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The Pompeiu problem for groups

BY ALAN L. CAREY,
Department of Pure Mathematics, University of Adelaide, Adelaide SA 5001, *Australia*

EBERHARD KANIUTH
Fachbereich Mathematik/Informatik, Universität Paderborn, D-4790 *Paderborn, Federal Republic of Germany*

AND WILLIAM MORAN
Department of Pure Mathematics, University of Adelaide, Adelaide SA 5001, *Australia*

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Introduction

The Pompeiu problem has its origins in classical analysis in $\mathbb{R}^n$ (see [**2**, **3**, **4**, **8**] for a discussion and some history). In this context it may be stated as follows. Let $D \subset \mathbb{R}^n$ be a bounded measurable set of positive Lebesgue measure and f a locally integrable function on $\mathbb{R}^n$. Then, if $\int_{\sigma(D)} f = 0$ for all rigid motions σ of $\mathbb{R}^n$, is $f = 0$?

In this generality the answer, even for $\mathbb{R}^2$, is no: D a disc and f the solution to a certain Neumann problem [**2**] provide a counter-example. The question of which regions in $\mathbb{R}^n$ or in symmetric spaces have this Pompeiu property (i.e. $\int_{\sigma(D)} f = 0$ implies $f = 0$) has thus received considerable attention not least because of its connections with other interesting problems (see the preceding references). Apart from the articles mentioned above, useful insight into why this question is of importance is given in the survey [**28**].

In this paper we are interested in the following variant of the problem which has links with harmonic analysis on locally compact groups and with their representation theory. For which locally compact groups G is it true that for every relatively compact Borel set $E \subset G$ of positive Haar measure and $f \in L^1(G)$,

$$\int_{xEy} f = 0 \quad \text{for all } x, y \in G \quad \text{implies} \quad f = 0\,? \tag{$*$}$$

If this holds for G then we call G a *Pompeiu group*. Notice that we have restricted our class of allowed functions f to $L^1(G)$, so that it follows that in this sense $\mathbb{R}^n$ is a Pompeiu group [**1**]. If we were to allow other classes of functions than L^1 then the results are strikingly different. We demonstrate in the last section that most groups fail to be Pompeiu when L^1 is replaced by L^∞. In a slightly different direction we show in Section 1 that for any locally compact group G there exist sets $E \subset G$ with the property ($*$) above.

This question seems to have been first studied in some detail in [**24**] where partial results are obtained: principally, semisimple Lie groups are not Pompeiu in general

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while the Heisenberg group and the motion group of $\mathbb{R}^2$ are Pompeiu. In [**24**] the question is posed whether these results are special cases of more general theorems. It turns out that this is indeed the case. Our purpose here is to resolve this Pompeiu problem for large classes of locally compact groups including: motion groups (Theorem 2·3), discrete groups (Corollary 2·7), connected simply connected solvable Lie groups (Corollary 3·4) and general nilpotent groups (Theorem 3·5). Our main result is however Theorem 3·1: if N is a normal subgroup of G such that G/N is abelian with no compact subgroups then if N is Pompeiu so too is G. This paper subsumes all previous results on the problem obtained in [**24**] and recently in [**26**], except for the semisimple and compact Lie cases [**24**]. Our proofs exploit fairly extensive knowledge of the representation theory and structure of the groups involved.

1. *Preliminaries*

For the reader's convenience we collect together in this section some definitions and basic facts which are used frequently later. Throughout the paper we suppose that G is a locally compact group, restricting to special classes of groups at various stages of our discussion. We fix a left Haar measure μ on G. If f is a function on G we denote by ${}_x f_y$ the function $z \mapsto f(xzy)$.

Definition 1·1. A relatively compact Borel subset E of G is called a *Pompeiu set* if for all $f \in L^1(G)$, the condition $\int_{xEy} f\,d\mu = 0$ for all $x, y \in G$ implies $f = 0$. We say G is a *Pompeiu group* if every relatively compact Borel set with $\mu(E) > 0$ is a Pompeiu set. Equivalently a relatively compact subset $E \subset G$ is a Pompeiu set if for all $f \in L^1(G)$ the condition $f * {}_x\chi_{E^{-1}} = 0$ for all $x \in G$ implies $f = 0$.

In what follows $\hat{G}$ denotes the dual space of G, that is the set of equivalence classes of irreducible unitary representations of G, and $\hat{G}_r$ the reduced dual, that is the support of the left regular representation. Both of these spaces are assumed to have the hull-kernel topology. We use the same letter, for example π, for a unitary representation of G and the corresponding $*$-representations of $L^1(G)$ and $C^*(G)$ and $\ker\pi$ always denotes the C^*-kernel of π. As a general reference to group representation theory and notation we mention [**9**]. In this connection we remark that as usual if H is normal in G we let G act on H by conjugation and hence G acts on $\hat{H}$. We write $G(\sigma)$ for the orbit of σ under this action on the dual.

We begin with an elementary negative result.

LEMMA 1·2. *Suppose that G contains a non-trivial compact normal subgroup. Then G is not a Pompeiu group.*

Proof. Let K be a non-trivial compact normal subgroup and let W be a compact symmetric neighbourhood of the identity in G/K. Let $E = \{x \in G \mid xK \in W\}$. Given any $\pi \in \hat{G}$, which is non-trivial on K, the restriction of π to K, $\pi|_K$, decomposes as a direct sum of irreducibles all of which belong to a single G-orbit. Thus the orthogonality relations for compact groups imply that for all ξ, η in the Hilbert space $\mathscr{H}_\pi$ of π we have:

$$\langle \pi(\chi_E)\,\xi, \eta\rangle = \int_W \int_K \langle \pi(k)\,\xi, \pi(x^{-1})\,\eta\rangle\,dk\,d\dot{x} = 0.$$

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Now choose any $\gamma \in \hat{K}, \gamma \neq 1_K$, with character t_γ and $g \in C_c(G)$ such that the function $f(x) = \int_K g(xy)\, t_\gamma(y^{-1})\, dy$, is non-zero. Then for each $\rho \in (G/K)^\wedge$,

$$\rho(f) = \rho(g)\,(\rho|_K)\,(t_\gamma) = 0.$$

Thus for any π in $\hat{G}$ and $x \in G$

$$\pi(f *_x \chi_E) = \pi(f)\,\pi(x)\,\pi(\chi_E) = 0.$$

This shows that $f *_x \chi_E = 0$ for all $x \in G$ so that E is not a Pompeiu set.

