into account and assume that the dust charge does not vary. The normalized momentum balance equation for ions (i) is

$$\partial_t \mathbf{P}_i = \mathbf{V}_i \times (\nabla \times \mathbf{P}_i) - \nabla \psi_i, \tag{2.1}$$

where $\mathbf{P}_i = \mathbf{V}_i + \mathbf{A}$, $\psi_i = p_i + \phi + V_i^2/2$, where $p_i = n_i T_i$ is the pressure of ions with temperature T_i and density n_i. For the two negatively charged ions ($s = 1, 2$), the normalized macroscopic evolution equations read as

$$\partial_t \mathbf{P}_s = \mathbf{V}_s \times (\nabla \times \mathbf{P}_s) - \nabla \psi_s, \tag{2.2}$$

$\mathbf{P}_s = \mathbf{V}_s - M_s \mathbf{A}$, $M_s = m_i/m_s$ and $\psi_s = D_s p_s - M_s \phi + V_s^2/2$, where $p_s = n_s T_s$ is the pressure, n_s and T_s are the number density and temperature of s-species of negative ions respectively, $D_s = \rho_i/\rho_s$ ($\rho_i = n_i m_i$, $\rho_s = n_s m_s$) is the ratio of mass density of ions ρ_i to mass density of one of the negative ions ρ_s. We have used the identity $(\mathbf{V} \cdot \nabla)\mathbf{V} = \nabla \mathbf{V}^2/2 - \mathbf{V} \times (\nabla \times \mathbf{V})$ and the relation $\mathbf{E} = -\nabla\phi - c^{-1}\partial_t \mathbf{A}$, c is the speed of light. The magnetic field ($\mathbf{B} = \nabla \times \mathbf{A}$) is normalized to B_0 which is an arbitrary value of the field. All lengths are measured in terms of the skin depth of positive ion which reads as $\lambda_i = c\sqrt{m_i}/\sqrt{4\pi n_i e^2}$, e is the electronic charge. Time is normalized to $m_i c/eB_0$ – the gyroperiod of positive ions. Velocities are normalized to Alfvén speed $V_A = B_0/(4\pi n_i m_i)^{1/2}$, pressures are described in terms of $B_0^2/4\pi$, the scalar potential ϕ and the vector potential $\mathbf{A}$ are normalized to $B_0^2/4\pi n_i e$ and $V_A m_i c/e$, respectively. The pressures are taken as functions of density that is $p = p(\rho)$, therefore on taking

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curl of (2.1) and (2.2), we obtain the following equations:

$$\partial_t (\nabla \times \mathbf{P}_i) = \nabla \times [\mathbf{V}_i \times (\nabla \times \mathbf{P}_i)], \tag{2.3}$$

$$\partial_t (\nabla \times \mathbf{P}_s) = \nabla \times [\mathbf{V}_s \times (\nabla \times \mathbf{P}_s)]. \tag{2.4}$$

One of the possible simplest steady-state solutions of above equations is the Beltrami condition that is the flows become parallel to the corresponding vorticities. The Beltrami conditions for above equations read as

$$\nabla \times \mathbf{P}_i = k_1 \mathbf{V}_i, \tag{2.5}$$

$$\nabla \times \mathbf{P}_s = k_2 \mathbf{V}_s, \tag{2.6}$$

where k_1 and k_2 are proportionality constants and called as Beltrami parameters. The above equations show that canonical (generalized) vorticities become parallel to the corresponding velocities to achieve the relaxation.

The quasi neutrality condition is $n_i = n_1 + n_2 + z_d n_d$, where n_d is the number density of dust grains. All the densities are taken to be constant. Using Ampere's Law and the expression of current density $\mathbf{J} = e(n_i \mathbf{V}_i - n_1 \mathbf{V}_1 - n_2 \mathbf{V}_2)$, the normalized velocity of positive ions is given by

$$\nabla \times \mathbf{B} = \mathbf{V}_i - \sum_{s=1,2} N_s \mathbf{V}_s, \tag{2.7}$$

where $N_s = n_s / n_i$. Multiply (2.6) by N_s and adding, we obtain

$$\nabla \times \sum_{s=1,2} N_s \mathbf{P}_s = k_2 \sum_{s=1,2} N_s \mathbf{V}_s. \tag{2.8}$$

