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Regular periodic decompositions for topologically transitive maps

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Abstract. One may often decompose the domain of a topologically transitive map into finitely many regular closed pieces with nowhere dense overlap in such a way that these pieces map into one another in a periodic fashion. We call decompositions of this kind *regular periodic decompositions* and refer to the number of pieces as the length of the decomposition. If f is topologically transitive but f^n is not, then f has a regular periodic decomposition of some length dividing n. Although a decomposition of a given length is unique, a map may have many decompositions of different lengths. The set of lengths of decompositions of a given map is an ideal in the lattice of natural numbers ordered by divisibility, which we call the decomposition ideal of f. Every ideal in this lattice arises as a decomposition ideal of some map. Decomposition ideals of Cartesian products of transitive maps are discussed and used to develop various examples. Results are obtained concerning the implications of local connectedness for decompositions. We conclude with a comprehensive analysis of the possible decomposition ideals for maps on 1-manifolds.

0. *Introduction*

Since no confusion should arise, we will refer to continuous *topologically transitive* maps simply as *transitive* maps. Transitivity forms part of a popular definition of chaos in discrete dynamical systems [**De**, §1.8, **BBCDS**]. In this setting, it is often used to express the notion of dynamical 'indecomposability' because it ensures that the domain of a map cannot be broken up into topologically non-trivial pieces which do not interact dynamically with each other. Despite this, transitive maps often admit a particular kind of decomposability in which their domains decompose into finitely many topologically non-trivial closed pieces which map into each other in a periodic fashion. In §2 it is shown that on *any* topological space a transitive map with a non-transitive higher iterate f^n always admits a decomposition of the type described above in which each of the pieces is *regular closed*. We call these *regular periodic decompositions*, and the number m of 'pieces' in a decomposition is called its *length*. The fact that the pieces are regular closed comes as a corollary of the proof technique and it has the useful consequence that

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f^m is transitive on each of the pieces. This in turn implies that a map f has at most one regular periodic decomposition length m. More general types of decomposition do not possess this uniqueness property. It is therefore argued that the notion of a regular periodic decomposition is the right one for transitive maps.

Although a regular periodic decomposition length m is unique, there may be regular periodic decompositions of many, sometimes infinitely many, different lengths. The set of lengths of regular periodic decompositions of a transitive map is an ideal in the lattice of natural numbers ordered by divisibility, as is the set of all n such that f^n is transitive. These sets, called the *decomposition ideal* and the *transitivity ideal* of f respectively, are the subject of §3. The decomposition ideal determines the transitivity ideal, but the reverse is not true. Every ideal in the lattice of natural numbers ordered by divisibility arises as a decomposition ideal of some transitive map. That this is so for infinite ideals is demonstrated in §4 by studying a class of homeomorphisms known as *adding machines*. Adding machines are basic models for transitive maps with infinite decomposition ideals. This claim is proven for maps whose domains are second countable Baire spaces. In §5 results about decomposition ideals of Cartesian products are used to give further examples of transitive maps with interesting decompositions and decomposition ideals, including maps with finite decomposition ideals on compact connected finite-dimensional manifolds. The implications of various topological assumptions about the domain for regular periodic decompositions and decomposition ideals are investigated in §6. This leads to a comprehensive analysis of the possible decomposition ideals for maps on 1-manifolds in §7. The much more difficult problem of determining possible decomposition ideals for maps on 2-manifolds is briefly discussed.

1. *Definitions and elementary properties*

Let X be any topological space and let $f : X \to X$ be continuous. The n-fold composite of f with itself is denoted by f^n and called the nth *iterate* of f. If A is any subset of X we denote the pre-image of A under f^n by $f^{-n}(A)$. A subset A of X will be called *invariant* if $f(A) \subseteq A$, and *backward invariant* if $f^{-1}(A) \subseteq A$. A set is invariant iff its complement is backward invariant and the closure of an invariant set is invariant. An element x of X is *periodic* if $f^n(x) = x$ for some $n \geq 1$ and the least n for which this happens is called the *period* of x. The generalisation of periodicity to subsets of X is a major theme of this paper.

The map f is *topologically transitive* or just *transitive*, if for any pair of non-empty open sets U and V, there is a $k \geq 1$ such that $f^k(U) \cap V \neq \varnothing$. If f is transitive, $f^k(X)$ is dense in X for $k \geq 1$, whence inverse images of non-empty open sets are non-empty. If f is a closed mapping, we have $f(X) = X$. This will be true, in particular, where X is compact Hausdorff. We denote the *orbit* $\{f^n(x) : n \geq 0\}$ of a point $x \in X$ by $O(x)$. If X is a second countable Baire space, and f is transitive, then the set of all points with dense orbit is residual in the Baire category sense [**Ma**, p. 69]. The following lemma encompasses some of the many equivalent characterisations of transitivity. Recall that a subset of a topological space is said to be *somewhere dense* if the interior of its closure is non-empty, and *nowhere dense* otherwise.

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LEMMA 1.1. *The following are equivalent for a map $f : X \to X$:*
(i) *f is topologically transitive;*
(ii) *any invariant subset of X is either dense or nowhere dense;*
(iii) *every proper closed invariant subset of X is nowhere dense;*
(iv) *every backward invariant subset of X with non-empty interior is dense.*

We will frequently have occasion to consider both forward and backward images of intersections and unions under iterates of a map. The following lemma is easily proven.

LEMMA 1.2. *For any map $f : X \to X$ and any subset A of X:*
(i) *if $f^n(A) \subseteq A$, then both $\bigcup_{i=0}^{n-1} f^i(A)$ and $\bigcap_{i=0}^{n-1} f^i(A)$ are invariant under f;*
(ii) *if $f^{-n}(A) \subseteq A$, then both $\bigcup_{i=0}^{n-1} f^{-i}(A)$ and $\bigcap_{i=0}^{n-1} f^{-i}(A)$ are backward invariant under f.*

We say that $f : X \to X$ is *topologically conjugate*, or just *conjugate*, to $g : Y \to Y$ if there is a homeomorphism $h : X \to Y$, called a *conjugacy*, such that $h \circ f = g \circ h$. Conjugacies preserve orbits, invariant sets and so on. If f and g are conjugate, then so are f^n and g^n for any $n \geq 1$. Where f and g are conjugate, f is transitive iff g is transitive. Properties such as these which are preserved by conjugacies are called *conjugacy invariants*. The weaker notion of *topological semi-conjugacy* is obtained by insisting only that h be a continuous surjection. Semi-conjugacies preserve orbits, invariant sets and so on. If g is *semi-conjugate* to f and f is transitive, then g is transitive, but the reverse implication does not necessarily hold. Suppose that $X/\sim$ is a quotient space such that $x \sim y \Rightarrow f(x) \sim f(y)$ for $x, y \in X$. Then we say that f is *compatible* with the quotient space and we can define the *natural induced map* $g : X/\sim \to X/\sim$ by putting $g([x]) = [f(x)]$. It is well known that the continuity of f guarantees the continuity of g. One may check that g is semi-conjugate to f via the *natural projection* $\pi : X \to X/\sim : x \mapsto [x]$, so if f is transitive then g is also transitive.

We will make use of some properties which are strictly stronger than transitivity. A map f is *topologically mixing*, or just *mixing*, if for any non-empty open sets U and V, there is a positive integer N such that for $n \geq N$, one has $f^n(U) \cap V \neq \varnothing$. Any Cartesian product of mixing maps is mixing. In §5 we give a generalisation of this result which is useful when constructing examples. A map f is said to be *weakly topologically mixing*, or just *weakly mixing*, if the Cartesian product $f \times f$ is transitive. If f is weakly mixing, then a finite Cartesian product of copies of f is also transitive [**Fu**]. This is equivalent to saying that if U_i and V_i are non-empty open sets for each $i \leq m$, then there is a $k \geq 1$ such that $f^k(U_i) \cap V_i \neq \varnothing$ for all $i \leq m$, which clearly implies that an infinite Cartesian product of copies of f is also weakly mixing. However, it is not easy to see whether every Cartesian product of weakly mixing maps is necessarily weakly mixing or even transitive. Since any Cartesian product of mixing maps is mixing and therefore weakly mixing, any counterexample to this proposition would have to take the form of a product in which at least one of the factors is weakly mixing but not mixing. Although maps which are weakly mixing but not mixing exist [**PS**], they are difficult to exhibit. A map is *minimal* if every orbit is dense. Following the terminology of [**PP**], a map f will

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be called *totally transitive* if f^n is transitive for every $n \geq 1$. The implications between the various conditions implying transitivity may be summarised as follows:

$$\text{mixing} \Rightarrow \begin{matrix}\text{weakly}\\ \text{mixing}\end{matrix} \Rightarrow \begin{matrix}\text{totally}\\ \text{transitive}\end{matrix} \Rightarrow \text{transitive} \Leftarrow \text{minimal}.$$

All of the implications in the diagram are completely elementary to check except the fact that weakly mixing maps are totally transitive. This may be proven using the result of [**Fu**] discussed above. It is an easy consequence of Lemma 1.1(iii) that if f^n is not transitive, then neither is f^{mn} for any $m \geq 1$. The final theorem of this section (due to Stacey [**St**]), makes it easier to prove that certain maps are weakly mixing than a direct approach would allow.

THEOREM 1.1. *A totally transitive map f with a dense set of periodic points is weakly mixing.*

Proof. Let U, V, G and H be any non-empty open subsets of X. We need to show that $(f \times f)^n(U \times V) \cap (G \times H) \neq \varnothing$ for some $n \geq 1$. By the transitivity of f, there is a $k \geq 1$ such that $W = U \cap f^{-k}(G)$ is open and non-empty. Hence W contains a point x of period m. For $j \geq 1$, one has $f^{mj+k}(x) = f^k(x) \in G$, whence $f^{mj+k}(U) \cap G \neq \varnothing$. Since $f^{-k}(H)$ is open and non-empty and f^m is transitive, there is a $j \geq 1$ such that $f^{mj}(V) \cap f^{-k}(H) \neq \varnothing$ whence $f^{mj+k}(V) \cap H \neq \varnothing$. Putting $n = mj + k$ gives the desired result. □

COROLLARY 1.1. *If f is totally transitive and has a dense set of periodic points, then f^n is weakly mixing for each $n \geq 1$.*

2. *Decompositions for transitive maps*

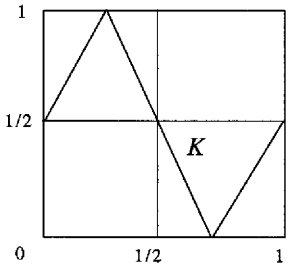
Some definitions are needed to make precise the notion of decomposition discussed in the introduction. Suppose we have a collection $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ of subsets of X such that $f(D_i) \subseteq D_{i+1(\mathrm{mod}\, n)}$ for $0 \leq i < n - 1$. We call $\mathcal{D}$ a *periodic orbit of sets*. Clearly, the union of the D_i is an invariant subset in these circumstances and f^n is invariant on each D_i. Also, $f^k(D_i) \subseteq D_{i+k(\mathrm{mod}\, n)}$ for all $k \geq 0$. We call $\mathcal{D}$ a *periodic decomposition* if the D_i are all closed, $D_i \cap D_j$ is nowhere dense whenever $i \neq j$, and the union of the D_i is X. The condition that the D_i have nowhere dense overlap is equivalent to saying that their interiors are disjoint. In the special case where the D_i are mutually disjoint, each D_i is clopen since its complement is a finite union of closed sets. The number of sets in a decomposition will be called the *length* of the decomposition. Clearly $\{X\}$ is always a periodic decomposition. We call this the *trivial decomposition*.

