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Martingale inequalities

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1. *Introduction.* The first result of this paper was proved in January 1975 in order to engage the interest of Professor J. E. Littlewood, who was in hospital at the time. This theorem is of some independent interest and we present it here with some related results.

The classical Littlewood–Khintchine inequality (see (5) and (6)) states that there is a constant A such that for every real sequence $(a_k)_1^n$ we have

$$\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \leqslant A \int_0^1 \left|\sum_{k=1}^{n} a_k r_k(t)\right| dt, \tag{1}$$

where $r_k(t)$ is the k-th Rademacher function: $r_k(t) = \operatorname{sign} \sin 2^k \pi t$. Denote by A_0 the minimal value of A that will do in (1). The constant A_0 was determined only recently by Szarek (7) (for a slightly weaker result see Hall (4)): A_0 turned out to be $\sqrt{2}$ and $a_1 = a_2 = 1$ $(n = 2)$ is essentially the only extremal sequence.

Burkholder (1) (see also (2)) proved the following martingale inequality related to (1). There is a constant C such that for every martingale $f = (f_1, f_2, \ldots)$ and every $\lambda > 0$ we have

$$P(S(f) \geqslant \lambda) \leqslant \frac{C}{\lambda} \|f\|_1, \tag{2}$$

where $S(f)$ is the square function of f and $\|f\|_1 = \sup_n \|f_n\|_1$ is the L_1 norm of f. Inequality (2), called Burkholder's weak L_1 inequality is easily seen to extend (1). Indeed, if

$$f_m = \sum_{k=1}^{m} d_k r_k(t),$$

then $f = (f_1, f_2, \ldots, f_n)$ is a martingale for which

$$S(f) \equiv \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} = \lambda,$$

say. Therefore by (2)

$$\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} = \lambda P(S(f) \geqslant \lambda) \leqslant C \int_0^1 \left|\sum_{k=1}^{n} a_k r_k(t)\right| dt = C\|f\|_1.$$

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Let C_0 be the minimal constant that will do in (2). By the remark above $\sqrt{2} = A_0 \leqslant C_0$ and in (2) Burkholder proved that $C_0 \leqslant 3$. One of the aims of this paper is to improve these bounds on C_0. We shall also investigate the best constant in an inequality more general than (1) and less general than (2).

Given $s \geqslant 0$ and $h \geqslant 0$ denote by $L(s,h)$ the infimum of the L_1 norm of a martingale $f = (f_1, f_2, \ldots)$ in which $|f_1| = h$ and $S(f) = (h^2+s^2)^{\frac{1}{2}}$. As usual, we put $f_0 = 0$ and $d_n = f_n - f_{n-1}$ if $n \geqslant 1$, so the conditions are

$$|d_1| = h \quad \text{and} \quad \sum_{k=2}^{\infty} d_k^2 = s^2.$$

Clearly $L(s,h)$ is positively homogeneous and $h \leqslant L(s,h) \leqslant s+h$. Furthermore, if $h \geqslant s$ then $L(s,h) = h$, as shown by a martingale $f = (f_1, f_2, \ldots)$ in which $|d_1| = h$, $|d_2| = s$ and $d_3 = d_4 = \ldots = 0$.

Our aim is to estimate $L(s,h)$. In order to prove that $L(s,h) \geqslant F(s,h)$ for some function F, it is sufficient to consider martingales $f = (f_1, f_2, \ldots, f_n)$ on $I = [0,1]$ in which each d_i is a step function $d_i = e_i^{(1)} \chi_{I_i^{(1)}} + e_i^{(2)} \chi_{I_i^{(2)}}$. Here $e_i^{(1)}$, $e_i^{(2)}$ are constants and $I_i^{(1)}$, $I_i^{(2)}$ are disjoint intervals such that each d_j, $j < i$, is constant on $I_i^{(1)} \cup I_i^{(2)}$. Of course, $e_i^{(1)}|I_i^{(1)}| + e_i^{(2)}|I_i^{(2)}| = 0$. We call these martingales *step martingales*. Denote by $L_n(s,h)$ the infimum of the L_1 norm of a step martingale $f = (f_1, f_2, \ldots, f_n)$ in which $|f_1| = h$ and $S(f) = (h^2+s^2)^{\frac{1}{2}}$. If $n \leqslant 2$ then there need not be a step martingale satisfying these conditions. If $n = 3$ then an essentially unique step martingale satisfies the conditions and shows that $L_3(s,h) = \frac{1}{2}\{h+s+|h-s|\}$. Clearly