From (2.7) and (2.8), we have

$$\nabla \times \mathbf{V}_i - k_2 \mathbf{V}_i = \nabla \times \nabla \times \mathbf{B} - k_2 \nabla \times \mathbf{B} + \sum_{s=1,2} N_s M_s \mathbf{B}. \tag{2.9}$$

Using (2.5) into above equation, we obtain the velocity of ions as given below

$$\mathbf{V}_i = \frac{1}{k_1 - k_2} \left[\nabla \times \nabla \times \mathbf{B} - k_2 \nabla \times \mathbf{B} + (1 + \sum_{s=1,2} N_s M_s) \mathbf{B} \right]. \tag{2.10}$$

Putting the ion velocity into (2.5), we obtain

$$\nabla \times \nabla \times \nabla \times \mathbf{B} - a \nabla \times \nabla \times \mathbf{B} + b \nabla \times \mathbf{B} - c \mathbf{B} = 0, \tag{2.11}$$

where $a = k_1 + k_2$, $b = 1 + k_1 k_2 + \sum_{s=1,2} N_s M_s$ and $c = k_2 + k_1 \sum_{s=1,2} N_s M_s$. This is the triple curl Beltrami equation because it is equivalent to a linear sum of three different Beltrami fields. The system self-organizes to triple Beltrami field because the Beltrami parameters which are the ratio of generalized vorticities to corresponding flows are taken to be equal for both the negative ions while the Beltrami parameter for positive ions is different. Due to immobility of charged dust grains, the characteristic scale length of the system comes out to be skin depth of the positive ion. This relaxed state is mathematically equivalent to the relaxed state for a plasma system consisting of a single negative charge, positive charge and immobile charged dust grains (Iqbal and Shukla 2012). In the present scenario if either the Beltrami parameters for both the negatively charged ions are taken to be different or the mobility of dust grains is taken into account, the self-organized state would not be a triple Beltrami field. On the other hand if we assume that all the bulk inertia of dusty plasma is provided by streaming dust particles, the self-organized magnetic field will be no more triple Beltrami field and the characteristic scale length of the system will become the dust skin depth as shown by Shukla and Mahajan (2004). If we assume that there are

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no dust grains in the system, the model is still valid and we will obtain the same self-organized state as given by (2.11).

3. Analytical structure

Any Beltrami magnetic field $\mathbf{G}$ satisfies the conditions

$$\begin{cases} \nabla \times \mathbf{G} = \alpha \mathbf{G} & (\text{in } \Omega), \\ \mathbf{n} \cdot \mathbf{G} = 0 & (\text{on } \partial\Omega), \end{cases} \tag{3.1}$$

where α is a scalar constant, Ω is the bounded domain having a smooth boundary $\partial\Omega$ and $\mathbf{n}$ is the unit normal vector on $\partial\Omega$. The solution of (3.1) in cylindrical coordinates can be expressed by Chandrasekhar–Kendall functions (Chandrasekhar and Kendall 1957). Let us consider three Beltrami fields $\mathbf{B}_j$ $(j = 1, 2, 3)$ which satisfy the following Beltrami conditions:

$$\nabla \times \mathbf{B}_1 = \alpha_1 \mathbf{B}_1, \tag{3.2a}$$

$$\nabla \times \mathbf{B}_2 = \alpha_2 \mathbf{B}_2, \tag{3.2b}$$

$$\nabla \times \mathbf{B}_3 = \alpha_3 \mathbf{B}_3, \tag{3.2c}$$

where $\alpha's$ are constant and represent the eigenvalues of the curl operator. Adding the above equations and considering

$$\mathbf{B} = c_1 \mathbf{B}_1 + c_2 \mathbf{B}_2 + c_3 \mathbf{B}_3, \tag{3.3}$$

where c_j $(j = 1, 2, 3)$ are arbitrary constants and can be evaluated using the boundary conditions, we can retrieve the triple Beltrami equation (TCB) as given in (2.11). We can write down (2.11) in terms of eigenvalues $(\alpha_1, \alpha_2, \alpha_3)$ as

$$(\nabla \times -\alpha_1)(\nabla \times -\alpha_2)(\nabla \times -\alpha_3)\mathbf{B} = 0, \tag{3.4}$$

