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Recurrence and pressure for group extensions

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Abstract. We investigate the thermodynamic formalism for recurrent potentials on group extensions of countable Markov shifts. Our main result characterizes recurrent potentials depending only on the base space, in terms of the existence of a conservative product measure and a homomorphism from the group into the multiplicative group of real numbers. We deduce that, for a recurrent potential depending only on the base space, the group is necessarily amenable. Moreover, we give equivalent conditions for the base pressure and the skew product pressure to coincide. Finally, we apply our results to analyse the Poincaré series of Kleinian groups and the cogrowth of group presentations.

1. *Introduction and statement of results*

In this paper, we investigate the central ideas of the thermodynamic formalism for recurrent potentials in the context of group extensions of countable Markov shifts (Σ, σ) with alphabet $I \subset \mathbb{N}$ and left shift map $\sigma : \Sigma \to \Sigma$. For a countable group G and for a semigroup homomorphism $\Psi : I^* \to G$, where I^* denotes the free semigroup generated by I, the skew product dynamical system $\sigma \rtimes \Psi : \Sigma \times G \to \Sigma \times G$, which is given by

$$(\sigma \rtimes \Psi)(\omega, g) = (\sigma(\omega), g\Psi(\omega_1)), \quad \omega = (\omega_1, \omega_2, \dots) \in \Sigma, \quad g \in G,$$

is called a *group-extended Markov system* (see §2.1 for details of Markov shifts). Let $\varphi : \Sigma \to \mathbb{R}$ denote a Hölder continuous potential, which extends to the potential $\varphi \circ \pi_1 : \Sigma \times G \to \mathbb{R}$, where $\pi_1 : \Sigma \times G \to \Sigma$ denotes the canonical projection on Σ. The *Perron–Frobenius operator* [**Rue68**, **Bow75**] is given by $\mathcal{L}_{\varphi\circ\pi_1}(f)(x) := \sum_{(\sigma\rtimes\Psi)(y)=x} e^{\varphi\circ\pi_1(y)} f(y)$. The potential $\varphi \circ \pi_1$ is called *recurrent* [**Sar99**, **Sar01**] if there exist $\rho > 0$ and a conservative measure ν on $\Sigma \times G$, finite and positive on cylindrical sets, such that $\mathcal{L}^*_{\varphi\circ\pi_1}(\nu) = \rho\nu$. Here, the measure is a σ-finite Borel measure on $\Sigma \times G$, and $\mathcal{L}^*_{\varphi\circ\pi_1}(\nu) = \rho\nu$ means that, for every $f \in L^1(\nu)$, we have $\nu(\mathcal{L}_{\varphi\circ\pi_1}(f)) = \rho\nu(f)$. In particular, the sum in the definition of $\mathcal{L}_{\varphi\circ\pi_1}(f)$ converges absolutely for ν-almost every x. If $\varphi \circ \pi_1$ is recurrent, then the logarithm of ρ is given by the *Gurevič pressure* $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ of $\varphi \circ \pi_1$ with respect to $\sigma \rtimes \Psi$. Moreover, there exists a continuous function $h : \Sigma \times G \to \mathbb{R}^+$ such that $\mathcal{L}_{\varphi\circ\pi_1}(h) = \rho h$, and the measure $h\, d\nu$

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(unique up to a constant multiple) is called the *Ruelle–Perron–Frobenius (RPF)* measure of $\varphi \circ \pi_1$. For details of these results of Sarig and their applications to the theory of equilibrium states, we refer the reader to §2.2 and [**Sar01**]. That the Ruelle–Perron–Frobenius measure of a recurrent potential is ergodic follows from work of Aaronson, Denker and Urbański [**ADU93**].

For a finitely primitive Markov shift Σ (see Definition 2.2) and a Hölder continuous potential $f : \Sigma \to \mathbb{R}$ with finite Gurevič pressure, there exists a unique σ-invariant Gibbs measure for f, which is denoted by μ_f (see Theorem 3). Note that if the alphabet I of Σ is infinite, then f is necessarily unbounded (see e.g. [**MU03**, Proposition 2.1.9]). It is therefore important to remark that our definition of Hölder continuity allows for unbounded potentials, since we require a Hölder condition only on cylindrical sets of length $n \geq 1$ (see (2.1)). Throughout, we let $\pi_2 : \Sigma \times G \to G$ denote the projection on G and we let λ denote the Haar measure on G.

We are now in the position to state our main result, which characterizes when $\varphi \circ \pi_1$ is recurrent.

