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Regularization and topological entropy for the spatial *n*-center problem

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Abstract. We show that the n-center problem in $\mathbb{R}^3$ has positive topological entropy for $n \geq 3$. The proof is based on global regularization of singularities and the results of Gromov and Paternain on the topological entropy of geodesic flows. The n-center problem in S^3 is also studied.

1. *Introduction*

Let $P = \{p_1, \ldots, p_n\}$ be a finite set in $\mathbb{R}^3$. The n-center problem is a Lagrangian system

$$\ddot{q} = -\nabla V(q), \quad q \in \mathbb{R}^3 \setminus P, \tag{1.1}$$

with the Lagrangian

$$L(q, \dot{q}) = \tfrac{1}{2}|\dot{q}|^2 - V(q), \quad V(q) = -\sum_{i=1}^{n} \frac{\mu_i}{|q - p_i|} + \varphi(q), \quad \mu_i > 0, \tag{1.2}$$

where $\varphi \in C^\infty(\mathbb{R}^3)$. Hence the potential energy V has n Newtonian singularities. We assume that $V < 0$ everywhere and there exists $R > 0$ such that

$$V(q) + \tfrac{1}{2}\langle q, \nabla V(q)\rangle < 0, \quad \text{for } |q| \geq R. \tag{1.3}$$

For the classical n-center problem $\varphi = 0$, and condition (1.3) is obviously satisfied. The classical two-center problem is a completely integrable system as shown by Euler. For Lagrange's two-center problem, $\varphi = -|q|^2/2$.

Let $H(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 + V(q)$ be the energy integral. Fix an energy level

$$\Sigma_E = \{(q, \dot{q}) \in (\mathbb{R}^3 \setminus P) \times \mathbb{R}^3 \mid H(q, \dot{q}) = E\}, \quad E \geq 0.$$

By Lagrange's equality and condition (1.3) for any solution of equation (1.1) of energy $E \geq 0$,

$$\frac{d^2}{dt^2}|q|^2 = 4(E - V - \tfrac{1}{2}\langle q, \nabla V(q)\rangle) > 0, \quad \text{for } |q| \geq R. \tag{1.4}$$

Hence all trajectories on Σ_E exiting the R-ball B escape to infinity, and non-trivial dynamics occurs in $\Sigma_E \cap \{q \in B\}$ only.

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THEOREM 1.1. *For $n \geq 3$ and $E \geq 0$ the topological entropy h_{top} of the n-center problem on Σ_E is positive.*

Note that no natural definition of the topological entropy for incomplete flows with singularities seems to exist. The topological entropy is well defined (independent of the metric) for flows on compact metric spaces [**6**]. To define the topological entropy in our case, we perform a reduction to a flow on a compact space.

By the Maupertuis principle [**1**], trajectories with energy E of the system (1.1) are geodesics of the Maupertuis metric $g_E(q, \dot{q}) = 2(E - V(q))|\dot{q}|^2$ on $\mathbb{R}^3 \setminus P$. Let

$$\Sigma = \{(q, \dot{q}) \in (\mathbb{R}^3 \setminus P) \times \mathbb{R}^3 \mid g_E(q, \dot{q}) = 1\}.$$

Then trajectories of (1.1) on Σ_E are mapped to trajectories of the geodesic flow $g^s : \Sigma \to \Sigma$ of the metric g_E by the scaling $(q, \dot{q}) \to (q, q')$, $q' = \dot{q}/\sqrt{2(E - V)}$, together with the time reparametrization $ds = \sqrt{2(E - V)}\, dt$. Instead of the system (1.1) on Σ_E, we will study the geodesic flow $g^s : \Sigma \to \Sigma$, and h_{top} means the topological entropy $h_{\text{top}}(\Sigma, g^t)$ of this flow.

The manifold Σ is non-compact and the flow $g^t : \Sigma \to \Sigma$ is non-complete due to collisions with the singularities p_i. Hence the topological entropy $h_{\text{top}}(\Sigma, g^t)$ is not well defined. However, it is well known that collisions can be regularized, so that g^t is extended to a smooth flow $\tilde{g}^t : \widetilde{\Sigma} \to \widetilde{\Sigma}$ without singularities on a manifold $\widetilde{\Sigma} \supset \Sigma$. Topologically, Σ is obtained from $\widetilde{\Sigma}$ by removing two-dimensional spheres $S_i^2 \subset \widetilde{\Sigma}$ corresponding to collisions with p_i. The trajectories of the flow $\tilde{g}^t$ cross S_i^2 transversely. The construction of $\widetilde{\Sigma}$ is described in §2. We define the topological entropy of the geodesic flow g^t as the topological entropy of the regularized flow $h_{\text{top}}(\Sigma, g^t) = h_{\text{top}}(\widetilde{\Sigma}, \tilde{g}^t)$.

The manifold $\widetilde{\Sigma}$ is non-compact at infinity. The flow $\tilde{g}^t$ is complete for $E > 0$, and also for $E = 0$ provided that $\liminf_{|q| \to \infty} |q|^2 V(q) < 0$. However, the topological entropy $h_{\text{top}}(\widetilde{\Sigma}, \tilde{g}^t)$ is well defined even if the flow $\tilde{g}^t$ is incomplete. Indeed, by Lagrange's inequality (1.4), the set $\Lambda \subset \widetilde{\Sigma}$ of non-wandering points for the flow $\tilde{g}^t$ is contained in $\widetilde{\Sigma} \cap \{q \in B\}$. Then the topological entropy $h_{\text{top}}(\widetilde{\Sigma}, \tilde{g}^t)$ is defined as the topological entropy $h_{\text{top}}(\Lambda, \tilde{g}^t)$ of the flow $\tilde{g}^t$ restricted to the compact invariant set Λ. Recall that since Λ is compact, the positiveness of the topological entropy is invariant under strictly monotone time reparametrization [**6**]. Hence the positiveness of the topological entropy is independent of the construction of the regularized flow. This means complicated behavior of the solutions of (1.1) on $\Sigma_E \cap \{q \in B\}$.

Remarks. (1) It was proved in [**2, 8**] that the n-center problem in $\mathbb{R}^2$ is non-integrable for $n \geq 3$, and has positive topological entropy. If $\varphi \equiv 0$ and all the centers in $\mathbb{R}^3$ lie in one plane, then the n-center problem in $\mathbb{R}^3$ contains the n-center problem in $\mathbb{R}^2$ as a subsystem and, hence, this also has positive topological entropy. However, this argument does not hold if the centers do not lie in a plane.

(2) Under the conditions of Theorem 1.1 the system (1.1) can be partly integrable. For example, when $\varphi = 0$ and the centers lie on a line, there is an integral of angular momentum.

(3) The condition $E \geq 0$ is necessary. For any n it is possible to construct an n-center problem satisfying all conditions of Theorem 1.1 that is integrable on Σ_E for some $E < 0$.

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For example, one can take V such that for large negative E the set $\{V \le E\}$ is a union of balls U_i around p_i and $V|_{U_i} = -1/|q - p_i|$.

(4) Theorem 1.1 holds also if the Lagrangian is replaced by $L = T(q, \dot{q}) - V(q)$, where the Riemannian metric T (kinetic energy) is conformally Euclidean near p_i and tends to the Euclidean metric at infinity, so that Lagrange's inequality holds for large $|q|$. The proof we give works also in this case.

