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On G_2-manifolds and geometry in dimensions 6 and 8

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Abstract
We study the geometry induced on the local orbit spaces of Killing vector fields on (Riemannian) G-manifolds, with an emphasis on the cases $G = \mathrm{Spin}(7)$ and $G = G_2$. Along the way, we classify the harmonic morphisms with one-dimensional fibres from G_2-manifolds to Einstein manifolds.

1. Introduction

It seems that, yet, there are no simple constructions of Riemannian manifolds with exceptional holonomy (but see [8] and [9]; also, see [4] and [19] for basic facts on these manifolds, and, also, Section 3, below, for fairly explicit descriptions of the exceptional holonomy Lie algebras representations). To us, the archetypical (simple) construction of a 'special' Riemannian metric is the classical Gibbons-Hawking construction (see [16], and the references therein) that characterises (for example), locally, the Ricci-flat anti-self-dual (equivalently, four-dimensional hyper-Kähler) manifolds endowed with a nowhere zero Killing vector field.

We are, thus, led to the study of the geometry induced on the local orbit spaces of Killing vector fields on G_2-manifolds and Spin(7)-manifolds (compare [1], [10], [11]). However, it is useful to start from a more general setting (see Section 2, below) whose easiest relevant case is provided by the, already mentioned, Ricci-flat anti-self-dual manifolds.

In Section 3, we present some results specific to dimension 6 such as (Proposition 2.1) the fact that, with respect to the orthogonal reductive decomposition $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{m}$, the adjoint representation of $\mathfrak{su}(3)$ on $\mathfrak{m}$ is equivalent to its canonical representation on $\mathbb{C}^3$ (this, presumably known, fact led us to the useful setting of Section 2). The results presented in that section are, firstly, applied to characterize the harmonic morphisms from G_2-manifolds to Einstein manifolds of dimension 6 (Corollary 2.4).

The results of Sections 2 and 3 are applied, in Sections 4 and 5, to obtain a better understanding of Killing vector fields on Riemannian manifolds with exceptional holonomy.

It is fairly well known that the (2-)harmonic morphisms are in the background of the Gibbons-Hawking construction. Here, (Proposition 3.5 and Remark 4.3(1)) we have 4-harmonic morphisms (see [2] for more on this notion).

In Section 6, some results on Killing vector fields on special Kähler manifolds are given.

Other work, similarly motivated, can be found in [1], [5], [7], [10] and [11]; see, also, [12] and [13].

2. A useful setting

In this paper, for simplicity, we work in the setting of complex-Riemannian manifolds (the 'real setting' means that we deal with (complexifications of) real-analytic Riemannian manifolds (note that, semi-Riemannian manifolds could be, also, considered)). For simplicity, we omit the word 'complex'.

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Let (P, M, G) be a principal bundle and let E be a vector bundle over M endowed with a (vector bundles) morphism $\rho{:}E \to TM$. Recall [15] that, by a ρ-connection on (P, M, G), we mean a morphism from E to TP/G, which, when composed with the canonical morphism from TP/G to TM, gives ρ.

We are interested in the following setting.

Remark 1.1. *Let $\mathfrak{g}$ be a Lie algebra endowed with a faithful orthogonal representation on an Euclidean space U and a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathfrak{m} \subseteq U$ is nondegenerate, and the induced representation of $\mathfrak{h}$ on $\mathfrak{m}$ is the restriction of the given representation of $\mathfrak{g}$ on U.*

Let G and H be the simply-connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then, if M is a manifold endowed with a reduction of its frame bundle to H through its representation on $\mathfrak{m}$ we, also, have a vector bundle E over M, with structural group G and typical fibre U, endowed with a morphism of vector bundles $\rho{:}E \to TM$ such that the following two facts hold:

- *Any ρ-connection on E corresponds to a connection on E and a section of $(\ker\rho)^* \otimes \operatorname{End} E$ (given by the orthogonal decomposition $U = \mathfrak{m} \oplus \mathfrak{m}^\perp$).*
- *Any connection on E (compatible with its structural group) corresponds to a connection on M and a $(1, 1)$-tensor field on M.*

A significant particular case of Remark 1.1 is the following. Let $\mathfrak{g} \subseteq \mathfrak{so}(n+1)$ be a Lie algebra endowed with a faithful orthogonal representation of dimension $n+1$, and let $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{so}(n)$, where $\mathfrak{so}(n) \subseteq \mathfrak{so}(n+1)$ is the Lie subalgebra preserving some nondegenerate hyperspace, where $n \in \mathbb{N} \setminus \{0\}$. Assuming $\mathfrak{g}$ nondegenerate, with respect to the Killing form of $\mathfrak{so}(n+1)$, and denoting by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, we obtain a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