where a, b, and c are taken as

$$a = \alpha_1 + \alpha_2 + \alpha_3, \tag{3.5a}$$

$$b = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3, \tag{3.5b}$$

$$c = \alpha_1\alpha_2\alpha_3. \tag{3.5c}$$

The eigenstates α_j $(j = 1, 2, 3)$ are the inverse of length and show the scale size of the magnetic field structures formed when the plasma gets the equilibrium state. The $\alpha's$ are real and may be complex if the domain under consideration is multiply connected (Yoshida and Giga 1990). The eigenstates are the roots of the following cubic equation:

$$\alpha^3 - a\alpha^2 + b\alpha - c = 0. \tag{3.6}$$

The roots of the above equation are

$$\alpha_1 = w - f + \frac{k_1 + k_2}{3}, \tag{3.7}$$

$$\alpha_2 = \frac{(f - w)}{2} - \frac{i\sqrt{3}(f + w)}{2} + \frac{k_1 + k_2}{3}, \tag{3.8}$$

$$\alpha_3 = \frac{(f - w)}{2} + \frac{i\sqrt{3}(f + w)}{2} + \frac{k_1 + k_2}{3}, \tag{3.9}$$

where $w = (-(g/2) + \sqrt{R})^{1/3}$, $f = ((g/2) - \sqrt{R})^{1/3}$ and $R = (g/2)^2 + (h/3)^3$. When $R = 0$, all roots are real and at least two are equal. When $R > 0$, one of the roots is real whereas other two are complex conjugate and for $R < 0$, all three roots are real.

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4. Bernoulli equation

Substituting (2.5) and (2.6) into (2.1) and (2.2) respectively, the resulting steady state equations are

$$\nabla\psi_i = 0, \tag{4.1}$$

$$\nabla\psi_s = 0. \tag{4.2}$$

Integrating the above equations, we obtain

$$\psi_i = C_i, \tag{4.3}$$

$$\psi_s = C_s, \tag{4.4}$$

where C_i and C_s are integration constants. Adding the above equations, we obtain

$$\psi_i + \textstyle\sum_s \psi_s = C, \tag{4.5}$$

where $C = C_i + \sum_s C_s$. This is the generalized Bernoulli equation as it relates the pressure, electric field and flow. This version of Bernoulli equation results through relaxation of multi-ion plasma and the constant is global whereas in case of ordinary fluids, the constant is the function of streamlines. Equation (4.5) suggests the possibility of creating a finite pressure through gradient of flows and electric potential.

5. Self-organized field structures

The solution of (2.11) is the triple Beltrami field which is equivalent to sum of three Beltrami fields. In an axisymmetric one-dimensional cylindrical plasma, the solution is given by

$$B_r = 0, \tag{5.1}$$

$$B_p = c_1 J_0(\alpha_1 r) + c_2 J_0(\alpha_2 r) + c_3 J_0(\alpha_3 r), \tag{5.2}$$

$$B_t = c_1 J_1(\alpha_1 r) + c_2 J_1(\alpha_2 r) + c_3 J_1(\alpha_3 r), \tag{5.3}$$

where J_0 and J_1 are zero and first order Bessel functions respectively, c_1, c_2 and c_3 are constants and can be determined using the boundary conditions. The cylinder is considered to be a limiting case of a large aspect ratio torus. Hence we have written $B_{p(t)}$ instead of $B_{\theta(z)}$, where $p(t)$ stands for poloidal and toroidal. If we take $B_t(r=0) = b_1$, $J_t = |(\nabla \times \boldsymbol{B})_t|_{r=0} = b_2$ and $J_p = |(\nabla \times \boldsymbol{B})_p|_{r=d} = b_3$, where d is the radius of cylinder, the constants read as