There is another version of Theorem 1.1 which is formulated in terms of the number of orbits with bounded action connecting two given points. Lagrange's inequality implies that any trajectory of energy E of the system (1.1) connecting two points $p, q \in B \setminus P$ lies in B. For any $\lambda > 0$ let $N_\lambda(p, q)$ be the number of such trajectories $\gamma : [a, b] \to B \setminus P$ with the Maupertuis action

$$\int_\gamma \sqrt{g_E}\, dt = \int_a^b |\dot{\gamma}(t)|^2\, dt \le \lambda.$$

THEOREM 1.2. *For $n \ge 3$ and $E \ge 0$ there exists $c > 0$ such that $N_\lambda(p, q) \ge e^{c\lambda}$ for almost all $p, q \in B \setminus P$.*

Thus the number of trajectories connecting two generic points grows exponentially with the action. Hence the geodesic entropy of the metric g_E on $B \setminus P$ is positive.

The definition of the geodesic entropy h_{geod} of a Riemannian metric on a manifold M of finite volume is as follows [**14**]. Take two generic points $p, q \in M$ that are not conjugate along any geodesic. For any $\lambda > 0$ let $N_\lambda(p, q)$ be the number of geodesics γ of length $\mathcal{L}(\gamma) \le \lambda$ connecting p and q. Then

$$h_{\text{geod}} = \limsup_{\lambda \to \infty} \frac{1}{\lambda} \log \int_{M \times M} N_\lambda(p, q)\, d\mu(p)\, d\mu(q), \tag{1.5}$$

where μ is the volume measure of the Riemannian metric.

For M a closed Riemannian manifold, Mané [**10**] and Paternain and Paternain [**14**] proved that $h_{\text{top}} = h_{\text{geod}}$. In general, such a result does not hold for non-compact manifolds. Define the geodesic entropy for the Maupertuis metric g_E on $M = B \setminus P$ by (1.5). We will see that $h_{\text{top}} \ge h_{\text{geod}}$ for $M = B \setminus P$ due to a special behavior of the metric at infinity.

Similar results hold for the n-center problem on a sphere S^3. Let $S^3 \subset \mathbb{R}^4$ be the standard sphere and $P = \{p_1, \dots, p_n\} \subset S^3$ a finite set. Define the potential by (1.2) with $|\cdot|$ the Euclidean metric in $\mathbb{R}^4$. We still assume that $V < 0$ everywhere. Consider a Lagrangian system on $S^3 \setminus P$ with the Lagrangian $L = T(q, \dot{q}) - V(q)$, where T is a Riemannian metric on S^3 that is conformally Euclidean near p_i, i.e. $T = a(q)|\dot{q}|^2/2$ for q close to p_i.

THEOREM 1.3. *For $n \ge 5$ and $E \ge 0$ the topological entropy of the system on Σ_E and the geodesic entropy of the Maupertuis metric g_E on $S^3 \setminus P$ are both positive.*

There exist counter examples for any $n \le 4$. We will prove the following.

PROPOSITION 1.1. *For $n \le 4$ there exist a C^∞ Riemannian metric T on S^3 conformally Euclidean in a neighborhood of P and a C^∞ potential $V < 0$ having Newtonian*

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singularities at P such that the n-center problem on S^3 with $L = T(q, \dot{q}) - V(q)$ is integrable on the energy level Σ_E for some energy $E > 0$, and the topological entropy on Σ_E is zero.

The proof of Theorem 1.3 for even n is similar but simpler than the proof of Theorems 1.1 and 1.2. Since we are mostly interested in the physical case of $\mathbb{R}^3$, we will prove Theorem 1.3 for even n only. The case of odd n is a little different and will be treated elsewhere.

Theorems 1.1–1.3 will be deduced from a general result on regularization of Newtonian singularities on a manifold which is a global version of the KS regularization [**17**]. Then we will use the results of Gromov [**5**] and Paternain [**13, 15**] on the topological entropy of geodesic flows on closed manifolds.

The paper is organized as follows. In §§2–4 the global regularization of Newtonian singularities in a three-dimensional manifold is discussed. In §5 the topology of the regularized configuration space is calculated. As a consequence, Proposition 1.1 is proved. In §6 relations between the topological entropy of a geodesic flow, growth rate of the number of geodesics and the topology of the manifold are recalled, following Gromov [**5**] and Paternain [**13, 15**]. These general results are applied to the n-center problem on S^3, and Theorem 1.3 is proved for even n. The n-center problem in $\mathbb{R}^3$ is treated in §7, where Theorems 1.1 and 1.2 are proved. To do this, we check that under certain assumptions the results of Gromov [**5**] and Paternain [**13, 15**] can be extended to the non-compact case.

Remark. Similar results hold for an n-center problem in an arbitrary two-dimensional manifold Q [**3, 7, 8**]. Then the topological entropy is always positive for n greater than twice the Euler characteristics of Q. We are unable to treat an n-center problem in an arbitrary three-dimensional manifold Q, because we are using the methods of Gromov and Paternain, and so have to assume that Q is simply connected. Similar results hold if the fundamental group of Q is large enough. For example, using the theorem of Taimanov [**16**] it is easy to show that the n-center problem on a closed manifold Q is non-integrable if $\dim H_1(Q, \mathbb{R}) \geq 5$.

2. *Global regularization*

Let Q be a connected three-dimensional Riemannian manifold, possibly with boundary, and $P = \{p_1, \ldots, p_n\}$ be a finite set in Q. Consider a Lagrangian system with the configuration space $Q \setminus P$ and the Lagrangian $L(q, \dot{q}) = T(q, \dot{q}) - V(q)$.

We assume that the C^∞ Riemannian metric T on Q (kinetic energy) is conformally Euclidean in a neighborhood U_i of any point $p_i \in P$. Hence there exist local coordinates $q : U_i \to \mathbb{R}^3$ such that the metric has a conformal form

$$T(q, \dot{q}) = \tfrac{1}{2} a_i(q) |\dot{q}|^2, \quad a_i > 0, \tag{2.1}$$

where $|\cdot|$ is the Euclidean metric on $\mathbb{R}^3$.

Let V be a C^∞ function on $Q \setminus P$ bounded above and having Newtonian singularities at the points $p_i \in P$. Hence in the conformal coordinates $q : U_i \to \mathbb{R}^3$,

$$V(q) = -\frac{f_i(q)}{|q|}, \quad f_i > 0, \tag{2.2}$$

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where f_i is a C^∞ function in a neighborhood of zero. We assume that $V < 0$ everywhere.

Let $H = T(q, \dot{q}) + V(q)$. Fix $E \geq 0$. We will study the system on the energy level $\Sigma_E = \{H = E\} \subset T(Q \setminus P)$. Trajectories with energy E are geodesics of the Maupertuis–Jacobi Riemannian metric [**1**] on $Q \setminus P$:

$$g_E(q, \dot{q}) = 4(E - V(q))T(q, \dot{q}) = \|\dot{q}\|^2_{g_E}. \tag{2.3}$$

More precisely, the map $\psi : \Sigma_E \to \Sigma$, where

$$\Sigma = T_1(Q \setminus P) = \{(q, \dot{q}) \in T(Q \setminus P) \mid g_E(q, \dot{q}) = 1\}$$

and $\psi(q, \dot{q}) = (q, \dot{q}/\sqrt{2(E - V)})$, takes the trajectories of the Lagrangian system on Σ_E to the trajectories of the geodesic flow $g^t : \Sigma \to \Sigma$ of the metric g_E (with time reparameterization).

We consider two cases:

- Q is non-compact or has a non-empty boundary;
- Q is a closed manifold (i.e. compact and without boundary). In this case we assume that n is even.