We, further, assume $\dim \mathfrak{m} = n$ and the representation of $\mathfrak{h}$ on $\mathfrak{m}$, induced by the adjoint representation of $\mathfrak{g}$, be (up to some nonzero factor) the given $\mathfrak{h} \subseteq \mathfrak{so}(n)$. Therefore, there exists a linear embedding $\mathfrak{m} \to \mathfrak{g} \subseteq \mathfrak{so}(n+1)$, given by

$$x \mapsto \begin{pmatrix} h(x) & x \\ -x^T & 0 \end{pmatrix},$$

for any $x \in \mathfrak{m}$, where $h{:}\mathfrak{m} \to \mathfrak{so}(n)$ is a linear map such that $[A, h(x)] = h(Ax)$, for any $A \in \mathfrak{h}$ and $x \in \mathfrak{m}$.

Therefore, if $\widetilde{A} = (A, x) \in \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then, as an element of $\mathfrak{so}(n+1)$,

$$\widetilde{A} = \begin{pmatrix} A + h(x) & x \\ -x^T & 0 \end{pmatrix}.$$

Now, we use Remark 1.1, with M a manifold endowed with an almost H-structure and $E = T(L \setminus 0)/(\mathbb{C} \setminus \{0\})$, where L is a line bundle over M endowed with a connection. In particular, $E = TM \oplus \mathbb{C}$, and we denote by $\mathbb{1}$ the section of E corresponding to $(0, 1)$, and by u the nowhere zero function on M such that $u\mathbb{1}$ is given by the infinitesimal generator of the action of $\mathbb{C} \setminus \{0\}$ on L.

It follows that any ρ-connection $\widetilde{\nabla}$ on E, where $\rho{:}E \to TM$ is the projection, corresponds to a ρ-connection ∇ on TM, and a section γ of $\operatorname{Hom}(E, TM)$. These are related through the involved section h of $\operatorname{Hom}(TM, \operatorname{End}(TM))$, as follows.

$$\begin{aligned}
&\widetilde{\nabla}_X Y = \nabla_X Y + h(\gamma(X))Y - <\gamma(X), Y>\mathbb{1},\\
&\widetilde{\nabla}_X \mathbb{1} = \gamma(X),\\
&\widetilde{\nabla}_{\mathbb{1}} X = \nabla_{\mathbb{1}} X + h(\gamma(\mathbb{1}))X - <\gamma(\mathbb{1}), X>\mathbb{1},\\
&\widetilde{\nabla}_{\mathbb{1}} \mathbb{1} = \gamma(\mathbb{1}),
\end{aligned} \tag{1.1}$$

for any (local) vector fields X and Y on M, where $<\cdot,\cdot>$ denotes the Riemannian metric on M (and where, for simplicity, we did not mention the compatibilities with the structural groups; for example, h is, in fact, a 1-form on M with values in the adjoint bundle of the underlying Riemannian structure on M, which is equivariant with respect to the actions of the adjoint bundle of the almost H-structure on M, where H is the simply-connected Lie group with Lie algebra $\mathfrak{h}$).

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Proposition 1.2. *The torsion $\widetilde{T}$ of $\widetilde{\nabla}$ is given by*

$$\widetilde{T}(X,Y) = T(X,Y) + h(\gamma(X))Y - h(\gamma(Y))X + \big(u\Omega(X,Y) - <\gamma(X),Y> + <\gamma(Y),X>\big)\mathbb{1},$$
$$\widetilde{T}(X,\mathbb{1}) = \gamma(X) - \nabla_{\mathbb{1}}X - h(\gamma(\mathbb{1}))X + \big(u^{-1}X(u) + <\gamma(\mathbb{1}),X>\big)\mathbb{1},$$

for any $X, Y \in TM$, where T is the torsion of the connection given by ∇ (and the connection on L), and Ω is the curvature form of the connection on L.

Proof. This follows quickly from (1.1). □

Let $G \subseteq O(k)$ be a Lie subgroup and M a Riemannian manifold, $\dim M = k$, $k \in \mathbb{N} \setminus \{0\}$. We say that M is a G-manifold if its orthonormal frame bundle admits a reduction (P, M, G) endowed with a connection that gives the Levi-Civita connection of M. Note that, without the connection assumption, we retrieve the basic notion of 'almost G-structure'.

We consider only nowhere isotropic Killing vector fields V on G-manifolds M which are infinitesimal automorphisms of the given G-structure; that is, their flow preserves the reduction (P, M, G) and its connection. We, further, assume that the frames from P whose first vector is given by V (locally) normalised form a principal bundle preserved by the local flow of V.

Let $\mathbb{C}^m$, $(m \in \mathbb{N} \setminus \{0\})$, be endowed with its canonical Riemannian structure, given by the canonical symmetric bilinear form $<\cdot,\cdot>$. The unit sphere, of dimension $m-1$, is the submanifold of $\mathbb{C}^m$ formed of those $u \in \mathbb{C}^m$ such that $<u,u>=1$.

