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SRB measures for almost Axiom A diffeomorphisms

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Abstract. We consider a diffeomorphism f of a compact manifold M which is *almost Axiom A*, i.e. f is hyperbolic in a neighborhood of some compact f-invariant set, except in some singular set of neutral points. We prove that if there exists some f-invariant set of hyperbolic points with positive *unstable* Lebesgue measure such that for every point in this set the stable and unstable leaves are 'long enough', then f admits an SRB (probability) measure.

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1. *Introduction*

This paper addresses the question of ensuring the existence of Sinai–Ruelle–Bowen (SRB) probability measure under weak conditions of hyperbolicity. It improves results from [**10**] which dealt with some non-uniformly hyperbolic dynamical systems. Under a weaker hyperbolicity condition we are able to deduce a stronger conclusion: the SRB measure is actually a probability measure.

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Beyond the technical result, we point out that studying how non-uniform hyperbolicity yields or does not yield the existence of SRB measures highlights how rich are the possibilities of leaving the world of uniformly hyperbolic diffeomorphisms. Relaxing assumptions on hyperbolicity also allows us to construct or study new examples; see §4. We observe that the conditions yielding the existence of SRB measures for non-uniformly hyperbolic systems have been much studied since the 1990s (see, for example, [**2**, **3**, **5**, **7**]) but still remain not entirely solved. The main reason for that is that there is no general theory for constructing the particular Gibbs states that constitute these SRB measures. This lack of general theory is probably due to the large number of ways of degenerating the uniform hyperbolicity. It thus shows that the main hypotheses and difficulties of constructing SRB measures are hidden in the definition of non-uniform hyperbolicity.

Recall that SRB measures are special *physical* measures, which are measures with observable generic points: for a smooth dynamical system (M, f), meaning that M is a compact smooth Riemannian manifold and f is a C^{1+} diffeomorphism acting on M, the set G_μ of generic points for an f-invariant ergodic probability measure μ on M is the set of points x such that for every continuous function $\phi : M \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) = \int \phi \, d\mu. \tag{1.1}$$

This set has full μ measure. Though holding for a large set of points in terms of μ measure, this convergence can be actually observed only when G_μ has strictly positive measure with respect to the volume of the manifold, that is, with respect to the Lebesgue measure on M. This volume measure will be denoted by Leb_M. In that context, μ is said to be *physical* if $\mathrm{Leb}_M(G_\mu) > 0$.

On the other hand, the definition of SRB measure or u-Gibbs state needs the notion of hyperbolicity. Roughly speaking, hyperbolicity means the existence of an integrable splitting $T_x M = E^u_x \oplus E^s_x$ where, in some sense that has to be made precise, df expands vectors in E^u and contracts vectors in E^s. If W^u and W^s are the associated integral manifolds, they are expected to form a local product structure. With these settings, a measure is said to be SRB if its disintegration along the unstable leaves W^u is absolutely continuous with respect to their Lebesgue measure Leb^u.

Here, the source of non-uniform hyperbolicity is the presence of parabolic points. The kind of systems we are studying are higher-dimensional and invertible extensions of Manneville–Pomeau like maps. Hyperbolic systems with 'parabolic' points have already been studied, for instance in [**7**, **8**, **10**], and the present paper is an improvement of [**10**].

For these systems, we want to find weak reasonable conditions yielding the existence of some SRB probability. In our context, we show that a sufficient condition to ensure the existence of SRB probability is that the unstable Lebesgue measure Leb^u 'sees' some hyperbolicity; see Theorem A below.

1.1. *Statement of results.* We start by defining the class of dynamical systems that we will consider.

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Definition 1.1. Given $f : M \to M$ a C^{1+} diffeomorphism and $\Omega \subset M$ a compact f-invariant set, we say that f is *almost Axiom A on* Ω if there exists an open set $U \supset \Omega$ such that:

(1) for every $x \in U$ there is a df-invariant splitting (invariant where it makes sense) of the tangent space $T_x M = E^u(x) \oplus E^s(x)$ with $x \mapsto E^u(x)$ and $x \mapsto E^s(x)$ Hölder continuous (with uniformly bounded Hölder constant);

(2) there exist continuous non-negative real functions $x \mapsto k^u(x)$ and $x \mapsto k^s(x)$ such that (for some choice of Riemannian norm $\|\cdot\|$ on M) for all $x \in U$,
   (a) $\|df(x)v\|_{f(x)} \leq e^{-k^s(x)}\|v\|_x$, for all $v \in E^s(x)$,
   (b) $\|df(x)v\|_{f(x)} \geq e^{k^u(x)}\|v\|_x$, for all $v \in E^u(x)$;

(3) the *exceptional set*, $S = \{x \in U, k^u(x) = k^s(x) = 0\}$, satisfies $f(S) = S$ and k^u and k^s are positive in $U \setminus S$.

From here on we assume that f is almost Axiom A on Ω. The sets S and $U \supset \Omega$ and the functions k^u and k^s are fixed as in the definition, and the splitting $T_x M = E^u(x) \oplus E^s(x)$ is called the *hyperbolic splitting*.

Definition 1.2. A point $x \in \Omega$ is called a *point of integration* of the hyperbolic splitting if there exist $\varepsilon > 0$ and C^1-disks $D^u_\varepsilon(x)$ and $D^s_\varepsilon(x)$ of size ε centered at x such that $T_y D^i_\varepsilon(x) = E^i(y)$ for all $y \in D^i_\varepsilon(x)$ and $i = u, s$.

By definition, the set of points of integration is invariant by f. As usual, having defined the two families of local stable and unstable manifolds, we define

$$\mathcal{F}^u(x) = \bigcup_{n \geq 0} f^n D^u_{\varepsilon(-n)}(f^{-n}(x)) \quad \text{and} \quad \mathcal{F}^s(x) = \bigcup_{n \geq 0} f^{-n} D^s_{\varepsilon(n)}(f^n(x)),$$

where $\varepsilon(n)$ is the size of the disks associated to $f^n(x)$. They are called the global stable and unstable manifolds, respectively.

We emphasize that Hölder regularity of the splitting shows a non-uniform local product structure: $D^u_{\varepsilon(x)}(x)$ and $D^s_{\varepsilon(x)}(x)$ are transverse and, more generally, every $D^u_{\varepsilon(x)}(x)$ is transversal to every $D^s_{\varepsilon(y)}(y)$ if y is close to x and if both disks are large enough to get non-empty intersection.

Given a point of integration of the hyperbolic splitting x, the manifolds $\mathcal{F}^u(x)$ and $\mathcal{F}^s(x)$ are also immersed Riemannian manifolds. We denote by d^u and d^s the Riemannian metrics, and by Leb^u_x and Leb^s_x the Riemannian measures, respectively in $\mathcal{F}^u(x)$ and $\mathcal{F}^s(x)$. If a measurable partition is subordinated to the unstable foliation $\mathcal{F}^u$ (see [**9**, **11**]), any f-invariant measure admits a unique system of conditional measures with respect to the given partition.

Definition 1.3. We say that an invariant ergodic probability having absolutely continuous conditional measures on unstable leaves $\mathcal{F}^u(x)$ with respect to Leb^u_x is a *Sinai–Ruelle–Bowen measure*.

Definition 1.4. Given $\lambda > 0$, a point $x \in \Omega$ is said to be λ-*hyperbolic* if

$$\liminf_{n \to +\infty} \frac{1}{n} \log \|df^{-n}(x)_{|E^u(x)}\| \leqslant -\lambda \quad \text{and} \quad \liminf_{n \to +\infty} \frac{1}{n} \log \|df^n(x)_{|E^s(x)}\| \leqslant -\lambda.$$

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Definition 1.5. Given $\lambda > 0$ and $\varepsilon_0 > 0$, a point $x \in \Omega$ is called (ε_0, λ)-*regular* if the following conditions hold:

(1) x is λ-hyperbolic and a point of integration of the hyperbolic splitting;
(2) $\mathcal{F}^i(x)$ contains a disk $D^i_{\varepsilon_0}(x)$ of size ε_0 centered at x, for $i = u, s$.

An f-invariant compact set $\Lambda \subset \Omega$ is said to be an (ε_0, λ)-*regular set* if all points in Λ are (ε_0, λ)-regular.

In this setting, we can state our first result and make precise the meaning of Leb^u 'seeing' hyperbolicity.

THEOREM A. *Let Λ be an (ε_0, λ)-regular set. If there exists some point $x_0 \in \Lambda$ such that $\mathrm{Leb}_{D^u_{\varepsilon_0}(x_0)}(D^u_{\varepsilon_0}(x_0) \cap \Lambda) > 0$, then f has an SRB measure.*

The existence of stable and unstable leaves is crucial to defining the SRB measures. Such existence is equivalent to the existence of integration points for a hyperbolic splitting. This is well known when the hyperbolic splitting is dominated, because it yields the existence of an uniform spectral gap for the derivative. A very close result is also well known in the Pesin theory, but only on a set of full measure, and then the precise topological characterization of the set of points of integration given by Pesin theory depends on the choice of the invariant measure.

In our case the splitting is not dominated and we have no given invariant measure to use Pesin theory. Nevertheless, using the graph transform, we prove integrability even in the presence of indifferent points for a set of points whose precise characterization does not depend on the ergodic properties of some invariant probability measure which would be given *a priori*.

Definition 1.6. A λ-hyperbolic point $x \in \Omega$ is said to be of *bounded type* if there exist two increasing sequences of integers (s_k) and (t_k) with

$$\limsup_{k\to+\infty} \frac{s_{k+1}}{s_k} < +\infty \quad \text{and} \quad \limsup_{k\to+\infty} \frac{t_{k+1}}{t_k} < +\infty$$

such that

$$\lim_{k\to+\infty} \frac{1}{s_k} \log \|df^{-s_k}(x)_{|E^u(x)}\| \leqslant -\lambda \quad \text{and} \quad \lim_{k\to+\infty} \frac{1}{t_k} \log \|df^{t_k}(x)_{|E^s(x)}\| \leqslant -\lambda.$$

Observe that the last requirement on these sequences is an immediate consequence of Definition 1.4. We emphasize that the assumption of being of bounded type is weak. Unless the sequence (s_k) is factorial it is of bounded type. An exponential sequence, for instance, is of bounded type. Thus, this assumption does not yield that the sequence (s_k) has positive density.

THEOREM B. *Every λ-hyperbolic point of bounded type is a point of integration of the hyperbolic splitting.*

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1.2. *Discussion of results.* SRB measures for hyperbolic systems with parabolic points have already been studied in [**7**, **8**]. Both are two-dimensional diffeomorphisms.

In [**8**] Hu and Young consider the case where the stable direction is uniformly contracting, and non-uniform behavior occurs only in the unstable direction. In particular, this means that the splitting is dominated and existence of long stable and unstable manifolds is well known in this case. Moreover, they also consider $\mathcal{C}^2$ regularity. In some sense, in their case, the diffeomorphism is almost a skew product over a classical one-dimension Manneville–Pomeau like map, and in that case (see [**12**]) the non-existence of finite SRB measure is well known.

In [**7**], Hu considers $\mathcal{C}^4$-regularity and his assumptions on hyperbolicity are different (neither stronger nor weaker than ours). First, he considers the non-degenerate case, which means that he has an assumption on the germ of the map close to the critical set† S. In contrast, we consider the $\mathcal{C}^{1+}$-regularity and have no assumption on the germ close to S. On the other hand, concerning hyperbolicity, Hu only has an assumption on cones, whereas our assumption needs to know all the orbit to be checked. Another point is that Hu considers the two-dimensional case and this automatically solves the problem of the existence of the stable and unstable manifolds. This does not apply in our case.

As we said, this paper is an improvement of [**10**]. The first and main improvement is that we show that the SRB measure is a probability measure, whereas all that was proved in [**10**] is the existence of a finite or σ-finite SRB measure. As an illustration, we give an example in §4. We believe that its interest lies in the finiteness of the measure because the way it is constructed implies that it is reasonable to expect it admits some σ-finite SRB measure.

The second improvement is that the lim sup in the definition of hyperbolicity has been replaced by lim inf. This technical point may appear as a minor improvement, in particular because the existence of a probability SRB measure allows us—*a posteriori*—to use the Pesin theory‡, but as a bi-product, it allows us to construct examples, which is not negligible because this theory of non-uniformly hyperbolic diffeomorphisms has few known and explicit examples, even if general results prove the abundance of such systems (see [**4**] and its introduction for a brief survey on $\mathcal{C}^1$-generic hyperbolicity). Our present example does not illustrate this improvement; we leave this for future work.