For Q a closed manifold and n odd, the global regularization of singularities is more complicated and will be discussed in another paper. We may assume that Q is oriented. If not, it is sufficient to pass to the two-sheet orienting covering. Then the number of centers will be even automatically. The next theorem is a global version of the KS-regularization.

THEOREM 2.1. *Let $n \geq 1$. For closed Q let n be even. There exist a four-dimensional manifold M, a smooth group action $\Phi_t : M \to M$, $t \bmod 2\pi$, of a circle S^1 on M, a smooth surjective map $\phi : M \to Q$ and a C^∞ Riemannian metric $G = \langle \cdot, \cdot \rangle_G$ on M such that:*

- *$\phi \circ \Phi_t = \phi$;*
- *for any point $p_i \in P$ the set $\phi^{-1}(p_i)$ consists of one point $s_i \in M$. Let $S = \phi^{-1}(P) = \{s_1, \ldots, s_n\}$;*
- *the group action is free on $M \setminus S$, and the map $\phi|_{M \setminus S} : M \setminus S \to Q \setminus P$ is a fiber bundle with the structure group S^1;*
- *G is invariant under the group action Φ_t;*
- *let $v(x) = (d/dt)|_{t=0}\Phi_t(x)$ be the Killing vector field generating the isometry group Φ_t. Then $\|\dot{x}\|_G = \|\phi'(x)\dot{x}\|_{g_E}$ for any $\dot{x} \in T_x M$ such that $\langle v(x), \dot{x} \rangle_G = 0$.*

COROLLARY 2.1. *Let $F(x, \dot{x}) = \langle v(x), \dot{x} \rangle_G$ be the momentum integral for the geodesic flow of the metric G. Then the map ϕ takes geodesics of the metric G on $M \setminus S$ with zero value of the momentum F into geodesics of the metric g_E on $Q \setminus P$.*

More precisely, let

$$T_1 M = \{(x, \dot{x}) \in TM \mid \|\dot{x}\|_G = 1\}, \quad \Gamma = \{(x, \dot{x}) \in T_1 M \mid F(x, \dot{x}) = 0\}. \tag{2.4}$$

Then Γ is a six-dimensional manifold invariant under the geodesic flow $G^t : T_1 M \to T_1 M$ of the metric G. The flows G^t and Φ'_s commute on Γ. Let $\hat{\Gamma} = \Gamma \cap T(M \setminus S)$. Then $\phi'(\hat{\Gamma}) = \Sigma$ and $g^t \circ \phi' = \phi' \circ G^t$ on $\hat{\Gamma}$, where $\phi' : TM \to TQ$ is the derivative of ϕ, and $g^t : \Sigma \to \Sigma$ is the geodesic flow of the metric g_E.

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In other words, the geodesic flow of the metric g_E on $Q \setminus P$ is obtained from the geodesic flow of the metric G on M by means of a reduction with respect to the symmetry group Φ_t [**1**]. The geodesics of G passing through one of the points $s_i \in S$ are projected to geodesics of g_E that collide with p_i and are reflected in the opposite direction.

The regularized energy level $\widetilde{\Sigma}$ is defined as a quotient space Γ/Φ'_t over the group action $\Phi'_t : \Gamma \to \Gamma$. Since the group action Φ'_t is free on Γ, $\widetilde{\Sigma}$ is a smooth manifold. There is a natural embedding $\Sigma \subset \widetilde{\Sigma}$ as $\hat{\Gamma}/\Phi'_t$, and

$$\widetilde{\Sigma} \setminus \Sigma = \cup_{i=1}^{n} T_{1_{p_i}} M/\Phi'_t = \cup_{i=1}^{n} S_i^2.$$

is a union of two-spheres. In fact $T_{1_{p_i}} M = S^3$ and Φ'_t acts on S^3 as in the Hopf fibration. The geodesic flow $G^t : \Gamma \to \Gamma$ of the metric G defines a quotient flow $\tilde{g}^t : \widetilde{\Sigma} \to \widetilde{\Sigma}$ such that $\tilde{g}^t|_\Sigma = g^t$. Hence $\tilde{g}^t$ is a regularization of the geodesic flow $g^t : \Sigma \to \Sigma$.

Remarks. (1) There is an analog of Theorem 2.1 for Q a two-dimensional manifold that is a global version of the Levi-Civita regularization [**3**]. Then M is also a two-dimensional manifold, and the map $\phi : M \to Q$ is a two-sheeted ramified covering ($\mathbb{Z}_2$ fibration) except for the case when Q is a closed oriented surface and n is odd. Indeed, an obvious Euler characteristics argument shows that if ϕ is a two-sheeted ramified covering over a closed surface, then the number n of branch points is even. For n odd the procedure should be iterated, and the regularizing map $\phi : M \to Q$ becomes a four-sheeted ramified covering.

Similarly, it will be shown in §4 that if $\phi : M \to Q$ is a singular S^1 fibration over a closed three-dimensional manifold satisfying the condition of Theorem 2.1, then necessarily n is even. If n is odd, then, as for the case of surfaces, the procedure should be iterated, and the dimension of the regularizing manifold raises to five.

(2) There are several other ways to regularize double collisions. The most famous is Moser's regularization [**12**]. Locally, near the singularities, these regularizations give isomorphic regularized flows $\tilde{g}^t : \widetilde{\Sigma} \to \widetilde{\Sigma}$. However, other known regularizations do not preserve the Lagrangian structure of the system (for example, Moser's regularization interchanges coordinates and momenta), and hence are not suitable for our purposes.

3. *KS-regularization*

For the proof of Theorem 2.1 we reformulate the KS-regularization [**17**] in a more geometric way. Consider a one-center problem in $\mathbb{R}^3$ with the kinetic and potential energies as in (2.1) and (2.2). Trajectories of energy $E > 0$ are geodesics of the Maupertuis metric

$$g_E(q, \dot{q}) = 2(E + f(q)/|q|)a(q)|\dot{q}|^2. \tag{3.1}$$

The KS-regularization introduces new *dependent* coordinates $x \in \mathbb{R}^4$ such that the singularity at zero disappears. Let $h : \mathbb{R}^4 \to \mathbb{R}^3$ be a quadratic map (the Hopf map) such that:

- $|h(x)| = |x|^2$;
- $h(e^{Jt}x) = h(x)$ for any $t \in \mathbb{R}$, where $J : \mathbb{R}^4 \to \mathbb{R}^4$ is a skew-symmetric matrix such that $J^2 = -1$;
- if $\langle Jx, \dot{x} \rangle = 0$, then $|h'(x)\dot{x}| = 2|x||\dot{x}|$.

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There are several equivalent constructions of the Hopf map. If $\mathbb{R}^4$ is identified with the set $\mathbf{H} = \{x = x_0 + ix_1 + jx_2 + kx_3\}$ of quaternions, the map $h : \mathbf{H} \to \mathbb{R}^3 = \{x \in \mathbf{H} \mid x_0 = 0\}$ can be defined as $h(x) = xi\bar{x}$, and $Jx = xi$.