$$c_j = \frac{u_j}{q}, \qquad (j = 1, 2, 3), \tag{5.4}$$

where

$$\begin{aligned}
u_1 &= b_3(\alpha_3 - \alpha_2) + (b_2 - b_1\alpha_3)\alpha_2 J_1(\alpha_2 d) + (b_1\alpha_2 - b_2)\,\alpha_3 J_1(\alpha_3 d),\\
u_2 &= b_3(\alpha_1 - \alpha_3) + (b_1\alpha_3 - b_2)\alpha_1 J_1(\alpha_1 d) + (b_3 - b_1\alpha_1)\,\alpha_3 J_1(\alpha_3 d),\\
u_3 &= b_3(\alpha_2 - \alpha_1) + (b_3 - b_2\alpha_2)\alpha_1 J_1(\alpha_1 d) + (b_1\alpha_1 - b_2)\,\alpha_2 J_1(\alpha_2 d),\\
q &= (\alpha_3 - \alpha_2)\alpha_1 J_1(\alpha_1 d) + (\alpha_1 - \alpha_3)\alpha_2 J_1(\alpha_2 d) + (\alpha_2 - \alpha_1)\alpha_3 J_1(\alpha_3 d).
\end{aligned}$$

In order to show the radial profiles of toroidal and poloidal magnetic fields, we consider a dusty plasma which contains one positive ion $\left(Ar^+\right)$ and two negative ions $\left(F^- - SF_6^-\right)$ in addition to heavy dust grains (El-Tantawy and Moslem 2012). In this case, $M_1 = m_i/m_1 = 40/19$ where m_i is the mass of Ar^+, m_1 is the mass of F^- and $M_2 = m_i/m_2 = 40/146$, where m_2 represents the mass of SF_6^-, respectively.

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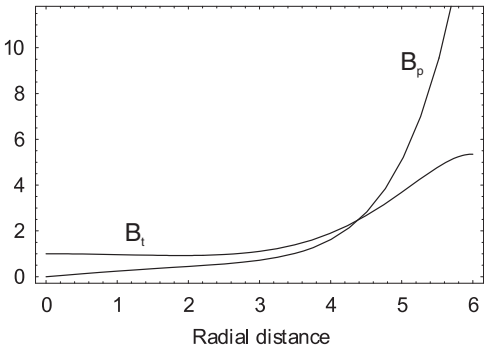


FIGURE 1. The radial profiles of magnetic fields for $k_1 = 1.0$ and $k_2 = 0.3$. The system size is $6\lambda_i$.

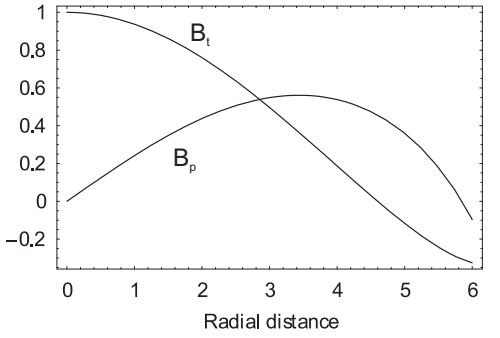


FIGURE 2. The radial profiles of magnetic field for $k_1 = 1.01$ and $k_2 = 0.09$. The system size is $6\lambda_i$.

We assume that $N_j = n_j/n_i = 0.25$, that is the number density of negative ions is one fourth of the positive ion. The size of the system is $6\lambda_i$ and the boundary values are taken as $b_1 = 1.0$, $b_2 = 0.5$ and $b_3 = 0.1$.

Figure 1 shows the field profiles for $k_1 = 1.0$ and $k_2 = 0.3$, where k_1 is the ratio of generalized vorticity of ions to their flows (see (2.5)) and k_2 represents the ratio of generalized vorticities of negative ions to their respective flows. For this case, the scale parameters are $\alpha_1 = 0.606997$, $\alpha_2 = 0.346502 + 1.16366i$ and $\alpha_3 = 0.346502 - 1.16366i$, and we observe that relaxed fields are diamagnetic. Figure 2 shows that magnetic fields are paramagnetic and these profiles are for $k_1 = 1.01$ and $k_2 = 0.09$. The scale parameters are $\alpha_1 = 0.498429$, $\alpha_2 = 0.300786 + 1.13815i$ and $\alpha_3 = 0.300786 - 1.13815i$. In both the cases one of the scale parameters is real whereas other twos are complex conjugate. It is evident from Figs 1 and 2, that the ratio of the generalized vorticities to flows plays an important role to self-organize the magnetic field towards paramagnetism or diamagnetism.