The last improvement is that results hold for $\mathcal{C}^{1+}$ maps instead of $\mathcal{C}^{1+1}$ maps but it is a small technical improvement.

1.3. *Overview.* The proofs are all based on the same key point: the estimates are all uniformly hyperbolic outside some fixed bad neighborhood $B(S, \varepsilon_1)$ of S. Roughly speaking, we study an induced map on the complement of $B(S, \varepsilon_1)$. We show that a λ-hyperbolic point cannot stay too long in this fixed neighborhood; see, for example, Lemmas 2.2 and 5.5. Therefore, an incursion in $B(S, \varepsilon_1)$ cannot spoil too much the (uniformly) hyperbolic estimates on contractions or expansions. Several steps are necessary to define this first-return map with good hyperbolic properties.

† Which is finite in his case, but not necessarily in ours.
‡ And then lim sup = lim = lim inf.

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Another technical point is that this first-return hyperbolic map will be defined on a non-compact set. The SRB measure we construct is obtained via considering an accumulation point for a special sequence of measures. At some point, we will thus have to show tightness for this sequence.

The rest of this paper proceeds as follows: in §2 we construct three different generations of rectangles satisfying some Markov property. The last generation is a covering of a 'good region' sufficiently far away from the critical region, and points there have good hyperbolic behavior.

In §3 we prove Theorem A. Namely, we construct a measure for the return into the cover of rectangles of third generation. This is where the inducing scheme is defined. Then we define a sequence of measures, prove tightness and then obtain the SRB measure for the induced map. Finally, we prove that this measure can be *opened out* to an SRB measure for f.

In §4 we construct an explicit example of a diffeomorphism satisfying the assumption of Theorem A.

In §5 we prove Theorem B, that is, the existence of stable and unstable manifolds.

2. *Markov rectangles*

2.1. *Local product structure.* The construction of Markov rectangles is strongly related to some local product structure. We emphasized above that Hölder regularity of the splitting shows that local stable and unstable disks are transverse. Nevertheless their size is not necessarily uniformly bounded from below, and this is why we need to consider some (ε_0, λ)-regular set Λ. We may still decrease the size of ε_0 (this will usually increase the size of Λ) and consider some minimal distance, say δ_0, such that for any pair of (ε_0, λ)-regular points x and y, $D^u_{\varepsilon_0}(x)$ and $D^s_{\varepsilon(0)}(y)$ have a transversal intersection (at a single point) if $d(x, y) < \delta_0$. We recall that in that case $[x, y]$ denotes $D^u_{\varepsilon_0}(x) \cap D^s_{\varepsilon(0)}(y)$.

2.2. *Neighborhood of the critical region.* Let Λ be some fixed (ε_0, λ)-regular set satisfying the hypothesis of Theorem A. The goal of this section is to construct Markov rectangles covering a large part of Λ and then to use Young's method; see, for example, [**8**]. This type of construction has already been implemented in [**10**]. We adapt it here and focus on the steps where the new assumption (namely lim inf instead of lim sup) produces some changes.

To control the lack of hyperbolicity close to the critical set S we will introduce several constants. Some are directly related to the map f, some are related to the set Λ and others depend on previous ones. Their dependence will be established in §2.5.

Given $\varepsilon_1 > 0$, we define (see Figure 1)

$$B(S, \varepsilon_1) = \{y \in M, d(S, y) < \varepsilon_1\},$$

and

$$\Omega_0 = \Omega \backslash B(S, \varepsilon_1), \quad \Omega_1 = \Omega_0 \cap f(\Omega_0) \cap f^{-1}(\Omega_0) \quad \text{and} \quad \Omega_2 = \Omega_0 \backslash \Omega_1.$$

The main idea is to consider a new dynamical system, which is in spirit equal to the first return to Ω_0. This system will be obtained as the projection of a subshift with countable

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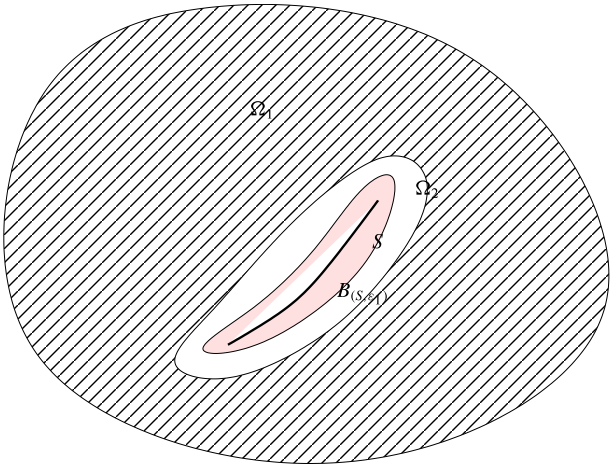


FIGURE 1. The sets Ω_i.

alphabet, this countable shift being obtained via a shadowing lemma. This will define rectangles of first generation. To deal with these countably many rectangles we will need to do some extra work to be able to 'cut' them. This will be done by defining new rectangles called rectangles of second and third generation.

We assume that $\varepsilon_1 > 0$ has been chosen sufficiently small so that for every x in Ω_2,

$$1 \le \|df(x)_{|E^u(x)}\| < e^{2(\lambda/3)} \quad \text{and} \quad 1 \le \|df^{-1}(x)_{|E^s(x)}\| < e^{2(\lambda/3)}.$$

Then we fix $0 < \varepsilon_2 < 1$ and define $\Omega_3 = \Omega_3(\varepsilon_2)$ as the set of points $x \in \Omega$ such that

$$\min\{\log \|df_{|E^u(x)}\|, -\log \|df^{-1}_{|E^u(x)}\|, -\log \|df_{E^s(x)}\|, \log \|df^{-1}_{|E^s(x)}\|\} \ge \varepsilon_2\lambda. \quad (2.1)$$

Definition 2.1. Let x in Ω_0. If $f(x) \in B(S, \varepsilon_1)$, we define the *forward length of stay* of x in $B(S, \varepsilon_1)$ as

$$n^+(x) = \sup\{n \in \mathbb{N} : f^k(x) \in B(S, \varepsilon_1),\ \forall 0 < k < n\}.$$

If $f^{-1}(x) \in B(S, \varepsilon_1)$, we define the *backward length of stay* of x in $B(S, \varepsilon_1)$ as

$$n^-(x) = \sup\{n \in \mathbb{N} : f^{-k}(x) \in B(S, \varepsilon_1),\ \forall 0 < k < n\}.$$

Observe that any of these numbers may be equal to $+\infty$.

2.3. *First generation of rectangles.* A first class of rectangles is constructed by adapting the classical shadowing lemma to our case. As mentioned above, this was done in [**10**, §3]; see Proposition 3.4. It just needs local properties of f and the dominated splitting $E^u \oplus E^s$, not requiring any other hyperbolic properties than the local stable and unstable leaves. Therefore, the new assumption with lim inf instead of lim sup in the definition of λ-hyperbolic point does not produce any change in that construction. We summarize in the five properties below the essential properties for these rectangles:

- There is a subshift of finite type (Σ_0, σ) with a countable alphabet $\mathcal{A} := \{a_0, a_1, \ldots\}$ and a map $\Theta : \Sigma_0 \to M$ such that the following holds:

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(1) $\Theta(\Sigma_0)$ contains $\Lambda \cap \Omega_0$;
(2) the dynamical system (Σ_0, σ) induces a dynamical system $(\Theta(\Sigma_0), F)$ commuting the diagram

$$\begin{array}{ccc} \Sigma_0 & \xrightarrow{\sigma} & \Sigma_0 \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta} \\ \Theta(\Sigma_0) & \xrightarrow{F} & \Theta(\Sigma_0) \end{array}$$

(3) for each $x \in \Omega_0$ we have $F(x) = f(x)$ if $x \in \Omega_1$, $F(x) = f^{n^+(x)}(x)$ if $x \in \Omega_2 \cap f^{-1}(B(S, \varepsilon_1))$, and $F^{-1}(x) = f^{-n^-(x)}(x)$ if $x \in \Omega_2 \cap f(B(S, \varepsilon_1))$.

Roughly speaking, F is the first return map into Ω_0. Actually, it is defined on the larger set $\Theta(\Sigma_0)$ and σ is a symbolic representation of this dynamics.

- We define the *rectangles of first generation* as the sets $T_{a_n} = \Theta([a_n])$, where $[a_n]$ is the 1-cylinder in Σ_0 associated to the symbol a_n, i.e. $[a_n]$ is the set of all sequences $\underline{x} = (x(k))_{k\in\mathbb{Z}}$ in Σ_0 with $x(0) = a_n$.

With this definition of the T_i we make precise what 'rectangle' means. First, we recall that the shift Σ_0 has a local product structure: for sequences in Σ_0 in the same 1-cylinder, $[\underline{y}, \underline{z}]$ denotes the sequence with the same past as $\underline{y}$ and the same future as $\underline{z}$. This defines local stable and unstable sets: $W^u_{\mathrm{loc}}(\underline{y})$ denotes the set of points in the same 1-cylinder as $\underline{y}$ and with the same past. $W^s_{\mathrm{loc}}(\underline{y})$ is the same for the future. Therefore,

$$[\underline{y}, \underline{z}] = W^u_{\mathrm{loc}}(\underline{y}) \cap W^s_{\mathrm{loc}}(\underline{z}).$$

It turns out (see [**10**, Proposition 3.6]) that every point in $\Theta(\Sigma_0)$ is a point of integration for the subbundles of the hyperbolic splitting and that sets $\Theta(W^{u,s}_{\mathrm{loc}}(\underline{y}))$ are pieces of $E^{u,s}$-integral curves containing $y := \Theta(\underline{y})$. Then, *rectangle* means that Θ respects both local product structures in Σ_0 and in M:

$$\text{for all } \underline{y}, \underline{z} \in [a_n], \quad \Theta([\underline{y}, \underline{z}]) = [\Theta(\underline{y}), \Theta(\underline{z})].$$

- For $x \in T_{a_n}$ with $x = \Theta(\underline{x})$ we define $W^{u,s}(x, T_{a_n}) := \Theta(W^u_{\mathrm{loc}}(\underline{x}))$. Then, the rectangles of first generation are *almost Markov* (see [**10**, Proposition 3.7]): if $x = \Theta(\underline{x})$ with $\underline{x} = x(0)x(1)x(2)\ldots$ where all the $x(i)$ are in the alphabet $\mathcal{A}$, then for every $j \geqslant 0$ we have

$$F^j(W^u(x, T_{x(0)})) \supset W^u(F^j(x), T_{x(j)})$$

and

$$F^j(W^s(x, T_{x(0)})) \subset W^s(F^j(x), T_{x(j)}).$$

We point out that this last property does not give a full Markov property because it depends on the existence of some code $\underline{x}$ connecting the two rectangles $T_{x(0)} \ni x$ and $T_{x(j)} \ni F^j(x)$.

- We may take the rectangles of first generation to be as small as want, say $\delta > 0$.
- For every n, the set of points in Ω_0 with forward and backward length of stay smaller than n are covered by finitely many rectangles.

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2.4. *Second generation of rectangles.* To get a full Markov property we need to cut the rectangles of first generation as in [**6**]. Due to the non-uniformly hyperbolic settings, we first need to reduce the rectangles to a set of good points with good hyperbolic properties.

Roughly speaking, the second generation of rectangles we construct here is based on the following process. We select points in the rectangles T_j which are:

- λ-hyperbolic;
- Leb^u density points with this property.

We show in Lemma 2.2 that these points satisfy some good property that allows us later to use Pliss's lemma. A similar version of Lemma 2.2 was stated in [**10**], but here exchanging the assumption lim sup for lim inf plays a role.