Theorem 1.3. *Let M be a G-manifold, with G connected and acting transitively on the unit sphere. Let $H \subseteq G$ be the connected Lie subgroup whose Lie algebra is formed of those matrices of the Lie algebra of G whose first rows and columns are zero.*

Let M be endowed with a Killing vector field such that H is the structural group of the bundle of frames whose first vector is given by V normalised.

Then, locally, M is of the form $N \times \mathbb{C}$ with the metric $<\cdot,\cdot> + u^2(\mathrm{d}t + A)^2$, where $(N, <\cdot,\cdot>)$ is an almost H-manifold endowed with a compatible connection ∇, a section $\mathcal{B}$ of its adjoint bundle, a nowhere zero function u, and a 1-form A such that, on denoting by γ, the $(1,1)$-tensor field on N given by $\gamma(X) = \mathcal{B}X - u^{-1}h(\operatorname{grad} u)X$, for any $X \in TN$, the following relations hold

$$T(X,Y) = -h(\gamma(X))Y + h(\gamma(Y))X,$$
$$\mathrm{d}A(X,Y) = 2u^{-1} <\gamma(X), Y>, \tag{1.2}$$

for any $X, Y \in TN$, where T is the torsion of ∇; in particular, $\nabla + h \circ \gamma$ is the Levi-Civita connection of $(N, <\cdot,\cdot>)$.

Proof. This follows quickly from Propositions 1.2, where the ρ-connection (which we denote by the same ∇) is given by the connection ∇ and $\mathcal{B}$. □

3. Basic facts in the geometry in dimension 6

The starting point of this section is the following known fact (see [6]), where $\mathfrak{g}_2$ is the simple Lie algebra of dimension 14 (see [17] for an explicit description and, also, the proof of Proposition 2.1, below).

Proposition 2.1. *Let $\mathfrak{g}_2 \subseteq \mathfrak{so}(7)$ be given by the fundamental representation of dimension 7 of $\mathfrak{g}_2$. Then under the equality of Lie algebras $\mathfrak{sl}(3) = \mathfrak{g}_2 \cap \mathfrak{so}(6)$, the (adjoint) representation of $\mathfrak{sl}(3)$ on its orthogonal complement in $\mathfrak{g}_2$ can be identified with the representation induced by the canonical representation of $\mathfrak{so}(6)$ (and with the direct sum of its 3-dimensional fundamental representations).*

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Proof. We have to show that we are in the setting of Section 2 with $\mathfrak{g} = \mathfrak{g}_2 \subseteq \mathfrak{so}(7)$, and $\mathfrak{sl}(3) = \mathfrak{g}_2 \cap \mathfrak{so}(6)$.

We choose to deduce this by giving an explicit description, in matrix form, of the Lie algebras embeddings $\mathfrak{sl}(3) \subseteq \mathfrak{g}_2 \subseteq \mathfrak{so}(7)$. For this, if $x \in \mathbb{C}^3$, we denote by $[x]$ the element of $\mathfrak{so}(3)$ such that $[x]y = x \times y$, for any $y \in \mathbb{C}^3$, where $\times$ denotes the quaternionic cross product (the well known explicit description of the isomorphism of Lie algebras between $(\mathbb{C}^3, \times)$ and $\mathfrak{so}(3)$). Also, we use the characterisation of $\mathfrak{sl}(3) \subseteq \mathfrak{so}(6)$ as formed of all the matrices $\begin{pmatrix} [x] & -y \\ y & [x] \end{pmatrix}$, with $x \in \mathbb{C}^3$ and y a 3×3 trace-free symmetric matrix.

Now, we use the obvious embedding of $\mathfrak{so}(6)$ into $\mathfrak{so}(7)$ whose image has the fourths' rows and columns equal to zero. Also, we embed $\mathbb{C}^6$ into $\mathfrak{so}(7)$ as follows: to any $(a, b) \in \mathbb{C}^6 = \mathbb{C}^3 \oplus \mathbb{C}^3$, we associate $\{a\}^1 + \{b\}^2$, where

$$\{a\}^1 = \begin{pmatrix} 0 & 2a & [a] \\ -2a^T & 0 & 0 \\ [a] & 0 & 0 \end{pmatrix}, \quad \{b\}^2 = \begin{pmatrix} [b] & 0 & 0 \\ 0 & 0 & -2b^T \\ 0 & 2b & -[b] \end{pmatrix}.$$

From [17], we deduce that $\mathfrak{g}_2$ is, inside $\mathfrak{so}(7)$, the direct sum of $\mathfrak{sl}(3)$ and $\mathbb{C}^6$, and the proof follows. □

Remark 2.2. *1) There is another way to look at Proposition 2.1. We have* $\mathfrak{so}(6) + \mathfrak{g}_2 = \mathfrak{so}(7)$, *and* $\mathfrak{so}(6) \cap \mathfrak{g}_2 = \mathfrak{sl}(3)$. *Consequently, the representation of* $\mathfrak{sl}(3)$ *on its orthogonal complement in* $\mathfrak{g}_2$ *is given, by restriction, by the representation of* $\mathfrak{so}(6)$ *on its orthogonal complement in* $\mathfrak{so}(7)$.