This lemma leads in the subsequent discussion to various assumptions on our groups which preclude the existence of compact normal subgroups. In this connection it is worthwhile recalling that the term *compact element of* G means that the closed subgroup generated by the element is compact. We say that G has no compact elements if e is the only compact element in G.

We continue with a variant of proposition 3·2 of [**24**].

LEMMA 1·3. (*a*) *If* $g \neq 0$ *is a* L^1*-function on* G *and if there exists a dense subset* D *of the reduced dual* $\hat{G}_r$ *of* G *such that* $\pi(g) \neq 0$ *for all* $\pi \in D$, *then for* $f \in L^1(G)$ *the condition* $f *_x g = 0$ *for all* $x \in G$ *implies* $f = 0$.

(*b*) *If the hypothesis of* (*a*) *holds for all* $g \in C_c(G)$, $g \neq 0$, *then* G *is a Pompeiu group.*

(*c*) *A group* G *is Pompeiu if and only if for every relatively compact Borel subset* E *in* G *with* $\mu(E) \neq 0$ *the hypothesis of* (*a*) *holds for* $g = \chi_E * \chi_E^*$.

Proof. (*a*) We include this argument for completeness even though it is essentially the same as that of [**24**]. Let π be any irreducible representation in the set D. Then $\pi(f)\,\pi(x^{-1})\,\pi(g) = 0$ for all $x \in G$. As $\pi(g) \neq 0$ this forces $\pi(f) = 0$. By the density of D in $\hat{G}_r$ we have $\pi(f) = 0$ for all $\pi \in \hat{G}_r$. Thus f acts as the zero operator in the regular representation of G which means $f = 0$.

(*b*) Given a relatively compact set $E \subset G$ with $\mu(E) \neq 0$ choose $h \in C_c(G)$ such that $g = \chi_{E^{-1}} * h \neq 0$. It follows immediately from (*a*) that E is a Pompeiu set.

(*c*) This is immediate from

$$\|\pi(\chi_E)\|^2 = \|\pi(\chi_E * \chi_E^*)\|.$$

This result indicates already that this L^1 Pompeiu problem is inextricably linked to representation theory. Our next preliminary result concerns the Fourier transform of compactly supported functions.

LEMMA 1·4. *Let* G *be an abelian group without compact elements and let* $g \neq 0$ *be a compactly supported* L^1*-function on* G. *Then the Fourier transform of* g *vanishes only on a set of Haar measure zero in* $\hat{G}$.

Proof. By the structure theorem for locally compact abelian groups, G is of the form $V \times D$ where V is a vector group and D is discrete and torsion free. Choose a finite subset A of D such that g vanishes outside $V \times A$, and let F denote the subgroup generated by A and $H = V \times F$. Then for $\lambda \in \hat{G}$,

$$\hat{g}(\lambda) = (g|_H)^\wedge(\lambda|_H).$$

Now the restriction map $\lambda \mapsto \lambda|_H$ from $\hat{G}$ onto $\hat{H}$ maps Haar measure on $\hat{G}$ onto Haar measure on $\hat{H}$. Hence the lemma is proved once we know that the Fourier transform

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of $h = g|_H$ vanishes only on a set of measure zero in $\hat{H}$. Now, if $\alpha \in \hat{H}$, $\alpha = \beta \times \gamma$, $\beta \in \hat{V}$, $\gamma \in \hat{F}$, then

$$\hat{h}(\alpha) = \sum_{a \in A} \gamma(a) \int_V h(x,a)\,\beta(x)\,dx = \sum_{a \in A} \gamma(a)\,(h_a|_V)\hat{}\,(\beta).$$

As $h_a|_V$ is a compactly supported L^1-function on V, its Fourier transform vanishes only on a set of measure zero. Moreover, $F = \mathbb{Z}^d$ for some d and $\gamma \mapsto \gamma(a)$ is a trigonometric polynomial on $\hat{F} = \mathbb{T}^d$. This completes the proof.

Our notation for induced representations is fairly standard: for the definition we follow Blattner [5] and we write $\mathrm{ind}_H^G \sigma$ for the representation of G induced by the representation σ of the closed subgroup H. We will need the following technical result due to Ludwig ([**20**], lemma 1).

LEMMA 1·5. *Let N be normal in G, $f \in L^1(G)$ and $g \in C_c(G)$. Then*

(a) $\mathrm{ind}_N^G \sigma(g) = 0$ *if and only if* $\sigma({}_x g|_N) = 0$ *for all* $x \in G$;

(b) there exists $a \in G$ *such that* ${}_a f|_N \in L^1(N)$ *and for any representation* σ *of* N, $\sigma({}_a f|_N) \neq 0$ *implies* $\mathrm{ind}_N^G \sigma(f) \neq 0$.

We omit the proof because a careful reading shows that Ludwig's proof already covers the statement of the lemma provided one replaces null sets by locally null sets. We conclude this section with our first theorem (which generalizes a result of Rana [**22**]) showing that every group contains many Pompeiu sets.

THEOREM 1·6. *If G is a first countable locally compact group then every compact subset of G of positive Haar measure contains a Pompeiu set for $L^1(G)$.*

Proof. We remark firstly that G may be assumed to be compactly generated. This is because if $C \subseteq G$ is compact with Haar measure $\lambda(C) > 0$, we can introduce the open subgroup H of G generated by C and suppose that we have already shown the existence of a Pompeiu set E for $L^1(H)$ contained in C. Then it is easily verified that E is Pompeiu for $L^1(G)$ as well. In particular, we can assume G is σ-compact (and first countable), hence separable. Therefore $\hat{G}$ contains a countable dense subset D. It suffices to show that there exists a Borel subset E of C such that $\pi(\chi_E) \neq 0$ for all $\pi \in D$.

Denote by $\mathscr{B}$ the set of all Borel subsets of C and for $E, F \in \mathscr{B}$ their symmetric difference by $E \triangle F$. Introduce an equivalence relation on $\mathscr{B}$: $E \sim F$ if and only if $\lambda(E \triangle F) = 0$. Then on the space $\tilde{\mathscr{B}}$ of equivalence classes there is a metric: $d(\tilde{E}, \tilde{F}) = \lambda(E \triangle F)$ which (as is well known) makes $(\tilde{\mathscr{B}}, d)$ a complete metric space. Now for fixed $\pi \in D$ consider

$$\tilde{\mathscr{B}}_\pi = \{\tilde{E} \in \tilde{\mathscr{B}} \mid \pi(\chi_E) \neq 0\}.$$

Then $\tilde{\mathscr{B}}_\pi$ is open in $\tilde{\mathscr{B}}$ and we claim that it is also dense. Once this has been shown, Baire's theorem suffices to show that $\bigcap_{\pi \in D} \tilde{\mathscr{B}}_\pi$ is dense in $\tilde{\mathscr{B}}$ which in turn proves the theorem.