THEOREM 1. *Let Σ be finitely primitive and let $(\Sigma \times G, \sigma \rtimes \Psi)$ be a topologically mixing group-extended Markov system. Let $\varphi : \Sigma \to \mathbb{R}$ be Hölder continuous with $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) < \infty$. Then $\varphi \circ \pi_1$ is recurrent if and only if there exists a (unique) group homomorphism $c : G \to (\mathbb{R}^+, \cdot)$ such that $\mathcal{P}(\varphi_c, \sigma) < \infty$ and $\mu_{\varphi_c} \times \lambda$ is conservative, where φ_c is given by $\varphi_c(x) := \varphi(x) - \log c(\Psi(x_1))$ for $x = (x_1, x_2, \dots) \in \Sigma$. Moreover, if $\varphi \circ \pi_1$ is recurrent, then the following four statements hold.*

(1) $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi_c, \sigma)$.

(2) *The unique (up to a constant multiple) continuous function $h : \Sigma \times G \to \mathbb{R}^+$, which is fixed by $e^{-\mathcal{P}(\varphi\circ\pi_1, \sigma\rtimes\Psi)}\mathcal{L}_{\varphi\circ\pi_1}$ and which is bounded on cylindrical sets, is given by $h = (h_1 \circ \pi_1)(c \circ \pi_2)$, where $h_1 : \Sigma \to \mathbb{R}^+$ is the unique Hölder continuous fixed point of $e^{-\mathcal{P}(\varphi_c,\sigma)}\mathcal{L}_{\varphi_c}$.*

(3) *The unique (up to a constant multiple) measure ν on $\Sigma \times G$, which is fixed by $e^{-\mathcal{P}(\varphi\circ\pi_1, \sigma\rtimes\Psi)}\mathcal{L}^*_{\varphi\circ\pi_1}$ and which is positive and finite on cylindrical sets, is given by $\nu = (c \circ \pi_2)^{-1}\, d(\nu_1 \times \lambda)$, where ν_1 is the unique probability measure fixed by $e^{-\mathcal{P}(\varphi_c,\sigma)}\mathcal{L}^*_{\varphi_c}$.*

(4) *The Ruelle–Perron–Frobenius measure $h\, d\nu$ of $\varphi \circ \pi_1$ is given by $\mu_{\varphi_c} \times \lambda$.*

The first assertion of the following corollary is a generalization of a result of Kesten, who pointed out that, for an irreducible recurrent random walk on a countable group G, the group is necessarily amenable [**Kes67**, p. 73]. Here, the random walk is called irreducible if it is possible to get from any element $g_1 \in G$ to any other element $g_2 \in G$ with positive probability. In order to state the second assertion of the corollary, recall that $\varphi \circ \pi_1$ is *positive recurrent* if the Ruelle–Perron–Frobenius measure $h\, d\nu$ of $\varphi \circ \pi_1$ is a finite measure (see Definition 2.6 and Theorem 4).

COROLLARY 1.1. *Under the hypotheses of Theorem 1, we have the following.*

(1) *If $\varphi \circ \pi_1$ is recurrent, then G is amenable.*

(2) *$\varphi \circ \pi_1$ is positive recurrent $\iff$ G is finite $\iff$ $(\Sigma \times G, \sigma \rtimes \Psi)$ is finitely primitive.*

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Proof. We start with the proof of (1). By Theorem 1(4), the ergodic RPF measure of $\varphi \circ \pi_1$ is the product measure $\mu_{\varphi_c} \times \lambda$. Since the natural extension of $(\Sigma, \sigma, \mu_{\varphi_c})$ is an ergodic $\mathbb{Z}$-space, it follows from [**Zim78**, Theorem 2.1] that $(\Sigma, \sigma, \mu_{\varphi_c})$ is an amenable $\mathbb{Z}$-space (see Zimmer [**Zim78**] for the definition). Now, the ergodicity of $\mu_{\varphi_c} \times \lambda$ implies that G is amenable by [**Zim78**, Theorem 3.1] (see also [**AW04**]).

Let us now prove (2). If $\varphi \circ \pi_1$ is positive recurrent, then the RPF measure of $\varphi \circ \pi_1$ is finite. Hence, by Theorem 1(4), the measure $\mu_{\varphi_c} \times \lambda$ is finite, which implies that G is finite. On the other hand, if G is finite, then $(\Sigma \times G, \sigma \rtimes \Psi)$ is finitely primitive, which implies that $\varphi \circ \pi_1$ is positive recurrent (see [**Sar03**, Corollary 2]). The remaining implication, that finite primitivity of $(\Sigma \times G, \sigma \rtimes \Psi)$ implies the finiteness of G, is straightforward and therefore omitted. □

Remark. By Theorem 1(4), the RPF measure of a recurrent potential $\varphi \circ \pi_1$ satisfies the Gibbs property (2.2) with respect to the potential $\varphi_c \circ \pi_1$.