$$L_3(s,h) \geqslant L_4(s,h) \geqslant \ldots \geqslant h$$

and $L(s,h) = \lim_{n\to\infty} L_n(s,h)$. The main advantage of considering step martingales is that one may apply induction on n. Thus, in order to prove

$$L(s,h) \geqslant F(s,h)$$

for some function F, we apply induction on n to prove

$$L_n(s,h) \geqslant F(s,h)$$

for $n \geqslant 3$. To start the induction we have to check that

$$L_3(s,h) = \tfrac{1}{2}\{s+h+|s-h|\} \geqslant F(s,h).$$

What do we need for the proof of the induction step? Suppose $n \geqslant 4$ and

$$L_{n-1}(s,h) \geqslant F(s,h).$$

Let $f = (f_1, f_2, \ldots, f_n)$ be a step martingale with $|f_1| = h$ and $S(f) = (s^2+h^2)^{\frac{1}{2}}$. Let $d_2 = e_1 \chi_{I_1} - e_2 \chi_{I_2}$, where $e_i > 0$, and assume, as we may, that $d_1 = h$ on $I_1 \cup I_2$. Then on I_1 we have

$$f_2 = h + e_1 \quad \text{and} \quad \sum_{k=3}^{n} d_k^2 = s^2 - e_1^2,$$

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and on I_2

$$f_2 = h - e_2 \quad \text{and} \quad \sum_{k=3}^{n} d_k^2 = s^2 - e_2^2.$$

Since $f' = (f_2, f_3, \ldots, f_n)$ restricted to I_i $(i = 1, 2)$ is a martingale,

$$\int_{I_i} |f_n|\, dw \geqslant |I_i|\, L_{n-1}\big((s^2 - e_i^2)^{\frac{1}{2}}, \quad |h - (-1)^i e_i|\big).$$

Furthermore, as d_2 is zero outside $I_1 \cup I_2$,

$$\int_{I - I_1 \cup I_2} |f_n|\, dw \geqslant |I - I_1 \cup I_2|\, L_{n-1}(s, h).$$

Consequently the induction step follows if we show that

$$F(s,h) \leqslant \sum_{i=1}^{2} |I_i|\, L_{n-1}\big((s^2 - e_i^2)^{\frac{1}{2}}, \quad |h - (-1)^i e_i|\big) + |I - I_1 \cup I_2|\, L_{n-1}(s, h).$$

Since $e_1|I_1| = e_2|I_2|$, by the induction hypothesis this inequality holds if

$$F(s,h) \leqslant \sum_{i=1}^{2} \frac{e_j}{e_1 + e_2} L_{n-1}\big((s^2 - e_i^2)^{\frac{1}{2}}, \quad |h - (-1)^i e_i|\big),$$

where $j \in \{1, 2\}$, $j \neq i$. In turn, by the induction hypothesis this inequality holds if

$$F(s,h) \leqslant \sum_{i=1}^{2} \frac{e_j}{e_1 + e_2} F\big((s^2 - e_i^2)^{\frac{1}{2}}, \quad |h - (-1)^i e_i|\big). \tag{3}$$

Depending on how much work we are willing to put into proving (3) and our ability and luck in choosing a function $F(s,h)$, we can get various lower bounds for $L(s,h)$. The estimate proved in 1975 can be obtained without any effort.

THEOREM 1. *If* $0 \leqslant h < s$ *then* $L(s,h) \geqslant \frac{1}{2}(s + h^2/s)$, *otherwise* $L(s,h) = h$.