LEMMA 2.2. *There exists $\zeta > 0$ such that for every λ-hyperbolic point x,*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|df^{-1}_{|E^u(f^j(x))}\| < -\zeta. \tag{2.2}$$

Proof. Let κ be a bound for $\|df\|$ and $\|df^{-1}\|$. For $0 < \varepsilon_2 < \frac{1}{3}$ we have defined Ω_3 as the set of points $x \in \Omega$ such that

$$\min\{\log \|df_{|E^u(x)}\|, -\log \|df^{-1}_{|E^u(x)}\|, -\log \|df_{E^s(x)}\|, \log \|df^{-1}_{|E^s(x)}\|\} \geq \varepsilon_2\lambda.$$

Let x be a λ-hyperbolic point. We set

$$\delta_{\varepsilon_2} = \liminf_{n\to\infty} \frac{1}{n} \#\{0 \leq k < n,\ f^k(x) \in \Omega_3\},$$

where $\#A$ stands for the number of elements in a finite set A. The key point in the proof is that any λ-hyperbolic point must visit Ω_3 forward with a positive frequency otherwise it cannot get a strong exponential contraction in the stable direction.

Let x be λ-hyperbolic. There exist infinitely many values of n such that

$$\log \|df^n_{|E^s(x)}\| \leqslant -\frac{\lambda}{2} n.$$

The smallest frequency of visit to Ω_3 is obtained when the point has maximal contraction as the point lies in Ω_3 and the biggest possible contraction as the point is outside Ω_3. There exists $\varepsilon_3 = \varepsilon_3(\varepsilon_2)$ such that for every $y \notin \Omega_3$,

$$\max\{\log \|df_{|E^u(y)}\|, -\log \|df^{-1}_{|E^u(y)}\|, -\log \|df_{E^s(y)}\|, \log \|df^{-1}_{|E^s(y)}\|\} \leq \varepsilon_3\lambda.$$

Then we have $\lambda/2 < \varepsilon_3\lambda(1 - \delta_2) + \delta_2 \log \kappa$. This yields

$$\delta_{\varepsilon_2} \geq \frac{(1/2) - \varepsilon_3}{(\log \kappa/\lambda) - \varepsilon_3} > 0, \tag{2.3}$$

which also gives

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|df^{-1}_{|E^u(f^j(x))}\| < -\delta_{\varepsilon_2}\varepsilon_2\lambda.$$

Take $\zeta = \delta_{\varepsilon_2}\varepsilon_2\lambda$. □

Remark 2.3. We emphasize that if (2.2) holds for x, then it holds for every $y \in \mathcal{F}^s(x)$.

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Now the main idea of this part is to restrict the F-invariant set $\bigcup_{i\in\mathbb{N}} T_{a_i}$ $(= \Theta(\Sigma_0))$ to some subset satisfying some good properties. Using Lemma 2.2, we arrive at the same conclusion as [**10**, Proposition 3.8].

PROPOSITION 2.4. *There exists some λ-hyperbolic set $\Delta \subset \Lambda$ such that:*
(1) $\mathrm{Leb}^u(\Delta) > 0$;
(2) *every $x \in \Delta$ is a density point of Δ for Leb^u_x;*
(3) *there exists some $\zeta > 0$ such that for every $x \in \Delta$,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|df^{-1}_{|E^u(f^j(x))}\| < -\zeta. \tag{2.4}$$

Proof. Lemma 2.2 gives a value ζ which works for every point in Λ. Defining

$$\Delta_0 = \bigcup_{i\in\mathbb{N}} \{x \in T_{a_i} : \exists (y, z) \in \Lambda^2 \text{ such that } x \in \mathcal{F}^u(y) \cap \mathcal{F}^s(z)\}.$$

Remark 2.3 shows that Δ_0 is an F-invariant set of λ-hyperbolic points in $\bigcup_{i\in\mathbb{N}} T_{a_i}$ such that (2.4) holds for every $x \in \Delta_0$. Moreover, $\mathrm{Leb}^u(\Delta_0) > 0$.

We now recall that the stable holonomy is absolutely continuous with respect to Lebesgue measure on the unstable manifolds. Therefore, if x is a density point of Δ_0 for Leb^u_x, every y in $\mathcal{F}^s(x) \cap \Delta_0$ is also a density point of Δ_0 for Leb^u_y. Let Δ be the set of density points in Δ_0 for Leb^u. This set is F-invariant and stable by intersections of stable and unstable leaves. Hence, every point $y \in \Delta$ is also a density point of Δ for Leb^u_y. □

For each $i \in \mathbb{N}$, let S_{a_i} be the restriction of the rectangle T_{a_i} to Δ, i.e.

$$S_{a_i} = T_{a_i} \cap \Delta.$$

Given $x \in S_{a_i}$ we set

$$W^u(x, S_{a_i}) = W^u(x, T_{a_i}) \cap \Delta \quad \text{and} \quad W^s(x, S_{a_i}) = W^s(x, T_{a_i}) \cap \Delta.$$

By construction of Δ, if x and y are in S_{a_i}, then $[x, y]$ is also in S_{a_i}. As previously for the T_i, we still have

$$\{[x, y]\} = W^s(x, S_{a_i}) \cap W^u(y, S_{a_i}).$$

These sets S_{a_i} are called *rectangles of second generation* and their collection is denoted by $\mathcal{S}$.

2.5. *Third generation of rectangles.* From the beginning we have introduced several constants. It is now time to fix some of them. For that purpose we want first to summarize how these constants depend on one another.

2.5.1. *The constants.* The set Λ is fixed and so the two constants ε_0 and λ are also fixed. Note that ε_0 imposes a constraint to define the local product structure for points in Λ.

We fix ζ as in (2.4) and pick $\varepsilon > 0$ small compared to λ and ζ, namely $\varepsilon < \zeta/10$.

We choose $\varepsilon_2 > 0$ sufficiently small such that every $x \in \Delta$ with $\log \|df^{-1}_{|E^u(x)}\| < -\zeta/3$ necessarily belongs to Ω_3. This is possible because the exponential expansions or contractions in the unstable and stable directions k^u and k^s vanish at the same time.

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We take $\varepsilon_1 > 0$ sufficiently small such that Ω_3 is a closed subset of Ω not intersecting $\overline{\Omega_2 \cup B(S, \varepsilon_1)}$. Therefore, there exists some $\rho > 0$ such that $d(\Omega_3, \overline{\Omega_2 \cup B(S, \varepsilon_1)}) > \rho$. Then, ε_1 being fixed, we choose the size $\delta > 0$ for rectangles of first generation as small as we wish. In particular, we assume that $2\delta < \rho/10$.

Each symbol a_i is associated to a point $\xi_{a_i} \in \Lambda \cap \Omega_0$ (used for the pseudo-orbit); if $\xi_{a_i} \in \Omega_1$, then we say that the rectangle has *order* 0; if $\xi_{a_i} \in \Omega_2$ and $n^+(\xi_{a_i}) = n > 1$ or $n^-(\xi_{a_i}) = n > 1$, then we say that the rectangle has *order* n.

Remark 2.5. We observe that for each $n \geqslant 0$, there are only finitely many rectangles of order n; the union of rectangles of order 0 covers $\Lambda \cap \Omega_1$ and the union of rectangles of orders greater than 1 covers $\Lambda \cap \Omega_2$. The union of rectangles of order 0 can be included in a neighborhood of size 2δ of Ω_1; the union of rectangles of higher order can be included in a neighborhood of Ω_2 of size 2δ.

2.5.2. *The rectangles.* Recall that each rectangle of second generation $S_{a_i} \in \mathcal{S}$ is obtained as a thinner rectangle of T_{a_i}. It thus inherits the order of T_{a_i}. As $\delta > 0$ is taken small, none of the rectangles of second generation and of order 0 which intersect Ω_3 can intersect some rectangle of order $n > 1$. We say that a rectangle of order 0 that intersects Ω_3 is of order 00.

Therefore, each rectangle of order 00 intersects only with a finite number of other rectangles. Hence, we can cut them as in [**6**]. Let S_{a_i} be a rectangle of order 00 and S_{a_j} be any other rectangle such that $S_{a_i} \cap S_{a_j} \neq \emptyset$. We set

$$S^1_{a_i a_j} = \{x \in S_{a_i} : W^u(x, S_{a_i}) \cap S_{a_j} = \emptyset \text{ and } W^s(x, S_{a_i}) \cap S_{a_j} = \emptyset\},$$
$$S^2_{a_i a_j} = \{x \in S_{a_i} : W^u(x, S_{a_i}) \cap S_{a_j} \neq \emptyset \text{ and } W^s(x, S_{a_i}) \cap S_{a_j} = \emptyset\},$$
$$S^3_{a_i a_j} = \{x \in S_{a_i} : W^u(x, S_{a_i}) \cap S_{a_j} \neq \emptyset \text{ and } W^s(x, S_{a_i}) \cap S_{a_j} \neq \emptyset\},$$
$$S^4_{a_i a_j} = \{x \in S_{a_i} : W^u(x, S_{a_i}) \cap S_{a_j} = \emptyset \text{ and } W^s(x, S_{a_i}) \cap S_{a_j} \neq \emptyset\}.$$

Define $\mathcal{S}_0$ as the set of points in M belonging to some S_{a_i} of order 00. For $x \in \mathcal{S}_0$ we set

$$\mathcal{R}(x) = \{y \in M, \forall i,\ j \in \mathbb{N}, x \in S^k_{a_i a_j} \Rightarrow y \in S^k_{a_i a_j}\}.$$

This defines a partition $\mathcal{R}$ of $\mathcal{S}_0$. By construction this partition is finite and each of its elements is stable under the map $[\,\cdot\,,\,\cdot\,]$. In other words, each element of $\mathcal{R}$ is a rectangle. These sets are called *rectangles of third generation*. If $x \in R_i \in \mathcal{R}$, we set

$$W^u(x, R_i) = D^u_{\varepsilon_0}(x) \cap R_i \quad \text{and} \quad W^s(x, R_i) = D^s_{\varepsilon_0}(x) \cap R_i.$$

By construction, if x is a point in a rectangle of second generation $S_{a_{i_1}}, \ldots, S_{a_{i_p}}$, then

$$W^{u,s}(x, R_i) = W^{u,s}(x, S_{a_{i_k}}) \cap R_i \quad \text{for } 1 \leq k \leq p.$$

Moreover, the next result follows as in [**10**, Proposition 3.9], which means that the family $\mathcal{R}$ is a Markov partition of $\mathcal{S}_0$.

PROPOSITION 2.6. *Let R_i and R_j be rectangles of third generation, $n \geqslant 1$ be some integer and $x \in R_i \cap f^{-n}(R_j)$. Then*

$$f^n(W^u(x, R_i)) \supset W^u(f^n(x), R_j) \quad \text{and} \quad f^n(W^s(x, R_i)) \subset W^s(f^n(x), R_j).$$

We also emphasize that, by construction, every point in a rectangle of third generation $\mathcal{R}$ is a density point for $\mathcal{R}$ and with respect to the unstable Lebesgue measure Leb^u.

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3. *Proof of Theorem A*

In this section we prove the existence of a finite SRB measure. The measure is constructed for a map $g : \mathcal{S}_0 \to \mathcal{S}_0$. The map g is of the form $g(x) := f^{\tau(x)}(x)$, and the last step of this section is to prove that the return time τ is integrable. This allows us to define the f-invariant probability SRB measure.

The map g needs good hyperbolic properties. Namely, we need to control the distortion of $\log J^u$ along orbits. For this we use the notion of hyperbolic times. This is explained in §3.1.

Later in this section we construct an invariant SRB measure for the map g. For this we consider a sequence of pushforwards of Leb^u on some piece of unstable manifold. We need to prove some tightness and thus to extend g on some $\overline{\mathcal{S}_0}^u \subset \overline{\mathcal{S}_0}$.

3.1. *Hyperbolic times.* Observe that every point in $\mathcal{S}_0$ is a λ-hyperbolic point and returns infinitely many often in Ω_3. Hence, every point in $\mathcal{S}_0$ returns infinitely many often in $\mathcal{S}_0$.

Let $x \in \mathcal{S}_0$ and $n \in \mathbb{N}^*$ be such that $f^n(x) \in \mathcal{S}_0$. Then there exist two rectangles of third generation R_l and R_k such that $x \in R_l$ and $f^n(x) \in R_k$. Thus, the Markov property given by Proposition 2.6 implies

$$f^{-n}(W^u(f^n(x),\, R_k)) \subset W^u(x,\, R_l).$$

To achieve our goal we need some uniform distortion bound on

$$\prod_{k=0}^{n-1} \frac{J^u(f^k(x))}{J^u(f^k(y))},$$

where $y \in f^{-n}(W^u(f^n(x),\, R_k))$ and $J^u(z)$ is the unstable Jacobian $\det df_{|E^u(z)}(z)$. To obtain distortion bounds we will use the notion of hyperbolic times introduced in [**1**].