2) Similarly, let $\mathfrak{so}(7)_0$ *and* $\mathfrak{so}(7)_1$ *be the images of the embeddings of* $\mathfrak{so}(7)$ *into* $\mathfrak{so}(8)$ *given by the canonical representation and the fundamental representation of dimension* 8 *of* $\mathfrak{so}(7)$, *respectively.*

Then $\mathfrak{so}(7)_0 + \mathfrak{so}(7)_1 = \mathfrak{so}(8)$, *and* $\mathfrak{so}(7)_0 \cap \mathfrak{so}(7)_1 = \mathfrak{g}_2$. *Moreover, both embeddings of* $\mathfrak{g}_2$ *into* $\mathfrak{so}(7)_j$, $j = 0, 1$, *are given by the fundamental representation of dimension* 7 *of* $\mathfrak{g}_2$. *Consequently, the representation of* $\mathfrak{so}(7)_1$ *on* $\mathfrak{so}(8)$ *decomposes as the direct sum of its adjoint and canonical representations. Furthermore, it, also, follows that the reductive decompositions induced by the embeddings of* $\mathfrak{so}(7)_j$ *into* $\mathfrak{so}(8)$, $j = 0, 1$, *are the same (symmetric decomposition).*

3) Also, the embedding of $\mathfrak{so}(7)$ *into* $\mathfrak{so}(8)$, *corresponding to the* 8*-dimensional fundamental representation of the former, admits an explicit description, similar to the one given, in Proposition 2.1, for* $\mathfrak{g}_2 \subseteq \mathfrak{so}(7)$. *This follows, for example, from the latter and the formula for h given in Section 5, below.*

Let N be a Riemannian manifold of dimension 6 endowed with a reduction P of its orthonormal frame bundle to GL(3), corresponding to an almost Hermitian structure J. Similarly to Remark 1.1, the Levi-Civita connection ∇ of N corresponds to a connection on P and a $(1, 1)$-tensor field A on N.

Then (N, J) is nearly-Kähler (that is, $(\nabla_X J)(X) = 0$, for any $X \in TN$) if and only if, pointwisely, A is in the space generated by Id_{TN} and J.

Unless otherwise stated, any nondegenerate hypersurface in an almost G_2-manifold will be considered endowed with the almost Hermitian structure given (up to sign) by the octonionic cross product.

Corollary 2.3 ([5]). *A nondegenerate hypersurface of a* G_2*-manifold is nearly-Kähler/Kähler if and only if it is umbilical/geodesic.*

Proof. This is a straightforward consequence of Proposition 2.1. □

Corollary 2.4. *Let* φ *be a real harmonic morphism with* 1*-dimensional fibres from a* G_2*-manifold to an Einstein manifold.*

Then, the codomain of φ *is nearly-Kähler and its fibres are geodesics orthogonal to an umbilical foliation by hypersurfaces.*

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Proof. This follows from [18] and Corollary 2.3. □

4. Killing vector fields on G_2-manifolds

Approaches different to the one of this section can be found in [1] and [11].

To give an almost SL(3)-structure, through the direct sum of its fundamental representations, on a six-dimensional Riemannian manifold N is the same as to specify the bundle of nondegenerate 'associative' spaces contained by TN. Here, associativity is induced by the octonionic cross product on the bundle $E = TN \oplus \mathbb{C}$, obtained through the setting of Section 2 (and with M replaced by N), by taking $\mathfrak{g} = \mathfrak{g}_2$ and $\mathfrak{h} = \mathfrak{sl}(3)$ (see, also, Proposition 2.1 whose explicit formulae for the embedding $\mathfrak{sl}(3) \subseteq \mathfrak{g}_2$ require a choice of a nondegenerate associative space).

Corollary 3.1. *Any G_2-manifold endowed with a Killing vector field is locally of the form $N \times \mathbb{C}$ with the metric $< \cdot, \cdot > + u^2(\mathrm{d}t + A)^2$, where $(N, < \cdot, \cdot >)$ is an almost* SL(3)*-manifold endowed with a compatible connection ∇, a section $\mathcal{B}$ of its adjoint bundle, a nowhere zero function u, and a* 1*-form A such that, on denoting by γ the* (1, 1)*-tensor field on N given by $\gamma(X) = \mathcal{B}X - u^{-1}h(\mathrm{grad}\, u)X$, for any $X \in TN$, the following conditions hold*

(i) $\nabla + h \circ \gamma$ is the Levi-Civita connection of $(N, < \cdot, \cdot >)$,
(ii) $\mathrm{d}A(X, Y) = 2u^{-1} < \gamma(X), Y >$, for any $X, Y \in TN$.