Define a C^∞ Riemannian metric G on $\mathbb{R}^4$ as

$$G(x, \dot{x}) = u(h(x))|\dot{x}|^2, \quad u(q) = 8(E|q| + f(q))a(q). \tag{3.2}$$

This metric has an isometry group $x \to e^{Jt}x$ corresponding to the Killing vector field $v(x) = Jx$. Hence the geodesic flow of G has a first integral of momentum

$$F(x, \dot{x}) = \langle \dot{x}, v(x) \rangle_G = u(x)\langle \dot{x}, Jx \rangle.$$

By the third property of the Hopf map and (3.1)–(3.2),

$$g_E(h(x), h'(x)\dot{x}) = G(x, \dot{x}), \quad \text{for } F(x, \dot{x}) = 0.$$

This implies that geodesics of the Riemannian metric G corresponding to the zero level $\{F = 0\}$ of the integral F are projected by the map h to geodesics of the Riemannian metric g_E. The metric g_E on $\mathbb{R}^3 \setminus \{0\}$ is obtained from the metric G on $\mathbb{R}^4 \setminus \{0\}$ by means of a reduction with respect to the isometry group $e^{Jt} : \mathbb{R}^4 \to \mathbb{R}^4$. Since G has no singularity at $x = 0$, the geodesic flow of G on $\{F = 0\}$ regularizes the geodesic flow of g_E. This yields Theorem 2.1 for a one-center problem in $\mathbb{R}^3$.

In the next section we will use the connection form [**9, 11**] of the Hopf fibration $h : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$. This fibration can be equipped with two structures of a fiber bundle with the transformation group $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ acting as $x \to e^{\pm Jt}x$ respectively. The first one is the standard Hopf fibration, and the second one its conjugate. The connection 1-form on $\mathbb{R}^4 \setminus \{0\}$ for the Hopf fibration is

$$\lambda_x = \frac{\langle Jx, dx \rangle}{|x|^2}. \tag{3.3}$$

Indeed, λ is Φ_t-invariant and $\lambda_x(Jx) \equiv 1$. On the horizontal plane $H_x = \{\lambda_x = 0\}$, the differential $d\lambda$ equals

$$d\lambda|_{H_x} = \frac{(J\,dx) \wedge dx}{|x|^2}.$$

This form is the pull back $h^*\Omega$ of the closed 2-form

$$\Omega = \frac{(q \times dq) \wedge dq}{2|q|^3} \tag{3.4}$$

on $\mathbb{R}^3 \setminus \{0\}$, which is the curvature form of the Hopf fibration. The Chern class of the fiber bundle is defined as the cohomology class $[\Omega/(2\pi)] \in H^2(\mathbb{R}^3 \setminus \{0\}, \mathbb{R})$. Since $\Omega/2\pi$ is a monopole 2-form:

$$\int_{S^2} \Omega = 2\pi,$$

$[\Omega/(2\pi)]$ is a generator of $H^2(\mathbb{R}^3 \setminus \{0\}, \mathbb{Z})$. The form $-\Omega$ is the curvature form of the conjugate bundle corresponding to the group action e^{-Jt}.

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4. *Proof of Theorem 2.1*

First we assume that Q is diffeomorphic to a closed ball B. Let $q : U_i \to \mathbb{R}^3$ be conformal coordinates in a neighborhood U_i of the center $p_i \in B$.

LEMMA 4.1. *There exists a four-dimensional manifold M, a group action $\Phi_t : M \to M$ and a map $\phi : M \to B$ such that:*

- *ϕ and Φ_t satisfy the conditions of Theorem 2.1;*
- *in a neighborhood U_i of any point $p_i \in P$, ϕ is the Hopf fibration. More precisely, there exists a chart $x : V_i = \phi^{-1}(U_i) \to \mathbb{R}^4$ such that $q \circ \phi|_{V_i} \circ x^{-1} = h$ and $x \circ \Phi_t|_{V_i} \circ x^{-1} = e^{(-1)^i J t}$;*
- *for n even, $\partial M = S^2 \times S^1$ and the fibration $\phi|_{\partial M}$ is trivial over the boundary $\partial B = S^2$;*
- *for n odd, $\partial M = S^3$ and $\phi|_{\partial M}$ is isomorphic to the Hopf fibration $h : S^3 \to S^2$.*

Proof. Fiber bundles $\phi : N \to B \setminus P$ with the transformation group S^1 are classified up to isomorphism by their Chern classes $c(\phi) \in H^2(B \setminus P, \mathbb{Z})$ [**11**]. Since $H^2(B \setminus P, \mathbb{Z}) = \mathbb{Z}^n$ has no torsion, the Chern class of ϕ can be defined as $c(\phi) = [\omega/(2\pi)] \in H^2(B \setminus P, \mathbb{R})$, where ω is the curvature 2-form on $B \setminus P$ for some connection of the fibration ϕ [**9, 11**]. To construct a fibration $\phi|_{M \setminus S} : M \setminus S \to B \setminus P$, we construct its curvature form ω on $B \setminus P$. It should be an integer 2-form, i.e. $\int_{\partial U_i} \omega \in 2\pi\mathbb{Z}$ for any i.

By using the monopole 2-form (3.4) and conformal coordinates $q : U_i \to \mathbb{R}^3$, define a closed 2-form $\Omega_i = q^*\Omega$ in $U_i \setminus \{p_i\}$. □

LEMMA 4.2. *There exists a closed 2-form ω on $B \setminus P$ such that $\omega|_{U_i \setminus \{p_i\}} = (-1)^i \Omega_i$. For n even, ω vanishes in a neighborhood of ∂B. For n odd, $\int_{\partial B} \omega = 2\pi$.*

Proof. The form Ω_i can be extended to a closed 2-form on $B \setminus \{p_i\}$ such that $\Omega_i|_{U_j} = 0$ for $i \neq j$. Put

$$\omega = \sum_{i=1}^{n} (-1)^i \Omega_i. \tag{4.1}$$

For n odd, by Stokes theorem, $\int_{\partial B} \omega = 2\pi$.

If n is even, then $\int_{\partial B} \omega = -\sum_{i=1}^{n} 2\pi(-1)^i = 0$, and so $\omega = d\beta$ in a neighborhood of the sphere ∂B. Extend β to a smooth 1-form on B vanishing on $\cup_{i=1}^{n} U_i$ and put $\tilde{\omega} = \omega - d\beta$. Then $\tilde{\omega}$ satisfies the condition of Lemma 4.2. □

Since S^1-bundles are classified by their Chern classes [**11**], there exists a fiber bundle $\phi : N \to B \setminus P$ with the transformation group $\Phi_t : N \to N$, $t \in S^1$, having a connection with the curvature form ω. Its restriction to $\phi^{-1}(U_i \setminus \{p_i\})$ is isomorphic to the restriction of the Hopf fibration $h : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ (or its conjugate) to a ball $B^4 = h^{-1}(B^3)$ with B^3 a ball around $0 \in \mathbb{R}^3$. The restriction of ϕ to ∂N is isomorphic to the Hopf fibration or a trivial fibration depending on if n is odd or even.

Set $M = N \cup S$, $S = \{s_1, \ldots, s_n\}$, and introduce a smooth structure on M by identifying the set $\phi^{-1}(U_i \setminus \{p_i\}) \cup \{s_i\} \subset M$ with B^4. Then M is a smooth manifold with boundary $S^2 \times S^1$ or S^3 and $\phi : M \to B$ is a smooth map which is isomorphic to the Hopf fibration in a neighborhood of the point s_i. The extension of the S^1-action Φ_t to M defined by $\Phi_t(s_i) \equiv s_i$ is smooth. The remaining conditions in Lemma 4.1 follow immediately. □

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Now let Q be an arbitrary oriented 3-manifold. Take a closed ball $B \subset Q$ containing P in its interior and apply Lemma 4.1. If n is even, $\phi : M \to B$ is a trivial fibration over ∂B and hence it can be extended to a smooth map $\tilde{\phi} : \widetilde{M} \to Q$ which is a trivial fibration over $\overline{Q \setminus B}$.