A closed set is called *regular closed* if it is the closure of its interior or, equivalently, it is the closure of an open set. Elementary topological arguments show that if D is regular closed in X and C is regular closed in D, then C is regular closed in X and that finite unions of regular closed sets are regular closed. Any open set intersecting a regular closed set also intersects its interior. A periodic decomposition is *regular* if all of its elements are regular closed. Since clopen sets are regular closed, a periodic decomposition is always regular in the case where $D_i \cap D_j$ is empty for $i \neq j$. Suppose

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that a transitive map f admits a regular periodic decomposition $\mathcal{D}$ of length n. Since f^n is invariant on each element of $\mathcal{D}$ and each element of $\mathcal{D}$ has non-empty interior, f^n is not transitive. Hence if f is totally transitive, it has no regular periodic decompositions. To motivate the discussion of regular periodic decompositions we consider the following piecewise linear map K on $[0, 1]$ (from [**BM1**]).

$$K(x) = \begin{cases} 2x + \frac{1}{2} \text{ if } x \in [0, \frac{1}{4}] \\ \frac{3}{2} - 2x \text{ if } x \in (\frac{1}{4}, \frac{3}{4}] \\ 2x - \frac{3}{2} \text{ if } x \in (\frac{3}{4}, 1] \end{cases}$$



It is easy to see that $\mathcal{D} = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ forms a regular periodic decomposition for K, so K^2 cannot be transitive on $[0, 1]$. Various techniques may be employed to check that K^2 is mixing and hence totally transitive on each element of $\mathcal{D}$. The transitivity of K will be established shortly by Theorem 2.1. Comments preceding Theorem 1.1 show that no even iterate of K is transitive. This does not mean, however, that K has a regular periodic decomposition of length $2n$ for each $n \geq 1$. The only decompositions for K turn out to be the trivial one $\{[0, 1]\}$ and the decomposition $\mathcal{D}$ discussed above. The next lemma collects together some basic properties of regular periodic decompositions.

LEMMA 2.1. *If $\{D_0, D_1, \ldots, D_{n-1}\}$ is a regular periodic decomposition for a transitive map f, then:*

(i) $\overline{f^k(D_i)} = D_{i+k(\mathrm{mod}\, n)}$ *for all* $0 \leq i \leq n-1$ *and* $k \geq 1$;

(ii) $f^k(D_i) \subseteq D_i$ *iff* $k = 0$ *(mod n);*

(iii) $f^{-j}(\mathrm{int}(D_i)) \subseteq \mathrm{int}(D_{i-j(\mathrm{mod}\, n)})$ *for* $0 \leq i \leq n-1$ *and* $j \geq 0$;

(iv) $D = \bigcup_{i \neq j} D_i \cap D_j$ *is invariant and nowhere dense in* X.

Proof. (i) Since $D_{i+k(\mathrm{mod}\, n)}$ is closed, $\overline{f^k(D_i)} \subseteq D_{i+k(\mathrm{mod}\, n)}$. Suppose $U = D_{i+k(\mathrm{mod}\, n)} \backslash \overline{f^k(D_i)}$ is non-empty. Then U is open in the regular closed set $D_{i+k(\mathrm{mod}\, n)}$, whence $V = U \cap \mathrm{int}(D_{i+k(\mathrm{mod}\, n)})$ is open and non-empty in X. But D_i is the only element of the decomposition mapping to $\mathrm{int}(D_{i+k(\mathrm{mod}\, n)})$ under f^k, so $f^{-k}(V) = \varnothing$ under f^k contradicting the fact that $f^k(X)$ is dense in X.

(ii) If $f^k(D_i) \subseteq D_i$ but $j = i + k \pmod{n} \neq i$, then since $f^k(D_i) \subseteq D_{i+k(\mathrm{mod}\, n)} = D_j$, we have $f^k(D_i) \subseteq D_i \cap D_j$. But this implies that $f^k(D_i)$ is nowhere dense contradicting part (i). The other implication follows from the definition.

(iii) We assume that $i \geq j$. The proof for $i < j$ is similar. If $f^{-j}(\mathrm{int}(D_i)) \cap D_k \neq \varnothing$ for some $k \neq i - j$, we would have a point of D_k mapping into $\mathrm{int}(D_i)$ under f^j. But $f^j(D_k) \cap \mathrm{int}(D_i) = \varnothing$ because $f^j(D_k) \neq D_i$. This contradiction establishes that $f^{-j}(\mathrm{int}(D_i)) \cap D_k = \varnothing$ for $k \neq i - j$, whence $f^{-j}(\mathrm{int}(D_i)) \subseteq D_{i-j}$. Since $f^{-j}(\mathrm{int}(D_i))$ is open we have $f^{-j}(\mathrm{int}(D_i)) \subseteq \mathrm{int}(D_{i-j})$.

(iv) It suffices to show that D is invariant by Lemma 1.1(ii). If $i \neq j \pmod{n}$ then $i + 1 \neq j + 1 \pmod{n}$, whence $f(D) \subseteq \bigcup_{i \neq j} f(D_i) \cap f(D_j) \subseteq \bigcup_{i \neq j} D_{i+1(\mathrm{mod}\, n)} \cap D_{j+1(\mathrm{mod}\, n)} \subseteq D$. □

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If f is a closed mapping, part (i) of the lemma asserts that $f^k(D_i) = D_{i+k(\mathrm{mod}\, n)}$ for all $k \geq 0$. This will be true when X is compact Hausdorff. In the statement of part (iii) of the lemma, the restriction to the interiors of elements of the decomposition is necessary since it is not true in general that $f^{-j}(D_i) \subseteq D_{i-j(\mathrm{mod}\, n)}$. For instance $K^{-1}([\frac{1}{2}, 1]) = [0, \frac{1}{2}] \cup \{1\}$, where K is the map discussed above.

THEOREM 2.1. *Let $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ be a regular periodic decomposition for a map f. Then f is transitive iff f^n is transitive on each D_i.*

Proof. Let f be transitive and let U and V be open sets intersecting D_i. Since D_i is regular closed, $\mathrm{int}(D_i) \cap U$ and $\mathrm{int}(D_i) \cap V$ are non-empty open subsets of X. By the transitivity of f, there is a $k \geq 1$ such that $f^k(\mathrm{int}(D_i) \cap U) \cap (\mathrm{int}(D_i) \cap V) \neq \varnothing$, whence $\mathrm{int}(D_i) \cap f^k(\mathrm{int}(D_i)) \neq \varnothing$ and so $\mathrm{int}(D_i) \cap f^{-k}(\mathrm{int}(D_i)) \neq \varnothing$. By Lemma 2.1(iii), $f^{-k}(\mathrm{int}(D_i)) \subseteq \mathrm{int}(D_{i-k(\mathrm{mod}\, n)})$. Since the interiors of the elements of $\mathcal{D}$ are disjoint, $i = i - k \pmod n$ whence k is a multiple of n. This establishes the transitivity of f^n on D_i.

Conversely, suppose that f^n is transitive on each D_i. Let U and V be non-empty open sets intersecting $\mathrm{int}(D_i)$ and $\mathrm{int}(D_j)$ respectively. Clearly, $W = f^{-k}(V \cap \mathrm{int}(D_j))$ is a non-empty open subset of D_i for some $0 \leq k \leq n-1$, so there is an $m \geq 1$, such that $f^{mn}(U) \cap W \neq \varnothing$ by the transitivity of f^n on D_i whence $f^{mn+k}(U) \cap V \neq \varnothing$. □

The regularity of the decomposition is essential in the proof given here, since it is obvious that $\mathcal{C} = ([0, \frac{1}{2}] \cup \{1\}, \{0\} \cup [\frac{1}{2}, 1])$ is a periodic decomposition of length 2 for the map K discussed above, but K^2 is not transitive on either element of $\mathcal{C}$. A non-transitive map may have many regular periodic decompositions of the same length. For example, the map on the unit circle given by a rotation through the angle π has $\{[\theta, \theta + \pi], [\theta + \pi, \theta]\}$ as a regular periodic decomposition of length 2 for each $\theta \in S^1$. Transitivity ensures uniqueness in this regard, however.

THEOREM 2.2. *A transitive map $f : X \to X$ has at most one regular periodic decomposition of length n.*

Proof. Let $\mathcal{C} = \{C_0, C_1, \ldots, C_{n-1}\}$ and $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ be regular periodic decompositions. We want to show that $\mathcal{C} = \mathcal{D}$. Clearly $C_i \cap D_j$ has non-empty interior in X for some $0 \leq i, j \leq n-1$. Hence $C_i \cap D_j$ is closed and has non-empty interior in C_i and D_j. Now $f^n(C_i \cap D_j) \subseteq C_i \cap D_j$ and f^n is transitive on both C_i and D_j by Theorem 2.1. Lemma 1.1(iii) shows that $C_i = C_i \cap D_j = D_j$. Lemma 2.1(i) yields $C_{i+k(\mathrm{mod}\, n)} = \overline{f^k(C_i)} = \overline{f^k(D_j)} = D_{j+k(\mathrm{mod}\, n)}$ for $1 \leq k \leq n-1$. □

Again, here the regularity of the decomposition is indispensable. By [**Si**, p. 357], the map K discussed earlier has a dense set of periodic points. Each periodic orbit P except $\{\frac{1}{2}\}$ intersects both $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ and gives rise to a distinct periodic decomposition $\{[0, \frac{1}{2}] \cup P, [\frac{1}{2}, 1] \cup P\}$ so K has infinitely many distinct periodic decompositions of length 2. The fact that there is a uniqueness theorem for regular periodic decompositions suggests strongly that this is the 'right' concept for transitive maps. This is reinforced by the fact that regularity arises in a natural way in the existence proof to which we now turn. We show that a transitive map f having a non-transitive higher iterate f^n admits

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a regular periodic decomposition of some length dividing n, thus proving an assertion stated without proof for compact metric spaces and described as folklore in [**CM**].

LEMMA 2.2. *If f is transitive and $G \subseteq X$ is open with $G, f^{-1}(G), \ldots, f^{-(n-1)}(G)$ pairwise disjoint and $f^{-n}(G) \subseteq G$, then $\{\bar{G}, \overline{f^{-(n-1)}(G)}, \ldots, \overline{f^{-1}(G)}\}$ is a regular periodic decomposition.*

Proof. Since the $f^{-i}(G)$ are disjoint and open, the $\overline{f^{-i}(G)}$ have nowhere dense intersections. For $0 \leq i \leq n-1$, the continuity of f yields $f(\overline{f^{-i}(G)}) \subseteq \overline{f(f^{-i}(G))} \subseteq \overline{f^{-(i-1)}(G)}$. It suffices to show that f maps $\bar{G}$ into $\overline{f^{-(n-1)}(G)}$, since this implies that $\bigcup_{i=0}^{n-1} \overline{f^{-i}(G)}$ is a closed invariant set with non-empty interior, whence $\bigcup_{i=0}^{n-1} \overline{f^{-i}(G)} = X$ by the transitivity of f. We check that the open set $U = G \setminus f^{-1}(\overline{f^{-(n-1)}(G)})$ is empty. Since $U \subseteq G$, it is disjoint from $f^{-i}(G)$ for $1 \leq i \leq n-1$ by hypothesis. It is disjoint from $f^{-n}(G)$ by definition. Thus $U \subseteq X \setminus V$, where

$$V = \bigcup_{i=1}^{n} f^{-i}(G) = \bigcup_{i=0}^{n-1} f^{-i}(f^{-1}(G))$$

is clearly non-empty and open. Since $f^{-1}(G)$ is backward invariant under f^n, Lemma 1.2 establishes that V is backward invariant under f and therefore dense by Lemma 1.1(iv). Hence $X \setminus V$ has empty interior and U is empty, whence $G \subseteq f^{-1}(\overline{f^{-(n-1)}(G)})$. Thus the closed set $\overline{f^{-(n-1)}(G)}$ contains $f(G)$, whence $f(\bar{G}) \subseteq \overline{f(G)} \subseteq \overline{f^{-(n-1)}(G)}$ as required. □

We will show that the hypotheses of Lemma 2.2 are satisfied when f is transitive, but has a non-transitive iterate f^p for p prime, giving a regular periodic decomposition of length p. Lemma 1.1(iv) furnishes a non-empty, non-dense open subset H of X such that $f^{-p}(H) \subseteq H$. Unfortunately, there will be many ways of choosing H and not all choices will yield the desired disjoint pre-images.

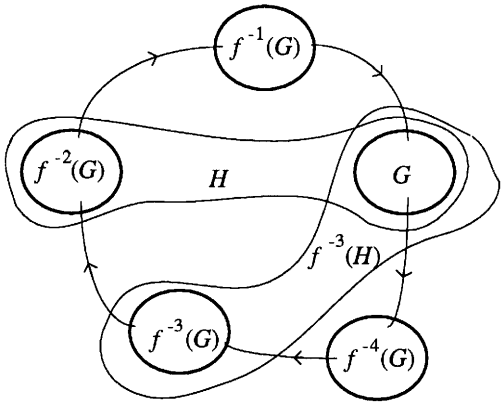


FIGURE 1.

