J. Plasma Physics (1998), *vol.* 59, *part* 4, *pp.* 671–682. Printed in the United Kingdom
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Non-universality of anomalous transport

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(Received 9 October 1997 and in revised form 9 November 1997)

For low-dimensional Hamiltonian systems with chaotic dynamics, we discuss the differences in the kinetic description, and the necessity of windowing of the time and parameter space, which is imposed by the differences in the phase-space topology and by the stickiness of trajectories at island boundaries.

1. Introduction

After Boltzmann's revolutionary work, the investigation of the kinetic description of physical systems became an important part of general physics research. Such research was stimulated not only by attempts to resolve the enigma of entropy growth, but, more importantly, by the development of a new kind of universal description of physical systems that obey statistical laws of behaviour. Along this path, the well-known work of Prigogine (1962) and Balescu (1997) created a basic approach to the generalized form of kinetic equation called the master equation. Typically, the kinetic description includes different approximations, in which the time scale arrangement, smallness of perturbations and, sometimes, large number of degrees of freedom play crucial roles. The theory of the kinetic description of physical processes is fairly complicated. Nevertheless, some of its important features can be mentioned without describing the details. The existence of a sufficiently small time scale of collisions and sufficiently rapid decay of correlations are responsible for the memory of dynamical processes. Short-memory dynamics is an important feature of the randomness that particle trajectories should obey.

It was found fairly recently that numerous physical processes do not satisfy the principle of short-memory randomness and that long power-like tails in distributions of different time scales can frequently occur in physical systems. The corresponding kinetic description for such cases involves new notions, such as Lévy processes, Lévy random walks, fractal Brownian motion, etc. (Montroll and Shlesinger 1984). Numerous applications of the new concept in kinetics were discussed in Shlesinger *et al.* (1993), where a notion of 'strange kinetics' was used. It seems that strange kinetics can be applied to a wide class of phenomena related to chaotic dynamics. The corresponding 'fractional' kinetic equation was derived in Zaslavsky (1994*a*, *b*) and applied to Hamiltonian dynamical systems with few degrees of freedom and intrinsic chaotic dynamics (Zaslavsky *et al.* 1997; Zaslavsky and Niyazov 1997). For cases where fractional kinetics can be applied, there is a fractal support in the phase-space–time description of the system. The origin of the fractal support lies in a strong non-uniformity of the phase space, where special zone(s) of particle

https://doi.org/10.1017/S002237789800659X Published online by Cambridge University Press

trapping exist. We call these zones 'traps' or 'quasi-traps'. The latter zones are related to Hamiltonian systems where absolute traps do not exist. The (quasi-)traps can have fractal structures, and so can the set of time intervals that the trajectory spends in the traps (Zaslavsky *et al.* 1993, 1997; Zaslavsky and Niyazov 1997). This is how we lose the universality of the kinetic description of a system with chaotic dynamics, because it now depends on the local space–time properties of the support, or, more generally speaking, on the phase-space topology. The topology of dynamical systems does not persist when a critical parameter exhibits a finite change, and only a special condition of the new approximation can bring us again to the universal kinetics (Zaslavsky *et al.* 1997; Zaslavsky and Niyazov 1997). Some comments on this issue are the subject of this paper.

2. Topological zoo

A typical structure of the phase space of a Hamiltonian system with finite chaotic motion consists of a so-called 'stochastic sea' and 'islands' imbedded in this sea. In fact, there are also other structures, such as cantori and stable/unstable manifolds. As a result, we see what is presented in Figs 1–3. Figure 1 (Zaslavsky *et al.* 1997; Zaslavsky and Niyazov 1997) shows the mapping of a trajectory for the web map

$$u_{n+1} = v_n, \quad v_{n+1} = -u_n - K \sin v_n \quad (u_n, v_n = \mathrm{mod}\ 2\pi), \tag{2.1}$$

with the only parameter being K. The dark area near the islands corresponds to 'stickiness' (Meiss and Ott 1985), a theory of which does not yet exist. This zone can be considered a singular one, in which a particle is trapped for a longer time. The 'trap' has a fine structure (Fig. 1 *b*) that displays a new set of islands, again with sticky boundaries. This hierarchy of islands continues to infinity, creating a fractal or multifractal structure of the trap. The remaining part of the stochastic sea outside the islands appears uniform, although it only requires higher resolution to see more structures.

The second example (Fig. 2) is for the standard map

$$p_{n+1} = p_n - K \sin x_n, \quad x_{n+1} = x_n + p_{n+1} \pmod{2\pi}, \tag{2.2}$$

where the trapping zone looks as a set of strips (Figs 2 *a*, *b*) which disappear near the bifurcation point (Fig. 2 *c*). Only a fairly small change of the parameter K is needed to make such a big difference in the topology.