To prove the claim let $E \in \mathscr{B}$ with $\pi(\chi_E) = 0$ and let $\epsilon > 0$ be given. We have to find $F \in \mathscr{B}$ such that $\lambda(E \triangle F) < \epsilon$ and $\pi(\chi_F) \neq 0$. For every $a \in G$ there exists an open neighbourhood V_a of a and a unit vector ξ_a in the Hilbert space of π with the following properties:

$$\lambda(V_a) < \epsilon, \quad \langle \pi(a)\,\xi_a, \xi_a \rangle \neq 0,$$

and

$$|\langle \pi(x)\,\xi_a, \xi_a \rangle - \langle \pi(a)\,\xi_a, \xi_a \rangle| \leqslant \tfrac{1}{2} |\langle \pi(a)\,\xi_a, \xi_a \rangle|$$

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for all $x \in V_a$. For some $a \in G$, $Q = V_a \cap C$ has positive measure. With $\xi = \xi_a$ we obtain

$$\left|\int_Q \langle \pi(x)\xi, \xi\rangle\, dx\right| \geqslant \left|\int_Q \langle \pi(a)\xi, \xi\rangle\, dx\right| - \left|\int_Q (\langle \pi(x)\xi, \xi\rangle - \langle \pi(a)\xi, \xi\rangle)\, dx\right|$$

$$\geqslant \lambda(Q)|\langle \pi(a)\xi, \xi\rangle| - \int_Q |\langle \pi(x)\xi, \xi\rangle - \langle \pi(a)\xi, \xi\rangle|\, dx \geqslant \tfrac{1}{2}\lambda(Q)|\langle \pi(a)\xi, \xi\rangle|.$$

Thus $\pi(\chi_Q) \neq 0$. Now let $F = Q \cup E$ so that $\pi(\chi_F) = \pi(\chi_Q) \neq 0$ and $\lambda(E \triangle F) \leqslant \lambda(Q) < \epsilon$.

We conclude this section with the remark that replacing $L^1(G)$ in the definition of Pompeiu set by the algebra of bounded measures leads to the same conclusions so that either form of the definition may be used.

2. *Motion groups and IN-groups*

The purpose of this section is to give for groups G in these two classes of locally compact groups necessary and sufficient conditions for G to be a Pompeiu group. We begin by establishing some notational conventions. If S and T are sets of unitary representations then S is weakly contained in T ($S \prec T$) if

$$\bigcap_{\sigma \in S} \ker \sigma \supseteq \bigcap_{\tau \in T} \ker \tau$$

and S and T are weakly equivalent ($S \sim T$) if $S \prec T$ and $T \prec S$ (we refer to [11] for results on weak containment). For any unitary representation π the support of π is the closed subset $\operatorname{supp} \pi = \{\tau \in \hat{G} \mid \tau \prec \pi\}$ of $\hat{G}$. In particular the support of the left regular representation is the reduced dual $\hat{G}_r$. Recall that $\hat{G}_r = \hat{G}$ if and only if G is amenable. It is also convenient to use the notation $S \otimes T$ for the set of all tensor product representations $\sigma \otimes \tau$, $\sigma \in S$, $\tau \in T$.

LEMMA 2·1. *Let N be an abelian normal subgroup of G without compact elements and let $g \in C_c(G), g \neq 0$. Suppose that for λ in some dense subset W of $\hat{N}$, $\operatorname{ind}_N^G \lambda$ is weakly equivalent to some $\rho_\lambda \in \hat{G}$. Then there exists a dense subset D in $\hat{G}_r$ such that $\pi(g) \neq 0$ for all $\pi \in D$. In particular G is a Pompeiu group.*

Proof. Fix $a \in G$ so that $h = {}_a g|_N \neq 0$. By Lemma 1·4 the set

$$V = \{\lambda \mid \hat{h}(\lambda) \neq 0\}$$

is dense and open in $\hat{N}$. By Lemma 1·5, $\hat{h}(\lambda) \neq 0$ implies $\operatorname{ind}_N^G \lambda(g) \neq 0$. Hence $V \cap W$ is dense in $\hat{N}$ and so the set

$$\{\operatorname{ind}_N^G \lambda \mid \lambda \in V \cap W\}$$

is weakly equivalent to the left regular representation of G. Thus $D = \{\rho_\lambda \mid \lambda \in V \cap W\}$ is dense in $\hat{G}_r$, and, by what we have seen above, $\rho_\lambda(g) \neq 0$ for all $\lambda \in D$.

To deal with motion groups we will employ the theory of transformation group C^*-algebras $C^*(K, X) = C^*(K, C_0(X))$ where K is a compact group and X is a locally compact Hausdorff space. Recall that for a semidirect product $G = K \ltimes N$ of a compact group K with an abelian normal subgroup N,

$$C^*(G) = C^*(K, C^*(N)) = C^*(K, C_0(\hat{N})).$$

The authors are indebted to S. Echterhoff for showing them the transformation group C^*-algebra arguments relevant to the proof of Lemma 2·2 below.

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Before proceeding we need some further notation. In the above situation we let S_λ denote the stability subgroup of $\lambda \in \hat{N}$. For each $\lambda \in \hat{N}$ and $\sigma \in \hat{S}_\lambda$, let $\tilde{S}_\lambda = S_\lambda \ltimes N$ and

$$\pi_{\lambda,\sigma} = \mathrm{ind}_{\tilde{S}_\lambda}^{G}(\sigma \times \lambda).$$

Every $\pi_{\lambda,\sigma}$ is irreducible and conversely every irreducible representation of G arises this way. We denote by $\mathscr{S}(K)$ the set of all closed subgroups of K endowed with Fell's compact-open topology [12].