Now let n be odd and Q non-closed. Then $H^3(Q, \mathbb{Z}) = 0$. Hence the Chern class $c \in H^2(B \setminus P, \mathbb{Z})$ of the fibration $\phi : M \setminus S \to B \setminus P$ is the restriction of a cohomology class $\tilde{c} \in H^2(Q \setminus P, \mathbb{Z})$. Extend the fibration ϕ to a fibration $\tilde{\phi} : \widetilde{M} \setminus S \to Q \setminus S$ with the Chern class $\tilde{c}$. The topological part of Theorem 2.1 is proved.

Remarks. (1) For even n the construction of the regularizing manifold M is unique. For odd n and non-closed Q the topology of M is non-unique, because the class $\tilde{c}$ was defined up to an element of $H^2(Q, \mathbb{Z})$. If $H^2(Q, \mathbb{Z}) = 0$, the construction above gives a unique M. For non-closed Q one can take arbitrary signs in (4.1) yielding different M, but we do not use this possibility.

(2) For an odd number of centers in a closed manifold we can not construct M in a similar way. Indeed, for a singular S^1 fibration such that $\phi|_{U_i}$ is a Hopf fibration or its conjugate, the curvature form ω is a monopole form on $U_i \setminus \{p_i\}$. By Stokes theorem,

$$0 = \sum_{i=1}^{n} \int_{\partial U_i} \omega = 2\pi \sum_{i=1}^{n} (\pm 1).$$

Hence n is necessarily even. For an odd number of centers in a closed manifold the regularizing procedure should be iterated and hence the regularizing manifold must be five-dimensional.

Next we define a Riemannian metric G on M satisfying the conditions of Theorem 2.1. This metric is similar to the Kaluga–Klein metric.

Let ω be a curvature 2-form of the fibration ϕ satisfying Lemma 4.2. Choose some connection 1-form λ on $M \setminus S$ corresponding to ω. Then $\omega|_{U_i \setminus \{p_i\}}$ is the monopole 2-form (3.4), and we can assume that $\lambda|_{V_i \setminus \{s_i\}}$ is the connection form (3.3) for the Hopf fibration.

Let w be a smooth positive function on Q such that in local coordinates $q : U_i \to \mathbb{R}^3$, $w|_{U_i} = u_i(q)|q|$, where u_i is the function (3.2) corresponding to the neighborhood U_i. Let $H_x = \{\dot{x} \in T_x M \mid \lambda_x(\dot{x}) = 0\}$ be the horizontal space. Define a Riemannian metric G on $M \setminus S$ in such a way that

$$\|v(x)\|_G^2 = w(\phi(x)), \quad \|\dot{x}^\perp\|_G = \|\phi'(x)\dot{x}\|_{g_E}, \tag{4.2}$$

where $\dot{x}^\perp \in H_x$ is the horizontal component of $\dot{x}$. Hence $v(x) \perp H_x$ in the metric G. Locally in $V_i \setminus \{s_i\}$, $\phi = h$ is a Hopf map, and so

$$\begin{aligned} G(x, \dot{x}) &= u_i(h(x))|x|^2 \lambda_x^2(\dot{x}) + g_E(x, h'(x)\dot{x}^\perp) \\ &= u_i(h(x))|x|^2 \left(\frac{\langle Jx, \dot{x} \rangle}{|x|^2} \right)^2 + 2a_i(h(x)) \left(E + \frac{f_i(h(x))}{|x|^2} \right) 4|x|^2 |\dot{x}^\perp|^2 \\ &= u_i(h(x))|\dot{x}|^2. \end{aligned} \tag{4.3}$$

Hence G can be extended to a smooth metric on M. Theorem 2.1 is proved.

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Remark. The Riemannian metric G is non-unique. Indeed, the connection form λ is non-unique, and also the function w on $Q \setminus \cup U_i$ can be arbitrary (provided w is smooth and positive). We will use this non-uniqueness in the proof of Theorem 1.1 to satisfy a convexity condition for the set $\phi^{-1}(B)$.

5. *Topology of the regularized configuration space*

Calculating the topology of the regularized manifold M is a simple exercise in the four-dimensional topology, but for the convenience of the reader we include all the details. We are interested in the cases $Q = \mathbb{R}^3$ and $Q = S^3$. For $Q = S^3$, we assume that n is even.

One center in $\mathbb{R}^3$ *or* B^3. Then the regularized manifold is obtained by the KS-regularization, and so $M = \mathbb{R}^4$ or B^4 respectively.

Two centers in S^3. Take two closed balls B_1^3 and B_2^3 in S^3 such that S^3 is obtained by gluing them together along the boundary, and B_1^3, B_2^3 contain the centers p_1, p_2 respectively. Then the regularized manifold for B_i^3 is B_i^4. Hence the regularized manifold M for two centers in S^3 is obtained by gluing the balls B_1^4 and B_2^4 along the boundary S^3. Hence $M = S^4$.

Two centers in $\mathbb{R}^3$ *or* B^3. Embed $Q = \mathbb{R}^3$ in S^3 as an open ball $S^3 \setminus B^3$. The regularized manifold for two centers in S^3 equals S^4. Since the map $\phi : S^4 \to S^3$ is a trivial fibration over $B^3 = S^3 \setminus Q$, the regularized manifold M for $Q = \mathbb{R}^3$ is obtained from S^4 by deleting the trivial fiber bundle $B^3 \times S^1$:

$$M = S^4 \setminus (B^3 \times S^1) = S^2 \times \mathbb{R}^2.$$

For $Q = B^3$, $M = S^2 \times D^2$, where D^2 is a closed disk.

Four centers in S^3. The sphere S^3 is obtained by gluing together two balls B_1^3 and B_2^3, each containing two centers p_1, p_2 and p_3, p_4 with different signs of the curvature form in (4.1). We proved that the regularized manifold for two centers in B^3 is $S^2 \times D^2$. Thus M is obtained by gluing together two copies of $S^2 \times D^2$ along the boundary $S^2 \times S^1$. Hence $M = S^2 \times S^2$.

Now we prove a general result. Fix an oriented 3-manifold Q. We assume that if Q is closed, then n is even. For odd n and Q non-closed, we assume that $H^2(Q, \mathbb{Z}) = 0$. Then the regularized manifold M corresponding to n centers in Q is uniquely defined. It will be denoted by M_n.

PROPOSITION 5.1. *For $n \geq 1$ the manifold M_{n+2} is a connected sum of M_n and $S^2 \times S^2$:*

$$M_{n+2} = M_n \# (S^2 \times S^2).$$

The connected sum $M\#N$ of two oriented manifolds M, N of the same dimension is defined as follows. Delete from M a ball B_1, and from N a ball B_2, and glue $\overline{M \setminus B_1}$ and $\overline{N \setminus B_2}$ together by an orientation reversing diffeomorphism of the spheres ∂B_1 and ∂B_2. Being more careful, the topological manifold $M\#N$ can be made a smooth manifold which, up to a diffeomorphism, does not depend on the construction.

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Proof. Let us show that the manifold M_{n+2} is obtained from M_n by surgery: $M_{n+2} = s(M_n)$, where the operation s is defined as follows. Take a simple contractible closed curve in M_n and its tubular neighborhood diffeomorphic to $B^3 \times S^1$. Delete $B^3 \times S^1$ from M_n and attach instead $S^2 \times D^2$ along the boundary $\partial(S^2 \times D^2) = \partial(B^3 \times S^1) = S^2 \times S^1$. We claim that the obtained manifold $s(M_n)$ is homeomorphic to M_{n+2}.