6. Summary

A relaxation model of multi-ion dusty plasma is developed where all the inertial and non-inertial effects are considered. It is shown that generalized vorticities and flows have a strong impact on creation of equilibrium magnetic field structures. For different Beltrami parameters of positive and negative ion fluids (k_1 and k_2), the magnetic field self-organizes to triple Beltrami field. The triple Beltrami field being

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the sum of three Beltrami fields is characterized by three scale parameters $(\alpha_1, \alpha_2, \alpha_3)$. It depicts the possibility of the creation of three relaxed vortices of different scale lengths. A small value scale parameter will give rise to a large structure because the size of relaxed structures depend on inverse of the scale parameter. The other two structures corresponding to remaining scale parameters will be of the order of ion skin depth. On the other hand, when the Beltrami parameters are same that is $k_1 = k_2 = k$, the relaxed magnetic field will be double Beltrami (sum of two force-free states) as shown in the previous work (Iqbal 2014). The present work is important to study the formation of large scale structures in multi-ion plasmas. As the positive ions and negative ions in addition to negative charge dust impurities are found in different space settings, the present results could be employed to analyze the magnetic field structures formed over there and to understand the underlying physics. The present work only deals with the formation of self-organized magnetic fields but it is important to study the stability of the self-organized structures. The work is underway and will be published somewhere else.

REFERENCES

Brinca, A. L. and Tsurutani, B. T. 1987 *Astron. Astrophys.* **187**, 311.
Burlaga, L. et al. 1998 *J. Geophys. Res.* **103**, 277.
Chaizy, P. et al. 1991 *Nature* **349**, 393.
Chandrasekhar, S. and Kendall, P. C. 1957 *Astrophys. J.* **126**, 457.
El-Tantawy, S. A. and Moslem, W. M. 2012 *Astrophys. Space Sci.* **337**, 219.
Hultqvist, B., Øieroset, M., Paschmann, G. and Treumann, R. 1999 *Magnetospheric Plasma Sources and Losses.* Dordrecht: Kluwer.
Iqbal, M. 2014 *J. Plasma Phys.* **80**, 59.
Iqbal, M., Berezhiani, V. I. and Yoshida, Z. 2008 *Phys. Plasmas* **15**, 032 905.
Iqbal, M. and Shukla P. K. 2012 *Phys. Plasmas* **19**, 033 517
Kim, E.-H., Johnson, J. R., Lee, D.-H. and Pyo, Y. S. 2013 *Earth Planets Space* **65**, 447.
Liang, P. E., Wilks, S. C. and Tabak, M. 1998 *Phys. Rev. Lett.* **81**, 4887.
Mahajan, S. M. and Yoshida, Z. 1998 *Phys. Rev. Lett.* **81**, 4863.
Mamun, A. A., Cairns, R. A. and Shukla, P. K. 1996 *Phys. Plasmas* **3**, 702.
Ohsaki, S. and Yoshida, Z. 2005 *Phys. Plasmas* **12**, 064 505.
Oohara, W., Date, D. and Hatakeyama, R. 2005 *Phys. Rev. Lett.* **95**, 175 003.
Oohara, W. and Hatakeyama, R. 2003 *Phys. Rev. Lett.* **91**, 205 005.
Oohara, W. and Hatakeyama, R. 2007 *Phys. Plasmas* **14**, 055 704.
Schwenn, R. and Marsch, E. 1990 *Physics of the Inner Helioshere: Large Scale Phenomena.* Berlin: Springer.
Shemansky, D. E. and Hall, D. T. 1992 *J. Geophys. Res.* **97**, 4143.
Stasiewics, K. 2004 *Phys. Rev. Lett.* **12** 125 004.
Shukla, P. K. and Mahajan S. M. 2004 *Phys. Lett.* A **328**, 185.
Taylor, J. B. 1974 *Phys. Rev. Lett.* **33**, 1139.
Tribeche, M., Younasi, S. and Zerguini, T. H. 2008 *Can. J. Phys.* **86**, 975.
Tsytovich, V. N., Sato, N. and Morfill, G. E. 2003 *New J. Phys.* **5**, 43.
Yoshida, Z. and Giga, Y. 1990 *Math. Z.* **204**, 235.
Yoshida, Z. and Mahajan, S. M. 1999 *J. Math. Phys.* **40**, 5080.
Zurbuchen, T. H. et al. 2008 *Science* **321**, 90.

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