For example, if the non-dense open set G satisfies the conditions of Lemma 2.2, we could choose H to be the union of any m pre-images of G, where $1 < m < p$. It is easy

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to check that H is then non-dense, open and $f^{-p}(H) \subseteq H$, but that the first p pre-images of H will not be disjoint. Figure 1 illustrates this problem in the case where $p = 5$. The next two lemmas enable us to start with an arbitrary choice for H, and obtain the required set G by taking intersections. Lemma 2.3 uses the fact that a non-zero element m of the additive group $\mathbb{Z}_n$ generates $\mathbb{Z}_n$ iff $\gcd(m, n) = 1$. In number theoretic terms one may say that $\gcd(m, n) = 1$ iff $\{jm(\mathrm{mod}\ n) : 0 \leq j \leq n-1\} = \{0, 1, \ldots, n-1\}$. Lemma 2.4 generalises Lemma 2.3 by induction, leading to a proof of the existence of a regular periodic decomposition for a transitive map f with a non-transitive *prime* iterate.

LEMMA 2.3. *Let f be transitive, let G_0 be non-empty, non-dense, open in X and backward invariant under f^n. For each $1 \leq m \leq n-1$ co-prime with n, there is a non-empty open $G \subseteq G_0$ such that $f^{-n}(G) \subseteq G$ and $f^{-m}(G) \cap G = \varnothing$.*

Proof. The open set $U = \bigcap_{i=0}^{n-1} f^{-i}(G_0)$ is backward invariant by Lemma 1.2 and hence empty by Lemma 1.1(iv). For $0 \leq r \leq n-1$ we have $jm = qn + r$ for some $q \geq 0$ and $0 \leq j \leq n-1$. Thus

$$f^{-jm}(G_0) = f^{-(r+qn)}(G_0) = f^{-r}(f^{-qn}(G_0)) \subseteq f^{-r}(G_0)$$

which gives $V = \bigcap_{j=0}^{n-1} f^{-jm}(G_0) \subseteq U = \varnothing$. Hence

$$k = \max\left\{ j \geq 0 : \bigcap_{i=0}^{j} f^{-im}(G_0) \neq \varnothing \right\} < n-1$$

and so $G = \bigcap_{i=0}^{k} f^{-im}(G_0)$ is a non-empty open subset of G_0. Since $f^{-n}(G_0) \subseteq G_0$, we have $f^{-n}(G) = \bigcap_{i=0}^{k} f^{-im}(f^{-n}(G_0)) \subseteq \bigcap_{i=0}^{k} f^{-im}(G_0) = G$ and by the maximal choice of k

$$f^{-m}(G) \cap G = \left[\bigcap_{i=1}^{k+1} f^{-im}(G_0)\right] \cap \left[\bigcap_{i=0}^{k} f^{-im}(G_0)\right] = \bigcap_{i=0}^{k+1} f^{-im}(G_0) = \varnothing. \qquad \square$$

LEMMA 2.4. *Let f be transitive, let G_0 be non-empty, non-dense, open in X and backward invariant under f^n. Then there is a non-empty open $G \subseteq G_0$ such that $f^{-n}(G) \subseteq G$ and $f^{-m}(G) \cap G = \varnothing$ for every $m \leq n-1$ co-prime with n.*

Proof. Lemma 2.3 yields a non-empty open $G_1 \subseteq G_0$ with $G_1 \cap f^{-1}(G_1) = \varnothing$ and $f^{-n}(G_1) \subseteq G_1$. Let $\{m \leq n-1 : \gcd(m, n) = 1\} = \{1, m_1, m_2, \ldots, m_k\}$. Suppose we have constructed a sequence $G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_j$ of non-empty open sets such that $G_i \cap f^{-m_i}(G_i) = \varnothing$ and $f^{-n}(G_i) \subseteq G_i$ for $i \leq j < k$. Then Lemma 2.3 yields a non-empty open $G_{j+1} \subseteq G_j$ satisfying $G_{j+1} \cap f^{-m_{j+1}}(G_{j+1}) = \varnothing$ and $f^{-n}(G_{j+1}) \subseteq G_{j+1}$. By induction we can extend this construction to a sequence $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_k$. Clearly $G = G_k$ has the required properties. $\square$

THEOREM 2.3. *Let f be transitive with f^p not transitive, where p is prime. Then f admits a regular periodic decomposition of length p.*

Proof. Lemma 1.1(iv) furnishes a non-empty, non-dense open subset H of X such that $f^{-p}(H) \subseteq H$. Since each $1 \leq m \leq p-1$ is co-prime with p, Lemma 2.4 gives

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a non-empty open subset G of H with $G \cap f^{-m}(G) = \varnothing$ for all $1 \leq m \leq p-1$ and $f^{-p}(G) \subseteq G$. One has $f^{-j}(G \cap f^{-(k-j)}(G)) = f^{-j}(G) \cap f^{-k}(G) = \varnothing$ because $G \cap f^{-(k-j)}(G) = \varnothing$ for $1 \leq j < k \leq p-1$. Hence the sets G, $f^{-1}(G), \ldots$ and $f^{-(p-1)}(G)$ are disjoint, so their closures form a regular periodic decomposition of length p by Lemma 2.2. □

The following pair of lemmas underpin the proof that if f is transitive but f^n is not transitive for composite n, then f^p is not transitive for some prime factor p of n. The lemmas cover the two possible cases where n is the product of two primes; either the primes are equal or they are distinct. Since treating both cases in a single proof leads to a messy proof logic, they are taken separately. In each case, the idea is to show that one of the prime iterates is not transitive by constructing a non-empty non-dense open set which is backward invariant under that iterate. The result then follows by Lemma 1.1(iv). The lemmas are extended by induction to cover the general case.

LEMMA 2.5. *If f is transitive with f^{p^2} not transitive for p prime, then f^p is not transitive.*

Proof. By Lemma 1.1(iv) there is a non-empty non-dense open set G_0 with $f^{-p^2}(G_0) \subseteq G_0$, so by Lemma 2.4 there is a non-empty open $G \subseteq G_0$ such that $f^{-p^2}(G) \subseteq G$ and $f^{-m}(G) \cap G = \varnothing$ for every $1 \leq m \leq p^2 - 1$ co-prime with p^2. Lemma 1.2 shows that the open set $V = \bigcup_{i=0}^{p-1} f^{-ip}(G)$ is backward invariant under f^p. We will show that V is not dense by establishing that $f^{-1}(G) \cap V = \varnothing$. We have already seen that $f^{-1}(G) \cap G = \varnothing$. Since each $ip-1$ is co-prime with p^2, we have $G \cap f^{-(ip-1)}(G) = \varnothing$, whence $f^{-1}(G \cap f^{-(ip-1)}(G)) = f^{-1}(G) \cap f^{-ip}(G) = \varnothing$ for $0 \leq i \leq p-1$ and hence $f^{-1}(G) \cap V = \varnothing$ as required. □

LEMMA 2.6. *Let f be transitive with f^{pq} not transitive where p and q are prime. Then either f^p is not transitive or f^q is not transitive.*

Proof. We suppose that f^p is transitive and show that f^q is not transitive. As in the previous lemma, there is a non-empty non-dense open set G such that $f^{-pq}(G) \subseteq G$ and $f^{-m}(G) \cap G = \varnothing$ for every $1 \leq m \leq pq - 1$ co-prime with pq. Since j and q are co-prime for $1 \leq j \leq q-1$ and $(f^p)^{-q}(G) = f^{-pq}(G) \subseteq G$, we apply Lemma 2.4 using the transitivity of f^p to obtain a non-empty open $H \subseteq G$ such that for $1 \leq j \leq q-1$, $(f^p)^{-q}(H) = f^{-pq}(H) \subseteq H$ and $(f^p)^{-j}(H) \cap H = f^{-jp}(H) \cap H = \varnothing$. By Lemma 1.2, the open set $W = \bigcup_{i=0}^{p-1} f^{-iq}(H)$ is backward invariant under f^q.

We show that W is not dense by establishing that $f^{-1}(H) \cap W = \varnothing$. Since $H \subseteq G$ we have $f^{-1}(H) \cap H = \varnothing$ and $H \cap f^{-(iq-1)}(H) = \varnothing$ if $iq - 1$ is co-prime with pq. If $iq - 1$ is not co-prime with pq, we have $iq - 1 = jp$ for some $1 \leq j \leq q-1$, whence $H \cap f^{-(iq-1)}(H) = H \cap f^{-jp}(H) = \varnothing$ by the construction of H. It now follows that $f^{-1}(H \cap f^{-(iq-1)}(H)) = f^{-1}(H) \cap f^{-iq}(H) = \varnothing$ for $1 \leq j \leq q-1$, whence $f^{-1}(H) \cap W = \varnothing$. □

THEOREM 2.4. *Let f be transitive with f^n not transitive for some composite n. Then f^p is not transitive for some prime p dividing n.*

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Proof. We write n as a product of primes $n = p_1 p_2 \cdots p_k$ and proceed by induction on k. If $k = 2$, Lemmas 2.5 and 2.6 establish the result. Suppose the result holds provided $k \leq m$, where $m \geq 2$. Since $m - 1 \geq 1$ we may put $f^{p_1 p_2 \cdots p_{m+1}} = (f^{p_1 p_2 \cdots p_{m-1}})^{p_m p_{m+1}}$. If $g = f^{p_1 p_2 \cdots p_{m-1}}$ is not transitive, then f^{p_i} is not transitive for some $i \leq m - 1$ by the inductive hypothesis. If g is transitive, we apply Lemma 2.5 or 2.6 to g to establish that either $g^{p_m} = f^{p_1 p_2 \cdots p_{m-1} p_m}$ is not transitive or $g^{p_{m+1}} = f^{p_1 p_2 \cdots p_{m-1} p_{m+1}}$ is not transitive. In either case the inductive hypothesis shows that f^{p_i} is not transitive for some $i \leq m+1$. □

COROLLARY 2.1. *For any transitive map f and $n > 1$, one has:*

(i) *f^n is transitive iff f^p is transitive for every prime p dividing n;*

(ii) *if f has one non-transitive higher iterate, the least non-transitive iterate must be prime;*

(iii) *f^n is non-transitive iff f has a regular periodic decomposition of length dividing n.*

An interesting result concerning minimal maps comes as an easy consequence of the results of this section. The lemma will also be useful when dealing with minimal maps in §4.

LEMMA 2.7. *The elements of a regular periodic decomposition for a minimal map f are pairwise disjoint and hence clopen.*

Proof. Let $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ be a decomposition for f. By Lemma 2.1(iv), $D = \bigcup_{i \neq j} D_i \cap D_j$ is closed, invariant and nowhere dense. Since f is minimal, D must be empty. This establishes that the D_i are disjoint and it follows that they are all clopen. □

THEOREM 2.5. *A minimal map f on a connected space X is totally transitive.*

Proof. If f is not totally transitive, it admits a non-trivial regular periodic decomposition $\mathcal{D}$ by Corollary 2.1. By Lemma 2.7 all of the elements of $\mathcal{D}$ are clopen so X cannot be connected. □

The above existence proofs apply in any topological space. There are a number of circumstances where the existence of regular periodic decompositions is much easier to prove. We consider the simplest example. Recall that *connected components* are the maximal connected subsets of X. They partition X, they are closed and they may or may not be open in X. If there are only finitely many of them they are all clopen and hence regular closed. Every component must map into a component under f. The following theorem, described as 'folklore' in [**BS**], imposes a considerable restriction on the types of disconnected spaces which admit continuous transitive maps.