The third example (Fig. 3) is for the perturbed pendulum model (Zaslavsky 1995)

$$\ddot{x} + \sin x = -\epsilon \sin(x - \nu t), \tag{2.3}$$

where one can see the stochastic layer with a strip-shaped trapping zone along the layer. The strip, in turn, is composed of multiple strips.

Despite the fact that these three examples correspond to three different models, all of them can occur in any model, depending on the parameter value. The dark domain in Figs 1–3 corresponds to a non-uniformity of phase space and consequently to a non-uniformity of the level of randomness of different parts of trajectories. This situation leads to absence of uniformity of the kinetics for different time scales and to different strategies in the statistical description of chaotic dynamics.

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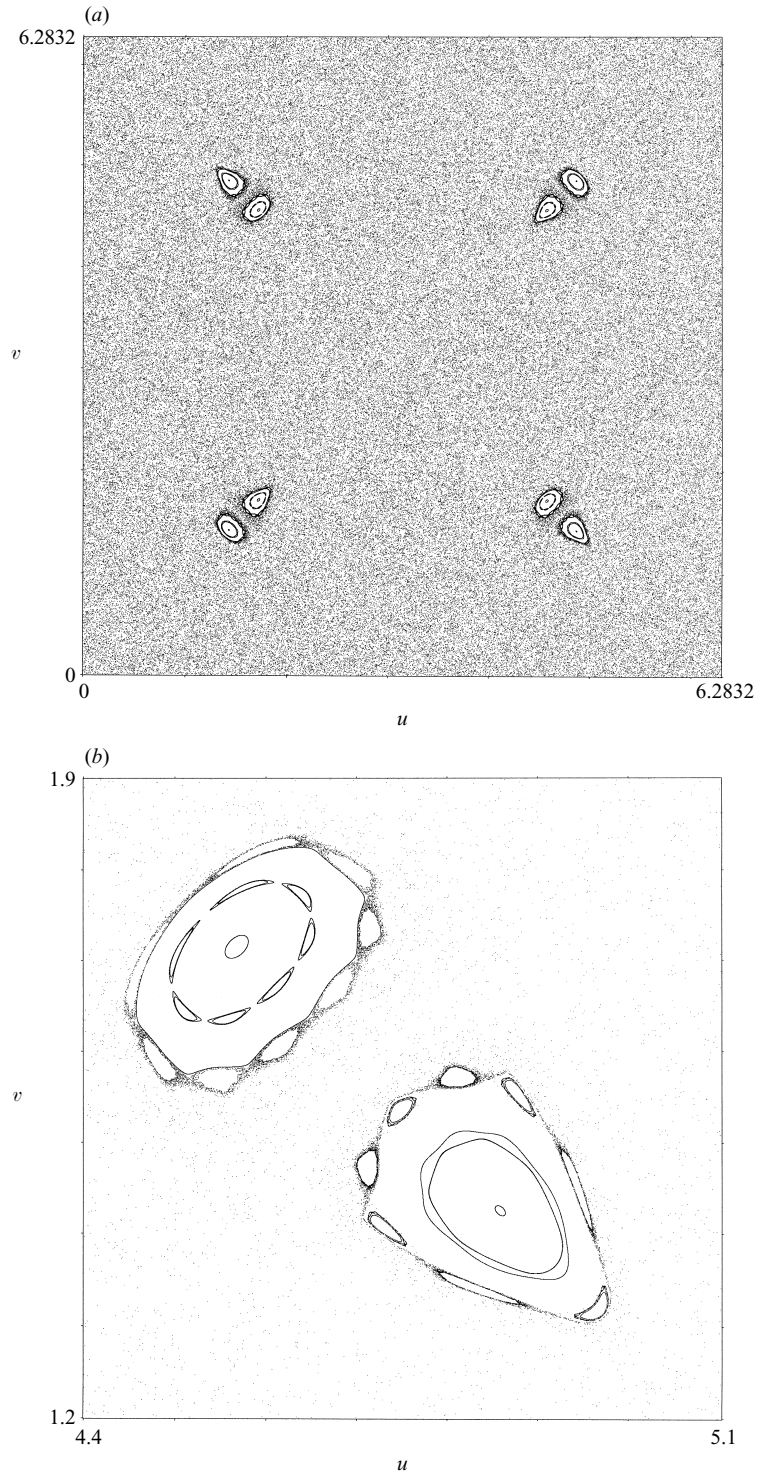


Figure 1. Phase portrait for the web map with parameter $K = 6.349\,972$: (*a*) full phase portrait; (*b*) magnification of the island set.