Definition 3.1. Given $0 < r < 1$, we say that n is a *r-hyperbolic time* for x if for every $1 \le k \le n$,

$$\prod_{i=n-k+1}^{n} \|df^{-1}_{|E^u(f^i(x))}\| \le r^k.$$

It follows from [**2**, Corollary 3.2] that if

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|df^{-1}_{|E^u(f^j(x))}(f^j(x))\| < 2 \log r,$$

then there exist infinitely many r-hyperbolic times for x. Therefore, by construction of Δ, taking $r_0 = e^{-(1/3)\zeta}$, for every x in $\mathcal{S}_0$ there exist infinitely many r_0-hyperbolic times for x.

Remark 3.2. Another important consequence we shall use later is that there is a set with positive density of r_0-hyperbolic times.

The next lemma shows that hyperbolic times allow to control the distortion for J^u in small (dynamical) balls.

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LEMMA 3.3. *There exists $\delta' > 0$ such that, if $\delta < \delta'$, then, for every x in $\mathcal{S}_0$, for every r_0-hyperbolic time for x, n, and for every y in $B_{n+1}(x, 4\delta)$, the integer n is also a $\sqrt{r_0}$-hyperbolic time for y.*

Proof. Pick $\varepsilon > 0$ small compared to ζ. By continuity of df and E^u, there exists some $\delta' > 0$ such that for all $x \in U$ and all $y \in B(x, 4\delta')$,

$$e^{-\varepsilon} < \frac{\|df^{-1}_{|E^u(x)}(x)\|}{\|df^{-1}_{|E^u(y)}(y)\|} < e^{\varepsilon}.$$

Let us assume that $\delta < \delta'$. Take $x_0 \in \mathcal{S}_0$ and n an r_0-hyperbolic time for x. Given $y \in B_{n+1}(x, 4\delta)$, then for every $0 \le k \le n$, we have that $f^k(y) \in B(f^k(x), 4\delta) \subset B(f^k(x), 4\delta')$, which means that for every $0 \le k \le n$,

$$e^{-\varepsilon} < \frac{\|df^{-1}_{|E^u(f^k(x))}(f^k(x))\|}{\|df^{-1}_{|E^u(f^k(y))}(f^k(y))\|} < e^{\varepsilon}. \tag{3.1}$$

The real number ε is very small compared to ζ, thus (3.1) proves that for every $0 \le k \le n$,

$$\|df^{k-n}_{E^u(f^k(y))}\| < (\sqrt{r_0})^{n-k}.$$

This proves that n is a $\sqrt{r_0}$-hyperbolic time for every y. □

From here on we assume that $\delta < \delta'$. The next lemma is a crucial point in the proof: *hyperbolic times are also return times.*

LEMMA 3.4. *If $x \in \mathcal{S}_0$ and $n \geqslant 1$ is an r_0-hyperbolic time for x, then $f^n(x) \in \mathcal{S}_0$.*

Proof. By definition of r_0-hyperbolic time, $\|df^{-1}_{|E^u(f^n(x))}\| < r_0 = e^{-(1/3)\zeta}$, which implies that $f^n(x) \in \Omega_3$ (by definition of ε_2). Hence, $f^n(x) \in \mathcal{S}_0$. □

Conversely, if x and $f(x)$ belong to $\mathcal{S}_0$, then 1 is an r_0-hyperbolic time for x. Therefore, we say that an r_0-hyperbolic time for x is a *hyperbolic return in* $\mathcal{S}_0$. Moreover, the Markov property of $\mathcal{R}$ proves that, if n is an r_0-hyperbolic time for x, then there exists $R_k \in \mathcal{R}$ such that $f^n(x)$ is in R_k and we have

$$f^{-n}(W^u(f^n(x), R_k)) \subset \mathcal{S}_0.$$

3.2. *The induced map g and its hyperbolic properties.* For our purpose, we need to make precise what 'itinerary' and 'cylinder' mean. The rectangles of third generation have been constructed by intersection of rectangles of second generation (of type 00). Rectangles of second generation are a restriction of rectangles of first generation. We recall that the set of rectangles of first generation satisfies an 'almost' Markov property.

We say, then, that two pieces of orbits $x, f(x), \ldots, f^n(x)$ and $y, f(y), \ldots, f^n(y)$ have the same itinerary if x and y have the same codes in Σ_0 up to the time which represents n. As the dynamics in Σ_0 is semiconjugated to that induced by F, this time may be different to n. Itineraries are thus not well defined for every value of n, but are well defined for every 'return' in $\mathcal{S}_0$.

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The points in the third generation of rectangles having the same itinerary until some return time into $\mathcal{S}_0$ define *cylinders*.

For $x \in \mathcal{S}_0$, we set $\tau(x) = n$ if n is the smallest positive integer such that n is an r_0-hyperbolic time for some point y having the same n-itinerary as x. We point out that Lemma 3.4 shows that for such a n, the itinerary is well defined. This allows us to define a map g from $\mathcal{S}_0$ to itself by $g(x) = f^{\tau(x)}(x)$.

Remark 3.5. As τ is not the first return time, the map g is *a priori* not one-to-one.

As usual we set

$$\tau^1(y) = \tau(y) \quad \text{and} \quad \tau^{n+1}(y) = \tau^n(y) + \tau(g^n(y)).$$

The n-cylinder for x will be the cylinder associated to $\tau^n(x)$.

We can now state the main hyperbolic properties for g.

LEMMA 3.6. *There is* $0 < \omega_0 < 1$ *such that for all* $0 \le k \le \tau(y)$, $y \in \mathcal{S}_0$ *and* $z \in f^{-\tau(y)}(W^u(g(y), \mathcal{R}(g(y))))$,

$$d^u(f^k(z), f^k(y)) \le \omega_0^{\tau(z)-k} d^u(g(z), g(y)).$$

Proof. If $y \in \mathcal{S}_0 \cap f^{-1}(\mathcal{S}_0)$, then $g(y) = f(y)$ and $y \in \Omega_3$ (far away from S). If $g(y) = f^{\tau(y)}(y)$ with $\tau(y) > 1$, then by definition of g, there exists y' such that:

(1) $\tau(y)$ is an r_0-hyperbolic time for y'; and

(2) y' is in $\mathcal{C}(y) \stackrel{\text{def}}{=} f^{-\tau(y)}(W^u(f^{\tau(y)}(y), \mathcal{R}(f^{\tau(y)}(y))))$.

The diameter of $\mathcal{R}(f^{\tau(y)}(y))$ is smaller than δ. Hence, by construction of the third generation of rectangles, $\mathcal{C}(y)$ is included in $B_{\tau(y)+1}(y, 4\delta)$, which gives that $\tau(y)$ is a $\sqrt{r_0}$-hyperbolic time for every point in $\mathcal{C}(y)$. Therefore, the map

$$f^{k-\tau(y)} : W^u(f^{\tau(y)}(y), \mathcal{R}(f^{\tau(y)}(y))) \to f^k(\mathcal{C}(y))$$

is a contraction and satisfies

$$\|df^{k-\tau(y)}_{|E^u}\| \le \sqrt{r_0}^{\tau(y)-k}.$$

Thus,

$$d^u(f^k(z), f^k(y)) \le r_0^{(\tau(y)-k/2)} d^u(g(z), g(y)). \qquad \square$$

LEMMA 3.7. *There exist some constants* $\chi_1 > 0$ *and* $0 < \omega < 1$ *such that for every* $n \ge 1$, *for every* $g^n(y)$ *in* $g^n(W^u(x, \mathcal{R}(x)))$, *for every* z *in* $f^{-\tau^n(y)}(W^u(g^n(y), \mathcal{R}(y)))$ *and for every* $m \le n$, *we obtain*

$$\sum_{j=0}^{\tau^m(y)-1} |\log(J^u(f^j(z))) - \log(J^u(f^j(y)))| \le \chi_1 \, \omega^{n-m}.$$

Proof. The map $x \mapsto J^u(x)$ is Hölder continuous because the map E^u is Hölder continuous; moreover, it takes values in $[1, +\infty[$. The map $t \mapsto \log(t)$ is Lipschitz continuous on $]1, +\infty[$. Thus, there exist some constants χ_2 and α, such that

$$\sum_{j=0}^{\tau^m(y)-1} |\log(J^u(f^j(z))) - \log(J^u(f^j(y)))| \le \chi_2 \sum_{j=0}^{\tau^m(y)-1} (d^u(f^j(z), f^j(y)))^\alpha. \quad (3.2)$$

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Hence, Lemma 3.6 and (3.2) give

$$\sum_{j=0}^{\tau^m(y)-1} |\log(J^u(f^j(z))) - \log(J^u(f^j(y)))| \le \chi_2 \left[\sum_{j=0}^{+\infty} r_0^{j\alpha/2}\right] (d^u(g^m(z), g^m(y)))^\alpha.$$

Lemma 3.6 also yields $d^u(g^m(z), g^m(y)) \le r_0^{(n-m)/2} \operatorname{diam}(\mathcal{R}(g^n(y)))$. □

3.3. *Extension to $\overline{\mathcal{S}_0}^u$. The SRB measure for g.*

3.3.1. *u-Closure.* The set of rectangles of order 00 is finite and defines a finite alphabet, say $\mathcal{A}$. As all these rectangles are disjoint, the g-forward itinerary of any point in $\mathcal{S}_0$ defines an infinite word in $\mathcal{A}^N$. Two points in the same stable leaf of a rectangle of order 00 define the same infinite word and expansion (together with hyperbolic times) shows that two points defining the same infinite word must be in the same stable leaf.

Let $\widehat{\mathcal{S}_0}$ be the set of words defined in such a way. It may be not closed in $\mathcal{A}^{\mathbb{N}}$. We take its closure and we claim that the correspondence between itineraries and words can be extend to this closure. This defines the g-invariant set in M, $\overline{\mathcal{S}_0}^u$. This set corresponds to the u-closure of $\mathcal{S}_0$, that is, the closure with respect to the topology in the unstable leaves.

3.3.2. *Definition of a sequence of measures and u-weak* topology.* We use the method in [**8**], which uses results from [**9**, §6]. Let x_0 be some point in $\mathcal{S}_0$ and set Leb_0^u as the restriction and renormalization of $\mathrm{Leb}_{x_0}^u$ to $W^u(x_0, \mathcal{R}(x_0))$. We define

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} g_*^i(\mathrm{Leb}_0^u).$$

We want to consider some accumulation points for μ_n. The next lemma shows they are well defined and g-invariant.

LEMMA 3.8. *Any weak* accumulation point for the u-closure topology for μ_n is g-invariant.*

Proof. For every n, the set $\mathcal{S}_0$ has full μ_n-measure μ_n. This means that the measure can be lifted into $\mathcal{A}^{\mathbb{N}}$ and that $\widehat{\mathcal{S}_0}$ has full measure for this lifted measure. Then any accumulation point in $\mathcal{A}^{\mathbb{N}}$ for the weak* topology can be projected in $\overline{\mathcal{S}_0}^u$ and is g-invariant. □

3.3.3. *Control of the distortion.* In the following we consider an accumulation point μ for μ_n as defined in Lemma 3.8.

We want to prove that μ is an SRB measure. First, let us make precise what SRB means for μ. We recall that being SRB means that the conditional measures are equivalent to the Lebesgue measure Leb^u on the unstable leaves. This can be defined for any measure, not necessarily the f-invariant ones, and only requires that the partition into pieces of unstable manifolds is measurable (see [**11**]).

To prove that μ is an SRB measure, it is sufficient (and necessary) to prove that there exists some constant χ, such that for every integer n, for every $y \in g^n(W^u(x_0, \mathcal{R}(x_0)))$

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and $y = f^m(x)$ for some $x \in W^u(x_0, \mathcal{R}(x_0))$, and for every $z \in W^u(y, \mathcal{R}(y))$,

$$e^{-\chi} \leq \frac{\prod_{i=0}^{m-1} J^u(f^{-i}(y))}{\prod_{i=0}^{m-1} J^u(f^{-i}(z))} \leq e^{\chi}. \tag{3.3}$$

Lemma 3.7 yields that (3.3) holds for every n, for every $y \in g^n(W^u(x_0, \mathcal{R}(x_0)))$ and for every $z \in W^u(y, \mathcal{R}(y))$, with $\chi := \chi_1 + 2\log\kappa$.