Proof. This follows quickly from Theorem 1.3 and Proposition 2.1. □

Remark 3.2. *In Corollary 3.1, ∇ is the Levi-Civita connection of N if and only if the Killing vector field corresponds, locally, to isometries between M and the Riemannian product of special Kähler (equivalently, Calabi-Yau) manifolds and open subsets of the complex line (then u is, necessarily, constant). Moreover, this is equivalent to (a) the integrability of the orthogonal complement distribution to the Killing vector field, and, in the real setting (similarly to Corollary 2.4), to (b) the orbits of the Killing vector field be geodesics.*

A similar statement holds in the setting of Theorem 1.3 if we assume $h{:}\mathfrak{m} \to \mathfrak{so}(n)$ injective. For example, by taking (1) $G = \mathrm{Spin}(7)$, with its fundamental representation of dimension 8*, and $H = G_2$, (2) $G = \mathrm{SL}(2)$ and H trivial, with G endowed with the representation on $\mathfrak{gl}(2)$, given by left multiplication. In the latter case, with the same notations as in Theorem 1.3, we have $h(X)Y = X \times Y$, for any $X, Y \in TN$, and $\gamma(X) = -u^{-1}h(\mathrm{grad}\, u)X$, for any $X \in TN$; consequently, the first relation of (1.2) is equivalent to the fact that $u^2 < \cdot, \cdot >$ is flat. Thus, in this case, Theorem 1.3 reduces to the classical Gibbons-Hawking construction (see [16], and the references therein).*

See Section 5, below, for the relevant h in case (1).

To make the conditions of Corollary 3.1 more explicit, we endow the tangent bundle of N with an 'associative (orthogonal) decomposition' $TN = T^+N \oplus T^-N$. That is, an orthogonal decomposition such that the horizontal lift of T^+N is nondegenerate and closed under the octonionic cross product.

Then, we can write $\mathcal{B} = \begin{pmatrix} [b] & -B \\ B & [b] \end{pmatrix}$, where b is a section of T^+N and B is a trace-free self-adjoint section of $\mathrm{Hom}(T^+N, T^-N)$, where we use the orientation preserving isometry between T^+N and T^-N given by the octonionic cross product on $TN \oplus \mathbb{C}$.

Also, we have $h\begin{pmatrix} X_+ \\ X_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} [X_-] & [X_+] \\ [X_+] & -[X_-] \end{pmatrix}$, for any $X_\pm \in T^\pm N$.

Consequently,

$$\gamma\begin{pmatrix} X_+ \\ X_- \end{pmatrix} = \begin{pmatrix} b \times X_+ - BX_- - \frac{1}{2}u^{-1}(\mathrm{grad}\, u)_- \times X_+ - \frac{1}{2}u^{-1}(\mathrm{grad}\, u)_+ \times X_- \\ BX_+ + b \times X_- - \frac{1}{2}u^{-1}(\mathrm{grad}\, u)_+ \times X_+ + \frac{1}{2}u^{-1}(\mathrm{grad}\, u)_- \times X_- \end{pmatrix}, \tag{3.1}$$

for any $X_\pm \in T^\pm N$.

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Hence, condition (ii) of Corollary 3.1 becomes

$$\begin{aligned}
&\mathrm{d}A(X_+, Y_+) = 2u^{-1}\det\big(b - \tfrac{1}{2}\,u^{-1}(\mathrm{grad}\,u)_-, X_+, Y_+\big),\\
&\mathrm{d}A(X_-, Y_-) = 2u^{-1}\det\big(b + \tfrac{1}{2}\,u^{-1}(\mathrm{grad}\,u)_-, X_-, Y_-\big),\\
&\mathrm{d}A(X_+, Y_-) = 2u^{-1}\big(<BX_+, Y_->\, - \tfrac{1}{2}\,u^{-1}\det\left((\mathrm{grad}\,u)_+, X_+, Y_-\right)\big),\\
&\mathrm{d}A(X_-, Y_+) = 2u^{-1}\big(-<BX_-, Y_+>\, - \tfrac{1}{2}\,u^{-1}\det\left((\mathrm{grad}\,u)_+, X_-, Y_+\right)\big),
\end{aligned}\tag{3.2}$$

for any $X_\pm, Y_\pm \in T^\pm N$, where, obviously, the last two equations give the same condition.