As a further corollary of our main theorem, we obtain the following equivalent conditions for the base pressure and the skew product pressure of a recurrent potential $\varphi \circ \pi_1$ to coincide.

COROLLARY 1.2. *Under the hypotheses of Theorem 1, suppose that $\varphi \circ \pi_1$ is recurrent. Then the following statements are equivalent.*

(1) *The group homomorphism $c : G \to (\mathbb{R}^+, \cdot)$ given by Theorem 1 is trivial.*
(2) $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$.
(3) *The function $h : \Sigma \times G \to \mathbb{R}^+$, given by Theorem 1(2), is independent of the second coordinate.*
(4) *The measure ν on $\Sigma \times G$, given by Theorem 1(3), is a product measure.*
(5) *The Ruelle–Perron–Frobenius measure of $\varphi \circ \pi_1$ is given by $\mu_\varphi \times \lambda$.*

Proof. If $c = 1$, then we have $\mathcal{P}(\varphi, \sigma) = \mathcal{P}(\varphi_c, \sigma) = \mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ by Theorem 1(1), which proves that (1) implies (2). Next, we prove that (2) implies (3). Since Σ is finitely primitive and φ is Hölder continuous, it follows from [**MU03**, Theorem 2.4.3] that there exists a bounded Hölder continuous function $h_1 : \Sigma \to \mathbb{R}^+$, which is fixed by $e^{-\mathcal{P}(\varphi,\sigma)}\mathcal{L}_\varphi$. Consequently, we have that

$$e^{-\mathcal{P}(\varphi,\sigma)}\mathcal{L}_{\varphi\circ\pi_1}(h_1 \circ \pi_1) = e^{-\mathcal{P}(\varphi,\sigma)}(\mathcal{L}_\varphi(h_1)) \circ \pi_1 = h_1 \circ \pi_1.$$

Since $\mathcal{P}(\varphi, \sigma) = \mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ by (2), we have $h = h_1 \circ \pi_1$, which proves (3). The proof that (1) is equivalent to (3) follows from Theorem 1(2). The equivalence of (3) and (4) follows from Theorem 1(2) and (3). To finish the proof, we verify that (1) and (5) are equivalent. If $c = 1$, then $\mu_\varphi \times \lambda$ is the RPF measure of $\varphi \circ \pi_1$ by Theorem 1(4). On the other hand, if $\mu_\varphi \times \lambda$ is the RPF measure of $\varphi \circ \pi_1$, then we have that $\mu_\varphi \times \lambda$ is conservative. Hence, that c is trivial follows from the uniqueness of c in Theorem 1. □

In order to state the next proposition, we give the following definition. For $k \in \mathbb{N}$, let Σ^k denote the set of admissible words of length k and, for $\omega \in \Sigma^k$, let $[\omega]$ denote the cylindrical set given by ω.

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Definition 1.3. Let $(\Sigma \times G, \sigma \rtimes \Psi)$ be an irreducible group-extended Markov system. Let $\varphi : \Sigma \to \mathbb{R}$ be Hölder continuous with $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) < \infty$. We say that φ is *symmetric on average with respect to* Ψ if

$$\sup_{g \in G} \limsup_{n \to \infty} \frac{\sum_{k=1}^{n} e^{-k\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)} \sum_{\omega \in \Sigma^k : \Psi(\omega) = g} e^{\sup S_k \varphi_{|[\omega]}}}{\sum_{k=1}^{n} e^{-k\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)} \sum_{\omega \in \Sigma^k : \Psi(\omega) = g^{-1}} e^{\sup S_k \varphi_{|[\omega]}}} < \infty.$$

PROPOSITION 1.4. *Under the hypotheses of Theorem 1, suppose that $\varphi \circ \pi_1$ is recurrent. Then we have that φ is symmetric on average with respect to Ψ if and only if $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$.*

Remark. For a finite group G, every homomorphism $c : G \to (\mathbb{R}^+, \cdot)$ is trivial. Furthermore, every Hölder continuous potential $\varphi \circ \pi_1 : \Sigma \to \mathbb{R}$ with finite Gurevič pressure is positive recurrent by Corollary 1.1(2). Hence, by combining the statement that (1) implies (2) from Corollary 1.2 and Proposition 1.4, we have that φ is symmetric on average with respect to Ψ.