3.3.4. *Non-escaping mass.* We consider an accumulation point as per Lemma 3.8. Therefore the final measure may be concentrated in $\overline{\mathcal{S}_0}\backslash\mathcal{S}_0$. We need to prove that this is not the case.

LEMMA 3.9. *Let $W^s_{\mathrm{loc}}(\mathcal{S}_0)$ denote the set of local stable leaves of points in $\mathcal{S}_0$. The measure μ can be chosen so that $\mu(W^s_{\mathrm{loc}}(\mathcal{S}_0)) = 1$.*

Proof. Let us pick some rectangle of order 00, say R_k, having positive μ measure. Due to the Markov property, $g^n_*(\mathrm{Leb}_0)_{|R_k}$ is given by the unstable Lebesgue measure supported on a finite number of unstable leaves of the form $W^u(z, R_k)$.

Now recall that stable and unstable leaves in R_k are in $\mathcal{S}_0$. Recall also that stable and unstable foliations are absolutely continuous. We can use the rectangle property of R_k to project all the Lebesgue measures for supp $g^n_*\,\mathrm{Leb}_0 \cap R_k$ on a fixed unstable leaf of R_k, say $\mathcal{F}_k$. All these measures project themselves on a measure absolutely continuous with respect to Lebesgue, which yields that the projection on $\mathcal{F}_k$ of μ_n can be written in the form $\varphi_{n,k} d\,\mathrm{Leb}_{\overline{\mathcal{F}_k}}$. We have to consider the u-closure $\overline{\mathcal{F}_k}$ to take into account the fact that μ may 'escape' from $\mathcal{S}_0$.

It is a consequence of the bounded variations stated above (see (3.3)) that all the $\varphi_{n,k}$ are uniformly (in n) bounded $\mathrm{Leb}_{\overline{\mathcal{F}_k}}$ almost everywhere. In other words, all the $\varphi_{n,k}$ belong to a ball of fixed radius for $L^\infty(\mathrm{Leb}_{\overline{\mathcal{F}_k}})$ and this ball is compact for the weak* topology.

Therefore, we can consider a converging subsequence (for the weak* topology). As there are only finitely many rectangles R_k, we can also assume that the sequence converges for every R_k. For simplicity we still write $\to_{n\to\infty}$ even if we are only considering a subsequence.

Now convergence means that for every $\psi \in L^1(\mathrm{Leb}(\overline{\mathcal{F}_k}))$,

$$\lim_{n\to\infty} \int \psi\varphi_{n,k}\, d\,\mathrm{Leb}_{\overline{\mathcal{F}_k}} = \int \psi\varphi_{\infty,k}\, d\,\mathrm{Leb}_{\overline{\mathcal{F}_k}}.$$

If π_k denotes the projection on $\mathcal{F}_k$, we can choose for ψ the function $\mathbb{1}_{\overline{\mathcal{F}_k}\backslash\mathcal{F}_k}$. Then for every n,

$$0 \leqslant \int \psi\varphi_{n,k}\, d\,\mathrm{Leb}_{\overline{\mathcal{F}_k}} = \int \psi\varphi_{n,k}\, d\,\mathrm{Leb}_{\mathcal{F}_k} \leqslant \mu_n(\overline{R_k}\backslash R_k) = 0.$$

This shows that $\mathcal{F}_k$ has full $\pi_{k*}\mu$ measure, and as this holds for every k, this shows that $W^s_{\mathrm{loc}}(\mathcal{S}_0)$ has full μ measure. □

Remark 3.10. An important consequence of Lemma 3.9 is that for μ- almost every point x, there exists a point $y \in \mathcal{S}_0$ in its local stable leaf. Note that every r_0-hyperbolic time for y is a return time for y, and then also for x.

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3.4. *The SRB measure for f.* The map g satisfies $g(z) = f^{\tau(z)}(z)$ for every z in $\mathcal{S}_0$. Let $\mathcal{T}(i) = \{z \in \mathcal{S}_0 : \tau(z) = i\}$ be the set of points in $\mathcal{S}_0$ such that the first return time equals i. We set

$$\widehat{m} = \sum_{i=1}^{+\infty} \sum_{j=0}^{i-1} f_*^j(\mu|\mathcal{T}(i)).$$

Then $\widehat{m}$ is a σ-finite f-invariant measure and it is finite if and only if τ is μ-integrable; see [**13**]. In this case, we may normalize the measure to obtain some probability measure. This will be an SRB measure.

We define for $n \geqslant 1$ the set

$$H_n = \{x \in M : n \text{ is an } r_0\text{-hyperbolic time for } y \text{ in } W^s_{\mathrm{loc}}(x)\}.$$

Observe that the following properties hold:

(a) if $x \in H_j$ for $j \in \mathbb{N}$, then $f^i(x) \in H_m$ for any $1 \leqslant i < j$ and $m = j - i$;

(b) there is $\theta > 0$ such that for μ-almost every x in $\mathcal{S}_0$,

$$\limsup_{n\to\infty} \frac{1}{n}\#\{1 \leq j \leq n : x \in H_j\} \geq \theta;$$

(c) $H_n \subset \{\tau \leqslant n\}$ for each $n \geqslant 1$.

Condition (a) is a direct consequence of the definition for H_n. Condition (b) holds by construction of $\mathcal{S}_0$, by Pliss's lemma (see, for example, [**2**]) and the property emphasized in Remark 3.10. Property (c) holds due to Lemma 3.4.

PROPOSITION 3.11. *The inducing time τ is μ-integrable.*

Proof. Suppose by contradiction that $\int \tau \, d\mu = \infty$. By Birkhoff's ergodic theorem,

$$\frac{1}{i}\sum_{k=0}^{i-1} \tau(g^k(x)) \to \int \tau \, d\mu = \infty,$$

for μ-almost every point $x \in \mathcal{S}_0$. Let $x \in \mathcal{S}_0$ be a μ-generic point. Define, for every $i \in \mathbb{N}$,

$$j_i = j_i(x) = \sum_{k=0}^{i-1} \tau(g^k(x)).$$

This means that $g^i(x) = f^{j_i}(x)$. We define

$$\mathcal{I} = \mathcal{I}(x) = \{j_1, j_2, j_3, \ldots\}.$$

Given $j \in \mathbb{N}$, there exists a unique integer $r = r(j) \geq 0$ such that $j_r < j \leq j_{r+1}$. Supposing that $x \in H_j$, then $g^r(x) \in H_m$, where $m = j - j_r$, by (a) above. For each n, we have

$$\frac{1}{n}\#\{j \leq n : x \in H_j\} \leq \frac{r(n)}{n}.$$

By construction, if $r(n) = i$, that is, $j_i < n \leq j_{i+1}$, then

$$\frac{j_i}{i} < \frac{n}{r(n)} \leq \frac{j_{i+1}}{i+1}\left(1 + \frac{1}{i}\right).$$

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Since

$$\frac{j_i}{i} = \frac{1}{i}\sum_{k=0}^{i-1} \tau(g^k(x)) \to \infty \quad \text{as } i \to +\infty,$$

it follows that

$$\lim_{n\to\infty} \frac{1}{n}\#\{1 \le j \le n : x \in H_j\} = \lim_{n\to\infty} \frac{r(n)}{n} = 0.$$

This contradicts (b), and so the inducing time function τ must be μ-integrable. □

4. *An example*

Here we describe the construction of a diffeomorphism f of the three-dimensional solenoid $M := \mathbb{S}^1 \times \mathbb{D}^2$. The construction has three steps. First, we carry out a local perturbation of a 3-adic Smale attractor. The main technical point is a change of the Riemannian metric. Then, we explain how we glue the local perturbation with the non-perturbed part. Finally, we show that the example satisfies hypotheses for Theorem A.

4.1. *The initial map.* We consider initially a 3-adic Smale attractor $\tilde{f}$ on M. By a change of coordinates we may assume that in a small neighborhood V centered at the fixed point $O = (0, 0, 0) \in M$ the diffeomorphism $\tilde{f}$ has a product form

$$(x, y, z) \longmapsto (\tilde{\phi}(x), \tilde{\psi}_1(y), \tilde{\psi}_2(z)) = (3x, \lambda^{-1}y, \lambda^{-1}z),$$

with $\lambda > 1$ and that $\{y = 0 \ \& \ z = 0\}$ is the local unstable manifold of O and $\{x = 0\}$ is the local stable manifold of 0. Note that by our assumption, O belongs to the attractor $\Lambda \subset M$.

Using polar coordinates in the vertical plan (y, z), we define a new map $F(x, r, \theta) := \tilde{f}(x, r\cos\theta, r\sin\theta)$. For simplicity we also assume that vertical slices are sent into vertical slices and that F is constant in θ. We can thus set $F(x, r)$ instead of $F(x, r, \theta)$. In the following, we shall either describe a point by three coordinates, (x, y, z) or (x, r, θ), or by two coordinates, (x, r), when the dependence in θ is not relevant.

4.2. *The local perturbation.* Now the idea is to modify F in V so that O becomes an indifferent fixed point for both dynamics on the stable and unstable directions, but still keeping the expanding behavior in $\{y = 0 \ \& \ z = 0\}$ and contracting behavior in $\{x = 0\}$. To do this we consider ϕ a Manneville–Pomeau $C^{1+\gamma}$ map in a neighborhood of 0, e.g.

$$\phi(x) = \frac{x}{(1 - x^\gamma)^{1/\gamma}} \quad \text{for some } 0 < \gamma < 1,$$

and ψ a contractive Manneville–Pomeau map; the main assumption on the ψ is that

$$\psi'(\xi) = 1 - \xi + \text{higher-order terms}.$$

with all negative higher-order terms. For instance, $\psi(\xi) = 2\xi + \log(1 - \xi)$ satisfies this assumption.

Note that ϕ admits an absolutely invariant continuous probability measure (see [**12**]).

We may consider the maps ϕ and ψ coinciding with $\tilde{\phi}$ and $\tilde{\psi}_j$, respectively, near the boundary of V. Hence, the map defined by ϕ and ψ in V can be extended to a map $f : M \to M$, by simply considering $f(x, y, z) = \tilde{f}(x, y, z)$ for $(x, y, z) \notin V$.

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4.3. *Change of metric.* The product map (ϕ, ψ) does not satisfies the assumption of Theorem A because the lines $x = 0$ and $r = 0$ have one neutral direction and one contracting or expanding direction. To transform the dynamics into a non-uniformly hyperbolic map with a single indifferent fixed point $\mathcal{O}$ we shall change the metric in $\widetilde{V} \Subset V$ as follows. For (x, y, z) in $\mathbb{R}^3$ we define a new scalar product

$$\langle u|v\rangle^* := \alpha(x, r)u \cdot v,$$

where $u \cdot v$ is the usual scalar product for the Euclidean norm and $r = \sqrt{y^2 + z^2}$. The function α needs to be $\mathcal{C}^\infty$ and positive.

We want df to be expanding in the horizontal direction and contracting in the vertical plane, except on O. The conditions are

$$\frac{\alpha(F(x, r))}{\alpha(x, r)}\phi'(x) > 1 \quad \text{for } (x, r) \neq (0, 0), \tag{4.1a}$$

$$0 < \frac{\alpha(F(x, r))}{\alpha(x, r)}\psi'(r) < 1 \quad \text{for } (x, r) \neq (0, 0), \tag{4.1b}$$

$$\alpha(0, 0) = 1. \tag{4.1c}$$

These conditions will be referred to as (4.1).

LEMMA 4.1. *Let $\varepsilon > 0$ and $\gamma' > \gamma$ be real numbers. The function defined by*

$$\widetilde{\alpha}(x, r) := \prod_{n=1}^{+\infty} \frac{1}{1 + (1 + (1/\gamma))(\phi^{-n}(x))^{\gamma'}} \prod_{n=0}^{+\infty}(1 - (\psi^n(r))^{1+\varepsilon})$$

satisfies conditions (4.1).