THEOREM 2.6. *Let $f : X \to X$ be transitive. If X has a connected component with non-empty interior, then it has only finitely many connected components, and they form a regular periodic decomposition. Hence if X has finitely many connected components, they form a regular periodic decomposition.*

Proof. Let C be a component of X with non-empty interior. By transitivity, there must be some $k \geq 1$ such that $f^k(C) \cap C \neq \varnothing$. We may assume that k is minimal. Now

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$f^i(C)$ must be contained in some component D_i of X for each $0 \leq i \leq k-1$ and $f^k(C)$ must be contained in $D_0 = C$. We cannot have $D_i = D_j$ for $0 \leq i < j \leq k-1$, as this would contradict the minimality of k. Clearly, the D_i map periodically into one another. Thus $D = \bigcup_{i=0}^{k-1} D_i$ is a closed invariant subset of X which has non-empty interior since it contains $\text{int}(C)$, whence $D = X$ by the transitivity of f. It now follows that each D_i is clopen and hence regular closed. The 'hence' follows from comments preceding the theorem. □

3. *Decomposition and transitivity ideals*

Given a transitive map f, we call the set of all $n \in \mathbb{N}$ such that f admits a regular periodic decomposition of length n the *decomposition ideal* of f. We denote this set by $DI(f)$. We have yet to justify the use of the term 'ideal'. For the example K of §2, it was asserted that the decomposition $\{[0, \frac{1}{2}], [\frac{1}{2}1]\}$ is the finest possible because K^2 is totally transitive on each of these intervals. Taking into account the trivial decomposition we see that $DI(K) = \{1, 2\}$. Clearly, the decomposition ideal of a transitive map f is $\{1\}$ precisely when f is totally transitive and the decomposition ideal of a transitive map is a conjugacy invariant. We call a decomposition $\mathcal{D}$ a *refinement* of $\mathcal{C}$ if every element of $\mathcal{D}$ is contained in some element of $\mathcal{C}$. One may show that each element of $\mathcal{C}$ contains the same number of elements of $\mathcal{D}$, so n must be a multiple of m. Each element D of $\mathcal{D}$ is contained in precisely one element C of $\mathcal{C}$. We call this unique element C the *parent* of D in $\mathcal{C}$, and, accordingly, we call all of the elements of $\mathcal{D}$ contained in C the *children* of C in $\mathcal{D}$. It follows from the next lemma that if n is an element of the decomposition ideal of f, then so is any divisor of n. By Theorem 2.2, the decomposition constructed in the lemma is the unique one of length m. Thus, given regular periodic decompositions $\mathcal{D}$ and $\mathcal{C}$ of length n and m respectively, $\mathcal{D}$ will be a refinement of $\mathcal{C}$ iff m divides n.

LEMMA 3.1. *Let $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ be a regular periodic decomposition for a transitive map f and let $m \geq 1$ with $n = mk$. Then putting $C_j = \bigcup_{i=0}^{k-1} D_{mi+j}$ for each $0 \leq j \leq m-1$ gives a regular periodic decomposition $\mathcal{C} = \{C_0, C_1, \ldots, C_{m-1}\}$ having $\mathcal{D}$ as a refinement.*

Proof. Since a finite union of regular closed sets is again regular closed, each element of $\mathcal{C}$ is regular closed. Intersections of distinct elements of $\mathcal{C}$ are clearly finite unions of intersections of distinct elements of $\mathcal{D}$ and therefore nowhere dense. For $0 \leq j \leq m-2$ one has

$$f(C_j) = \bigcup_{i=0}^{k-1} f(D_{mi+j}) \subseteq \bigcup_{i=0}^{k-1} D_{mi+j+1} = C_{j+1}$$

and since

$$f(C_{m-1}) \subseteq \left[\bigcup_{i=0}^{k-2} D_{m(i+1)}\right] \cup f(D_{mk-1}) \subseteq \left[\bigcup_{i=1}^{k-1} D_{mi}\right] \cup D_0 = \bigcup_{i=0}^{k-1} D_{mi} = C_0,$$

$\mathcal{C}$ is a regular periodic decomposition for f. □

LEMMA 3.2. *If f has regular periodic decompositions $\mathcal{D}$ of length n and $\mathcal{C}$ of length m, then f has a regular periodic decomposition of length $k = \text{lcm}(m, n)$ which is a refinement of both $\mathcal{D}$ and $\mathcal{C}$.*

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Proof. Let $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ and $\mathcal{C} = \{C_0, C_1, \ldots, C_{m-1}\}$. Since we must have $D_i \cap C_j \neq \varnothing$ for some $D_i \in \mathcal{D}$ and $C_j \in \mathcal{C}$, it is no loss of generality to assume that $G = \text{int}(D_0) \cap \text{int}(C_0) \neq \varnothing$. By Lemma 2.1(iii), $f^{-k}(\text{int}(D_0)) \subseteq \text{int}(D_0)$ and $f^{-k}(\text{int}(C_0)) \subseteq \text{int}(C_0)$ so $f^{-k}(G) \subseteq G$. If $f^{-i}(G) \cap f^{-j}(G) \neq \varnothing$ for some $0 \leq i \leq j \leq k-1$, one has

$$\varnothing \neq f^{-i}(\text{int}(D_0)) \cap f^{-j}(\text{int}(D_0)) \subseteq \text{int}(D_{n-i(\text{mod}\, n)}) \cap \text{int}(D_{n-j(\text{mod}\, n)})$$

and hence $i = j \pmod n$, whence n divides $j - i$. An analogous argument shows that m divides $j - i$ and since $j - i < k = \text{lcm}(m, n)$, we must have $j - i = 0$. Thus $G, f^{-1}(G), \ldots, f^{-(k-1)}(G)$ are disjoint, so their closures form a regular periodic decomposition for f by Lemma 2.2. □

Recall that the natural numbers $\mathbb{N}$ form a lattice with the partial order given by $m \leq n$ iff m divides n. The operations of meet and joint in this lattice are gcd and lcm respectively. We will refer to this simply as the lattice $\mathbb{N}$. Recall also that an *ideal* J in a lattice L is a subset (and in fact a sub-lattice) of L such that (i) if $a \in J$ and $b \leq a$ then $b \in J$ and (ii) if $a \in J$ and $b \in J$ then $a \vee b \in J$. The previous two lemmas clearly justify the use of the expression 'decomposition ideal'. The next lemma concerns the situation where one has a regular periodic decomposition of a space which is itself an element of a regular periodic decomposition. This necessarily involves the concept of relative interior. If X is any topological space and $C \subseteq D \subseteq X$, the interior of C relative to D will be denoted by $\text{int}_D(C)$, while the interior relative to the whole space X will be denoted $\text{int}_X(C)$. Under these conditions, one always has $\text{int}_X(C) \subseteq \text{int}_D(C)$.

LEMMA 3.3. *Let $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ be a regular periodic decomposition for f on X, let $\mathcal{C} = \{C_0, C_1, \ldots, C_{m-1}\}$ be a regular periodic decomposition for f^n on D_0 and let $G = \text{int}_X(C_0)$. Then $\{\tilde{G}, \overline{f^{-(mn-1)}(G)}, \overline{f^{-(mn-2)}(G)}, \ldots, \overline{f^{-1}(G)}\}$ is a regular periodic decomposition for f on X.*

Proof. We show that G satisfies the hypotheses of Lemma 2.2. Applying Lemma 2.1(iii) to $f^n|_{D_0}$ and using the fact that $\text{int}_X(C_0) \subseteq \text{int}_{D_0}(C_0)$ one has

$$f^{-nm}(G) = (f^n)^{-m}(\text{int}_X(C_0)) \subseteq (f^n)^{-m}(\text{int}_{D_0}(C_0)) \subseteq \text{int}_{D_0}(C_0) \subseteq C_0.$$

Since $f^{-nm}(G)$ is open in X, this yields $f^{-nm}(G) \subseteq \text{int}_X(C_0) = G$. It remains to show that if $f^{-i}(G) \cap f^{-j}(G) \neq \varnothing$, where $0 \leq i \leq j \leq mn - 1$, then $i = j$. Since $G \subseteq \text{int}_X(D_0)$, Lemma 2.1(iii) yields $f^{-i}(G) \subseteq f^{-i}(\text{int}_X(D_0)) \subseteq \text{int}_X(D_{-i(\text{mod}\, n)})$. Similarly, $f^{-j}(G) \subseteq \text{int}_X(D_{-j(\text{mod}\, n)})$. Hence $\text{int}_X(D_{-i(\text{mod}\, n)}) \cap \text{int}_X(D_{-j(\text{mod}\, n)}) \neq \varnothing$. Since $\mathcal{D}$ is a regular periodic decomposition for f on X, we have $i = pn + k$ and $j = qn + k$, where $0 \leq p \leq q \leq m - 1$. The assumption that $f^{-i}(G) \cap f^{-j}(G) \neq \varnothing$ then gives

$$\begin{aligned}
\varnothing &\neq f^{-(n-k)}(f^{-i}(G) \cap f^{-j}(G)) \\
&= f^{-(n-k)}(f^{-(pn+k)}(G)) \cap f^{-(n-k)}(f^{-(qn+k)}(G)) \\
&= f^{-(p+1)n}(\text{int}_X(C_0)) \cap f^{-(q+1)n}(\text{int}_X(C_0)) \\
&\subseteq f^{-(p+1)n}(\text{int}_{D_0}(C_0)) \cap f^{-(q+1)n}(\text{int}_{D_0}(C_0)) \\
&= (f^n)^{-(p+1)}(\text{int}_{D_0}(C_0)) \cap (f^n)^{-(q+1)}(\text{int}_{D_0}(C_0)).
\end{aligned}$$

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Applying Lemma 2.1(iii) to $f^n|_{D_0}$ yields $(f^n)^{-(p+1)}(\mathrm{int}_{D_0}(C_0)) \subseteq \mathrm{int}_{D_0}(C_{-(p+1)(\mathrm{mod}\, m)})$ and $(f^n)^{-(q+1)}(\mathrm{int}_{D_0}(C_0)) \subseteq \mathrm{int}_{D_0}(C_{-(q+1)(\mathrm{mod}\, m)})$. Since $\mathcal{C}$ is a regular periodic decomposition for $f^n|_{D_0}$, we see that $p+1$ and $q+1$ are congruent mod m, but since $0 \leq p \leq q \leq m-1$, we must have $p = q$, whence $i = j$ as required. □

Fortunately, the lemma enables us to dispense henceforth with the notation for relative interiors. We will call a regular periodic decomposition *terminal* if it is of maximal length.

THEOREM 3.1. *A regular periodic decomposition $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ for a transitive map f is terminal iff f^n is totally transitive on each D_i.*

Proof. Suppose f^n is not totally transitive on D_i. Since f^n is transitive on D_i, there is a regular periodic decomposition $\mathcal{C} = \{C_0, C_1, \ldots, C_{m-1}\}$ of length $m \geq 2$ for f^n on D_i. By Lemma 3.3, there is a decomposition of length $mn > n$ for f so $\mathcal{D}$ is not terminal. Conversely, if $\mathcal{D}$ is not terminal, there is a regular periodic decomposition of length $m > n$. By Lemma 3.2, $\mathcal{D}$ has a refinement of length $k = \mathrm{lcm}(m, n) > n$. Some element C of this refinement is a proper closed subset of D_i. Clearly C has non-empty interior and is invariant under f^k, whence f^k is not totally transitive on D_i. □

The construction of a transitive map with any desired finite decomposition ideal is straightforward, provided one does not demand that the domain be connected. If J is a finite ideal in $\mathbb{N}$, it is easy to see that $J = \{m \geq 1 : m \text{ divides } n\}$ for some $n \in \mathbb{N}$. Let $g : Y \to Y$ be any totally transitive map and let $X = Y \times \mathbb{Z}_n$, where $\mathbb{Z}_n$ is equipped with the discrete topology so that the $Y \times \{i\}$ are all clopen in X. Putting $f : X \to X : (s, i) \mapsto (g(s), i + 1 \ (\mathrm{mod}\ n))$, it is easy to see that $\mathcal{D} = (Y \times \{0\}, Y \times \{1\}, \ldots, Y \times \{n-1\})$ forms a regular periodic decomposition for f and that the restriction f^n to each $Y \times \{i\}$ is conjugate to g^n and hence totally transitive on each $Y \times \{i\}$. By Theorem 2.1, f is transitive and by Theorem 3.1, $\mathcal{D}$ is terminal, so f has decomposition ideal J.

Let J be an infinite ideal J in $\mathbb{N}$ and let p be prime. Then either $\mu(p) = \max\{k \geq 0 : p^k \in J\} \geq 0$ exists or $p^k \in J$ for all $k \geq 0$. If $\mu(p) = 0$ then no power of p is an element of J. Where $p^k \in J$ for all $k \geq 0$, we put $\mu(p) = \infty$, so that μ is a function from the set of prime numbers into $\mathbb{N} \cup \{\infty\}$ and we call $\mu(p)$ the *multiplicity* of p. This terminology is consistent with the usage in the classification of adding machines given in [**BS**], which we derive by a different method in §4. Clearly, two ideals are equal iff their multiplicity functions are identical. Indeed, the multiplicity function may be regarded as an alternative concept to that of decomposition ideal. Let J be an ideal in a lattice L. A subset A of J is said to *generate* J if J is the smallest ideal in L containing A. A generating set of the form $\{b_n : n \geq 1\}$ for an ideal J where $b_1 < b_2 < b_3 < \cdots$ will be called a *generating sequence* for J. The next lemma is a consequence of the well-known result from lattice theory [**DP**, p. 185] that $\{a \in L : a \leq \bigvee F \text{ for some finite } F \subseteq A\}$ is the smallest ideal containing A.