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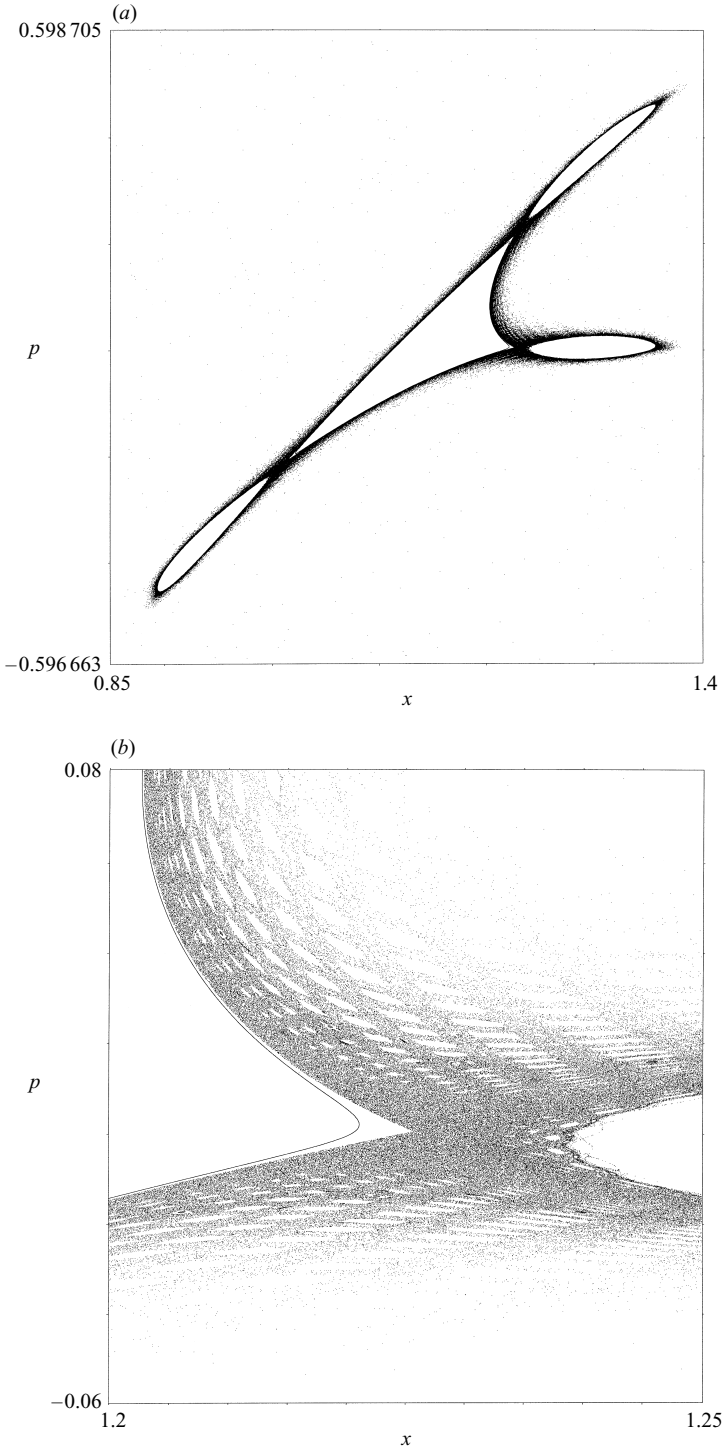


Figure 2 (a, b). For caption see facing page.

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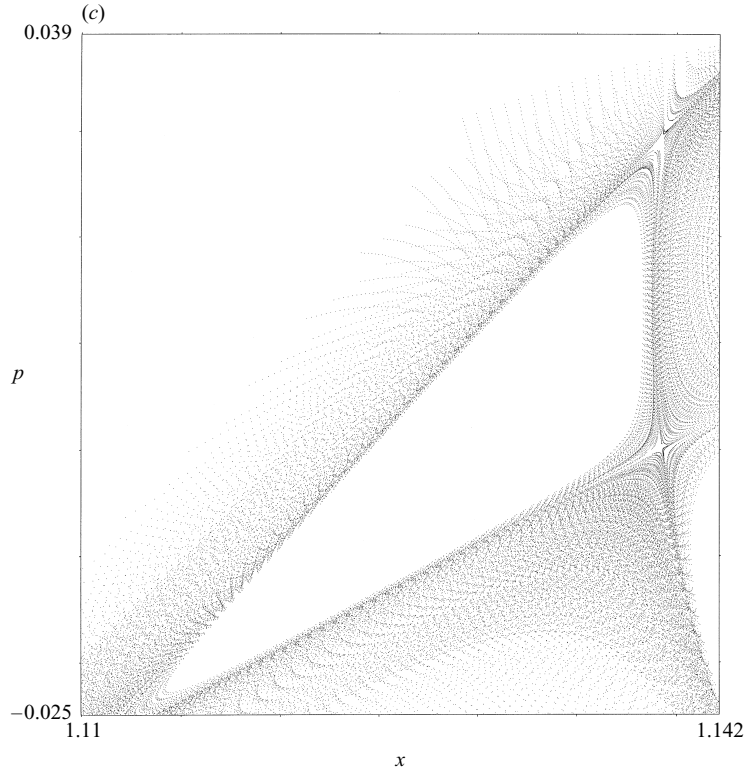