Remark 3.3. *1) If* $b = \frac{1}{2}\,u^{-1}(\mathrm{grad}\,u)_-$, $B = 0$, *and* $(\mathrm{grad}\,u)_+ = 0$ *then (3.1) becomes*

$$\gamma\begin{pmatrix} X_+\\ X_-\end{pmatrix} = \begin{pmatrix} 0\\ u^{-1}(\mathrm{grad}\,u)_- \times X_-\end{pmatrix},$$

for any $X_\pm \in T^\pm N$. *Therefore, in this case, condition (ii) of Corollary 3.1 becomes*

$$\begin{aligned}
&\mathrm{d}A(X_-, Y_-) = 2u^{-2}\det\left((\mathrm{grad}\,u)_-, X_-, Y_-\right),\\
&\mathrm{d}A(X_+, Y_\pm) = 0,
\end{aligned}\tag{3.3}$$

for any $X_\pm, Y_\pm \in T^\pm N$ *; equivalently,* $\mathrm{d}A = 2u^{-2} *_- \mathrm{d}u$, *where* $*_-$ *is the Hodge* $*$*-operator on* T^-N.

Now, we change $<\cdot,\cdot>$ *by the conformal factor* u^2 *along* T^-N *and by keeping it unchanged elsewhere; denote by* $<\cdot,\cdot>_1$ *this new metric on* N. *Then (3.3) becomes*

$$\mathrm{d}A = -*^1_-\,\mathrm{d}(u^{-2}),\tag{3.4}$$

where $*^1_-$ *is the Hodge* $*$*-operator on* T^-N, *with respect to* $<\cdot,\cdot>_1$.

If $T^\pm N$ *are integrable and* ∇ *is compatible with the decomposition* $TN = T^+N \oplus T^-N$ *then (i) of Corollary 3.1 holds if and only if the Levi-Civita connection of* $<\cdot,\cdot>_1$ *is given by*

$$X \mapsto \nabla_X + u^{-1}X(u)\mathrm{Id}_{TN} + \tfrac{1}{2}\begin{pmatrix} u^{-1}[(\mathrm{grad}\,u)_- \times X_-] & 0\\ 0 & u^{-1}[(\mathrm{grad}\,u)_- \times X_-]\end{pmatrix},$$

for any $X = X_+ + X_- \in TN = T^+N \oplus T^-N$. *Consequently,* $(N, <\cdot,\cdot>_1)$ *is a* $\mathrm{SL}(3)$*-manifold.*

Assume, further, that, up to a suitable (local) frame on T^+N $(= T^-N)$, *the local connection form of* ∇ *is given by the* $\mathfrak{so}(3)$*-valued* 1*-form* $-\frac{1}{2}\,u^{-1}[(\mathrm{grad}\,u)_-]$ *(under the obvious diagonal embedding* $\mathfrak{so}(3) \subseteq \mathfrak{sl}(3)$*). Then,* $T^\pm N$ *are (integrable and) totally geodesic with respect to* $<\cdot,\cdot>$*; consequently, the same holds for* $<\cdot,\cdot>_1$. *Thus, locally,* $(N, <\cdot,\cdot>_1)$ *is a product* $N^+ \times N^-$, *where* $N^\pm$ *are leaves of* $T^\pm N$. *Then (i) of Corollary 3.1 holds if and only if* $N^\pm$ *are flat. Consequently, by applying Corollary 3.1 we obtain a* G_2*-manifold which, locally, is a product of a flat Euclidean space of dimension* 3, *and a Ricci-flat anti-self-dual manifold obtained by applying the Gibbons-Hawking construction.*

2) If $b = 0$, $B = 0$, *and* $(\mathrm{grad}\,u)_+ = 0$ *(or* $(\mathrm{grad}\,u)_- = 0$*) then (i) of Corollary 3.1 is equivalent to* $(N, u<\cdot,\cdot>)$ *is Kähler.*

3) If $(\mathrm{grad}\,u)_+ = 0$ *(or* $(\mathrm{grad}\,u)_- = 0$*) then (i) of Corollary 3.1 is equivalent to the fact that the Levi-Civita connection of* $(N, u<\cdot,\cdot>)$ *is* $\nabla^1 + h \circ \mathcal{B}$, *where* ∇^1 *is a connection on* N *compatible with the almost Hermitian structure of* $(N, u<\cdot,\cdot>)$.

Corollary 3.4. *With the same hypotheses and notations as in Corollary 3.1, suppose that the set where* $\mathrm{grad}\,u$ *is an eigenvector of the underlying almost complex structure has empty interior.*

Then the Levi-Civita connection of $(N, u<\cdot,\cdot>)$ *is equal to* $\nabla^1 + h \circ \mathcal{B}$, *where* ∇^1 *is a connection on* $(N, u<\cdot,\cdot>)$ *compatible with its almost Hermitian structure; in particular, if* $\mathcal{B} = 0$ *then* $(N, u<\cdot,\cdot>)$ *is Kähler; moreover, in the latter case,* $u^{1/2}\omega$ *and* $u^{1/2}\widetilde{\omega}$ *are closed, for any forms* ω *and* $\widetilde{\omega}$, *of types* $(3,0)$ *and* $(0,3)$, *respectively, such that* $\nabla\omega = 0 = \nabla\widetilde{\omega}$.