LEMMA 3.4. *A sequence $b_1 \leq b_2 \leq b_3 \leq \cdots$ contained in an ideal J of a lattice L is a generating sequence for J iff $J = \{m \in L : m \leq b_n \text{ for some } n \geq 1\}$.*

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For an infinite ideal J in $\mathbb{N}$, there is always a generating sequence, as we now show. The set of primes in J having infinite multiplicity will be denoted by J_∞ and the set of primes in J having non-zero but finite multiplicity will be denoted by J_Φ. We may write the elements of J_∞ and J_Φ as strictly increasing sequences. It is possible that one or both of J_Φ and J_∞ is finite, but if J_∞ is empty then J_Φ must be infinite. If either J_∞ or J_Φ is finite, we extend it to an infinite sequence by adding an infinite string of 1's at the end. For example, if $J_\infty = \{2\}$ and $J_\Phi = \{3, 5, 7\}$, we obtain the sequences $(2, 1, 1, 1, \ldots)$ and $(3, 5, 7, 1, 1, 1, \ldots)$ respectively. These sequences are clearly unique. Let $(p_1, p_2, p_3, \ldots)$ and $(q_1, q_2, q_3, \ldots)$ be the sequences generated in this way from J_∞ and J_Φ respectively and, for $n \geq 1$, let

$$a_n = q_n^{\mu(q_n)}\left(\prod_{i=1}^{n} p_i\right) \quad \text{and let} \quad b_n = \prod_{i=1}^{n} a_i.$$

We call the sequences $(a_1, a_2, a_3, \ldots)$ and $(b_1, b_2, b_3, \ldots)$ the *multiplicity sequence* and the *canonical generating sequence* for J, respectively. We need to check that the latter sequence is in fact a generating sequence for J. It is clear that each b_n divides b_{n+1}. If $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where the p_i are distinct primes, we will call the $p_i^{k_i}$ the *terms* of the prime power decomposition. The terms of the prime power decomposition of b_n are precisely $q_i^{\mu(q_i)}$ and p_i^{n-i+1} for $i \leq n$. Since these terms are elements of J, and b_n is their least common multiple, each b_n must also be in J. The ideal generated by $(b_1, b_2, b_3, \ldots)$ is therefore contained in J. The terms of the prime power decomposition of an element m of J must be divisors of some $q_n^{\mu(q_n)}$ or some p_n^k. Clearly $q_n^{\mu(q_n)}$ divides b_i for $i \geq n$. By the definition of the multiplicity sequence, p_n divides a_i, for $i \geq n$. Hence p_n^k divides $b_{n+k} = \prod_{i=1}^{n+k} a_i$ and so p_n^k divides b_i for $i \geq n+k$. One may therefore find a sufficiently large $i \geq 1$ such that m divides b_i. Thus $\{b_n : n \geq 1\}$ generates J.

The set of all n such that f^n is transitive will be called the *transitivity ideal* of f and denoted by $TI(f)$. If P is a set of prime numbers, we denote by $\hat{P}$ the set of natural numbers whose prime factors, if any, are all contained in P. Thus $\hat{P}$ always contains 1 together with all products of elements of P.

THEOREM 3.2. *Let f and g be transitive. Then:*
(i) *$TI(f) = \hat{P}$, where P is the set of primes for which f^p is transitive;*
(ii) *$TI(f)$ is an ideal in $\mathbb{N}$ and $TI(f)_\Phi$ is empty;*
(iii) *$TI(f) = \hat{P}$, where P is the set of primes not in $DI(f)$;*
(iv) *$TI(f) = TI(g)$ iff $DI(f) \cap \mathbb{P} = DI(g) \cap \mathbb{P}$, where $\mathbb{P}$ is the set of prime numbers.*

Proof. (i) If $n \in \hat{P}$, then f^p is transitive for every prime factor p of n, whence f^n is transitive by Corollary 2.1(i). Conversely, if f^n is transitive, Corollary 2.1(i) shows that f^p is transitive for every prime factor p of n, whence $n \in \hat{P}$. Hence $TI(f) = \hat{P}$.

(ii) If $n \in \hat{P}$, every divisor of n is also in $\hat{P}$. The definition of $\hat{P}$ also makes it clear that if $m, n \in \hat{P}$ then $mn \in \hat{P}$, whence $\text{lcm}(m, n) \in \hat{P}$. Thus $TI(f) = \hat{P}$ is an ideal. Moreover, the definition of $\hat{P}$ makes it clear that $TI(f)_\Phi = \hat{P}_\Phi$ is empty.

(iii) For any prime p such that f^p is transitive, p cannot be in $DI(f)$. Conversely, if p is not in $DI(f)$, then f^p must be transitive; otherwise there would be a decomposition

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of length p by Theorem 2.3. Hence P is precisely the set of primes such that f^p is transitive, whence $TI(f) = \hat{P}$ by (i).

(iv) Follows easily from (iii). □

Thus if two maps have the same decomposition ideal, they also have the same transitivity ideal. The converse is far from true. The map K introduced in §2 has decomposition ideal $\{1, 2\}$, whence $TI(K)$ is the set of odd numbers. We will shortly show that every ideal in $\mathbb{N}$ is the decomposition ideal of some map. Thus there is a transitive map with decomposition ideal $\{2^k : k \geq 0\}$ which must have the same transitivity ideal as K. Theorem 3.2 makes it clear that the multiplicity of every prime in $TI(f)$ is either zero or infinite. Combining this fact with the examples given in the next section, one obtains the following theorem.

THEOREM 3.3. *An infinite ideal J in $\mathbb{N}$ is the transitivity ideal of some map iff J_Φ is empty.*

Thus transitivity ideals are typically infinite. The one special case where this is not is the trivial ideal $J = \{1\}$. We will see in §4 that there are transitive maps with decomposition ideal $\mathbb{N}$ and hence transitivity ideal $\{1\}$. They have the curious property of having no transitive higher iterates at all.

4. *Adding machines*

We now consider a class of transitive homeomorphisms with infinite decomposition ideals. They arise as topological group translations on the Cantor space. Different examples are generated by making the Cantor space into a topological group in different (non-isomorphic) ways. Let $(G, +)$ be an Abelian topological group. For any element u of G, the map $R_\mu : X \to X : x \mapsto x + u$, known as a *translation* of G by u, is a homeomorphism with inverse R_{-u}. The dynamics of a translation are comparatively simple. All orbits are homeomorphic. If there is a periodic point, all points are periodic with the same period. If there is a dense orbit, all orbits are dense so the translation is minimal. Thus, when G is a second countable Baire space, a translation will be transitive iff it is minimal. When testing for transitivity, we therefore need only check that the orbit of the identity element 0 is dense.

Let $\boldsymbol{a} = (a_1, a_2, a_3, \ldots)$ be a sequence of natural numbers with each $a_n \geq 2$. The topological group of $\boldsymbol{a}$-adic integers, denoted by $\boldsymbol{\Lambda}_{\boldsymbol{a}}$ has underlying space $\prod_{n\in\mathbb{N}} \mathbb{Z}_{a_n}$, where each $\mathbb{Z}_{a_n}$ is equipped with discrete topology and the whole space is given the Tychonoff product topology. By standard theorems, $\boldsymbol{\Lambda}_{\boldsymbol{a}}$ is homeomorphic to the Cantor space. We refer to $\boldsymbol{a}$ as the *defining sequence* of $\boldsymbol{\Lambda}_{\boldsymbol{a}}$. For natural numbers m and n we let $m(\text{div } n)$ denote the greatest integer less than or equal to m/n. Group addition on $\boldsymbol{\Lambda}_{\boldsymbol{a}}$ may be thought of as pointwise addition (mod a_i) with 'carry' to the right. That is $(s_1, s_2, s_3, \ldots) + (t_1, t_2, t_3, \ldots) = (u_1, u_2, u_3, \ldots)$, where $u_i = s_i + t_i + c_i$ (mod a_i) and the 'carry' c_i is given by $c_1 = 0$ and for $i \geq 2$, $c_i = (s_{i-1} + t_{i-1} + c_{i-1})$ (div a_i). We refer to the s_i as *digits*. With this addition, $\boldsymbol{\Lambda}_{\boldsymbol{a}}$ is a topological group and the constant sequence $\mathbf{0} = (0, 0, 0, \ldots)$ acts as the identity element [**HR**, §10]. The finite sequence consisting of the first n digits of an element of $\boldsymbol{\Lambda}_{\boldsymbol{a}}$ is called its *initial n-block*. A countable base of neighbourhoods for $\boldsymbol{\Lambda}_{\boldsymbol{a}}$ is given by $\mathcal{B} = \{U_n(s) : n \geq 1, s \in \boldsymbol{\Lambda}_{\boldsymbol{a}}\}$, where $U_n(s)$ is the

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set of all elements of $\mathbf{\Lambda}_a$ with the same initial n-block as s. One may check that $U_n(s)$ is clopen and hence regular closed and that $R_t(U_n(s)) = U_n(R_t(s)) = U_n(s+t)$ for any translation R_t. Since there are $b_n = \prod_{i=1}^{n} a_i$ distinct initial n-blocks for elements of $\mathbf{\Lambda}_a$ for $n \geq 1$, we call $(b_1, b_2, b_3, \ldots)$ the *initial block counting sequence* of $\mathbf{\Lambda}_a$.

We call translations of the form R_u on $\mathbf{\Lambda}_a$, where $\boldsymbol{u} = (1, 0, 0, \ldots)$ and $\boldsymbol{a}$ is some defining sequence *adding machines*. Since $\boldsymbol{u}$ will remain fixed and we obtain different adding machines by taking different defining sequences generating different domains, we will denote the adding machine R_u on $\mathbf{\Lambda}_a$ by L_a. The name *adding machine* has been reserved by some authors (see, for example [**De**, p. 137], [**BC** p. 133]) for the case where $\boldsymbol{a}$ is the constant sequence $(2, 2, 2, \ldots)$. This map is known to arise as the restriction of the logistic map $f(x) = \lambda x(1-x)$ to an invariant Cantor set, where $\lambda \approx 3.56995\ldots$ is the limiting parameter value of the period doubling sequence in the logistic family [**Mi1**, **Mi2**]. Adding machines also arise as attractors for a large class of maps on locally connected, locally compact, metric spaces [**BS**]. The next two lemmas are established by induction on n.

LEMMA 4.1. *If $n \geq 1$ and $k < a_{n+1}$, then $kb_n\boldsymbol{u} = (\underbrace{0, 0, \ldots, 0}_{n}, k, 0, 0, 0, \ldots)$.*

LEMMA 4.2. *For each initial n-block $(t_1, t_2, \ldots, t_n)$ in $\mathbf{\Lambda}_a$, there is a $0 \leq k < b_n$ such that $k\boldsymbol{u} = (t_1, t_2, \ldots, t_n, 0, 0, 0, \ldots)$.*

In order to establish that the orbit of $\mathbf{0}$ under L_a is dense, it is sufficient to show that for $n \geq 1$, the initial n-blocks of the elements of $\{R_a^k(\mathbf{0}) : 0 \leq k \leq b_n - 1\} = \{k\boldsymbol{u} : 0 \leq k \leq b_n - 1\}$ consist precisely of the b_n possible initial n-blocks of elements of $\mathbf{\Lambda}_a$. But this is a consequence of Lemma 4.2.

THEOREM 4.1. *Every adding machine is minimal.*

We have already noted that each $U_n(s)$ is clopen and hence regular closed. The fact that initial n-blocks of the elements of the set $\{ku : 0 \leq k \leq b_n - 1\}$ are precisely the b_n possible initial n-blocks of elements of $\mathbf{\Lambda}_a$ implies that $\mathcal{D}^n = \{U_n(\mathbf{0}), U_n(\boldsymbol{u}), U_n(2\boldsymbol{u}), \ldots, U_n((b_n - 1)\boldsymbol{u})\}$ forms a regular periodic decomposition of length b_n for L_a. This shows that $DI(L_a)$ contains the ideal generated by the initial block counting sequence $(b_1, b_2, b_3, \ldots)$. We complete our analysis of the decomposition ideals of adding machines by showing that $DI(L_a)$ is precisely the ideal generated by the initial block counting sequence. Observe that the elements of $\mathcal{D}^n$ are precisely the clopen sets of the form $U_n(s)$ for $n \geq 1$ and $s \in \mathbf{\Lambda}_a$. Hence the collection of all of the elements of the $\mathcal{D}^n$ is precisely the basis $\mathcal{B}$ for $\mathbf{\Lambda}_a$. This rather unusual situation enables us to calculate $DI(L_a)$ precisely using the following lemma.