LEMMA 2·2. *Let G be the semidirect product $G = K \ltimes N$ where K is a compact group and N is abelian. Suppose that K_1 is a closed subgroup of K and Λ an open K-invariant subset of $\hat{N}$ with the following properties*:

(i) *for $\lambda \in \Lambda$, the stability group S_λ is conjugate to K_1*;

(ii) *the map $\lambda \to S_\lambda$ is continuous from Λ into $\mathscr{S}(K)$.*

Let $\Lambda_1 = \{\lambda \in \Lambda \mid S_\lambda = K_1\}$ and $\Pi = \{\pi_{\lambda,\sigma} \mid \lambda \in \Lambda_1, \sigma \in \hat{K}_1\}$ and denote by H the normalizer of K_1 in K. Then inducing defines a homeomorphism between the quotient space $(\Lambda_1 \times \hat{K}_1)/H$ and Π.

Proof. Let $\mathscr{C}$ denote the subset of $\mathscr{S}(K)$ consisting of all subgroups of K that are conjugate to K_1. Then the canonical mapping

$$xH \mapsto xK_1x^{-1}$$

is a homeomorphism between K/H and $\mathscr{C}$. By continuity of $\lambda \mapsto S_\lambda$ we therefore obtain a continuous, K-equivariant mapping

$$\phi : \Lambda \to K/H$$

defined by $\phi(\lambda) = xH$ when $S_\lambda = xK_1x^{-1}$. Introduce the K-invariant ideal A of $C^*(N)$ with $\hat{A} = \Lambda$ and the H-invariant ideal I of A with $\hat{I} = \Lambda_1 = \phi^{-1}(\{eH\})$. Then by a theorem of Green ([15], theorem 17) (see also [23], situation 7) the crossed product C^*-algebras

$$C^*(K, A) = C^*(K, \Lambda) \quad \text{and} \quad C^*(H, A/I) = C^*(H, \Lambda_1)$$

are Morita equivalent. In particular, their dual spaces are homeomorphic. Finally notice that the dual space of $C^*(K, A)$ is just Π and that by [27], corollary 5·12 inducing defines a homeomorphism from the quotient space of H-orbits in $\hat{K}_1 \times \Lambda_1$ onto the dual space of $C^*(H, \Lambda_1)$.

Before stating our main result on motion groups we need to recall some further notions. A locally compact group is said to be $*$-regular if the canonical mapping which takes the kernel of an irreducible representation of $C^*(G)$ to its intersection with $L^1(G)$ is a homeomorphism of the primitive ideal space of $C^*(G)$ with the primitive ideal space of $L^1(G)$ [6]. Motion groups are polynomially growing and hence by a fundamental result of Boidol *et al.* ([6], Satz 1) have $*$-regular L^1-algebras.

THEOREM 2·3. *Suppose that G is a semidirect product $G = K \ltimes N$, where K is a compact Lie group and N is abelian. Then G is a Pompeiu group if and only if the following conditions hold*:

(i) *$N = \mathbb{R}^d \times D$, where D is discrete and torsion-free*;

(ii) *for all λ in some dense subset of $\hat{N}$, the stability subgroup of λ in K is trivial.*

Proof. For $\lambda \in \hat{N}$ let S_λ denote its stability group in K. We observe first that conditions (i) and (ii) are sufficient. Indeed, if $\lambda \in \hat{N}$ and $S_\lambda = \{e\}$, then $\mathrm{ind}_N^G \lambda \in \hat{G}$, so that by Lemma 2·1, (i) and (ii) imply that G is Pompeiu.

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Conversely, suppose that G is a Pompeiu group. Since every compact set of compact elements in N is contained in a compact normal subgroup of G, Lemma 1·2 shows that N has no compact elements. Thus N is of the form $V \times D$ where V is a vector group and D is discrete and torsion-free. The proof of (ii) requires considerably more intricate arguments. We suppose that (ii) fails so that the set $\{\lambda \in \hat{N} \mid S_\lambda \neq \{e\}\}$ has non-void interior U'.

We notice first that the decomposition $N = V \times D$ may be assumed to be stable (that is, V and D may be assumed invariant) under the action of K on N. This may be deduced from a general splitting theorem of Grosser and Moskowitz [**16**] as follows. Denoting by τ_a the action of $a \in K$ on N and by $\hat{\tau}_a$ the dual action on $\hat{N} = \hat{V} \times \hat{D}$, we have $\hat{\tau}_a(\hat{D}) = \hat{D}$, since $\hat{D}$ is just the subgroup of all compact elements in $\hat{N}$. Now by [**14**], theorem 1·1 there exists a closed subgroup W of $\hat{N}$ such that $\hat{N} = W \times \hat{D}$ and W is stable, i.e. $\hat{\tau}_a(W) = W$ for all $a \in K$. Clearly W is a vector group and the required decomposition of N is given by $N = \hat{D}^\perp \times W^\perp$. We may of course also assume that K acts by orthogonal transformations on V.

Next we show that there are only finitely many conjugacy classes of stability subgroups. If $\lambda = \alpha \times \beta \in \hat{N}$, $\alpha \in \hat{V}$, $\beta \in \hat{D}$, then $S_\lambda = S_\alpha \cap S_\beta$. Since K_0, the connected component of K, acts trivially on D, we have $K_0 \subseteq S_\beta$, and so there are only finitely many S_β as $[K:K_0] < \infty$. By [**7**], theorem 3·1 there exists a so-called principal stability subgroup for the action of K_0 on $\hat{V}$, i.e. there are an open dense subset U'' of $\hat{V}$ and a subgroup S of K_0 such that for every $\alpha \in U''$, $S_\alpha \cap K_0$ is conjugate to S in K_0. We may now take the union of conjugates of U'' under the K-action and hence assume that U'' is K-invariant. Thus there are finitely many subgroups $K_1, \ldots, K_r$ of K so that for every $\lambda \in U'' \times \hat{D}$, S_λ is conjugate in K to exactly one of them. Let $U = U' \cap (U'' \times \hat{D})$: then U is open, K-invariant and non-empty. After ordering appropriately, we have shown so far that there exist an open and K-invariant non-empty subset U of $\hat{N}$ and subgroups K_j, $1 \leqslant j \leqslant r$, of K such that

$$[K_1:S] \leqslant \ldots \leqslant [K_r:S] < \infty, \quad K_j \cap K_0 = S,$$

and for each $\lambda \in U$, S_λ is conjugate to exactly one K_j. By a result of Glimm ([**13**], theorem 3) we can in addition assume that the stability groups vary continuously on U. We set $U_j = \{\lambda \in U; S_\lambda = K_j\}$, so that U is the disjoint union

$$U = \bigcup_{j=1}^{r} \Big(\bigcup_{a \in K/K_j} \hat{\tau}_a(U_j) \Big).$$

Moreover, for $1 \leqslant j \leqslant r$ and $\sigma \in \hat{K}_j$, let

$$\hat{G}_{j,\sigma} = \{\pi_{\lambda,\sigma} = \operatorname{ind}_{K_j \prec N}^{G}(\sigma \times \lambda) \mid \lambda \in U_j\} \quad \text{and} \quad \hat{G}_j = \bigcup_{\sigma \in \hat{K}_j} \hat{G}_{j,\sigma}.$$

Then
$$V = \bigcup_{j=1}^{r} \hat{G}_j = \{\pi \in \hat{G} \mid \pi|_N \sim K(\lambda) \quad \text{for some} \quad \lambda \in U\}$$

is open in $\hat{G}$.