Figure 2. (*a*) Trapping zone (dark area) for the standard map with $K = 6.9011$. (*b*) Its magnification with slightly different parameter $K = 6.9009$. (*c*) Disappearance of the trapping zone for $K = 6.9500$.

3. Time and parameter windowing

Depending on the time of observation of the system (say 10^{11}), we can neglect some singular zones, i.e. the zones of stickiness that have too small a probability of capturing a trajectory. Only a few singular zones are responsible for the observable asymptotics of the trajectories and their characteristic time windows can overlap. Assuming, for example, that the dynamics in each singular zone can be characterized by a kind of self-similarity, one can consider the general situation of motion as multifractal in space and time. To avoid so high a level of complexity of motion, let us introduce parameter windows in such a way that for a given parameter window there are well-separated time scales of particle residence in singular zones, the i.e. the characteristic times of trapping into different domains of the phase space differ greatly. This provides an opportunity to select a parameter window and a corresponding time window for which an intermediate asymptotics can be considered, and a fractional kinetic equation can be introduced. Such a case is similar to fractal rather than multifractal dynamics. Some of the windows are described in Zaslavsky *et al.* (1997), Zaslavsky and Niyazov (1997) and Benkadda *et al.* (1997).

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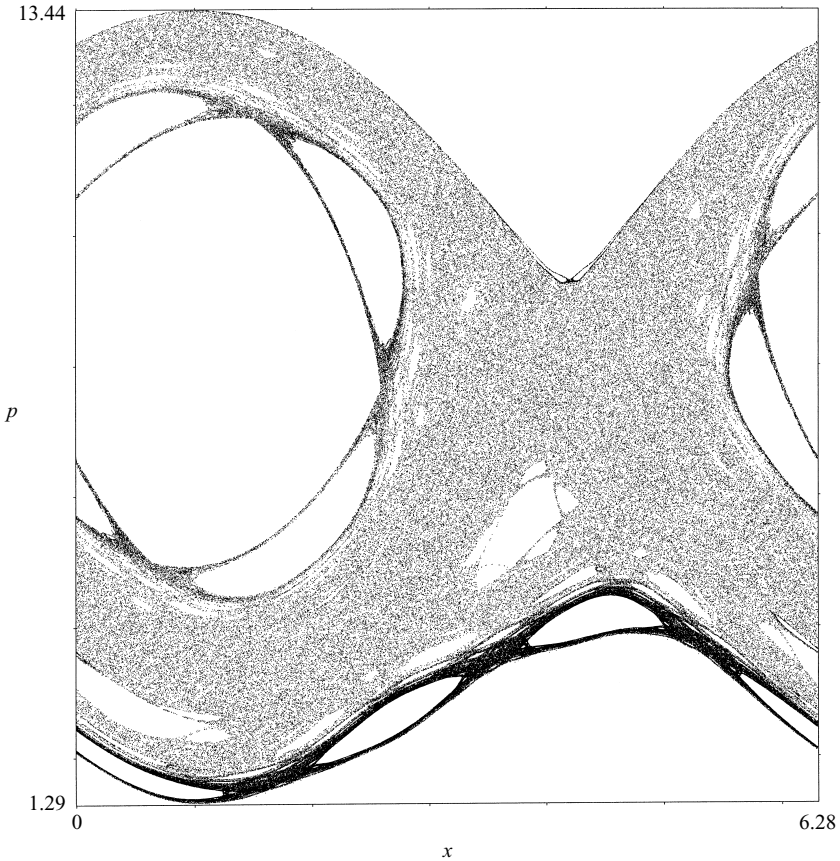


Figure 3. Trapping zone for the perturbed pendulum with $\epsilon = 7.860\,55$ and $\nu = 8.57$.