LEMMA 4.3. *Let f be a minimal map on a compact space X, let $\mathcal{D}^n$ be a decomposition of length b_n where $b_1 \leq b_2 \leq b_3 \leq \cdots$ and suppose that the collection $\mathcal{B}$ of all elements of all the $\mathcal{D}^n$ is a base of neighbourhoods for X. Then $\boldsymbol{b} = (b_1, b_2, b_3, \ldots)$ generates $DI(f)$.*

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Proof. Clearly, $DI(f)$ contains the ideal generated by $\boldsymbol{b}$. We need to show that if $\mathcal{C}$ is a regular periodic decomposition of length m for f, then m divides some b_n. Lemma 2.7 establishes that each $C \in \mathcal{C}$ is clopen and hence a union of elements of $\mathcal{B}$. Since C is compact we may put $C = \bigcup_{k\in K} D_{i_k}^{n_k}$, where K is finite and $D_{i_k}^{n_k}$ is an element of $\mathcal{D}^{n_k}$ for each k. Let $n = \max\{n_k : k \in K\}$. By hypothesis b_{n_k} divides b_n for $i \leq k$, whence by Lemma 2.1(ii)

$$f^{b_n}(C_0) = \bigcup_{k\in K} f^{b_n}(D_{i_k}^{n_k}) \subseteq \bigcup_{k\in K} D_{i_k}^{n_k} = C_0.$$

It now follows that m divides b_n by Lemma 2.1(ii), as required. □

The hypotheses of the lemma place severe restrictions upon the space X. A metrizable space satisfying the conditions given, is either the Cantor space or a finite discrete space. The lemma and comments preceding it clearly show that the decomposition ideal of $L_{\boldsymbol{a}}$ is precisely the ideal generated by the initial block counting sequence for $\boldsymbol{\Lambda}_{\boldsymbol{a}}$. It is now easy to see that for any infinite ideal J in the lattice $\mathbb{N}$, there is an adding machine having decomposition ideal J. Let $\boldsymbol{a} = (a_1, a_2, a_3, \ldots)$ be the multiplicity sequence of J and let $(b_1, b_2, b_3, \ldots)$ be the canonical generating sequence for J, as described in the previous section. By construction, $(b_1, b_2, b_3, \ldots)$ is precisely the initial block counting sequence for $L_{\boldsymbol{a}}$, whence $DI(L_{\boldsymbol{a}}) = J$. We record these results as a theorem.

THEOREM 4.2. *If $\boldsymbol{a}$ is the canonical generating sequence of an infinite ideal J in $\mathbb{N}$, them $DI(L_{\boldsymbol{a}}) = J$ and $DI(L_{\boldsymbol{a}})$ is the ideal generated by the initial block counting sequence for $\boldsymbol{\Lambda}_{\boldsymbol{a}}$.*

Adding machines may be thought of as basic models for transitive maps with infinite decomposition ideals. They describe the way in which the elements of the decompositions map in terms of the parent/child relationships of the elements of the decompositions. One may make this insight precise in general, but the machinery required is somewhat cumbersome [**Ba**, §4.1]. For a transitive map f on a second countable Baire space X, however, one may construct a semi-conjugacy from $f|_G$, where G is the set of all points in X with dense orbit, to a dense restriction of an adding machine $L_{\boldsymbol{a}}$. It will be apparent from the construction that the existence of this semi-conjugacy guarantees that the elements of the decompositions of f and $L_{\boldsymbol{a}}$ map into one another in essentially the same way with the same relations of containment. For any regular periodic decomposition $\mathcal{D}$, we denote the union of the interiors of the elements of $\mathcal{D}$ by $\text{int}(\mathcal{D})$. The following lemma is a simple consequence of Lemma 2.1.

LEMMA 4.4. *Let $\mathcal{D}$ be a refinement of a regular periodic decomposition $\mathcal{C}$ of length m such that $D_0 \subseteq C_0$. Then $D_i \subseteq C_j$ iff $j = i$ (*mod *m).*

THEOREM 4.3. *Let f be a transitive map on a second countable Baire space X with $DI(f)$ infinite and let $G = \{x \in X : \overline{O(x)} = X\}$. Then there is a dense restriction of an adding machine which is semi-conjugate to $f|_G$.*

Proof. Let $\boldsymbol{a} = (a_1, a_2, a_3, \ldots)$ and $(b_1, b_2, b_3, \ldots)$ be the multiplicity sequence and the canonical generating sequence for $DI(f)$ respectively, and for each $n \in \mathbb{N}$ let $\mathcal{D}^n$ denote

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the regular periodic decomposition of length b_n for f. We may label the elements of the $\mathcal{D}^n$ so that $D_0^{n+1} \subseteq D_0^n$. By Lemma 2.1(iv) any point in the intersection of two elements of $\mathcal{D}^n$ has a nowhere dense orbit, whence $G \subseteq \bigcap_{n=1}^{\infty} \operatorname{int}(\mathcal{D}^n)$. Thus for any $x \in G$ and any $n \geq 1$, there is a unique $0 \leq j_n(x) \leq b_n - 1$ such that $x \in \operatorname{int}(D^n_{j_n(x)})$, and for $i \leq n$ we may denote by y_i the ith digit of $j_n(x)\boldsymbol{u}$ in $\boldsymbol{\Lambda}_{\boldsymbol{a}}$, where $\boldsymbol{u} = (1, 0, 0, \ldots)$ as usual. Our semi-conjugacy $h : G \to \boldsymbol{\Lambda}_{\boldsymbol{a}}$ is given by $h(x) = (y_1, y_2, y_3, \ldots)$. To see that this is consistent, it suffices to show that the first n digits of $j_n(x)\boldsymbol{u}$ and $j_{n+1}(x)\boldsymbol{u}$ agree. Since $x \in \operatorname{int}(D^{n+1}_{j_{n+1}(x)}) \cap \operatorname{int}(D^n_{j_n(x)})$ we must have $D^{n+1}_{j_{n+1}(x)} \subseteq D^n_{j_n(x)}$, whence $j_{n+1}(x) = j_n(x) + kb_n$ for some $k < a_{n+1}$ by Lemma 4.4. Hence by Lemma 4.1

$$j_{n+1}(x)\boldsymbol{u} = j_n(x)\boldsymbol{u} + kb_n\boldsymbol{u} = j_n(x)\boldsymbol{u} + (\underbrace{0, \ldots, 0}_{n}, k, 0, 0, \ldots),$$

which establishes agreement between the first n digits of $j_n(x)\boldsymbol{u}$ and $j_{n+1}(x)\boldsymbol{u}$. For any basic clopen set $U_n(h(x))$ about $h(x)$ and any $s \in \operatorname{int}(D^n_{j_n(x)})$ one has $j_n(s) = j_n(x)$, whence $j_n(s)\boldsymbol{u} = j_n(x)\boldsymbol{u}$ so $h(s) \in U_n(h(x))$, which shows that h is continuous. To prove that $L_{\boldsymbol{a}}(h(x)) = h(x) + \boldsymbol{u}$ and $h(f(x))$ agree up to the nth digit for every $n \geq 1$, it will suffice to show that $j_n(x)\boldsymbol{u} + \boldsymbol{u}$ and $j_n(f(x))\boldsymbol{u}$ agree up to the nth digit. Since $G \subseteq \operatorname{int}(\mathcal{D}^{b_n})$ and $x \in \operatorname{int}(D^n_{j_n(x)})$ we have $f(x) \in \operatorname{int}(D^n_{j_n(x)+1(\operatorname{mod} b_n)})$, whence $j_n(f(x)) = j_n(x) + 1(\operatorname{mod} b_n)$. There are two cases. If $j_n(x) < b_n - 1$, then $j_n(f(x)) = j_n(x) + 1$, whence $j_n(f(x))\boldsymbol{u} = j_n(x)\boldsymbol{u} + \boldsymbol{u}$. If $j_n(x) = b_n - 1$, then $j_n(f(x)) = 0$, whence $j_n(f(x))\boldsymbol{u} = \boldsymbol{0}$ and by Lemma 4.1

$$j_n(x)\boldsymbol{u} + \boldsymbol{u} = b_n\boldsymbol{u} - \boldsymbol{u} + \boldsymbol{u} = b_n\boldsymbol{u} = (\underbrace{0, \ldots, 0}_{n}, 1, 0, 0, \ldots).$$

In either case the first n digits of $j_n(x)\boldsymbol{u}$ and $j_{n+1}(x)\boldsymbol{u}$ agree, whence $h \circ f = L_{\boldsymbol{a}} \circ h$ which implies that $L_{\boldsymbol{a}}$ is invariant on the non-empty set $h(G)$. Since $L_{\boldsymbol{a}}$ is minimal, $h(G)$ must be dense in $\boldsymbol{\Lambda}_{\boldsymbol{a}}$. □

Of course, if f is itself minimal, we have $G = X$. If in addition X is compact, then h is a closed mapping, whence $h(G) = \boldsymbol{\Lambda}_{\boldsymbol{a}}$ so we have a semi-conjugacy from f to $L_{\boldsymbol{a}}$. This will be true in particular for compact metric spaces.

THEOREM 4.4. *Let f be a minimal map on a compact metric space with $DI(f)$ infinite. Then there is an adding machine which is semi-conjugate to f.*

Suppose that $L_{\boldsymbol{a}}$ and $L_{\boldsymbol{b}}$ have the same decomposition ideal J. The semi-conjugacy h from $L_{\boldsymbol{a}}$ and $L_{\boldsymbol{b}}$ given in Theorem 4.3 is a continuous surjection and has compact Hausdorff domain and codomain. It will therefore be a homeomorphism provided only that it is injective. For x and y in $\boldsymbol{\Lambda}_{\boldsymbol{a}}$ with $x \neq y$, there is an $n \geq 1$ such that $x_n \neq y_n$, whence $j_n(x) \neq j_n(y)$. Since $j_n(x), j_n(y) < b_n$ we must have $j_n(x)\boldsymbol{u} \neq j_n(y)\boldsymbol{u}$, whence $h(x) \neq h(y)$. This yields a complete classification for adding machines.

THEOREM 4.5. *Adding machines are conjugate iff they have the same decomposition ideal.*

An immediate corollary of this theorem is that adding machines are conjugate iff they have the same multiplicity functions. This is the result obtained by Buescu and

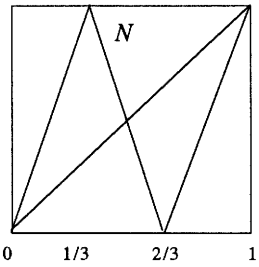
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Stewart [**BS**], who derive the multiplicity functions directly from the defining sequences of the $\boldsymbol{a}$-adic topological groups. Their proof makes use of the spectral theory for homeomorphisms of compact metric spaces.

5. *Constructing examples using Cartesian products*

We now give a generalisation of mixing for a map f with a regular periodic decomposition $\mathcal{D}$ of length n. We say that f is *mixing relative to* $\mathcal{D}$ and that f is *relatively mixing* if f^n is mixing on every element of $\mathcal{D}$. If this condition holds, f is transitive by Theorem 2.1 and $\mathcal{D}$ is terminal by Theorem 3.1. One may construct a relatively mixing map on the circle having any desired finite decomposition ideal. Given $n > 1$, we divide the circle into n closed arcs $\mathcal{A} = \{A_0, A_1, \ldots, A_{n-1}\}$ of equal length, that is, $A_i = [2\pi i/n, 2\pi(i+1)/n]$. Since A_0 is homeomorphic to a compact interval, there is a map $\phi : A_0 \to A_0$ which is topologically conjugate to the piecewise linear map $N : [0, 1] \to [0, 1]$ given by

$$N(x) = \begin{cases} 3x & \text{if } x \in [0, \frac{1}{3}) \\ 2 - 3x & \text{if } x \in [\frac{1}{3}, \frac{2}{3}) \\ 3x - 2 & \text{if } x \in [\frac{2}{3}, 1] \end{cases}$$



It is easy to show that N and hence ϕ are mixing. If h is the automorphism of the circle given by rotation through an angle of $2\pi/n$, we define $Q_n : S^1 \to S^1$ by putting $Q_n(\theta) = h(\phi(\theta))$ if $\theta \in A_0$ and $Q_n(\theta) = h(\theta)$ otherwise. It is easy to check that $\mathcal{A}$ is a regular periodic decomposition for Q_n, that Q_n is continuous and mixing relative to $\mathcal{A}$, because the restriction of Q_n^n to each A_i is conjugate to ϕ via an iterate of h.