Let $\{p_1, \dots, p_{n+2}\} \subset Q$ be the singular set. Take a ball B^3 containing the centers p_{n+1}, p_{n+2} with the opposite signs of the curvature form ω_{n+2} in (4.1). It can be assumed that ω_{n+2} vanishes in the neighborhood of the sphere $S^2 = \partial B^3 \subset Q$. The restriction of the regularizing map $\phi_{n+2} : M_{n+2} \to Q$ to this sphere is a trivial fibration. By what we have already shown, $\phi_{n+2}^{-1}(B^3) = S^2 \times D^2$.

Let ω_n be a closed 2-form coinciding with ω_{n+2} in $Q \setminus B^3$ and vanishing in B^3. Then the fibration ϕ_n corresponding to ω_n regularizes the n-center problem with the singularity set $\{p_1, \dots, p_n\}$. Hence the restrictions of the maps $\phi_n : M_n \to Q$ and $\phi_{n+2} : M_{n+2} \to Q$ are isomorphic fibrations over $Q \setminus B^3$. However, the restriction of ϕ_n to B^3 is trivial, and so $\phi_n^{-1}(B^3) = B^3 \times S^1$. This proves the claim.

Since surgery can be performed inside an embedded ball $B^4 \subset M_n$, Proposition 5.1 follows from the equality

$$s(B^4) = \overline{(S^2 \times S^2) \setminus B^4} = B^4 \# (S^2 \times S^2).$$

Equivalently, $s(S^4) = S^2 \times S^2$. Incidentally, we already proved this well-known equality when regularizing the four-center problem on S^3. Proposition 5.1 is proved. □

Hence for $Q = S^3$ and $k \geq 2$,

$$M_{2k} = \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{k-1}. \tag{5.1}$$

For $Q = \mathbb{R}^3$ and $k \geq 1$,

$$M_{2k+1} = \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{k} \setminus B^4, \tag{5.2}$$

$$M_{2k} = \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{k-1} \# (S^2 \times \mathbb{R}^2). \tag{5.3}$$

For $n \geq 1$ centers in $\mathbb{R}^3$, M_n is homotopically equivalent to $\underbrace{S^2 \vee \cdots \vee S^2}_{n-1}$.

COROLLARY 5.1. *For an even number $n \geq 2$ of centers in S^3,*

$$H_0(M, \mathbb{Z}) = H_4(M, \mathbb{Z}) = \mathbb{Z}, \quad H_1(M, \mathbb{Z}) = H_3(M, \mathbb{Z}) = 0, \quad H_2(M, \mathbb{Z}) = \mathbb{Z}^{n-2}.$$

For $n \geq 1$ centers in $\mathbb{R}^3$,

$$H_0(M, \mathbb{Z}) = \mathbb{Z}, \quad H_1(M, \mathbb{Z}) = H_3(M, \mathbb{Z}) = H_4(M, \mathbb{Z}) = 0, \quad H_2(M, \mathbb{Z}) = \mathbb{Z}^{n-1}.$$

Next we construct an integrable on an energy level four-center problem in S^3.

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Proof of Proposition 1.1. For the four-center problem in S^3, the regularized manifold M_4 is $S^2 \times S^2$. We need to construct a Riemannian metric on $S^2 \times S^2$ possessing a S^1 isometry group with four fixed points of the same type as in the Hopf fibration and such that the geodesic flow is integrable. Let $\psi_t : S^2 \to S^2$, $t \bmod 2\pi$, be the rotation group with two fixed points a_1, a_2. Take a metric of revolution on S^2 which is Euclidean near $a_{1,2}$. Then define the isometry group $\Phi_t : S^2 \times S^2 \to S^2 \times S^2$ as $\Phi_t(x_1, x_2) = (\psi_t(x_1), \psi_t(x_2))$. It has four fixed points (a_i, a_j). It is easy to see that $(S^2 \times S^2)/\Phi_t$ is homeomorphic to S^3. Near any fixed point, the group action Φ_t is isomorphic to the group action of the Hopf fibration, and the product metric on $S^2 \times S^2$ is Euclidean near the fixed points. Hence for the zero level of the momentum integral, the reduced metric on S^3 with four points deleted is the Maupertuis metric for a system on S^3 with four Newtonian singularities. The kinetic energy is conformally Euclidean near each singularity. Since the geodesic flow of the Maupertuis metric is integrable, Proposition 1.1 is proved for $n = 4$.

For $n = 1, 2$ Proposition 1.1 is obvious. We skip the case $n = 3$, when the regularized manifold is S^5. □

6. *The topological entropy of the regularized flow*

In this section we prove Theorem 1.3 using the results of Gromov [**5**] and Paternain [**13, 15**] on the topological entropy of geodesic flows.

Suppose that the assumptions of Theorem 2.1 are satisfied. Then the map $\phi' : T_1M \to T_1Q$ takes trajectories of the geodesic flow G^t of the metric G on the invariant manifold Γ that belong to $\hat{\Gamma} = \Gamma \cap T(M \setminus S)$ to trajectories of the geodesic flow $g^t : \Sigma \to \Sigma$ of the metric g. The regularized energy level $\widetilde{\Sigma}$ is the quotient of Γ over the group $\Phi'_t : \Gamma \to \Gamma$. Let $\pi : \Gamma \to \widetilde{\Sigma}$ be the natural projection.

The goal is to estimate the topological entropy for the flow $g^t : \Sigma \to \Sigma$, or, equivalently, for the regularized flow $\tilde{g}^t : \widetilde{\Sigma} \to \widetilde{\Sigma}$, by using the geodesic flow $G^t : \Gamma \to \Gamma$. The difficulty in applying the results of Gromov and Paternain is that only a subsystem of the geodesic flow $G^t : T_1M \to T_1M$ corresponds to the flow $\tilde{g}^t : \widetilde{\Sigma} \to \widetilde{\Sigma}$.

The Riemannian metric G on M defines a Riemannian metric on T_1M in a standard way. Denote by d the corresponding Φ'_t-invariant distance on Γ. The distance $\tilde{d}$ on $\widetilde{\Sigma}$ is the quotient of the metric d under the group action Φ'_t:

$$\tilde{d}(\pi(x), \pi(y)) = \min_t d(\Phi'_t(x), y) \le d(x, y), \quad x, y \in \Gamma. \tag{6.1}$$

In this section we assume that Q is a closed manifold. Then Γ is a closed manifold. Recall the definition of the topological entropy $h_{\text{top}}(K, G^t)$ for a compact set $K \subset \Gamma$ and a flow $G^t : \Gamma \to \Gamma$ [**6**]. Let

$$d_T(x, y) = \sup_{0 \le t \le T} d(G^t(x), G^t(y)).$$

A set $A \subset K$ is said to (T, ε)-span K if for any $x \in K$ there exists $y \in A$ such that $d_T(x, y) \le \varepsilon$. Let $S_T(\varepsilon, K)$ be the minimal cardinality of an (T, ε)-spanning set. Then

$$h_{\text{top}}(K, G^t) = \lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{1}{T} \log S_T(\varepsilon, K). \tag{6.2}$$

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The topological entropy $h_{\text{top}}(K, \tilde{g}^t)$ for the regularized flow $\tilde{g}^t : \widetilde{\Sigma} \to \widetilde{\Sigma}$ and a compact set $K \subset \widetilde{\Sigma}$ is defined similarly with d replaced by $\tilde{d}$. Then

$$h_{\text{top}}(K, \tilde{g}^t) = h_{\text{top}}(\pi^{-1}(K), G^t). \tag{6.3}$$

Indeed, since Φ_s' is an isometry commuting with G^t, by (6.1),

$$S_T(\varepsilon, K) \le S_T(\varepsilon, \pi^{-1}(K)) \le 2 \max_{x \in K} \operatorname{diam}(\pi^{-1}(x))\, S_T(\varepsilon, K).$$