We claim that $\hat{G}_1$ is open in V and hence in $\hat{G}$. Suppose there is a net $\pi_i = \pi_{\lambda_i, \sigma_i}$, $i \in I$, in $V \setminus \hat{G}_1$ converging to some $\pi = \pi_{\lambda,\sigma} \in \hat{G}_1$. It follows that

$$K(\lambda_i) \sim \pi_i|_N \to \pi|_N \sim K(\lambda),$$

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so that there are $a_i \in K$, $i \in I$, such that $\hat{\tau}_{a_i}(\lambda_i) \to \lambda$ in $\hat{N}$. We can assume that $\lambda_i \in U_j$ for some $1 \leqslant j \leqslant r$ and all $i \in I$, and since K is compact, $a_i \to a^{-1}$ for some $a \in K$. Hence

$$\lambda_i = \hat{\tau}_{a_i^{-1}}(\hat{\tau}_{a_i}(\lambda_i)) \to \hat{\tau}_a(\lambda),$$

and this implies $K_j = S_{\lambda_i} \subseteq S_{\hat{\tau}_a(\lambda)} = aS_\lambda a^{-1}$. In particular,

$$S = K_j \cap K_0 \subseteq aS_\lambda a^{-1} \cap K_0 = a(K_1 \cap K_0)a^{-1} = aSa^{-1}.$$

Thus $aSa^{-1} = S$ and then by the minimality of $[K_1 : S]$, $aS_\lambda a^{-1} = K_j$, i.e. $\hat{\tau}_a(\lambda) \in U_j$ and $\pi_{\lambda,\sigma} \in \hat{G}_j \subseteq \hat{V} \backslash \hat{G}_1$, a contradiction. We now set

$$W = \bigcup \{\hat{G}_{1,\sigma} \mid \sigma \in \hat{K}_1, \sigma \neq 1_{K_1}\}.$$

By Lemma 2·2, $G_1 = W \cup \hat{G}_{1,1_{K_1}}$ is homeomorphic to the quotient space $(\hat{K}_1 \times U_1)/H$, where H denotes the normalizer of K_1 in K. Of course, $\{1_{K_1}\} \times U_1$ is H-invariant and its image in $(\hat{K}_1 \times U_1)/H$ is closed. This implies that W is open in $\hat{G}$.

To finish the proof of the theorem, we find a compact symmetric subset E of G with positive measure and an $f \in L^1(G)$, $f \neq 0$, such that $\pi(f) = 0$ for all $\pi \in \hat{G} \backslash W$ and $\pi(\chi_E) = 0$ for all $\pi \in W$. This implies $f *_x \chi_E = 0$ for all $x \in G$, so that E is not a Pompeiu set. The existence of an f with the desired property follows by *-regularity of $L^1(G)$.

Finally, let $E = K \times X$, where X is the unit ball in V. Now, for $\lambda \in U_1$ and $\sigma \in \hat{K}_1$, $\sigma \neq 1_{K_1}$,

$$\pi_{\lambda,\sigma}(\chi_E) = \operatorname{ind}_H^G(\sigma \times \lambda)(\chi_E) = 0$$

if for all $s, t \in G$, $(\sigma \times \lambda)({}_s(\chi_E)_t|_H) = 0$ (by [**20**], lemma 1). Write $s = (b^{-1}, y^{-1})$ and $t = (c, z^{-1})$, $b, c \in K_1$, $y, z \in N$, and let $\xi, \eta \in \mathscr{H}_\sigma = \mathscr{H}_{\sigma \times \lambda}$. Since Haar measure on N is K-invariant and $K(X) = X$, we get

$$\begin{aligned}(\sigma \times \lambda)({}_s(\chi_E)_t|_H) &= \int_{K_1} \int_N \lambda(x) \langle \sigma(a)\xi, \eta \rangle \chi_E(b^{-1}ac, y^{-1}\tau_{b^{-1}}x \cdot \tau_{b^{-1}a}z^{-1})\, dx\, da \\ &= \int_{K_1} \langle \sigma(a)\xi, \eta \rangle \int_N \lambda(\tau_b(yx\tau_{b^{-1}a}z))\, \chi_E(b^{-1}ac, x)\, dx\, da \\ &= \lambda(\tau_b y)\lambda(z) \int_X \lambda(\tau_b x)\, dx \cdot \int_{K_1} \langle \sigma(a)\xi, \eta \rangle\, da = 0\end{aligned}$$

by the orthogonality relations for compact groups since $\sigma \neq 1_{K_1}$.

COROLLARY 2·4. *Let G be a semidirect product $G = K \ltimes \mathbb{R}^d$, with K a compact group. Then G is a Pompeiu group if and only if K is Lie and the set of all characters in $\mathbb{R}^d$ with trivial stability subgroup is dense in $\mathbb{R}^d$.*

Proof. According to Theorem 2·3 it only remains to observe that K being Lie is a necessary condition. But since G is compactly generated it contains a compact normal subgroup C so that G/C is Lie. By Lemma 1·2, $C = \{e\}$.

Remark 2·5. If $G = K \ltimes N$ where K is compact (not necessarily Lie) and N abelian, then conditions (i) and (ii) imply that G is a Pompeiu group. The assumption that G is Lie is only used in the proof that (i) and (ii) are necessary.