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Proof. It is sufficient to prove that $(N, u<\cdot,\cdot>)$ is Kähler, at least, outside a closed set with empty interior.

If u is constant, then the proof follows from Corollary 3.1. Consequently, we may assume that u has no critical points. Then, at least outside the set S where grad u is isotropic, locally, we may endow N with an associative decomposition $TN = T^+N \oplus T^-N$, such that $(\mathrm{d}u)|_{T^-N} = 0$. Furthermore, the same holds on the interior of S, and the proof follows from (2) and (3) of Remark 3.3. □

The following result can be, also, found in [1, Proposition 1].

Proposition 3.5. *Let (N, k) be a Kähler manifold,* $\dim N = 6$*, endowed with a nowhere isotropic form α of type $(3,0)\oplus(0,3)$. Suppose that $v^{-1/3}\alpha$ and $\iota_{v^{-1}\mathrm{grad}\, v}\alpha$ are closed, where $v^2 = k(\alpha,\alpha)$.*

Then, the locally defined, $\left(\mathbb{C}\times N, v^{2/3}k + v^{-4/3}(\mathrm{d}\,t + A)^2\right)$ *is a G_2-manifold, where A is a 1-form on N such that $\mathrm{d}A = -\frac{4}{3}\,\iota_{v^{-1}\mathrm{grad}\, v}\alpha$. Moreover, any G_2-manifold endowed with a Killing vector field is, locally, obtained this way if, with the same notations as in Corollary 3.1, $\mathcal{B} = 0$ and the set where* grad u *is an eigenvector of the underlying complex structure has empty interior.*

Proof. This follows from Corollaries 3.1 and 3.4. □

Remark 3.6. *In Proposition 3.5 we, also, have* $\mathrm{div}(\mathrm{grad}\, v) + v^{-1}k(\mathrm{d}\, v, \mathrm{d}\, v) = 0$*; equivalently, the locally defined,* log v *is harmonic.*

Further, if (N, k) is flat special Kähler, then such α can be found, but only covariantly constants in the real setting.

5. Killing vector fields on Spin(7)-manifolds

With the same notations as in Section 2, here, we consider the case $G = \mathrm{Spin}(7)$ and $H = G_2$ (compare [10]). Then a result similar to Corollary 3.1 follows from Theorem 1.3. Furthermore, with U the space of imaginary octonions, $h{:}U \to \mathfrak{so}(U)$ is given by $h(X) = -\frac{1}{3}X\times(\cdot)$, for any $X\in U$, where $\times$ denotes the octonionic cross product.

Similarly to Section 4, an orthogonal decomposition $U = U_+ \oplus U_-$ with U_+ (nondegenerate and) associative, will be called *associative*.

Lemma 4.1. *Let $U = U_+ \oplus U_-$ be an associative decomposition, and let $u\in U_+$ and $X\in U$. Then $(u\times X)\times(\cdot)$ is given by $-3(u_aX_b - u_bX_a)_{a,b}$, up to an element of $\mathfrak{g}_2(\subseteq\mathfrak{so}(U))$, where u_a and X_a are the components of u and X, respectively, with respect to an octonionic basis of U, adapted to the given associative decomposition.*

Proof. This follows from a straightforward computation. □

Corollary 4.2. *Under the hypotheses and notations of Theorem 1.3, with $G = \mathrm{Spin}(7)$ and $H = G_2$, assume $\mathcal{B} = 0$.*

Then (locally) $(N, u^{2/3}<\cdot,\cdot>)$ is a G_2-manifold which, if u is nonconstant, is the product of a special Kähler manifold, of dimension 6, and an open subset of $\mathbb{C}$.

Proof. It is sufficient to prove that $(N, u^{2/3}<\cdot,\cdot>)$ is a G_2-manifold, at least, outside a closed set with empty interior.

If u is constant, then the proof follows from Theorem 1.3. Consequently, we may assume that u has no critical points. Then, at least outside the set S where grad u is isotropic, locally, we may endow N with an associative decomposition $TN = T^+N \oplus T^-N$, such that $(\mathrm{d}u)|_{T^-N} = 0$. Furthermore, the same

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holds on the interior of S, and from Theorem 1.3 and Lemma 4.1 it follows that $(N, u^{2/3} < \cdot, \cdot >)$ is a G_2-manifold, at least, outside the frontier of S.