Proof. First, $\widetilde{\alpha}(0, 0) = 1$. Let us set $g(x, r) := (\widetilde{\alpha}(F(x, r))/\widetilde{\alpha}(x, r))$. We get

$$g(x, r) = \frac{1}{(1 + (1 + (1/\gamma))x^{\gamma'})(1 - r^{1+\varepsilon})}. \tag{4.2}$$

It remains to prove that both infinite products converge. Note that

$$\phi^{-n}(x) = \frac{x}{(1 + nx^\gamma)^{1/\gamma}},$$

which yields

$$\log\left(1 - \left(1 + \frac{1}{\gamma}\right)(\phi^{-n}(x))^{\gamma'}\right) \in O(n^{-(\gamma'/\gamma)})$$

and we use $\gamma' > \gamma$ to get the convergence of the logarithm of the product, hence convergence of the product. Similarly,

$$r = \psi^{-n} \circ \psi^n(r) \sim \frac{\psi^n(r)}{1 - n\psi^n(r)},$$

which yields

$$\psi^n(r) = \frac{r}{1 + nr} + O\left(\frac{1}{n}\right),$$

and then

$$\log(1 - (\psi^n(r))^{1+\varepsilon}) \in O\left(\frac{1}{n^{1+\varepsilon}}\right),$$

and the second infinite product also converges. □

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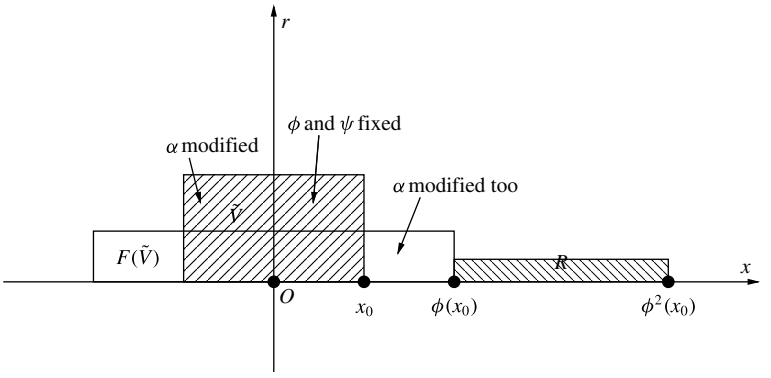


FIGURE 2. Modification of the Riemannian norm.

We claim that convergence for both infinite products is normal in a small neighborhood of $(0, 0)$, which shows that $\widetilde{\alpha}$ is continuous. If the change of the metric is done in a very small neighborhood of O, $\widetilde{\alpha}$ is positive, upper-bounded by 1 and lower-bounded by $1 - \varepsilon'$ with $\varepsilon' > 0$ small. If α is not $\mathcal{C}^\infty$ we can approximate it by a $\mathcal{C}^\infty$ positive function which still satisfies (4.1).

4.4. *Coherence with the non-perturbed part.* The perturbation has been done in the fundamental domain $\widetilde{V}$ (see Figure 2). However, to realize conditions (4.1) the metric α has also been modified in $F(\widetilde{V})\backslash\widetilde{V}$. We emphasize that the modification of the metric propagates itself to ensure hyperbolicity. As we want to glue the perturbation f with $\widetilde{f}$ itself, we need to decrease the modification of the metric near the boundary of V.

We now explain how we change the metric in R (see Figure 2) so as to authorize connections with the dual metric outside V. First, we emphasize that α is less than 1 but can be made greater than $1 - \varepsilon$ by choosing $\widetilde{V}$ very small. We still want a single parabolic point, thus conditions (4.1) hold in $F(\widetilde{V})\backslash\widetilde{V}$ and in R, but with a more general form of F. That is to say, we need to get for (x, r) in $F(\widetilde{V})\backslash\widetilde{V}$,

$$\frac{\alpha(F(x,r))}{\alpha(x,r)}\frac{\partial F}{\partial x} > 1 \quad \text{and} \quad 0 < \frac{\alpha(F(x,r))}{\alpha(x,r)}\frac{\partial F}{\partial r} < 1.$$

Remember that $(\partial F/\partial x) > 1$ because we are far away from O but $(\partial F/\partial r)(x_0, 0) = \psi'(0) = 1$ (and $(\partial F/\partial r) \geqslant 1$ elsewhere). On the other hand, we can impose that on the right-hand side of $F(\widetilde{V})\backslash\widetilde{V}$, F coincides with $\widetilde{f}$, which implies $(\partial F/\partial x) = 3$ and $(\partial F/\partial r) = \frac{1}{10}$ close to the right-hand-side border of $F(\widetilde{V})\backslash\widetilde{V}$.

This means that there is room to define $\alpha \circ F$ in $F(\widetilde{V})\backslash\widetilde{V}$ (or in other words, to define α in R) such that it is $\mathcal{C}^\infty$ and that $\alpha \equiv 1$ close to the right-hand-side border of R (see Figure 3).

Note that (4.1a) shows that $\alpha(x_0, 0)/(\partial F/\partial x)(x_0, 0) < \alpha(F(x_0, 0))$, which allows us to have α continuous at $F(x_0, 0)$.

The same kind of arguments allow us to modify α in $F^{-1}(F^{-1}(\widetilde{V})\backslash\widetilde{V})$, and we finally get a single parabolic point O in the Smale attractor Λ.

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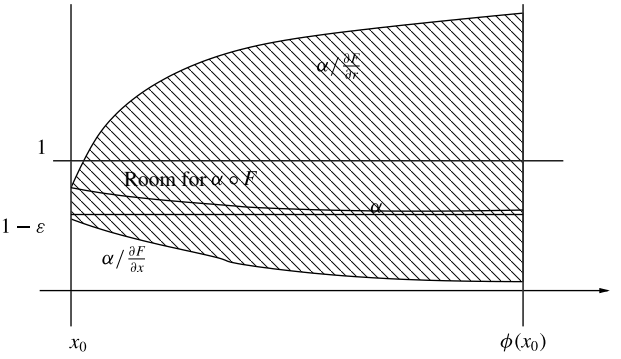


FIGURE 3. Room to modify α in R.

4.5. *Assumptions of Theorem A.* By construction, there is a df-invariant splitting in Λ and f is almost Axiom-A. Moreover, unstable manifolds have infinite length. The only assumption to check is that Leb^u sees some λ-hyperbolic point.

Concerning backward iterates (and contraction in the unstable direction), any periodic point has a one-dimensional local unstable manifold and every point in such an unstable manifold has backward exponential contractions in the unstable direction.

Therefore we just have to check that Leb^u sees forward contractions in the stable direction. For that, we first claim that $f : \Lambda \to \Lambda$ projects on $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$. By construction, φ is a three-to-one map; two branches are equal to $x \mapsto 3x \bmod 1$ and the branch at 0 is expanding with an indifferent point at the origin with the same germ as ϕ (the dynamics corresponds to ϕ close to 0).

Therefore φ admits an absolutely invariant probability measure (see [**12**]), and Lebesgue almost every point in $\mathbb{S}^1$ is expanding for φ. This implies that Lebesgue almost every point returns with positive frequency in $\mathbb{S}^1 \setminus]-\varepsilon, \varepsilon[$ (with $\varepsilon > 0$ small). Consequently, Leb^u-almost every point in Λ returns in $M \setminus]-\varepsilon, \varepsilon[\times \mathbb{D}^2$ with positive frequency, which shows that Leb^u-almost every point has exponential contraction in the stable direction.

5. *Proof of Theorem B*

Let $\lambda > 0$ be fixed. We consider a large positive constant $K \gg 2$ whose magnitude will be adjusted at the end of the proof of Theorem B. Remember that we defined above

$$B(S, \varepsilon_1) = \{y \in M : d(S, y) < \varepsilon_1\} \quad \text{and} \quad \Omega_0 = \Omega \setminus B(S, \varepsilon_1).$$

We pick $\varepsilon_1 > 0$ (precise conditions will be stated along the way) small enough so that, in particular, $B(S, \varepsilon_1) \subset U$. By continuity, we can also choose $\varepsilon_1 > 0$ small enough such that for every $x \in B(S, \varepsilon_1)$,

$$1 \le \|df(x)_{|E^u(x)}\| < e^{\lambda/K} \quad \text{and} \quad 1 \le \|df^{-1}(x)_{|E^s(x)}\| < e^{\lambda/K}. \tag{5.1}$$

Moreover, there exist $\lambda^u > 0$ and $\lambda^s > 0$ such that, for every $x \in \Omega \setminus B(S, \varepsilon_1)$, the following statements hold:

(1) for all $v \in E^u(x) \setminus \{0\}$,

$$\|df(x)v\|_{f(x)} > e^{\lambda^u}\|v\|_x \quad \text{and} \quad \|df^{-1}(x)v\|_{f^{-1}(x)} < e^{-\lambda^u}\|v\|_x;$$

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(2) for all $v \in E^s(x)\backslash\{0\}$,

$$\|df^{-1}(x)v\|_{f^{-1}(x)} > e^{\lambda^s}\|v\|_x \quad \text{and} \quad \|df(x)v\|_{f(x)} < e^{-\lambda^s}\|v\|_x.$$

5.1. *Graph transform.*

5.1.1. *Constants to control lack of hyperbolicity.* We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^N$, where $N = \dim M$. By continuity, we can assume that the maps $x \mapsto \dim E^*(x)$ are constant, for $* = u, s$. From now on, we will denote by $\mathbb{R}^u$ the space $\mathbb{R}^{\dim E^u} \times \{0\}^{\dim E^s}$. In the same way $\mathbb{R}^s$, $B^u(0, \rho)$ and $B^s(0, \rho)$ will denote respectively the spaces $\{0\}^{\dim E^u} \times \mathbb{R}^{\dim E^s}$, $B^{\dim E^u}(0, \rho) \times \{0\}^{\dim E^s}$ and $\{0\}^{\dim E^u} \times B^{\dim E^s}(0, \rho)$.

PROPOSITION 5.1. *Let $\varepsilon > 0$ be small compared to λ, λ^u or λ^s. There are constants $\rho_1 > 0$, $0 < K_1 < K_2$, a positive function $\bar{\rho}$, and a family of embeddings $\phi_x : B_x(0, \rho_1) \subset \mathbb{R}^N \to M$ with $\phi_x(0) = x$ such that†:*

(1) *$d\phi_x(0)$ maps $\mathbb{R}^u$ and $\mathbb{R}^s$ onto $E^u(x)$ and $E^s(x)$, respectively;*

(2) *if $\widehat{f}_x = \phi^{-1}_{f(x)} \circ f \circ \phi_x$ and $\widehat{f}_x^{-1} = \phi^{-1}_{f^{-1}(x)} \circ f^{-1} \circ \phi_x$, then:*

(a) *if $x \in \Omega\backslash B(S, \varepsilon_1)$, then:*

(i) *for all $v \in \mathbb{R}^u\backslash\{0\}$,*

$$|d\widehat{f}_x(0) \cdot v| > e^{\lambda^u}|v| \quad \text{and} \quad |d\widehat{f}_x^{-1}(0) \cdot v| < e^{-\lambda^u}|v|;$$

(ii) *for all $v \in \mathbb{R}^s\backslash\{0\}$,*

$$|d\widehat{f}_x^{-1}(0) \cdot v| > e^{\lambda^s}|v| \quad \text{and} \quad |d\widehat{f}_x(0) \cdot v| < e^{-\lambda^s}|v|;$$

(b) *if $x \in B(S, \varepsilon_1)$, then:*

(i) *for all $v \in \mathbb{R}^u\backslash\{0\}$,*

$$|v| \le |d\widehat{f}_x(0) \cdot v| < e^{\lambda/K}|v| \quad \text{and} \quad |v| \ge |d\widehat{f}_x^{-1}(0) \cdot v| > e^{-(\lambda/K)}|v|;$$

(ii) *for all $v \in \mathbb{R}^s\backslash\{0\}$,*

$$|v| \le |d\widehat{f}_x^{-1}(0) \cdot v| < e^{\lambda/K}|v| \quad \text{and} \quad |v| \ge |d\widehat{f}_x(0) \cdot v| > e^{-(\lambda/K)}|v|;$$

(3) *$0 < \bar{\rho}(x) \le \rho_1$ for every $x \in B(S, \varepsilon_1)$ and $\bar{\rho}(x) = \rho_1$ for every $x \in \Omega\backslash B(S, \varepsilon_1)$;*

(4) *on the ball $B_x(0, \bar{\rho}(x))$ we have $\mathrm{Lip}(\widehat{f}_x - d\widehat{f}_x(0)) < \varepsilon$ and $\mathrm{Lip}(\widehat{f}_x^{-1} - d\widehat{f}_x^{-1}(0)) < \varepsilon$;*

(5) *for every x and for every $z, z' \in B_x(0, \rho_1)$,*

$$K_1|z - z'| \le d(\phi_x(z), \phi_x(z')) \le K_2|z - z'|.$$

This is a simple consequence of the exponential map exp defined for every Riemannian manifold. However, it is important for the rest of the paper to understand that the two constants K_1 and K_2 do not depend on ε_1. They result from the distortion due to the angle between the two subspaces E^u and E^s, plus the injectivity radius. These quantities are uniformly bounded.