By Theorem 2·3 the motion group of $\mathbb{R}^n$, i.e. $\mathrm{SO}(n) \ltimes \mathbb{R}^n$, where $n \in \mathbb{N}$, is a Pompeiu group exactly for $n = 1, 2$. Theorem 2·2 answers the question posed in [**24**], §4(*d*) and explains and generalizes the results [**24**], corollary 4·4 and [**26**], theorem 7. It is worth mentioning that our construction of non-Pompeiu sets is as explicit as the one in [**26**].

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We now turn to *IN*-groups. Recall that a locally compact group G is said to be an *IN*-group if G contains a compact neighbourhood V of e with $x^{-1}Vx = V$ for all $x \in G$.

THEOREM 2·6. *Let G be an IN-group and G_F the open normal subgroup consisting of all elements with relatively compact conjugacy classes. Then G is a Pompeiu group if and only if $G_F = \mathbb{R}^d \times D$ where D is discrete, abelian and torsion-free.*

Proof. Suppose first that G is a Pompeiu group. The set G_F^c of all compact elements in G_F, is a closed normal subgroup of G, and

(i) every compact subset of G_F^c is contained in a compact normal subgroup of G ([**16**], theorem 3·20);

(ii) G_F/G_F^c is abelian and has no compact elements ([**16**], theorem 3·16).

It follows from (i) and Lemma 1·2 that $G_F^c = \{e\}$. Hence by (ii), $G_F = \mathbb{R}^d \times D$, where D is discrete abelian and torsion-free.

Conversely, let G_F be of this form. Note that since $\mathbb{R}^d$ is open in G, G actually has arbitrarily small invariant neighbourhoods of the identity ($G \in [SIN]$ in the notation of [**16**]). Let E be a relatively compact Borel set in G with $\mu(E) > 0$ and $f \in L^1(G)$ such that $\int_{aEb} f(x)\, d\mu(x) = 0$ for all $a, b \in G$. By [**19**], lemma 3 there exists a σ-compact open subgroup H of G such that $f \in L^1(H)$, $E \subset H$ and $H_F = G_F \cap H$. In particular, H_F contains no compact elements. As $\mathbb{R}^d$ is open in G, H is separable. Suppose that we have already shown that H is a Pompeiu group. It follows then that $f = 0$. This reduction argument shows that we can henceforth assume that G is separable, and this will enable to apply Lemma 2·1.

In what follows we have to employ the theory of characters on G. Let N be a closed normal subgroup of G and $K(N, G)$ the set of all continuous positive definite functions ϕ on N with $\phi(e) \leqslant 1$ that satisfy the equation $\phi(x^{-1}nx) = \phi(n)$ for all $n \in N$, $x \in G$. $K(N, G)$ is convex and weak $*$-compact, and the set $E(N, G)$ of its non-zero extreme points is called the space of G-characters on N. In case $N = G$ we simply write $E(G)$ instead of $E(G, G)$. Moreover, for any positive definite function ϕ on G, we denote by π_ϕ the cyclic representation of G associated to ϕ. It is well known that for $\phi \in E(G)$, π_ϕ is a factor representation.

We now take $N = G_F$, and for every $\gamma \in E(G_F, G)$ we define its trivial extension $\tilde{\gamma}$ on G by $\tilde{\gamma}|_{G_F} = \gamma$ and $\tilde{\gamma}|_{G \setminus G_F} = 0$. Then $\pi_{\tilde{\gamma}}$ is unitarily equivalent to the induced representation $\mathrm{ind}_{G_F}^G \pi_\gamma$. Generalizing a result of Thoma [**25**], Hauenschild has shown in [**17**], Satz 1 that the set Γ of all $\gamma \in E(G_F, G)$ such that $\tilde{\gamma} \in E(G)$ is dense in $E(G_F, G)$, provided we endow $E(G_F, G)$ with the topology of uniform convergence on compact subsets of G_F. Now, since $G \in [SIN]$ and since the conjugacy classes of elements in G_F are relatively compact, by a theorem of Grosser and Moskowitz ([**16**], theorem 0·1) G acts as a relatively compact automorphism group on G_F, and hence on the dual group $\hat{G}_F$. Therefore there is an 'averaging map'

$$q : \hat{G}_F \to E(G_F, G),$$

which is continuous and open (see [**21**], theorem 5·8). For $\gamma \in E(G_F, G)$, we have $q^{-1}(\gamma) = \overline{G(\lambda)}$. This implies that for $\lambda \in q^{-1}(\gamma)$,

$$\mathrm{ind}_{G_F}^G \lambda \sim \mathrm{ind}_{G_F}^G \pi_\gamma = \pi_{\tilde{\gamma}}.$$

Since q is open, $W = q^{-1}(\Gamma)$ is dense in $\hat{G}_F$. Finally, recall that if A is a separable C^*-algebra, then the kernel of every factor representation of A is a primitive ideal in A

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(see [9], (3·9·1), (c) and (5·7·6), (b) and (d)). Since G is second countable, this applies to prove that $\pi_{\tilde{\gamma}}$ is weakly equivalent to some irreducible representation of G for every $\gamma \in \Gamma$. We have thus verified that the set W defined above satisfies the hypothesis of Lemma 2·1. This proves that G is a Pompeiu group.

COROLLARY 2·7. *A discrete group is a Pompeiu group if and only if its finite conjugacy class subgroup is torsion-free.*

3. *Nilpotent and solvable groups*

Scott and Sitaram [24] proved that the Heisenberg groups are Pompeiu groups and asked whether this holds for any connected, simply connected nilpotent Lie group. Weit [26] recently improved their result to the extent that semidirect products of vector groups are Pompeiu. In this section we obtain a far reaching generalization (Theorem 3·1 below). In particular, it turns out that every connected and simply connected solvable Lie group is a Pompeiu group and that a general nilpotent group is Pompeiu if and only if it contains no compact elements.

THEOREM 3·1. *Let N be a closed normal subgroup of G such that G/N is abelian and has no compact elements. If N is Pompeiu, then G is also Pompeiu.*

Proof. We first show that given any open non-empty subset V of $\hat{N}_r$, there exists $\pi \in \hat{G}_r$ such that $\operatorname{supp}(\pi|_N) \cap V$ is non-empty. To this end we employ Green's method of extending ideals (see [15], proposition 9). Let J denote the G-invariant ideal in $C_r^*(N)$ whose dual is $G(V)$, the union of all the G-orbits of elements of V. Denote by m the map which injects $C_r^*(N)$ into the multiplier algebra of $C_r^*(G)$. Then for every representation π of $C_r^*(G)$ and $f \in C_r^*(N)$,

$$\pi(m(f)) = (\pi|_N)(f).$$

The extension of J is the ideal in $C_r^*(G)$ generated by $\{m(f)\,a \mid f \in J, a \in C_r^*(G)\}$. Thus, assuming that $\operatorname{supp}(\pi|_N) \subseteq \hat{N}_r \setminus V$ we necessarily have $\operatorname{supp}(\pi|_N) \subseteq \hat{N}_r \setminus G(V)$ for all $\pi \in \hat{G}_r$, and so we deduce that π annihilates the extension of J for all $\pi \in \hat{G}_r$, a contradiction because the extension is a non-trivial ideal.