If u is nonconstant then, as $\mathcal{B} = 0$, we have on $(N, u^{2/3} < \cdot, \cdot >)$ a nontrivial covariantly constant vector field, thus, completing the proof. □

Remark 4.3. *1) In Corollary 4.2, if $(N, u^{2/3} < \cdot, \cdot >)$ is given by a product of a hyper-Kähler manifold, of dimension 4, and a domain of an Euclidean space, of dimension 3, with u constant along the submanifolds given by the former, then, up to a constant, $u^{-5/3}$ must be linear.*

Conversely, by starting with a linear function defined on a suitable domain of the space of imaginary quaternions then, by using Theorem 1.2 and Corollary 4.2, we can build a Spin(7)*-manifold.*

More generally, we can start from a product as in Corollary 4.2.

2) Without the assumption $\mathcal{B} = 0$, the conclusion of Corollary 4.2 is that the Levi-Civita connection of $(N, u^{2/3} < \cdot, \cdot >)$ is $\nabla^1 + h \circ \mathcal{B}$, where ∇^1 is a connection on N compatible with the almost G_2-structure of $(N, u^{2/3} < \cdot, \cdot >)$.

6. Killing vector fields on special Kähler manifolds

With the same notations as in Section 2, here, we consider the case $G = \mathrm{SL}(n)$ and $H = \mathrm{SL}(n-1)$, where the former is considered with the direct sum of its fundamental representations of dimensions $n \geq 2$. Similarly to the previous sections, these are the eigenspaces of an orthogonal complex structure J on an Euclidean space $\widetilde{U}$, $\dim \widetilde{U} = 2n$. Furthermore, the induced representation space of $\mathrm{SL}(n-1)$ is a codimension 1 subspace $U \subseteq \widetilde{U}$ admitting an orthogonal decomposition $U = \mathbb{C}\, u_0 \oplus U_+ \oplus U_-$, where $u_0 \in U$ is such that $< u_0, u_0 > = 1$, $Ju_0 \in U^\perp$, and $J|_{U_\pm} : U_\pm \to U_\mp$ are isometries; in particular, $\dim U_\pm = n - 1$. Note that $\mathrm{SL}(n-1)$ acts trivially on u_0, and 'canonically' on $U_+ \oplus U_-$ (the latter also holds for $\mathrm{SL}(n)$, with respect to $\widetilde{U}_+ = U^\perp \oplus U_+$ and $\widetilde{U}_- = \mathbb{C}\, u_0 \oplus U_-$).

Consequently, the resulting $h : U \to \mathfrak{so}(U)$ is characterised by the following:

(i) $h(u_0)|_{U_+ \oplus U_-} = -\frac{1}{n-1} J|_{U_+ \oplus U_-}$, $h(u_0)u_0 = 0$.
(ii) $h(u_+)u_0 = Ju_+$, $h(u_+)v_+ = 0$, $h(u_+)u_- = - < Ju_+, u_- > u_0$, for any $u_+, v_+ \in U_+$ and $u_- \in U_-$.
(iii) $h(u_-)u_0 = Ju_-$, $h(u_-)v_- = 0$, $h(u_-)u_+ = - < Ju_-, u_+ > u_0$, for any $u_-, v_- \in U_-$ and $u_+ \in U_+$.

Remark 5.1. *1) The corresponding map $h : U \times U \to U$ is skew-symmetric if and only if $n = 2$ is which case, as mentioned in Remark 3.2, it is the quaternionic cross product.*

2) Conditions (ii) and (iii) can be written $h(u)u_0 = Ju$ and $h(u)v = - < Ju, v > u_0$, for any $u, v \in u_0^\perp$. Furthermore, as $h(u)u_0 \in u_0^\perp$, for any $u \in U$, we have that (iii) implies (ii).

Corollary 5.2. *If $n \geq 3$ and $\mathcal{B} = 0$, the distribution $u_0^\perp$ on $(N, < \cdot, \cdot >)$ is integrable, umbilical and its mean curvature is equal to $\frac{1}{n-1}\, u_0(u)u_0$. Consequently, locally, $u_0^\perp$ is geodesic with respect to $u^{2/n-1} < \cdot, \cdot >$.*

Proof. This follows from conditions (i), (ii), (iii), above, and Theorem 1.3. □

Corollary 5.3. *Let M, $\dim M \geq 4$, be a special Kähler manifold endowed with a Killing vector field V. Let $\mathscr{V}$ be the foliation on M given by V and JV.*

If $\mathcal{B} = 0$ then $\mathscr{V}$ is a conformal foliation.

Proof. If $\dim M = 4$ this is well known (and fairly obvious; see [2]).

If $\dim M \geq 6$, by Corollary 5.2, locally, $\mathscr{V}$ is given by the composition of a Riemannian submersion followed by a horizontally conformal submersion, with nondegenerate fibres. □

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