THEOREM 5.1. *Let $f_i : X_i \to X_i$ be mixing relative to $\mathcal{D}^i = \{D_0^i, D_1^i, \ldots, D_{m_i-1}^i\}$ for $i \le k$, let the m_i be pairwise co-prime, let $M = m_1 \cdot m_1 \cdots m_k$ and let $f = \prod_{i=1}^k f_i$. Then f is relatively mixing and $DI(f) = \{m \ge 1 : m$ divides $M\}$.*

Proof. We first check that $\mathcal{D} = \{D_0, D_1, \ldots, D_{M-1}\}$ is a regular periodic decomposition for f, where $D_j = D^1_{j(\mathrm{mod}\, m_1)} \times D^2_{j(\mathrm{mod}\, m_2)} \times \cdots \times D^k_{j(\mathrm{mod}\, m_k)}$ for $0 \le j \le M-1$. It is a straightforward matter to show that the elements of $\mathcal{D}$ are regular closed in X and that $f(D_j) \subseteq D_{j+1(\mathrm{mod}\, M)}$ for $0 \le j \le M-1$. For $0 \le p \ne q \le M-1$, there is an $i \le k$ such that $p \ne q \pmod{m_i}$. Otherwise, m_i divides $q - p$ for all $i \le k$, whence $q - p$ is a multiple of $M = \mathrm{lcm}\{m_i : i \le k\}$. Thus $\mathrm{int}(D^i_{p(\mathrm{mod}\, m_i)}) \cap \mathrm{int}(D^i_{q(\mathrm{mod}\, m_q)}) = \varnothing$ and hence $\mathrm{int}(D_p) \cap \mathrm{int}(D_q) = \varnothing$.

To see that f is mixing relative to $\mathcal{D}$, consider a pair $U = U_1 \times U_2 \times \cdots \times U_k$ and $V = V_1 \times V_2 \times \cdots \times V_k$ of basic open subsets of D_j for $0 \le j \le M-1$. Since $f_i^{m_i}$ is mixing on $D^i_{j(\mathrm{mod}\, m_i)}$ for $1 \le i \le k$, there is an $N_i \ge 1$ such that $f_i^{qm_i}(U_i) \cap V_i \ne \varnothing$ for all $q \ge N_i$. Let $N = \max\{N_i : 1 \le i \le k\}$ and let $n \ge N$. Since $n(M/m_i) \ge N \ge N_i$ one has $f_i^{nM}(U_i) \cap V_i = f_i^{n(M/m_i)m_i}(U_i) \cap V_i \ne \varnothing$, whence $f^{nM}(U) \cap V \ne \varnothing$. Hence

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f^n is mixing on D_j. The calculation of $DI(f)$ now follows from comments following Theorem 3.1. □

Theorem 5.1 enables the construction of relatively mixing maps with a terminal decomposition of any given length $m \geq 1$ on a large class of finite-dimensional manifolds. Let B^n denote the n-dimensional closed unit ball. We make use of the 'tent map' $T : [0, 1] \to [0, 1] : x \mapsto 1 - |1 - 2x|$ which is well known to be mixing. We have already seen that S^1 admits a relatively mixing map Q_n with terminal decomposition of length m. Suppose that for some $n \geq 2$ we have a relatively mixing map f on S^{n-1} with terminal decomposition of length m. The map $g = f \times T : S^{n-1} \times [0, 1] \to S^{n-1} \times [0, 1]$ is relatively mixing and has terminal decomposition of length m by Theorem 5.1. Since $T(0) = 0$, the map g is compatible with the quotient space $X = (S^{n-1} \times [0, 1])/(S^{n-1} \times \{0\})$ and therefore induces a relatively mixing map h on X with terminal decomposition of length m. Since $T(1) = 0$ and $\partial(X) = S^{n-1} \times \{1\}$, h is compatible with the quotient space $Y = X/\partial(X) = X/(S^{n-1} \times \{1\})$ and sends $\partial(X)$ to the point $S^{n-1} \times \{0\}$ of X. It therefore induces a relatively mixing map on Y with a terminal decomposition of length m. The following result now follows by induction since X is homeomorphic to B^n and Y is homeomorphic to S^n.

LEMMA 5.1. *For any $m \geq 1$ and any $n \geq 2$, there is a relatively mixing map on B^n which is constant on $\partial(B^n)$ and has a terminal decomposition of length m.*

Let f be the map on B^n given by the construction of Lemma 5.1, so that f has a terminal decomposition of length m. Since f is constant on $\partial(B^n)$, it is compatible with any quotient space X which identifies only points in $\partial(B^n)$. The map induced by f on any such space X will be relatively mixing and have a terminal decomposition of length m. Since it is well known that every compact connected and triangulable finite-dimensional manifold can be obtained in this way, we have the following theorem.

THEOREM 5.2. *Let M be a compact connected and triangulable finite-dimensional manifold and let $m \geq 1$, then there is a relatively mixing map on M with a terminal decomposition of length m.*

Theorem 5.2 applies to all compact connected topological 2-manifolds [**Bl**, p. 68] and all compact connected finite-dimensional smooth manifolds [**KS**, p. 308]. The decomposition ideal for a finite Cartesian product function as given by Theorem 5.1 is the smallest ideal in $\mathbb{N}$ which contains the decomposition ideals of all of the factors. It is not clear whether one can compute the decomposition ideal in such a direct way for a countably infinite product. The intersection of the decomposition ideals of a pair of transitive maps always contains 1. Where $DI(f) \cap DI(g) = \{1\}$, we say that $DI(f)$ and $DI(g)$ are *co-prime*. This condition holds iff m and n are co-prime for every m and n in $DI(f)$ and $DI(g)$ respectively.

THEOREM 5.3. *Let $f_i : X_i \to X_i$ be relatively mixing for every $i \geq 1$. If $DI(f_i)$ and $DI(f_j)$ are co-prime for $i \neq j$, then $f = \prod_{i=1}^{\infty} f_i$ is transitive and $DI(f_i) \subseteq DI(f)$ for $i \geq 1$.*

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Proof. We denote the terminal decomposition for f_i by $\mathcal{D}^i = \{D_0^i, D_1^i, \ldots, D_{m_i-1}^i\}$ so that $\mathcal{D}^i$ has length m_i. Let $U = U_1 \times \cdots \times U_\alpha \times X_{\alpha+1} \times X_{\alpha+2} \times \cdots$ and $V = V_1 \times \cdots \times V_\beta \times X_{\beta+1} \times X_{\beta+2} \times \cdots$ be basic open sets in X. For each $i \leq \beta$, there are $s_i, t_i \leq m_i$ such that both $G_i = U_i \cap \text{int}(D_{s_i}^i)$ and $V_i \cap \text{int}(D_{t_i}^i)$ are non-empty open subsets of X_i. Putting $r_i = t_i - s_i \pmod{m_i}$ one sees that both G_i and $H_i = f_i^{-r_i}(V_i \cap \text{int}(D_{t_i}^i))$ are non-empty open subsets of $D_{s_i}^i$. Since $f_i^{m_i}$ is mixing on $D_{s_i}^i$, there is an $N_i \geq 1$ such that for all $n \geq N_i$, one has $f^{nm_i}(G_i) \cap H_i \neq \varnothing$. Let $N = \max\{N_i m_i : i \leq \beta\}$. Since the m_i are pairwise co-prime for $i \leq \beta$ the additive group $\prod_{i=1}^{\beta} \mathbb{Z}_{m_i}$ is cyclic with generator $(\underbrace{1, 1, \ldots, 1}_{\beta})$. There is therefore a $k \geq N$ such that $k \pmod{m_i} = r_i$ for $i \leq \beta$. Thus $k = q_i m_i + r_i$ for some $q_i \geq N_i$. Since $f_i^{q_i m_i}(G_i) \cap H_i \neq \varnothing$ for all $i \leq \beta$, one has

$$\varnothing \neq f_i^{r_i}(f_i^{q_i m_i}(G_i)) \cap f_i^{r_i}(H_i) = f_i^k(G_i) \cap f_i^{r_i}(f_i^{-r_i}(V_i \cap \text{int}(D_{k_i}^i))) \subseteq f_i^k(U_i) \cap V_i$$

and for $i > \beta$, one has $f_i^k(U_i) \cap V_i = f_i^k(U_i) \cap X_i \neq \varnothing$. It follows that $f^k(U) \cap V \neq \varnothing$ and f is transitive. To prove that $DI(f_i) \subseteq DI(f)$ for $i \geq 1$ one may show that the collection $\mathcal{D} = \{D_0, D_1, \ldots, D_{m_i-1}\}$, where $D_j = X_1 \times X_2 \times \cdots \times X_{i-1} \times D_j^i \times X_{i+1} \times \cdots$, is a regular periodic decomposition for f as in Theorem 5.1. □

One may use the maps Q_n defined earlier to construct examples of maps on connected spaces with infinite decomposition ideals. Let $(a_1, a_2, a_3, \ldots)$ be any sequence of pairwise co-prime integers with each $a_i > 1$. Then $Q = \prod_{i=1}^{\infty} Q_{a_i}$ is transitive with infinite decomposition ideal by Theorem 5.3. The domain of Q is compact, connected, locally connected, path connected, locally path-connected, metrizable and homogeneous. This suggests that it will be difficult to find elementary topological features which prevent a space from admitting transitive maps with infinite decomposition ideals. In §7 we will see that such restrictions may exist for specific spaces.

6. *Topology and dynamics of decompositions*

A regular periodic decomposition will be called *connected* if each element of the decomposition is connected. Thus a *connected refinement* is a refinement in which each element is connected. For example, the circle map Q_6 of §5 has a connected terminal decomposition comprising of 6 closed arcs. The elements of the decompositions of lengths 2 and 3 are not connected, since they are unions of 3 and 2 disjoint closed arcs respectively, but they have a mutual connected refinement of length 6. The existence of this connected refinement is no accident. Recall that a space is *locally connected* if every neighbourhood of every point contains a connected neighbourhood of that point and that in such a space, the connected components of any open set are open.

THEOREM 6.1. *Let $f : X \to X$ be transitive, where X is locally connected. Then every regular periodic decomposition for f has a connected refinement.*

Proof. Let $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ be any regular periodic decomposition for f. Since X is locally connected and D_0 has non-empty interior there is a connected component of D_0 with non-empty interior in X. Theorem 2.1 shows that f^n is transitive on D_0, so by Theorem 2.6 there are finitely many components in D_0 and they form a regular

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periodic decomposition $\mathcal{C} = \{C_0, C_1, \ldots, C_{m-1}\}$ for f^n on D_0. Lemma 3.3 asserts that $\mathcal{E} = \{\bar{G}, \overline{f^{-(mn-1)}(G)}, \overline{f^{-(mn-2)}(G)}, \ldots, \overline{f^{-1}(G)}\}$ is a regular periodic decomposition for f on X, where $G = \text{int}_X(C_0)$, and that $\mathcal{E}$ is a refinement of $\mathcal{D}$. Since C_0 is regular closed in X, we see that $\bar{G} = C_0$, so $\bar{G}$ is connected. By Lemma 2.1(i) each element of $\mathcal{E}$ is the closure of an image of $\bar{G}$ under an iterate of f, whence each element of $\mathcal{E}$ is connected. □

COROLLARY 6.1. *A terminal decomposition for a transitive map on a locally connected space is connected.*

The following result on compactness of elements of regular periodic decompositions in Hausdorff spaces is easily established.

THEOREM 6.2. *Let $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ be a regular periodic decomposition for $f : X \to X$, where X is Hausdorff. Either X and every D_i is compact or X and every D_i is non-compact.*

There are many restrictions upon the way in which elements of a regular periodic decomposition can intersect one another. One type of restriction is imposed by existence of periodic orbits.