Moreover, as Ω is a compact set in U, we can choose $\rho_1 > 0$ such that $B(\Omega, \rho_1) \subset U$. As the maps $x \mapsto E^u(x)$ and $x \mapsto E^s(x)$ are continuous, we can also assume that ρ_1 is small enough so that for every $x \in \Omega$ and $y \in \phi_x(B(0, \rho_1))$, the slope of $d\phi_x^{-1}(y)(E^u(y))$ in $\mathbb{R}^N = \mathbb{R}^u \oplus \mathbb{R}^s$ is smaller than $1/2$.

† The notation B_x is to remind us that the ball defined in R^N has its image centered at x.

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5.1.2. *General graph transform.* Here, we recall some well-known basic statements for the graph transform. Let E be some Banach space and $T : E \to E$ be a linear map such that there exists a T-invariant splitting $E = E_1 \oplus E_2$. We set $T_i = T_{|E_i}$ and we assume that the norm on E is adapted to the splitting, i.e. $\|\cdot\|_E = \max(\|\cdot\|_{E_1}, \|\cdot\|_{E_2})$. We also assume that there exist $\lambda_2 < 0 < \lambda_1$ such that:

(1) $\|T_1^{-1}v\|_{E_1} \leq e^{-\lambda_1}\|v\|_{E_1}$, for every v in E_1;
(2) $\|T_2 v\|_{E_2} \leq e^{\lambda_2}\|v\|_{E_2}$, for every v in E_2.

PROPOSITION 5.2. *Let ρ be a positive number, $\varepsilon > 0$ be small compared to λ_1 and $-\lambda_2$, and $F : E \to E$ be a C^1 map such that $F(0) = 0$, $\mathrm{Lip}(F - T) < \varepsilon$ and $\mathrm{Lip}(F^{-1} - T^{-1}) < \varepsilon$ on the ball $B(0, \rho)$. Then:*

(1) *the image by F of the graph of any map $g : B_1(0, \rho) \to B_2(0, \rho)$ satisfying $g(0) = 0$ is a graph of some map $\Gamma(g) : B_1(0, \rho e^{\lambda_1 - 2\varepsilon}) \to B_2(0, \rho e^{-\lambda_2 + 2\varepsilon})$;*
(2) *F induces some operator Γ on the set Lip_1 of 1-Lipschitz continuous maps $g : B_1(0, \rho) \to B_2(0, \rho)$ with $g(0) = 0$;*
(3) *Γ is a contraction on Lip_1 (with the standard norm on the space of Lipschitz continuous maps), and thus it admits some unique fixed point.*

For the proof of Theorem B, it is important to keep in mind the two key points in the proof of Proposition 5.2. On the one hand, the fact that Γ is a contraction on Lip_1 is essentially due to the spectral gap of dF. This is obtained by the properties of the T_i and Lipschitz proximity of F and T. On the other hand, the fact that the image by F of any graph (from $B_1(0, \rho)$ to $B_2(0, \rho)$) extends beyond the boundary of the ball $B(0, \rho)$ is essentially due to expansion on E_1.

5.1.3. *Adaptations to our case.* We want to use Proposition 5.2 in the fibered case of $\widehat{f}_x$. If x is in $\Omega_0 \cap f^{-1}(\Omega_0)$ the spectral gap of $d\widehat{f}_x(0)$ is uniformly bounded from below in $B_x(0, \rho_1)$, and so we can apply the result with $d\widehat{f}_x(0)$, $\widehat{f}_x$ and $\rho = \rho_1$. In $B(S, \varepsilon_1)$, the value of $\bar{\rho}(x)$ has to decrease to 0 when x tends to S: the spectral gap of df tends to 0 as x tends to S because $k^u(x) + k^s(x)$ tends to 0. The idea is to apply Proposition 5.2 for $d\widehat{f}_x^n(0)$ and $\widehat{f}_x^n$ for some good n. First, we have to check that the assumptions of the proposition hold.

PROPOSITION 5.3. *Let $x \in \Omega_0$ and $n \geq 2$ be such that $f^n(x) \in \Omega_0$ and $f^k(x) \in B(S, \varepsilon_1)$ for all $1 \leqslant k \leqslant n - 1$. There exists some constant $C > 0$ such that for every $0 < r \leq 1$, Proposition 5.2 holds for $F = \widehat{f}_x^n$, $T = dF(0)$ and $\rho = Cre^{-(9n\lambda/2K)}$, and also for $F = \widehat{f}_{f^n(x)}^{-n}$, $T = d\widehat{f}_{f^n(x)}^{-n}(0)$ and $\rho = Cre^{-(9n\lambda/2K)}$.*

Proof. We refer the reader to [**10**, Proposition 2.3]. We emphasize that that proof uses $d^2 f$ to actually get Lipschitz continuity for df. This is now a standard computation for exchanging Lipschitz estimates with Hölder estimates. □

Remark 5.4. We checked that the constant 9 that appeared in [**10**, Proposition 2.3] also works if we only use a Hölder continuity condition on df. We point out that these constants are very common in the Pesin theory and have to be understood 'in spirit': we take a very small fraction of the global minimal expansion ratio.

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Let $x \in \Omega_0$ and $C > 0$ be given by Proposition 5.3. We recall that $n^{\pm}(x)$ were defined in Definition 2.1. If $f(x) \in B(S, \varepsilon_1)$ and $n^+(x) < +\infty$, then we set

$$l^f(x) = Ce^{-(9\lambda/2K)n^+(x)}. \tag{5.2}$$

Analogously, if $f^{-1}(x) \in B(S, \varepsilon_1)$ and $n^-(x) < +\infty$, then we set

$$l^b(x) = Ce^{-(9\lambda/2K)n^-(x)}. \tag{5.3}$$

If $x \in \Omega_0 \cap f^{-1}(\Omega_0)$, the graph transform due to Proposition 5.2 with $F = \widehat{f_x}$, $T = d\widehat{f_x}(0)$ and $\rho = \rho_1$ will be called the *one-step graph transform*. If $x \in \Omega_0 \cap f^{-1}(B(S, \varepsilon_1))$, the graph transform due to Proposition 5.3 with $F = \widehat{f_x}^{n^+(x)}$, $T = dF(0)$ and $\rho = Cre^{-(9\lambda/2K)n^+(x)}$ will be called the *$n^+(x)$-step graph transform*. Both will be denoted by Γ_x.

5.2. *Truncations.* We will show that if $x \in \Omega_0$ is a λ-hyperbolic point of bounded type, then we are able to construct a piece of unstable leaf as some set $\phi_x(\text{graph}(g_x))$, where g_x will be some special map from $B^u_x(0, l(x))$ to $B^s_x(0, l(x))$ satisfying $g_x(0) = 0$ for some $l(x) \le \rho_1$. We will use the graph transform along the backward orbit of x.

The next result shows that every λ-hyperbolic point must return infinitely often to Ω_0, both in the future and in the past.

LEMMA 5.5. *Let $\xi \in \Omega_0$ be some λ-hyperbolic point such that $f(\xi) \notin \Omega_0$ (respectively, $f^{-1}(\xi) \notin \Omega_0$). Then $n^+(\xi) < +\infty$ (respectively, $n^-(\xi) < +\infty$).*

Proof. Assume that $f(\xi) \notin \Omega_0$ and $n^+(\xi) = +\infty$. This means that $f^n(\xi) \in B(S, \varepsilon_1)$ for all $n \geqslant 1$. This implies that for all $n \ge 1$ and all $v \in E^s(f^n(\xi))$,

$$\|df(f^n(\xi))v\| \ge e^{-(n\lambda/K)}\|v\|.$$

Hence we get

$$\liminf_{n\to\infty} \frac{1}{n} \log \|df^n_{|E^s}(\xi)\| \ge -\frac{\lambda}{K},$$

which contradicts the fact that ξ is λ-hyperbolic. The other case is proved similarly. □

Let $x \in \Omega_0$ be a λ-hyperbolic point. Given $n \geqslant 0$, we define $x_n = f^{-n}(x)$. By Lemma 5.5 there exist integers $0 \le q_0 < p_0 \le q_1 < p_1 \le \cdots$ such that for all $i \geqslant 1$:

- $x_k \in \Omega_0$ for all $0 \le k \le q_0$;
- $x_k \in B(S, \varepsilon_1)$ for all $q_i < k < p_i$;
- $x_k \in \Omega_0$ for all $p_{i-1} \le k \le q_i$.

For $i \geqslant 0$ we set

$$y_i = x_{q_i} = f^{-q_i}(x), \quad z_i = x_{p_i} = f^{-p_i}(x) \quad \text{and} \quad m_i = p_i - q_i. \tag{5.4}$$

We define Γ^n_x as the composition of the graph transforms along the piece of orbit $x_n, \ldots, x_1$, where we take the one-step graph transform Γ_{x_k} if x_k and x_{k-1} lie in Ω_0, and the m_i-step graph transform if x_k is one of the z_i.

Our goal is to prove that the sequence of maps $\Gamma^n_x(\widehat{0}_{x_n})$ converges to some map, where $\widehat{0}_x$ denotes the null map from $B^u_x(0, \rho_1)$ to $B^s_x(0, \rho_1)$. This is well known for

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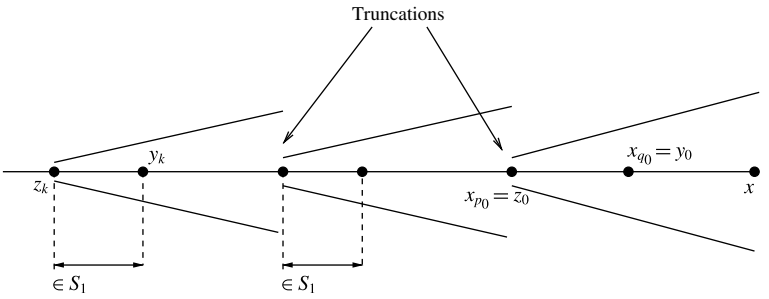


FIGURE 4. Variations of the size of the graphs along one orbit. Increase and truncations due to excursion close to S_1.

uniformly hyperbolic dynamical systems, but here the critical set S influences the graph transform: by construction, Γ_x^n is a contraction of ratio smaller than

$$\exp\left(-\left(n - \sum_{i,\, p_i \le n} (m_i - 1)\right)\left(\lambda^u + \lambda^s - 4\varepsilon\right)\right).$$

But it is not clear that the length of the graph associated to $\Gamma_x^n(\widehat{0}_{x_n})$ is uniformly (in n) bounded away from 0.

Let us now explain how S influences the graph transform and specially the length of the graphs associated to the $\Gamma_x^n(\widehat{0}_{x_n})$. Given $x_n \in \Omega_0$, there exists an integer k such that $p_k \le n \le q_{k+1}$. For $p_k + 1 \le i \le n$ we just apply the one-step graph transform, and so we get some graph in $B_{z_k}(0, \rho_1)$. At this point we apply the m_k-step graph transform to the graph restricted to the ball $B_{z_k}(0, l^f(z_k))$, and we get some graph in $B_{y_k}(0, l^b(y_k))$. Note that $l^f(z_k)$ and $l^b(y_k)$ may be much smaller than ρ_1. Thus we must restrict the transformation to a much smaller piece of graph. We call this phenomenon a *truncation*. See Figure 4.

For $p_{k-1} \le i \le q_k$, we can again apply the one-step graph transform, but to the small piece of graph with length $l^b(y_k)$. However, the length increases along this piece of orbit (as long as it is smaller than $2\rho_1$) because, at each step, we can take the entire part of the image graph. For $i = p_{k-1}$ three cases may occur:

(i) The length of the graph is $2\rho_1$ and so there is a new truncation.