Now let E be a relatively compact Borel set in G with positive Haar measure. We are going to prove the existence of a dense subset Π of $\hat{G}_r$ such that $\pi(\chi_E) \neq 0$ for all $\pi \in \Pi$. According to Lemma 1·5 we may choose $a \in G$ such that ${}_a\chi_E|_N \neq 0$ in $L^1(N)$ and for every representation σ of N, $\sigma({}_a\chi_E|_N) \neq 0$ implies $\operatorname{ind}_N^G \sigma(\chi_E) \neq 0$. Let $F = a^{-1}E \cap N$ so that $\chi_F = {}_a\chi_E|_N$. Let

$$\Sigma = \{\sigma \in \hat{N}_r \mid \sigma(\chi_F) \neq 0\}.$$

Then by assumption on N, Σ is dense (and open) in $\hat{N}_r$. By what we have seen above

$$\Sigma \cap \bigcup_{\pi \in \hat{G}_r} \operatorname{supp}(\pi|_N)$$

is dense in Σ. Let us consider a pair of representations $\pi \in \hat{G}_r$ and $\sigma \in \hat{N}_r$ with the properties $\pi|_N \succ \sigma$ and $\sigma(\chi_F) \neq 0$. We claim that the set of all $\lambda \in (G/N)^\wedge$ such that $\pi \otimes \lambda(\chi_E) \neq 0$ is dense in $(G/N)^\wedge$. Notice firstly that $\pi \otimes \lambda_0(\chi_E) \neq 0$ for some $\lambda_0 \in (G/N)^\wedge$. In fact, $\operatorname{ind}_N^G \sigma(\chi_E) \neq 0$ because $\sigma(\chi_F) \neq 0$, and

$$\operatorname{ind}_N^G \sigma \prec \operatorname{ind}_N^G(\pi|_N) = \pi \otimes \operatorname{ind}_N^G 1_N,$$

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(where 1_N denotes the trivial representation) which in turn is weakly equivalent to $\pi \otimes (G/N)^\wedge$. Choose $\xi \in \mathscr{H}_\pi = \mathscr{H}_{\pi \otimes \lambda_0}$ so that $\langle \pi \otimes \lambda_0(\chi_E)\xi, \xi\rangle \neq 0$. For arbitrary $\lambda \in (G/N)^\wedge$ we have by Weil's formula

$$\begin{aligned}\langle \pi \otimes \lambda(\chi_E)\xi, \xi\rangle &= \int_G \lambda(x)\chi_E(x)\langle \pi(x)\xi, \xi\rangle\, dx \\ &= \int_{G/N} \lambda(\dot{x}) \int_N \chi_E(xn)\langle \pi(n)\xi, \pi(x^{-1})\xi\rangle\, dn\, d\dot{x} \\ &= \int_{G/N} \lambda(\dot{x})\langle \pi({}_x\chi_E|_N)\xi, \pi(x^{-1})\xi\rangle\, d\dot{x}.\end{aligned}$$

The function f_ξ on G/N defined by

$$f_\xi(\dot{x}) = \langle \pi({}_x\chi_E|_N)\xi, \pi(x^{-1})\xi\rangle$$

is an L^1-function with compact support on G/N and $\hat{f_\xi}(\lambda_0) \neq 0$. By Lemma 1·4, $\hat{f_\xi}$ vanishes only on a set of Haar measure zero in $(G/N)^\wedge$. In particular, the set of all $\lambda \in (G/N)^\wedge$ where

$$\langle (\pi \otimes \lambda)(\chi_E)\xi, \xi\rangle \neq 0$$

is dense in $(G/N)^\wedge$. This proves the above claim.

Now fix $\sigma \in \Sigma$. Then there are nets $\pi_i \in \hat{G}_r$ and $\sigma_i \in \hat{N}_r$, $i \in I$, such that $\sigma_i \in \operatorname{supp}(\pi|_N)$, $\sigma_i \in \Sigma$ and $\sigma_i \to \sigma$ in $\hat{N}_r$. By our previous arguments, for every i there is a dense subset Λ_i of $(G/N)^\wedge$ such that $(\pi \otimes \lambda)(\chi_E) \neq 0$ for every $\lambda \in \Lambda_i$. Next observe that by continuity of forming tensor products

$$\pi_i \otimes \widehat{G/N} \sim \pi_i \otimes \Lambda_i$$

and by continuity of inducing $\operatorname{ind}_N^G \sigma_i \to \operatorname{ind}_N^G \sigma$. Combining all these facts we obtain

$$\operatorname{ind}_N^G \sigma \prec \{\operatorname{ind}_N^G \sigma_i \mid i \in I\} \prec \{\operatorname{ind}_N^G(\pi_{i|_N}) \mid i \in I\} \sim \bigcup_{i \in I} \pi_i \otimes (G/N)^\wedge \sim \bigcup_{i \in I} \pi_i \otimes \Lambda_i$$

and this last set is contained in $\Pi = \{\pi \in \hat{G} \mid \pi(\chi_E) \neq 0\}$. Finally, as Σ is dense in $\hat{N}_r$,

$$\bigcup_{\sigma \in \Sigma} \operatorname{supp}(\operatorname{ind}_N^G \sigma)$$

is dense in $\hat{G}_r$. This shows that Π is dense in $\hat{G}_r$.

Remark 3.2. We note that a result similar to this, and with possible applications in a different context, may be proved by essentially the same method as Theorem 3·1. This is the fact that if N is a closed normal subgroup of G such that G/N is abelian and has no compact elements then if the hypothesis of Lemma 1·3 (a) holds for all $h \in C_c(N)$ it holds for all $g \in C_c(G)$.

The following corollary is a straightforward consequence of Theorem 3·1.