LEMMA 6.1. *Let $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$ be a regular periodic decomposition for a transitive map f and let p be a periodic point whose period is co-prime with n. Then $O(p) \subseteq D = \bigcap_{i=0}^{n-1} D_i$.*

Proof. Without loss of generality, we assume that $p \in D_0$. Since the period m of p is co-prime with n, there is a $0 \leq j \leq n-1$ such that $i = jm \pmod{n}$, whence $p = f^{jm}(p) \in f^i(D_0) \subseteq D_i$ and hence $p \in D$. Similar arguments show that the other elements of $O(p)$ are also in D. □

THEOREM 6.3. *Let $f : X \to X$ be a transitive map with a fixed point c and let X be locally connected. Then any regular periodic decomposition $\mathcal{D}$ for f is connected.*

Proof. By Theorem 6.1, $\mathcal{D}$ has a connected refinement $\mathcal{C}$ and by Lemma 6.1, every element of $\mathcal{C}$ contains c. Each $D \in \mathcal{D}$ is the union of its children in $\mathcal{C}$ by Lemma 3.1 and since the children of D in $\mathcal{C}$ are connected and have common point c, we conclude that $\mathcal{D}$ is connected. □

Another restriction on the intersections which can occur for a regular periodic decomposition concerns the relationship between elements of the decomposition and its refinements.

THEOREM 6.4. *Let f be transitive and let $\mathcal{C}$ be a refinement of a regular periodic decomposition $\mathcal{D} = \{D_0, D_1, \ldots, D_{n-1}\}$. Then $C \in \mathcal{C}$ intersects D_j iff the parent of C intersects D_j.*

Proof. If C intersects D_j then so does its parent, since C is a subset of its parent. Conversely, suppose that D_k intersects D_j and let $\mathcal{E} = \{C \in \mathcal{C} : C \subseteq D_k$ and

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$C \cap D_j \neq \varnothing\}$. Since D_k intersects D_j and D_k is the union of its children in $\mathcal{C}$ by Lemma 3.1, one of the children of D_k must also intersect D_j, whence $\mathcal{E} \neq \varnothing$. For any $C \in \mathcal{E}$ one has $f^n(C) \subseteq f^n(D_k) \subseteq D_k$ and $\varnothing \neq f^n(C \cap D_j) \subseteq f^n(C) \cap D_j$, so the element of $\mathcal{C}$ containing $f^n(C)$ is also an element of $\mathcal{E}$. This implies that $B = \bigcup_{C \in \mathcal{E}} C$ is invariant under f^n. Since B is a closed invariant subset of D_k with non-empty interior and f^n is transitive on D_k, we conclude that $B = D_k$. Since the interiors of the elements of $\mathcal{C}$ are disjoint it follows that every element of $\mathcal{C}$ having D_k as a parent is also in $\mathcal{E}$. □

7. *One-dimensional manifolds*

One may enumerate all of the possible regular periodic decompositions and decomposition ideals of maps of the four topological 1-manifolds. Most transitive maps on these spaces are relatively mixing. Many of the results of this section, or results implying them, already appear in the literature using different terminology (see [**CM**], [**BC**], [**BM1**] and [**BM2**]). The approach taken here emphasises the role of regular periodic decompositions and uses results of [**Si**] in conjunction with Theorem 1.1 and its corollary to obtain simpler proofs.

Suppose a transitive map f on a half-open interval $[a, b)$ has a non-trivial regular periodic decomposition. By Theorem 6.1 there is a connected refinement $\{I_0, I_1, \ldots, I_{n-1}\}$, where each I_j is an interval which is closed in $[a, b)$. Hence each I_j is compact, except for one of the form $[c, b)$ for some $c > a$. As this contradicts Theorem 6.2, we conclude that f is totally transitive. If f has a non-trivial regular periodic decomposition on an open interval (a, b) there must again be a connected refinement $\{I_0, I_1, \ldots, I_{n-1}\}$, this time containing precisely two non-compact intervals of the forms $(a, c]$ and $[d, b)$. Since the assumption that $n > 2$ would contradict Theorem 6.2, we conclude that the decomposition is precisely $\{(a, c], [d, b)\}$. Finally, consider a transitive map f on a compact interval $[a, b]$. Again, the existence of a regular periodic decomposition of length $m > 1$ yields a connected refinement $\{I_0, I_1, \ldots, I_{n-1}\}$ and the possibility that $n > 2$ is compatible with Theorem 6.2 since each I_j is compact. In this case, however, f must have a fixed point which would be in each I_j by Lemma 6.1. This would be impossible were $n > 2$. We summarise these results as a theorem.

THEOREM 7.1. *The decomposition ideal of a transitive map of a compact interval or an open interval is either* $\{1\}$ *or* $\{1, 2\}$. *The decomposition ideal of a transitive map of a half open interval is* $\{1\}$.

One may find examples of maps exhibiting all of the possible decomposition ideals for interval maps suggested by the theorem. The result for compact intervals may alternatively be deduced from a result in [**BC**, p. 156] with a little extra work. Similar techniques to [**BC**, p. 157] prove the following lemma which enables us to show that every transitive map on any subinterval of the real line is relatively mixing. The proof may be simplified by the use of Theorem 1.1 and Silvermans's result [**Si**, p. 357] that a transitive map on any interval has a dense set of periodic points (see [**Ba**]).

LEMMA 7.1. *A totally transitive map on an interval* I *is mixing.*

THEOREM 7.2. *Every transitive map* f *on an interval* I *is relatively mixing.*

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Proof. By Theorem 7.1, f has a terminal decomposition consisting of $n \leq 2$ intervals. Since f^n is totally transitive on each of these intervals, Lemma 7.1 asserts that f^n is mixing on each of them. □

THEOREM 7.3. *The decomposition ideal of a transitive map $f : S^1 \to S^1$ is finite.*

Proof. Suppose f has a regular periodic decomposition $\mathcal{D}$ of length $m \geq 3$. Then $\mathcal{D}$ has a connected refinement $\mathcal{A}$ of length $n \geq 3$. Clearly, each $A \in \mathcal{A}$ must be a closed arc and must intersect precisely two other arcs in $\mathcal{A}$, which we denote by A_- and A_+. Any refinement of $\mathcal{A}$ has a connected refinement $\mathcal{C}$. Now A is the parent of some element C of $\mathcal{C}$ and C must intersect both A_- and A_+ by Theorem 6.4. Since C is connected, we have $C = A$. Hence $\mathcal{A} = \mathcal{C}$, so $\mathcal{A}$ is terminal. □

One could alternatively have proven Theorem 7.3 by noting that since each element A of $\mathcal{A}$ is homeomorphic to a compact interval and f^n is transitive on A, Theorem 7.1 shows that f^n has a terminal decomposition of length either 1 or 2 on A. This would yield a terminal decomposition of length n or $2n$ for f. The above proof shows that there cannot be a terminal decomposition of length $2n$ in this situation. The following lemma, which enables us to prove an analogue of Theorem 7.2 for circle maps, is a consequence of a main theorem in [**CM**] but the proof given here is more elementary.

LEMMA 7.2. *A totally transitive map $f : S^1 \to S^1$ with a dense set of periodic points is mixing.*

Proof. Let J and K be non-empty open arcs and let p and q be periodic points in K with distinct orbits. If $f^N(J) \supseteq O(p)$ for some $N \geq 1$, then $f^n(J) \cap K \neq \varnothing$ for all $n \geq N$, so we assume that $f^N(J)$ does not contain $O(p)$ for any $N \geq 1$. Let m be the period of p. If $s \in O(p) \cap f^{jm}(J)$ for some $j \geq 1$, one has $s = f^{(i-j)m}(s) \in f^{im}(J)$ for all $i \geq j$. Therefore, since $f^{jm}(J)$ does not contain $O(p)$ for any $j \geq 1$, there exists $s \in O(p)$ which is not in $f^{jm}(J)$ for any $j \geq 1$. There is an open arc U containing s which does not intersect $O(q)$, and $U \setminus \{s\}$ consists of two non-empty open arcs V and W. Since f^m is weakly mixing by Corollary 1.1, there is a $k \geq 1$ such that $f^{km}(J)$ intersects V and W. Since $f^{km}(J)$ is connected and does not contain s, it must contain $S^1 \setminus U$. Since $S^1 \setminus U$ contains $O(q)$ and $q \in K$, we have $f^n(J) \cap K \neq \varnothing$ for all $n \geq km$. □

THEOREM 7.4. *A transitive map f of the circle is relatively mixing iff it is not conjugate to an irrational rotation.*

Proof. If f is not totally transitive, it is not conjugate to an irrational rotation and it has a connected terminal regular periodic decomposition $\mathcal{A}$ consisting of n closed arcs by Theorem 7.3. Now f^n is totally transitive on each element of $\mathcal{A}$ by Theorem 3.1 and hence mixing by Lemma 7.1. If f is totally transitive but not conjugate to an irrational rotation, then f has a dense set of periodic points by [**Si**, p. 367] and is mixing by Lemma 7.2. It is well known that irrational rotations are not mixing. □

The results of this section all hinge on the fact that the proper connected regular closed subsets of 1-manifolds are precisely the closed non-degenerate intervals. For 2-manifolds one has no such simple characterisation of connected regular closed sets. In

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this case, the topology of regular closed sets can be very complicated indeed. It is the author's conjecture, nonetheless, that Theorem 7.3 holds at least for 2-manifolds. It is conceivable that it holds for all finite-dimensional manifolds. The theorems of §6 give some information about regular periodic decompositions for transitive maps on manifolds. They must always have connected refinements by Theorem 6.1. Theorem 6.4 places strong restrictions upon the way in which the children of an element of a decomposition are positioned. For a transitive map with a fixed point, Theorem 6.3 guarantees that each regular periodic decomposition is connected and Lemma 6.1 guarantees that all of its elements have a common point. By the Brouwer fixed point theorem, this will apply to all transitive maps on B^n for any $n \geq 2$. Many other results of this type are possible and may shed some light on the problem of whether transitive maps on finite-dimensional manifolds can have infinite decomposition ideals.

REFERENCES

[Ba] J. Banks. Regular periodic decompositions for topologically transitive maps. *PhD Thesis*, La Trobe University, Australia, 1994.

[BBCDS] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey. On Devaney's definition of chaos. *Amer. Math. Monthly* **99** (1992), 332–334.

[BC] L. S. Block and W. A. Coppel. *Dynamics in One Dimension (Lecture Notes in Mathematics 1513)*. Springer, Berlin, 1992.

[Bl] R. W. Blackett. *Elementary Topology*. Academic, New York, 1982.

[BM1] M. Barge and J. Martin. Chaos periodicity and snakelike continua. *Trans. Amer. Math. Soc.* **289** (1985), 355–365.

[BM2] M. Barge and J. Martin. Dense periodicity on the interval. *Proc. Amer. Math. Soc.* **94** (1985), 731–735.

[BS] J. Buescu and I. Stewart. Liapunov stability and adding machines. *Ergod. Th. & Dynam. Sys.* **15** (1995), 271–290.

[CM] E. M. Coven and I. Mulvey. Transitivity and the centre for maps of the circle. *Ergod. Th. & Dynam. Sys.* **6** (1986), 1–8.

[De] R. L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley, Menlo Park, California, 1989.

[DP] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.

[Fu] H. Furstenberg. Disjointness in ergodic theory, minimal sets and a problem in diophantine approximation. *Math. Systems Theory* **1** (1967), 1–49.

[HR] E. Hewitt and K. Ross. *Abstract Harmonic Analysis*. Springer, Berlin, 1963.

[KS] R. C. Kirby and L. C. Siebenmann. *Foundational Essays in Topological Manifolds, Smoothings and Triangulations*. Princeton University Press, 1977.

[Ma] R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer, Berlin, 1987.

[Mi1] J. Milnor. On the concept of attractor. *Commun. Math. Phys.* **99** (1985), 177–195.

[Mi2] J. Milnor. On the concept of attractor: correction and remarks. *Commun. Math. Phys.* **102** (1985), 517–519.

[PP] J. Peters and T. Pennings. Chaotic extensions of dynamical systems by function algebras. *J. Math. Anal. Appl.* **159** (1991), 345–360.

[PS] K. Petersen and L. Shapiro. Induced flows. *Trans. Amer. Math. Soc.* **177** (1973), 375–390.

[Si] S. Silverman. On maps with dense orbits and the definition of chaos. *Rocky Mountain J. Math.* **22** (1992), 353–375.

[St] P. Stacey. Private communication. Unpublished, 1992.

https://doi.org/10.1017/S0143385797069885 Published online by Cambridge University Press