4. Fractal properties of Poincaré recurrences and exit time

The function

$$P_R(t) = \lim_{\Delta\Gamma \to 0} \frac{1}{\Delta\Gamma} P_R(t; \Delta\Gamma) \tag{4.1}$$

can be considered as a probability density of the first recurrence to an infinitesimal phase volume $\Delta\Gamma$ after a time interval t with the normalization condition

$$\int_0^\infty P_R(t)\, dt = 1. \tag{4.2}$$

The existence of the limit (4.1) for Hamiltonian systems is guaranteed by Kac's (1958) theorem if the dynamics is finite and the corresponding dynamically induced measure is non-zero. The last property has not yet been proved for typical Hamiltonian systems. From the same theorem, the mean recurrence time

$$\tau_R = \int_0^\infty t P_R(t)\, dt \tag{4.3}$$

is finite. If, for example, the asymptotic behaviour of $P_R(t)$ is

$$P_R(t) \sim \frac{1}{t^\gamma} \quad (t \to \infty) \tag{4.4}$$

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then it follows from (4.3) and the condition $\tau_R < \infty$, that

$$\gamma > 2. \tag{4.5}$$

The set of recurrence cycles

$$\{\tau_j\} = \{t_{j+1} - t_j\}, \tag{4.6}$$

where $t_1, t_2, \ldots$, is the set of the instants of recurrences, can possess fractal or multifractal properties. The corresponding analysis can be found in Afraimovich (1997), Afraimovich and Zaslavsky (1997), Zaslavsky *et al.* (1997) and Zaslavsky and Niyazov (1997). More precisely, one can say that, for example, for (2.1) and (2.2), there are windows of K for which the distribution (4.4) is valid and the corresponding value of γ can be expressed through the local characteristics of the singular zone related to the time window (Zaslavsky *et al.* 1997; Zaslavsky and Niyazov 1997).

A characteristic similar to $P_R(t, \Delta\Gamma)$ can be introduced for the escape of trajectories from $\Delta\Gamma$ at a time instant t. Let $\psi(t; \Delta\Gamma)$ be such a probability density, to escape from $\Delta\Gamma$, which is normalized as

$$\int_0^\infty \psi(t, \Delta\Gamma)\, dt = 1, \tag{4.7}$$

and let a simple power-like law be valid asymptotically:

$$\psi(t; \Delta\Gamma) \sim \frac{1}{t^{1+\beta}}, \quad (t \to \infty), \tag{4.8}$$

with some exponent β. For the case of one leading asymptotic in special parameter and time windows, we have trivially

$$\gamma = 1 + \beta \tag{4.9}$$

if $\Delta\Gamma$ in $\psi(t; \Delta\Gamma)$ is taken in the singular zone and $\Delta\Gamma$ in $P_R(t; \Delta\Gamma)$ is taken anywhere outside the singular zone (for more details see Zaslavsky *et al.* 1997; Zaslavsky and Niyazov 1997). This is so because a trajectory should visit and then escape from the singular zone for the large-t asymptotics, and thus the residence time in the sticky domain dominates the other part of the trajectory in the non-sticky domain.

A short comment here can help to understand the role of different values of the exponents β or γ. Consider the probability $\Psi(T; \Delta\Gamma)$ for the particle to escape from the domain $\Delta\Gamma$ during a time $t > T$. Using (4.8), we have asymptotically

$$\Psi(T; \Delta\Gamma) \sim \int_T^\infty dt\, \psi(t; \Delta\Gamma) \sim \frac{1}{T^\beta} \quad (T \to \infty) \tag{4.10}$$

The larger the value of β, the less is the probability for a particle to be trapped in the singular zone during the time $t > T$.

Another property describes the self-similarity of the escape probability $\Psi(t; \Delta\Gamma)$. Let us consider the renormalization transform

$$\hat{R}_\lambda(\Psi(T; \Delta\Gamma) = \Psi(\lambda T; \Delta\Gamma). \tag{4.11}$$

It follows asymptotically from (4.10) and (4.8) that

$$\Psi(\lambda T; \Delta\Gamma) = \lambda^\beta \Psi(T; \Delta\Gamma) \quad (T \to \infty) \tag{4.12}$$

The property (4.12) will be useful in the following consideration.

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5. Fractal time and quasi-traps

The concept of fractal time is based on the fractal properties of a time set $\{t_j\}$ where the instants t_j are related to the characteristic set of events, such as step length in a random walk (Montroll and Shlesinger 1984; Shlesinger 1988; Klafter *et al.* 1996). The application of the fractal time concept to Hamiltonian chaotic dynamics requires some modifications (Zaslavsky 1995; Afraimovich 1997; Afraimovich and Zaslavsky 1997; Zaslavsky *et al.* 1997; Zaslavsky and Niyazov 1997).