Proof. Put

$$F(s,h) = \begin{cases} \frac{1}{2}(s + h^2/s) & \text{if} \quad 0 \leqslant h < s \\ h & \text{if} \quad 0 \leqslant s \leqslant h. \end{cases}$$

In the proof of inequality (3) we may and shall assume that $0 \leqslant h \leqslant s$ and $s = 1$. The minimum of the right hand side of (3) is attained at e_1 and e_2 for which

$$F\big((1 - e_i^2)^{\frac{1}{2}}, \quad |h - (-1)^i e_i|\big) = \tfrac{1}{2}\left\{(1 - e_i^2)^{\frac{1}{2}} + \frac{(h - (-1)^i e_i)^2}{(1 - e_i^2)^{\frac{1}{2}}}\right\}.$$

(To see this simply take the derivatives of the functions on the right hand side with respect to e_1 and e_2, assuming that one of the functions is given by the second formula.) Thus (3) holds if and only if

$$(1 + h^2)(e_1 + e_2)(1 - e_1^2)^{\frac{1}{2}}(1 - e_2^2)^{\frac{1}{2}} \leqslant e_2(1 - e_1^2 + (h + e_1)^2)(1 - e_2^2)^{\frac{1}{2}} + e_1(1 - e_2^2 + (h - e_2)^2)(1 - e_1^2)^{\frac{1}{2}},$$

that is

$$Ah^2 - Bh + A \geqslant 0,$$

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where $A \geqslant 0$ and B are functions of e_1 and e_2. Hence if (3) fails for some h then it fails for $h = 1$ as well. However for $s = 1$ and $h = 1$ inequality (3) holds trivially since

$$F(1,1) = 1 \leqslant \frac{e_2}{e_1+e_2}(1+e_1)+\frac{e_1}{e_1+e_2}|1-e_2| \leqslant \sum_{i=1}^{2}\frac{e_j}{e_1+e_2}F\big((1-e_i^2)^{\frac{1}{2}},\ |1-(-1)^i e_i|\big).$$

COROLLARY 2. *If $f = (f_1, f_2, \ldots)$ is any martingale then*

$$P(S(f) \geqslant 1) \leqslant 2\|f\|_1.$$

Proof. The inequality follows from $L(1,0) \geqslant \frac{1}{2}$ by routine manipulations. Let $f = (f_1, f_2, \ldots, f_n)$ be a step martingale with $P(S(f) \geqslant 1) = p = c\|f\|_1 > 0$. Suppose $0 < p < 1$. Let $I_1, I_2, \ldots, I_m$ be maximal intervals on which each f_i is constant and $S(f) < 1$. For each I_j continue f by a copy of itself, suitably shrunk to I_j. Denote by g the new martingale. Clearly $P(S(g) \geqslant 1) \geqslant p + (1-p)p$ and

$$\|g\|_1 \leqslant \|f\|_1 + (1-p)\|f\|_1,$$

so

$$P(S(g) \geqslant 1) \geqslant p(2-p) = c\|f\|_1(2-p) \geqslant c\|g\|_1.$$

Continuing in this way we find a martingale h for which $P(S(h) \geqslant 1)$ is as near to 1 as we like and $P(S(h) \geqslant 1) \geqslant c\|h\|_1$. From this it is easily seen that for every $\epsilon > 0$ there is a martingale k with $S(k) \equiv 1$ and $1 \geqslant (c-\epsilon)\|h\|_1$. Hence by Theorem 1 we have $c - \epsilon \leqslant 2$, implying the result.

By applying the above corollary to f/λ instead of f, we obtain that $C_0 \leqslant 2$.