Fix a point $q \in Q \setminus P$ and let $N = \phi^{-1}(q)$ be the fiber over q. We will estimate the topological entropy of the set

$$K = N^{\perp} = \{(x, \dot{x}) \in T_1 M \mid x \in N, \dot{x} \perp T_x N\} = \pi^{-1}(T_{1_q} Q).$$

For any $x \in N$ the tangent space $T_x N$ is generated by $v(x)$, and so $(x, \dot{x}) \in N^{\perp}$ is equivalent to $\langle \dot{x}, v(x) \rangle_G = 0$. Hence $N^{\perp} \subset \Gamma$. Then (6.3) implies that

$$h_{\text{top}}(N^{\perp}, G^t) = h_{\text{top}}(T_{1_q} Q, \tilde{g}^t) \le h_{\text{top}}(\widetilde{\Sigma}, \tilde{g}^t).$$

Since, by definition, $h_{\text{top}}(\Sigma, g^t) = h_{\text{top}}(\widetilde{\Sigma}, \tilde{g}^t)$, we obtain

$$h_{\text{top}}(N^{\perp}, G^t) \le h_{\text{top}}(\Sigma, g^t). \tag{6.4}$$

Take a point $x \in M$ that is not focal for N and denote by

$$n_N(x, \lambda) = \#\{\text{geodesics } \gamma : [0, 1] \to M \mid \mathcal{L}(\gamma) \le \lambda, (\gamma(0), \dot{\gamma}(0)) \in N^{\perp}, \gamma(1) = x\}$$

the number of geodesics of length $\le \lambda$ leaving N orthogonally and terminating at x. For a geodesic flow on a closed Riemannian manifold M, Gromov [**5**] and Paternain [**13, 15**] proved that

$$\limsup_{\lambda \to \infty} \frac{1}{\lambda} \log \int_M n_N(x, \lambda)\, d\mu(x) \le h_{\text{top}}(N^{\perp}, G^t), \tag{6.5}$$

where μ is the Riemannian volume on M.

Let $Q = S^3$. Then M is a simply connected closed manifold, and we can use the following result of Gromov [**5**] and Paternain [**13, 14**]. Denote by $\Omega(N, x)$ the space of curves in $C^0([0, 1], M)$ connecting N with x. Let $b_i(\Omega(N, x)) = \dim H_i(\Omega(N, x), \mathbb{Q})$ be its Betti number.

LEMMA 6.1. *There exists a constant $C > 0$ such that for all $k \in \mathbb{N}$,*

$$\sum_{i=1}^{k} b_i(\Omega(N, x)) \le n_N(x, Ck). \tag{6.6}$$

This result can be extended to the case of M, a compact Riemannian manifold with geodesically convex boundary. We will use this in the next section.

Integrating inequality (6.6) and using (6.5) yields

$$\limsup_{k \to \infty} \frac{1}{k} \log \sum_{i=1}^{k} b_i(\Omega(N, x)) \le C h_{\text{top}}(N^{\perp}, G^t). \tag{6.7}$$

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Thus $h_{\text{top}}(N^{\perp}, G^t) = 0$ only when the sum of Betti numbers $\sum_{i=1}^{k} b_i(\Omega(N,x))$ grows sub-exponentially with k.

The homotopy type of the space $\Omega(N,x)$ is independent of $x \in M$. Take $x \in S$ and fix a point $y \in N$. The space $\Omega(N,x)$ is homeomorphic to $\Omega(y,x) \times S^1$, where $\Omega(y,x)$ is the space of curves connecting y with x (the loop space $\Omega(M)$ of M).

Indeed, any curve $\gamma \in \Omega(N,x)$, $\gamma : [0,1] \to M$, defines $t \in S^1$ by $\Phi_t(y) = \gamma(0)$ and $\omega \in \Omega(y,x)$ by $\omega(s) = \Phi_{-t}\gamma(s)$. We used that x is a fixed point for Φ_t.

Hence

$$b_i(\Omega(N,x)) = b_i(\Omega(M)) + b_{i-1}(\Omega(M)). \tag{6.8}$$

A simply connected space M is said to be rationally hyperbolic if the sum of Betti numbers $\sum_{i=1}^{k} b_i(\Omega(M))$ grows exponentially with k, and in the opposite case it is said to be rationally elliptic.

Then (6.4), (6.7) and (6.8) imply the following.

COROLLARY 6.1. *If M is rationally hyperbolic, the topological entropy $h_{\text{top}}(\Sigma, g^t)$ is positive.*

If $H_i(M,\mathbb{Z}) = 0$ for $i > m$, then rational ellipticity implies that [**4**]

$$\sum_k 2k \dim(\pi_{2k}(M) \otimes \mathbb{Q}) \le m, \quad \sum_k (2k-1) \dim(\pi_{2k-1}(M) \otimes \mathbb{Q}) \le m. \tag{6.9}$$

To prove Theorem 1.3 for $n = 2k$ centers in S^3, we must check that the regularized manifold (5.1) is rationally hyperbolic for $k \ge 3$. Since

$$\dim(\pi_2(M) \otimes \mathbb{Q}) = \dim H_2(M,\mathbb{Q}) = 2(k-1),$$

this follows from (6.9). For n even the statement of Theorem 1.3 about the topological entropy is proved.

Now we prove the part of Theorem 1.3 related to the geodesic entropy. The next lemma holds for arbitrary Q and n.

LEMMA 6.2. *For almost all pairs of points $q, p \in Q \setminus P$, $N_\lambda(q,p) = n_N(x,\lambda)$, where $N = \phi^{-1}(q)$, $x \in \phi^{-1}(p)$.*

Proof. Any geodesic connecting q and p in $Q \setminus P$ is lifted to a unique geodesic in M starting orthogonally to N and terminating at x. The geodesics on M passing through S do not correspond to geodesics in $Q \setminus P$.

However, for almost all $p, q \in Q \setminus P$, the geodesics emerging from N orthogonally and passing through x are projected into curves that do not pass through P. Indeed, the set A of points $(y,x) \in M \times M$ such that there exists a geodesic of the metric G connecting them and passing through S is a union of five-dimensional immersed manifolds

$$\{(\exp_{s_i}(vt), \exp_{s_i}(-vs)) \mid s,t \ge 0, v \in T_{s_i}M, \|v\|_G = 1\}.$$

Thus almost all points $(q,p) \in Q \times Q$ do not belong to $(\phi \times \phi)(A)$. □

We already showed that under the assumptions of Theorem 1.3, $n_N(x,\lambda)$ grows exponentially with λ. Hence Theorem 1.3 is proved for even n.

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7. *Non-compact case*

Now let $Q = \mathbb{R}^3$. Then the regularized manifold M is non-compact, and so to estimate the topological entropy as in the previous section we must extend the results (6.5) and (6.6) of Gromov and Paternain to non-compact Riemannian manifolds. In our case this generalization is straightforward due to Lagrange's inequality (1.4).

Take a closed ball $B \subset \mathbb{R}^3$ with radius R centered at zero. By Lagrange's inequality, B is geodesically convex in the Maupertuis metric, i.e. for all $p, q \in B \setminus P$ any geodesic connecting p, q in $Q \setminus P$ lies in B. The compact manifold $W = \phi^{-1}(B)$ with boundary is homotopically equivalent to M.