Let the finite phase space Γ_0 (the outermost square in Fig. 4) be filled almost uniformly by a chaotic trajectory except for a relatively small domain $\Gamma_1 \subset \Gamma_0$ (the intermediate square in Fig. 4) where the trajectory is almost uniform except for a smaller domain $\Gamma_2 \subset \Gamma_1$ (the innermost square in Fig. 4), and so on. We assume the existence of a set

$$\Gamma_0 \supset \Gamma_1 \supset \dots \tag{5.1}$$

with a space-scaling constant λ_Γ:

$$\lambda_\Gamma = \frac{\Gamma_{n+1}}{\Gamma_n} < 1 \quad \forall n. \tag{5.2}$$

It follows from (5.2) that

$$\Gamma_n = \lambda_\Gamma^n \Gamma_0. \tag{5.3}$$

Now introduce the time T_0 that the trajectory spends in $\Gamma_0 - \Gamma_1$, the time T_1 that the trajectory spends in $\Gamma_1 - \Gamma_2$, and so on. Assume the existence of a time-scaling constant λ_T:

$$\lambda_T = \frac{T_{n+1}}{T_n} > 1 \quad \forall n, \tag{5.4}$$

which means that the smaller the domain Γ_j, the larger the residence time of a particle in the domain. This is a way to construct a trap, which corresponds to the possibility of approaching a singular zone (or a set in phase space) where the Lyapunov exponent approaches zero.

Now let us make some simple transformations. It follows from (5.4) that

$$T_n = \lambda_T^n T_0, \tag{5.5}$$

or

$$n = \frac{\ln(T_n/T_0)}{\ln \lambda_T}. \tag{5.6}$$

The substitution of (5.6) into (5.3) gives

$$\frac{\Gamma_n}{\Gamma_0} = \left(\frac{T_0}{T_n}\right)^{\mu_\Gamma}, \tag{5.7}$$

with

$$\mu_\Gamma = \frac{|\ln \lambda_\Gamma|}{\ln \lambda_T}. \tag{5.8}$$

Let us now use the asymptotic formula (4.4) for the recurrence probability density and the asymptotic condition for the escape probability density,

$$\psi(t; \Delta\Gamma) \sim P_R(t) \sim \frac{1}{t^\gamma}, \tag{5.9}$$

where the location of the domain $\Delta\Gamma$ will be defined below. The phase volume G_n

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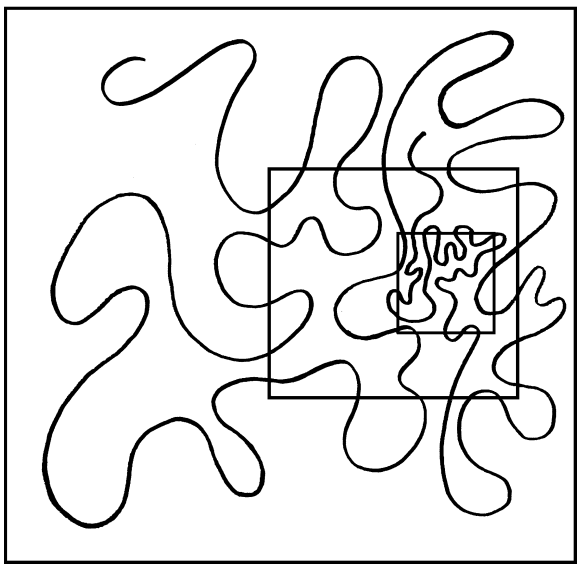


Figure 4. Sketch of the quasi-trap.

of particles that pass the domain Γ_n during the escape time $t > T_n$ is of the order (Meiss 1997)

$$G_n \sim \int_{T_n}^{\infty} t\psi(t;\Gamma_n)\,dt \sim \frac{1}{T_n^{\gamma-2}}, \tag{5.10}$$

in accordance with (5.9). These particles guarantee the recurrence after T_n. The mean time of such recurrences with $t > T_n$ is

$$t_{\rm rec} \sim \frac{1}{G_n} \sim T_n^{\gamma-2}. \tag{5.11}$$

On the other hand, this time is proportional to the inverse dimensionless phase volume Γ_n, i.e.

$$t_{\rm rec} \sim \frac{\Gamma_0}{\Gamma_n}. \tag{5.12}$$

Using (5.7) and comparing (5.10) and (5.11), we get in the limit $n \to \infty$

$$\gamma = 2 + \mu_\Gamma = 2 + \frac{|\ln \lambda_\Gamma|}{\ln \lambda_T}. \tag{5.13}$$

This result coincides with the result in Zaslavsky *et al.* (1997) and Zaslavsky and Niyazov (1997) obtained by the renormalization group approach. A qualitative way of obtaining the expression (5.12) shows how the quasi-trap can work in Hamiltonian systems. The described construction of the quasi-trap corresponds to the situation considered in Zaslavsky *et al.* (1997) and Zaslavsky and Niyazov (1997) when islands around islands make a fractal-type hierarchy.