We now give a lower bound for C_0 by constructing an appropriate martingale. For simplicity we define the differences $d_1, d_2, \ldots$ only on $[0, \frac{1}{2}]$ and extend them to $[0,1]$ by $d_n(t) = -d_n(1-t)$. Let $t_1 = 21 - 12\sqrt{3}$,

$$d_1(t) = -\tfrac{1}{3}\sqrt{3}, \quad 0 \leqslant 2t \leqslant 1,$$

$$d_n(t) = \begin{cases} (-\frac{1}{3})^n 4\sqrt{3} & \text{if} \quad 0 \leqslant 2t \leqslant t_1^{n-1}, \\ (-\frac{1}{3})^{n-1}(3-\sqrt{3})/2 & \text{if} \quad t_1^{n-1} < 2t \leqslant t_1^{n-2}, \\ 0 & \text{if} \quad t_1^{n-2} < 2t \leqslant 1 \end{cases}$$

for $n \geqslant 2$. With $f_n = \Sigma_{k=1}^{n} d_k$ the martingale $f = (f_1, f_2, \ldots)$ is such that $\|f\|_1 = \frac{2}{3}$ and the modulus of the step function $\lim_{n\to\infty} f_n(t)$ is exactly $(1 - S^2(f)(t))^{\frac{1}{2}}$. Hence there is a martingale g with $S(g) \equiv 1$ and $\|g\|_1 = \|f\|_1 = \frac{2}{3}$. This shows that $C_0 \geqslant \frac{3}{2}$.

It is possible to find considerably better examples than the one we have just given, but these martingales are not easily described. In particular, by making rather crude use of a computer, one can construct a martingale that shows $C_0 \geqslant 1{\cdot}61$.

To conclude this section we remark that it seems likely that

$$L(s,h) \geqslant s\exp\{(h^2-s^2)/2s^2\} \quad \text{if} \quad 0 \leqslant h \leqslant s.$$

This would imply, in particular, $C_0 \leqslant \sqrt{e} \sim 1{\cdot}6487$.

3. *Haar martingales.* We saw in the introduction that a sequence of constants can be viewed as a martingale in which the k-th difference d_k is a scalar multiple of the

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k-th Rademacher function or, equivalently, $|d_k|$ is constant. On the other hand, as far as inequalities are concerned, step martingales are the most general martingales we need to consider. Recall that in a step martingale d_k takes only two non-zero values. A martingale will be said to be a *Haar martingale* if $|d_k|$ takes only one non-zero value. To justify this terminology we note that an inequality holds for all Haar martingales if and only if it holds for all martingales in which d_k is the k-th Haar function on $[0, 1]$. By (2) there is a minimal $B_0 > 0$ such that

$$P(S(f) \geqslant \lambda) \leqslant B_0/\lambda \|f\|_1$$

whenever $\lambda > 0$ and f is a Haar martingale. By definition $A_0 \leqslant B_0 \leqslant C_0$. In this section we prove an analog of Theorem 1 for Haar martingales and estimate B_0.

Given $s \geqslant 0$ and $h \geqslant 0$ denote by $H(s, h)$ the infimum of the L_1 norm of a Haar martingale $f = (f_1, f_2, \ldots)$ in which $|f_1| = h$ and $S(f) = (s^2 + h^2)^{\frac{1}{2}}$. By imitating the proof of Theorem 1 it is easily proved that $H(s, h) \geqslant (2s + h^2/s)/3$ if $0 \leqslant h \leqslant s$, so $B_0 \leqslant \frac{3}{2}$. We do not include the details here since we shall prove a stronger result. For $x \geqslant 0$ put

$$\psi(x) = e^{-x^2/2} + x \int_0^x e^{-t^2/2}\, dt,$$

and define

$$G(s, h) = \begin{cases} s\psi(h/s)/\psi(1) & \text{if} \quad 0 \leqslant h \leqslant s, \\ h & \text{if} \quad 0 \leqslant s \leqslant h. \end{cases}$$

Note that

$$\psi'(x) = \int_0^x e^{-t^2/2}\, dt, \quad \psi''(x) = e^{-x^2/2},$$

and

$$\psi(x) = x\psi'(x) + \psi''(x). \tag{4}$$

THEOREM 3. $H(s, h) \geqslant G(s, h)$.