(ii) The length of the graph is strictly smaller than $2\rho_1$, but is bigger than $2l^f(z_{k-1})$, so again there is a new truncation.

(iii) The length of the graph, $2l$, is strictly smaller than $2l^f(z_{k-1})$. We apply the m_{k-1}-step graph transform with $\rho = l$, and we obtain some graph in $B_{y_{k-1}}(0, l)$. From x, we can see the truncation due to Γ_{z_k}.

We proceed in this way along the piece of orbit $y_{k-1}, \ldots, x_{-1}$. Hence, the length of the graph $\Gamma_x^n(\widehat{0}_{x_n})$ is (at least) equal to $2l^f(z_j)$, where j is such that p_j is the smallest integer where a truncation occurs. We set $i(n) := j$.

The main problem we have to deal with is knowing whether the sequence $i(n)$ may go to $+\infty$ or is bounded. In the first case, this means that the sequence of graphs $\Gamma_x^n(\widehat{0}_{x_n})$ converges to a graph with zero length in any ball $B(x, \rho)$. In contrast, the latter case produces a limit graph with positive length. In Proposition 5.8 below we show that, for λ-hyperbolic points of bounded type, it is possible to have the sequence $(i(n))_n$ bounded. For its proof we need two preliminary lemmas.

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Let $A = \max_{x\in M} \|df(x)\|$ and define

$$\gamma = \left(\log A - \frac{\lambda}{2}\right)\left(\frac{\lambda}{2} - \frac{\lambda}{K}\right)^{-1}.$$

We emphasize that γ converges to some finite positive number when K grows to infinity.

LEMMA 5.6. *Given a λ-hyperbolic x, let m be an integer such that $f^j(x) \in B(S, \varepsilon_1)$ for all $0 \leqslant j \leqslant m-1$.*

(1) *If $m > \gamma q$ for some integer $q \geqslant 1$, then*

$$\|df^{m+q}_{|E^u(x)}\| \leqslant e^{(m+q)\lambda/2}.$$

(2) *If $\|df^q_{|E^u(f^m(x))}\| \leqslant e^{q\lambda/2}$, then for every $0 \leqslant j \leqslant m-1$,*

$$\|df^{q+j}(x)\| \leqslant e^{(q+j)\lambda/2}.$$

Proof. Let $\mathbf{w}$ be a vector in $E^u(x)$. Using the fact that $f^j(x) \in B(S, \varepsilon_1)$ for $0 \leqslant j \leqslant m-1$ and also (5.1), we deduce

$$\begin{aligned}\|df^{m+q}(x)\cdot\mathbf{w}\| &= \|df^q(f^m(x))df^m(x)\cdot\mathbf{w}\| \\ &\leqslant A^q e^{m(\lambda/K)}\|\mathbf{w}\| \\ &\leqslant e^{q\log A + m(\lambda/K)}\|\mathbf{w}\|.\end{aligned}$$

Now, using the inequality $m > \gamma q$, we obtain

$$m\left(\frac{\lambda}{2} - \frac{\lambda}{K}\right) > q\left(\log A - \frac{\lambda}{2}\right)$$

and so

$$q\log A + m\frac{\lambda}{K} < (m+q)\frac{\lambda}{2}.$$

This concludes the proof of the first item.

Let us now prove the second item. We set $x_l := f^l(x)\cdot df(x_j)$ expands every vector in the unstable direction for $0 < j \leqslant m$ by a factor lower than $e^{\lambda/K} < e^{\lambda/2}$. Hence, the expansion in the unstable direction for $df^{q+j}(x)$ is strictly smaller than

$$e^{q(\lambda/2)+j(\lambda/K)} < e^{+(q+j)(\lambda/2)}. \qquad \square$$

LEMMA 5.7. *Given a λ-hyperbolic point x, let y_k, q_k, p_k and m_k be as in (5.4). Given $0 < \varepsilon < 1$, let $0 \leqslant s \leqslant q_k$ be the largest integer such that $\log \|df^{-s}(x)|_{E^u(x)}\| \leqslant -s\lambda(1-\varepsilon)$. If*

$$\log \|df^{q_k}(y_k)|_{E^u(y_k)}\| < \frac{9m_k\lambda}{2K},$$

then

$$m_k > \frac{2K}{9}s(1-\varepsilon).$$

Proof. If $s = 0$, then there is nothing to prove. Assume, then, that $s \geqslant 1$. As we are assuming that $\log \|df^{q_k}(y_k)_{|E^u(y_k)}\| < (9m_k\lambda/2K)$ and $s \leqslant q_k$, we must have

$$\log \|df^s(x_s)_{|E^u(x_s)}\| < \frac{9m_k\lambda}{2K},$$

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because df always expands in the unstable direction. Also, by the choice of s, we have $\log \|df^{-s}(x)|_{E^u(x)}\| \leqslant -s\lambda(1-\varepsilon)$, and so

$$s\lambda(1-\varepsilon) < \frac{9m_k\lambda}{2K}. \qquad \square$$

5.3. *Proof of Theorem B.* We can now state the key proposition for constructing the local unstable manifolds.

PROPOSITION 5.8. *There is a constant $K > 0$ such that if x is a λ-hyperbolic of bounded type, then the sequence of truncations $(i(n))_n$ is bounded.*

Proof. We consider an increasing sequence of integers $(s_n)_n$ such that

$$\lim_{n\to\infty} \frac{1}{s_n} \log \|df^{-s_n}(x)|_{E^u(x)}\| \leqslant -\lambda.$$

We also consider $0 < \varepsilon < \frac{1}{3}$ and N sufficiently large such that for every $n \geqslant N$,

$$\frac{1}{s_n} \log \|df^{-s_n}(x)|_{E^u(x)}\| < -\lambda(1-\varepsilon) \quad \text{and} \quad \frac{s_{n+1}}{s_n} < L,$$

for some real number L (which exists by definition of λ-hyperbolic of bounded type).

Let us pick K. Several conditions on K are stated along the way. First we assume that K is sufficiently large such that $1/K < 1-\varepsilon$.

Consider k such that $q_k \geqslant s_N$ and let $0 \leqslant s \leqslant q_k$ be as in Lemma 5.7. Note that we necessarily have $s \geqslant s_N$. Then consider the largest n such that $s_n \leqslant s$. Note that $n \geqslant N$, as $(s_n)_n$ is an increasing sequence.

Assume now, by way of contradiction, that

$$\log \|df^{q_k}(y_k)|_{E^u(y_k)}\| < \frac{9m_k\lambda}{2K}. \tag{5.5}$$

Then, by Lemma 5.7,

$$m_k > \frac{2K}{9} s(1-\varepsilon). \tag{5.6}$$

Now we distinguish the two possible cases.

(1) $s < q_k$. We claim that, in this case, s_{n+1} is larger than p_k. Indeed, by definition of s and n, we cannot have $s_n < s_{n+1} \leqslant q_k$. Therefore, s_{n+1} is larger than p_k and we get

$$\frac{s_{n+1}}{s_n} > \frac{q_k + m_k}{s} > 1 + \frac{2K}{9}(1-\varepsilon)$$

which can be made larger than L if K is chosen sufficiently large.

(2) $s = q_k$. It may happen that the contraction of $df^{-q_k}(x)$ in the unstable direction is so strong that $s_{n+1} = q_k + 1$. Actually, this property may endure for several iterates. In other words, there may be several s_j between q_k and p_k. Nevertheless, we can show that there will always be a (too) large gap between some of them.

Consider K sufficiently large that

$$\frac{2K}{9}(1-\varepsilon) > \gamma^2 + 2\gamma > \gamma.$$

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Using (5.6) and remembering that we are considering the case $s = q_k$, we obtain $m_k > \gamma q_k$. Then we apply Lemma 5.6 with $m = m_k$ and $q = q_k$, and this shows that $(\gamma + 1)q_k$ is not one of the s_j. Note that

$$m_k > \frac{2K}{9}(1-\varepsilon)q_k > (\gamma^2 + 2\gamma)q_k$$

holds if K is sufficiently large (remember that γ is bounded). This yields $m_k - \gamma \cdot q_k > \gamma(1+\gamma)q_k$. Now we are back to case (1) with $(\gamma+1)q_k$ instead of q_k and $m_k - \gamma \cdot q_k$ instead of m_k. Thus, there are no s_j between $(\gamma+1)q_k$ and $q_k + m_k$. Observe that this last interval has length greater than

$$\left(\frac{2K}{9}(1-\varepsilon) - 1 - \gamma\right)q_k.$$

If s_j is the largest term of the sequence (s_l) smaller than $(\gamma+1)q_k$, then $s_{j+1} > p_k = m_k + q_k$ and we get

$$\frac{s_{j+1}}{s_j} \geqslant \frac{p_k}{(1+\gamma)q_k} = \frac{m_k + q_k}{(1+\gamma)q_k} \geqslant \left(\frac{2K(1-\varepsilon)}{9} + 1\right)(1+\gamma)^{-1},$$

which can again be made larger than L by a convenient choice of $K > 0$.

Both cases yield a contradiction, which means that assumption (5.5) does not hold. This shows that for every k such that $q_k > s_N$,

$$i(p_k) \leqslant \max\{i(n),\, n \leqslant s_N\}.$$

As truncation only occurs for integers of the form p_k, the proposition is proved. □

With the previous notation, the length l_n of the graph associated to $\Gamma_x^n(\widehat{0}_{x_n})$ is smaller than $2\rho_1$ and is actually fixed by the finite number of truncations $\max\{i(n),\, n \leqslant s_N\}$, where N appears in the proof of Proposition 5.8. This proves that the sequence of lengths (l_n) is bounded away from 0. The family of maps $\Gamma_x^n(\widehat{0}_{x_n})$ converges to some map

$$g_x : B_x^u(0, l(x)) \to B_x^s(0, l(x))$$

with $g_x(0) = 0$, where $l(x)$ is such that $0 < l(x) \leq l_n$ for every n. We define

$$\mathcal{F}^u_{\mathrm{loc}}(x) = \phi_x(\mathrm{graph}(g_x)).$$

Furthermore, Lemma 5.5 proves that the backward orbit of x returns infinitely often to Ω_0. Therefore, if $f^{-k}(x)$ belongs to $B(S, \varepsilon_1)$ we set

$$\mathcal{F}^u_{\mathrm{loc}}(f^{-k}(x)) = f^n(\mathcal{F}^u_{\mathrm{loc}}(f^{-(n+k)}(x))),$$

where n is the smallest positive integer such that $f^{-(n+k)}(x)$ belongs to Ω_0. Then we set

$$\mathcal{F}^u(x) = \bigcup_{n,\, f^{-n}(x) \in \Omega_0} f^n(\mathcal{F}^u_{\mathrm{loc}}(f^{-n}(x))).$$

The uniqueness of the map g_x and its construction prove that $\mathcal{F}^u(x)$ is an immersed manifold.

To complete the proof of Theorem B, we must check that the $\mathcal{C}^1$-disks that have been constructed are tangent to the correct spaces. Let y be in $\mathcal{F}^u_{\mathrm{loc}}(x)$. By construction

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of $\mathcal{F}^u_{\mathrm{loc}}(x)$ we have for every n such that $f^{-n}(x) \in \Omega_0$, $f^{-n}(y) \in \mathcal{F}^u_{\mathrm{loc}}(f^{-n}(x))$. For such an integer n, we pick some map $g_{n,y} : B^u(0, \rho_1) \to B^s(0, \rho_1)$ such that $f^{-n}(y) \in \phi_{f^{-n}(x)}(\mathrm{graph}(g_{n,y}))$ and $T_{f^{-n}(y)}\phi_{f^{-n}(x)}(\mathrm{graph}(g_{n,y})) = E^u(f^{-n}(y))$. As the map g_x is obtained as some unique fixed point for the graph transform, the sequence $\Gamma^n_x(g_{n,y})$ converges to g_x. By df-invariance of E^u (until the orbit of y leaves U) we must have $T_y\mathcal{F}^u(x) = E^u(y)$.

We also can carry out the same construction with f^{-1} to obtain some immersed manifolds $\mathcal{F}^s(x)$. Then $x \in \mathcal{F}^u(x) \cap \mathcal{F}^s(x)$. It might be important to have an estimate for the length of $\mathcal{F}^u_{\mathrm{loc}}(x)$ or $\mathcal{F}^u(x)$. However, we are not able to give a lower bound for such estimates.

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