6. Fractional kinetics

There are a few ways to map a trajectory. It follows from (2.1) that the trajectory (u_n, v_n) can be mapped onto the torus $u_n \in (0, 2\pi)$, $v_n \in (0, 2\pi)$, as shown on the phase portrait in Fig. 1(*a*). The same trajectory can be mapped onto the infinite

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unbounded phase space $u_n \in (-\infty, \infty)$, $v_n \in (-\infty, \infty)$. In the latter case, one can speak about kinetics in infinite phase space. Both approaches to the dynamics are equivalent since they use the same information, namely the full set of points of the trajectory $(u_0, v_0), (u_1, v_1), \ldots$, but for the kinetics we use the infinite phase space. Intermediate cases can also be useful, depending on the physical problem to be studied. In (2.2), we consider a cylindrical phase space where $p_n \in (-\infty, \infty)$, $x_n \in (0, 2\pi)$. Such a map is convenient for considering diffusion along the momentum axis p. By lifting the operator $x(\mathrm{mod}\, 2\pi)$, we arrive at a doubly infinite phase space in which the kinetics in both the x and p directions can be studied. When speaking about kinetics, we shall always have in mind an unbounded domain along the diffusional variable(s).

The idea of a fractional kinetic equation (Zaslavsky 1994*a*, *b*) comes from the necessity to replace the 'normal' support by a fractal one when the balance is considered between temporal change of the distribution function $\delta_t F(t; x)$ and its phase-space variation $\delta_x F(t; x)$ averaged over the local volume of coarse graining:

$$\delta_t F(t, x) = F(t + \Delta t; x, x_0) - F(t; x, x_0), \tag{6.1a}$$

$$\delta_x F(t, x) = \sum_{\Delta x} \langle\langle [F(t; x + \Delta x, x_0) - F(t; x, x_0)] \rangle\rangle, \tag{6.1b}$$

where the distribution function $f(t; x, x_0)$ describes the transition from the point x_0 to the point x during t, the paths Δx provide the change $\delta_t F$, and the double angle brackets $\langle\langle \ldots \rangle\rangle$ denote an average over 'phases' of different paths one can expect in the 'normal' situation, in which the infinitesimal change Δt leads to $\delta_t F = O(\Delta t)$ and $\delta_x F = O(\Delta x)$, or, after averaging over the phases, $\delta_x F = O((\Delta x)^2)$. This is not the case when the support is fractal and the corresponding kinetic equation can be written in the form

$$(\Delta t)^\beta \frac{\partial^\beta F}{\partial t^\beta} = A \frac{\partial^\alpha F}{\partial x^\alpha}, \tag{6.2}$$

with

$$A = \sum_{\Delta x} \langle\langle (\Delta x)^\alpha \mathscr{D} \rangle\rangle, \tag{6.3}$$

where $\mathscr{D}$ is a constant dependent on phases, and α and β are some fractional parameters that define the support. A more general expression involving the derivatives $\partial^\alpha / \partial |x|^\alpha$ in a symmetrical case, and the combination of the fractal derivatives, can be found in Saichev and Zaslavsky (1997). The final expression is

$$\frac{\partial^\beta F}{\partial t^\beta} = \mathscr{A} \frac{\partial^\alpha F}{\partial |x|^\alpha}, \tag{6.4}$$

where we consider a symmetrical case for simplicity, with the notation

$$\mathscr{A} = \sum_{\Delta x} \frac{\langle\langle \mathscr{D} (\Delta x)^\alpha \rangle\rangle}{(\Delta t)^\beta}. \tag{6.5}$$

The crucial point is that in the limit $\Delta t \to 0$ the term $\mathscr{A}$ is finite and does not vanish. In the 'normal' case $\alpha = 2$ and $\beta = 1$, but in a general case we have $\alpha \leqslant 2$ (Saichev and Zaslavsky 1997) and an unclear situation for β.

Eventually, the parameters α and β should arise from the intrinsic properties of dynamical systems. Their values depend on time and the control parameter

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windows, on the symmetry of the phase space, etc. In fact, we do not have a universal recipe for determining α and β directly from the dynamics, and it is only occasionally possible to find the exponents α and β. It is possible to determine α and β for some special situations described in Zaslavsky *et al.* (1997), Venkataramani *et al.* (1997) and White *et al.* (1998). Let us demonstrate how one can estimate β when the exponent α corresponds to a 'normal' case, i.e. $\alpha = 2$.