LEMMA 7.1. *It is possible to construct the regularizing metric G in Theorem 2.1 in such a way that W is geodesically convex with respect to the metric G.*

Proof. Let $X = \phi^{-1}(\mathbb{R}^3 \setminus B)$. If n is even, the map $\phi : M \to \mathbb{R}^3$ is a trivial fibration over $\mathbb{R}^3 \setminus B$. Hence $X = (\mathbb{R}^3 \setminus B) \times S^1$ with the coordinates $q \in \mathbb{R}^3 \setminus B$, $s \bmod 2\pi$. The connection form λ on X can be assumed to be trivial: $\lambda = ds$. Putting $w|_{\mathbb{R}^3 \setminus B} = 1$ in the construction of G in (4.2), we get, as in (4.3), $G(x, \dot{x}) = \dot{s}^2 + g_E(q, \dot{q})$. Since B is geodesically convex in the metric g_E, convexity of $W = \phi^{-1}(B)$ in the metric G follows immediately.

The case of odd n is less trivial. Then the fibration $\phi : X \to \mathbb{R}^3 \setminus B$ is isomorphic to the restriction of the Hopf fibration $h : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ to $\mathbb{R}^3 \setminus B$. Without loss of generality we can assume that $\phi|_X = h$ and the connection 1-form λ on X is the standard form (3.3). Let $w(q) = 8(E - V(q))|q|$ on $\mathbb{R}^3 \setminus B$. Then by (4.2), $G|_X = U(x)|\dot{x}|^2$, where $U(x) = w(h(x))$ on X.

Geodesics of the metric $G|_X$ are, up to a reparametrization, solutions of the equation $\ddot{x} = \nabla U(x)$ such that $|\dot{x}|^2 = 2U(x)$. Let us show that for any such solution

$$\frac{d^2}{dt^2}|x|^2 = 4(U(x) + \tfrac{1}{2}\langle \nabla U(x), x\rangle) > 0.$$

Denote $u(q) = 8(E - V(q))$. Then $U(x) = |x|^2 u(h(x))$ and

$$U(x) + \tfrac{1}{2}\langle \nabla U(x), x\rangle = 2|x|^2(u(h(x)) + \tfrac{1}{2}\langle \nabla u(h(x)), h(x)\rangle) > 0$$

for $|x| \geq \sqrt{R}$ by condition (1.3). □

Proof of Theorem 1.2. Let us estimate the number $N_\lambda(q, p)$ of geodesics in $B \setminus P$ of length $\leq \lambda$ connecting generic points $p, q \in B \setminus P$. Set $N = \phi^{-1}(q)$ and take some point $x \in \phi^{-1}(p)$. Lemma 6.2 implies that $n_N(x, \lambda) = N_\lambda(q, p)$ for almost all $q, p \in B \setminus P$, where $n_N(x, \lambda)$ is the number of geodesics in W connecting N with x and orthogonal to N.

Lemma 6.1 holds also for M a simply connected compact manifold with geodesically convex boundary. The proof is essentially the same as in [**5, 14**]. Hence (6.6) holds for M replaced by W, where now $\Omega(N, x)$ is the space of curves in W joining N with x. Because M and W are homotopically equivalent, this space is equivalent to the similar space of curves in M. By (6.8), $n_N(x, Ck)$ grows exponentially in k if M is rationally hyperbolic.

By Corollary 5.1 and (6.9), M is rationally hyperbolic for $n \geq 3$. Hence $N_\lambda(p, q)$ grows exponentially with λ. This yields Theorem 1.2. □

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Proof of Theorem 1.1. Let $\Lambda \subset \widetilde{\Sigma}$ be the non-wandering set for the flow $\tilde{g}^t : \widetilde{\Sigma} \to \widetilde{\Sigma}$. Recall that the topological entropy for $\tilde{g}^t$ equals $h_{\text{top}}(\Lambda, \tilde{g}^t)$.

Let $q : \widetilde{\Sigma} \to \mathbb{R}^3$ be the natural projection. Since $q(\Lambda) \subset B$, the behavior of the flow outside $q^{-1}(B)$ does not change the entropy. Let f be a smooth non-negative function such that $f(r) = 1$ for $r \leq R$ and $f(r) = 0$ for $r \geq 2R$. By the reparameterizations $dt = f(|q|)\, ds$ and $dt = f(|q \circ \pi|)\, ds$, we replace the flows $\tilde{g}^t$ and G^t by the flows $\bar{g}^s$ and $\bar{G}^s$ respectively. These flows are trivial for $|q| \geq 2R$ and $|q \circ \pi| \geq 2R$ respectively and coincide with $\tilde{g}^t$ and G^t in $q^{-1}(B)$ and $T_1 W$. Hence $\bar{g}^t|_\Lambda = \tilde{g}^t|_\Lambda$. Except the set of equilibria for $|q| \geq 2R$, the non-wandering set for $\bar{g}^t$ equals Λ. For $\bar{G}^t$ the non-wandering set is contained in $T_1 W$, except the set of equilibria for $|q \circ \pi| \geq 2R$. Hence

$$h_{\text{top}}(\Lambda, \tilde{g}^t) = h_{\text{top}}(\Lambda, \bar{g}^t) = h_{\text{top}}(\widetilde{\Sigma}, \bar{g}^t) \geq h_{\text{top}}(T_{1q} Q, \bar{g}^t) = h_{\text{top}}(N^\perp, \bar{G}^t). \tag{7.1}$$

To prove Theorem 1.1, we must extend inequality (6.5) to the non-compact case. The difficulty is that the proofs in [**13, 14**] use Yomdin's theorem [**18**] which holds only for smooth flows on closed manifolds. However, the flow $G^t : \Gamma \to \Gamma$ is replaced by a smooth flow $\bar{G}^t : \Gamma \to \Gamma$ with compact support. Then we can extend $\bar{G}^t$ to a smooth flow on a closed manifold $\overline{\Gamma} \supset \Gamma$ putting $\bar{G}^t|_{\overline{\Gamma} \setminus \Gamma} = \text{id}$ and use Yomdin's theorem.

Let us show that

$$h_{\text{top}}(N^\perp, \bar{G}^t) \geq \limsup_{\lambda \to \infty} \frac{1}{\lambda} \log \int_W n_N(x, \lambda)\, d\mu(x). \tag{7.2}$$

We use a modification of an argument in [**14**]. Consider the function $n_N(x, \lambda)$ on the set of points $x \in W$ not focal for N. By convexity, any geodesic starting at N and terminating at x is contained in W. Hence

$$G^t(N^\perp) \cap T_1 W = \bar{G}^t(N^\perp) \cap T_1 W,$$

and so

$$\text{Vol}(G^t(N^\perp) \cap T_1 W) \leq \text{Vol}(\bar{G}^t(N^\perp)). \tag{7.3}$$

The n-dimensional volume is induced by the Riemannian metric on $\Gamma \subset T_1 M$ inherited from M. A calculation as in [**14**] yields

$$\int_W n_N(x, \lambda)\, d\mu(x) \leq \int_0^\lambda \text{Vol}(G^t(N^\perp) \cap T_1 W)\, dt. \tag{7.4}$$

Note that in [**14**] an equality holds, but this is not true in our case.

Now, by Yomdin's theorem [**18**],

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \log \text{Vol}(\bar{G}^\lambda(N^\perp)) \leq h_{\text{top}}(N^\perp, \bar{G}^t). \tag{7.5}$$

Equations (7.3)–(7.5) imply (7.2).

Theorem 1.1 follows from (6.6), (7.1) and (7.2). □

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