Proof. We proceed as in the proof of Theorem 1. Without loss of generality

$$0 \leqslant h \leqslant s = 1.$$

Since the martingales under consideration are Haar martingales, we have to prove (3) in the case when $e_1 = e_2 = \epsilon$, that is

$$G(1, h) \leqslant \tfrac{1}{2}\{G(1 - \epsilon^2)^{\frac{1}{2}}, h + \epsilon) + G(1 - \epsilon^2)^{\frac{1}{2}}, |h - \epsilon|). \tag{3'}$$

Elementary though tedious manipulations show that in the proof of (3′) we may assume that both functions on the right are given by the first formula. For instance, if

$$|h - \epsilon| \geqslant (1 - \epsilon^2)^{\frac{1}{2}},$$

then

$$0 \leqslant h \leqslant \epsilon - (1 - \epsilon^2)^{\frac{1}{2}},$$

and (3′) becomes

$$\psi(h) \leqslant \epsilon\psi(1).$$

This holds if it holds at $h = h(\epsilon) = \epsilon - (1 - \epsilon^2)^{\frac{1}{2}}$, and since $(d/d\epsilon)\,\psi(h(\epsilon)) < \psi(1)$, it suffices to check it at $\epsilon = \frac{1}{2}\sqrt{2}$ and $h = 0$, in which case it is trivial. Therefore (3′) can be written in the following form:

$$\psi(x) \leqslant \frac{(1 - \epsilon^2)^{\frac{1}{2}}}{2}\{\psi(x_1) + \psi(x_2)\}, \tag{5}$$

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where $$x_1 = \frac{x+\epsilon}{(1-\epsilon^2)^{\frac{1}{2}}} \leqslant 1 \quad \text{and} \quad x_2 = \frac{|x-\epsilon|}{(1-\epsilon^2)^{\frac{1}{2}}}.$$

Assume first $0 \leqslant \epsilon \leqslant x$. Inequality (5) does hold at $\epsilon = 0$ so it suffices to show that the partial derivative of the right hand side of (5) with respect to ϵ is non-negative. By (4) this holds if

$$-\epsilon(1-\epsilon^2)^{-\frac{1}{2}}\{x_1\psi'(x_1)+\psi''(x_1)+x_2\psi'(x_2)+\psi''(x_2)\} + (1-\epsilon^2)^{\frac{1}{2}}\left\{\psi'(x_1)\frac{1+\epsilon x}{(1-\epsilon^2)^{\frac{3}{2}}}+\psi'(x_2)\frac{-1+\epsilon x}{(1-\epsilon^2)^{\frac{3}{2}}}\right\} \geqslant 0,$$

which, in turn, is equivalent to

$$\psi'(x_1)-\psi'(x_2) \geqslant \frac{\epsilon}{(1-\epsilon^2)^{\frac{1}{2}}}\{\psi''(x_1)+\psi''(x_2)\},$$

that is $$\int_{x_2}^{x_1} e^{-t^2/2}dt \geqslant \tfrac{1}{2}(x_1-x_2)\{e^{-x_1^2/2}+e^{-x_2^2/2}\}.$$

This inequality does hold since $0 \leqslant x_2 \leqslant x_1 \leqslant 1$ so $e^{-t^2/2}$ is concave in the interval $[x_2, x_1]$.

The case $x \leqslant \epsilon$ can be dealt with exactly the same way, so the proof of Theorem 3 is complete.

Analogously to Corollary 2 we have the following consequence of Theorem 3.

COROLLARY 4.

$$B_0 \leqslant e^{-\frac{1}{2}} + \int_0^1 e^{-t^2/2}dt \sim 1{\cdot}4622.$$

Once again, with the aid of a computer one can find a Haar martingale that shows $B_0 \geqslant 1{\cdot}44$. In particular, we see that $A_0 < B_0 < C_0$.

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Added in proof. Using a different approach Burkholder has proved Corollary 2. (A sharp inequality for martingale transforms, *Annals of Probability* **7** (1979) 858–863.) He also conjectured that $C_0 = \sqrt{2}$ which we know to be false since $C_0 \geqslant 1{\cdot}61$.

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