Assume that there is only one singular zone and that a particle, being trapped in the singular zone, performs normal, Gaussian-type, symmetric diffusion along x. The time dependence of the motion in the singular zone is governed by the escape-time distribution function $\psi(t; \Delta\Gamma)$ given in (4.8). This function describes the properties of the one singular zone. The probability of staying for a time $t > T$ in the singular zone means that only the large-time asymptotics ($t > T$) will be considered. Such a probability $\Psi(t; \Delta\Gamma)$ is given in (4.10), and it possesses the scaling properties (4.12) under the transformation $T \to \lambda T$. The expression $\delta_t F$ in (6.1) written as the left-hand side of (6.2) obeys the same scaling as the probability $\Psi(t; \Delta\Gamma)$ under the transformation $\Delta t \to \lambda \Delta t$. This leads us to the conjecture that the exponents β in (6.4) and in (4.8), (4.10) are the same, i.e. that the corresponding kinetic equation will have the form

$$\frac{\partial^\beta F}{\partial t^\beta} = \mathscr{A} \frac{\partial^2 F}{\partial x^2}. \tag{6.6}$$

We shall discuss this issue in greater detail elsewhere. Equation (6.6) yields the transport equation

$$\langle x^2 \rangle \sim t^\beta, \tag{6.7}$$

where β can be obtained from the Hamiltonian or from observations.

7. Conclusions

Chaotic dynamics of low-dimensional systems displays different kinds of behaviour depending on the structure of the phase space. Although some typical structures can be specified, we have been challenged by a new situation where the kinetic description of the system depends on the windows in time and on the control-parameter(s) values used in considering the system.

Acknowledgements

This work was supported by the US Department of Energy, Grant DE-FG02-92ER54184, the US Department of Navy, Grant N00014-96-1-0055, and in part by the Institute des Hautes Études Scientifiques during a visit to the Institute.

References

Afraimovich, V. 1997 *Chaos* **7**, 12.

Afraimovich, V. and Zaslavsky, G. M. 1997 *Phys. Rev.* **E55**, 5418.

Balescu, R. 1997 *Statistical Dynamics: Matter Out of Equilibrium.* Imperial College Press, London, 1997.

Benkadda, S., Kassibrakis, S., White, R. B. and Zaslavsky, G. M. 1997 *Phys. Rev.* **E55**, 4909.

Kac, M. 1958 *Probability and Related Topics in the Physical Sciences. Proceedings of Summer Seminar, Boulder, CO, 1957.* Interscience, New York.

Klafter, J., Shlesinger, M. and Zumofen, G. 1996 *Phys. Today* **49**, 33.

Meiss, J. 1997 *Chaos* **7**, 39.

https://doi.org/10.1017/S002237789800659X Published online by Cambridge University Press

Meiss, J. D. and Ott, E. 1985 *Phys. Rev. Lett.* **55**, 2741.
Montroll, E. W. and Shlesinger, M. F. 1984 *Studies in Statistical Mechanics*, Vol. 11 (ed. J. Lebowitz and E. Montroll), p. 1. North-Holland, Amsterdam.
Prigogine, I. 1962 *Nonequilibrium Statistical Mechanics*. Wiley, New York.
Saichev, A. and Zaslavsky, G. M. 1997 *Chaos* **7**, 753.
Shlesinger, M. 1988 *Ann. Rev. Phys. Chem.* **39**, 269.
Shlesinger, M. F., Zaslavsky, G. M. and Klafter, J. 1993 *Nature* **363**, 31.
Venkataramani, S. C., Antonsen Jr, T. M. and Ott, E. 1997 *Phys. Rev. Lett.* **78**, 3864.
White, R., Benkadda, S., Kassibrakis, S. and Zaslavsky, G. M. 1998 *Chaos* (to appear).
Zaslavsky, G. M. 1994 *a Physica* **D76**, 110.
Zaslavsky, G. M. 1994 *b Chaos* **4**, 25.
Zaslavsky, G. M. 1995 *Chaos* **5**, 653.
Zaslavsky, G. M. and Niyazov, B. A. 1997 *Phys. Rep.* **283**, 73.
Zaslavsky, G. M., Stevens, D. and Weitzner, H. 1993 *Phys. Rev.* **E48**, 1683.
Zaslavsky, G. M., Edelman, M. and Niyazov, B. A. 1997 *Chaos* **7**, 159.

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