COROLLARY 3·3. *Suppose that G contains an ascending series*

$$\{e\} = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_r = G$$

of closed subgroups, where for each $1 \leqslant j \leqslant r$, N_{j-1} is normal in N_j and N_j/N_{j-1} is abelian and has no compact elements. Then G is a Pompeiu group.

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COROLLARY 3·4. *Every connected and simply connected solvable Lie group is a Pompeiu group.*

Proof. The nilradical $N = [G, G]$ of G is simply connected, so that the quotients in the ascending central series of N are vector groups. Moreover, G/N is a vector group (see [**16**], chapter 12, theorem 1·2).

The remainder of this section is devoted to nilpotent groups.

THEOREM 3·5. *Let G be an arbitrary locally compact nilpotent group. Then G is a Pompeiu group if and only if G contains no compact elements.*

Proof. Denote by G^c the set of all compact elements in G and by G_0 the connected component of G. We first collect some facts about G^c.

(i) G^c is a closed (normal) subgroup of G ([**14**], corollary 3·5·1 and lemma 3·8).

(ii) $G_0 G^c$ is open in G, and $G/G_0 G^c$ is torsion-free ([**14**], theorem 8·3).

(iii) If D is a torsion-free nilpotent (discrete) group and $Z(D)$ the centre of D, then as is well known, $D/Z(D)$ is again torsion-free.

Suppose now that $G^c = \{e\}$. Then in particular G_0 contains no non-trivial compact subgroup and hence is simply connected. Moreover, G_0 is open in G by (ii), and all factor groups in the ascending central series of G/G_0 are torsion-free by (ii) and (iii). Corollary 3·3 now implies that G is Pompeiu.

Conversely, assume $G^c \neq \{e\}$. It suffices to proves that G contains a non-trivial compact normal subgroup, since then G fails to be Pompeiu by Lemma 1·2. Let

$$Z_1 = Z(G) \subseteq Z_2 \subseteq \dots$$

denote the ascending central series of G. Let m be minimal with the property $G^c \cap Z_{m+1} \neq \{e\}$. Choose any $a \in G^c \cap Z_{m+1}, a \neq e$, and let K be the compact subgroup generated by a and set $H = KZ_m$. H is normal in G, hence so is H^c. But $H^c = K$ since H^c is a group and $G^c \cap Z_m = \{e\}$.

4. *Remarks on the L^∞ case*

We say that G is L^∞-Pompeiu if for all relatively compact Borel subsets $E \subset G$ of non-zero Haar measure and $f \in L^\infty(G)$ the condition $\int_{xEy} f = 0$ for all $x, y \in G$ implies $f = 0$.

PROPOSITION 4·1. *Suppose N is a first countable closed normal subgroup of a locally compact group G. If G/N is not L^∞-Pompeiu then neither is G.*

Proof. Notice first that as N is normal and first countable there exists a Borel cross-section $\beta: G/N \to G$ which maps compact sets to relatively compact sets (see [**10**], theorem 1 and remarks 3 and 6). Suppose $E \subset G/N$ is relatively compact of non-zero Haar measure and $f \in L^\infty(G/N), f \neq 0$, is such that

$$\int_{G/N} f(\dot{x})\, \chi_{\dot{a}E\dot{b}}(\dot{x})\, d\dot{x} = 0 \quad \text{for all } \dot{a}, \dot{b} \in G/N.$$

Let $\alpha: G \to N$ be defined by $\alpha(x) = x^{-1}\beta(\dot{x})$. Let $g \in L^\infty(G)$ be the function $g(x) = f(\dot{x})$ and choose a relatively compact open subset U of N. Then

$$(\chi_U \circ \alpha)(x)\, \chi_E(\dot{x}) = \chi_F(x)$$

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for some relatively compact $F \subset G$. To lighten the notation below we write xN rather than $\dot{x}$ which we used previously. Now let $\delta(b)$ be the modular function arising from the automorphism $n \mapsto n^b = bnb^{-1}$ of N and calculate:

$$\begin{aligned}
\int_G \chi_{a^{-1}Fb^{-1}}(x)\, g(x)\, dx &= \int_G \chi_U(\alpha(axb))\, \chi_E(axbN) f(xN)\, dx \\
&= \int_{G/N} \int_N \chi_U(b^{-1}n^{-1}x^{-1}a^{-1}\beta(axnbN))\, \chi_E(axbN) f(xN)\, dn\, d(xN) \\
&= \int_{G/N} \int_N \chi_U((n^{-1})^b(axb)^{-1}\beta(axbn^bN))\, \chi_E(axbN) f(xN)\, dn\, d(xN) \\
&= \int_{G/N} \delta(b) \int_N \chi_U(n^{-1}b^{-1}x^{-1}a^{-1}\beta(axbN))\, \chi_E(axb) f(xN)\, dn\, d(xN) \\
&= \int_{G/N} \delta(b) \int_N \chi_U(n^{-1}\alpha(axb))\, \chi_E(axbN) f(xN)\, dn\, d(xN) \\
&= \delta(b)\, k \int_{G/N} \chi_E(axbN) f(xN)\, d(xN)
\end{aligned}$$

where k is a constant independent of a, x, b and depending only on U. Thus we have

$$\int_G \chi_{a^{-1}Fb^{-1}}(x)\, g(x)\, dx = 0$$

for all $a, b \in G$ proving that F is not an L^∞-Pompeiu set.

LEMMA 4·2. *If G is a non-trivial abelian group then G is not L^∞-Pompeiu.*

Proof. We readily see that G cannot be L^∞-Pompeiu if it contains a non-trivial compact element. Indeed the L^1-function constructed in the proof of Lemma 1·2 was bounded. Thus we may assume that $G = \mathbb{R}^d \times D$ where D is discrete and torsion-free. By Proposition 4·1 it suffices to verify that neither $\mathbb{R}$ nor D is L^∞-Pompeiu. For $\mathbb{R}$ we take $E = [0, 1]$ and f to be any periodic continuous function such that $\int_0^1 f(t)\, dt = 0$. Next, for D, we choose $a \in D$, $a \neq e$ and denote by A the subgroup generated by a. Then take $E = \{e, a\}$ and f defined by $f(a^n) = (-1)^n$ for $n \in \mathbb{Z}$, and $f|_{D \setminus A} = 0$.

COROLLARY 4·3. *Any first countable locally compact group G such that $[G, G] \neq G$ is not L^∞-Pompeiu.*

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