Since it will be convenient for some of our applications to investigate irreducible and not necessarily aperiodic group extensions, we make the following remark.

Remark 1.5. The results of Corollary 1.1 also hold if $(\Sigma \times G, \sigma \rtimes \Psi)$ is irreducible with period $p > 1$. Moreover, regarding Proposition 1.4, we have $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$ if and only if

$$\sup_{g \in G_0} \limsup_{n \to \infty} \frac{\sum_{k=1}^{n} e^{-kp\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)} \sum_{\omega \in \Sigma^{kp} : \Psi(\omega) = g} e^{\sup S_{kp} \varphi_{|[\omega]}}}{\sum_{k=1}^{n} e^{-kp\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)} \sum_{\omega \in \Sigma^{kp} : \Psi(\omega) = g^{-1}} e^{\sup S_{kp} \varphi_{|[\omega]}}} < \infty,$$

where $G_0 := \bigcup_{n \in \mathbb{N}} \Psi(\Sigma^{np})$ is a subgroup of G.

In the next example, which illustrates Proposition 1.4, we consider asymmetric nearest-neighbour random walks on the additive integers $(\mathbb{Z}, +)$.

Example 1.6. For the alphabet $I := \{1, -1\}$, we consider the full shift $\Sigma := I^{\mathbb{N}}$ and the group-extended Markov system given by the canonical semigroup homomorphism $\Psi : I^* \to (\mathbb{Z}, +)$. By Theorem 1, each homomorphism $c : (\mathbb{Z}, +) \to (\mathbb{R}^+, \cdot)$ gives rise to a recurrent potential $\varphi \circ \pi_1$, given by $\varphi(x) := \log c(x_1)$, for which we have $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(0, \sigma) = \log 2$. Moreover, we have that

$$\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \log 2 \leq \log(c(1) + c(-1)) = \mathcal{P}(\varphi, \sigma),$$

with equality if and only if $c = 1$ (cf. Corollary 1.2). By Proposition 1.4, we have $c = 1$ if and only if φ is symmetric on average with respect to Ψ. Also, note that the gap between $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ and $\mathcal{P}(\varphi, \sigma)$ can be arbitrarily large. Further, $\varphi \circ \pi_1$ is null recurrent by Corollary 1.1(2).

Proposition 1.4 extends a result of Matsuzaki and Yabuki (Theorem 2) for Kleinian groups to the general framework of the thermodynamic formalism for group extensions of Markov shifts. In this way, we shed new light on Theorem 2. The result of Matsuzaki and Yabuki was used in [**Jae13**] to give a short new proof of a lower bound of the exponent of

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convergence for normal subgroups of Kleinian groups. By the results of Proposition 1.4, a similar lower bound can be proved in the setting of graph-directed Markov systems associated to free groups [**Jae14c**].

The proof of Theorem 1 relies on Sarig's thermodynamic formalism for recurrent potentials (see [**Sar01**] or Theorem 4), which characterizes recurrence in terms of the existence of a conservative eigenmeasure and a unique continuous eigenfunction of the Perron–Frobenius operator. Sarig's construction of this eigenmeasure is similar to the construction of the Patterson measure [**Pat76**]. The work of Aaronson *et al*. [**ADU93**], giving exactness and pointwise dual ergodicity of eigenmeasures of the Perron–Frobenius operator under certain assumptions, is also crucial for Sarig's results.

The paper is organized as follows. In §2, we collect the necessary preliminaries from the thermodynamic formalism for countable-state Markov shifts, which include the definitions of a Markov shift, Hölder continuous functions, the Gurevič pressure and Gibbs measures in §2.1. In §2.2, we give the definition of recurrent potentials and a characterization of these in terms of fixed points of the Perron–Frobenius operator, which is due to Sarig. In §2.3, we review some recent results on group-extended Markov systems and amenability. In §3, we give the proofs of our main results.

Let us end this introductory section with the following subsections. In §1.1, we show how to deduce Theorem 2 of Matsuzaki and Yabuki for a large class of Fuchsian groups by applying Proposition 1.4. In §1.2, we give an ergodic-theoretic proof of an estimate on the cogrowth of group presentations. In §1.3, we discuss how our main theorem relates to amenability of groups. In §1.4, we briefly relate our results to the classification of recurrent groups. We give an extension of this classification to locally constant potentials on irreducible group-extended Markov systems and we give a conjecture for the general case of Hölder continuous potentials.

1.1. *A related result of Matsuzaki and Yabuki for Kleinian groups.* In order to state the result of Matsuzaki and Yabuki, recall that a Kleinian group G is a discrete group of isometries acting on hyperbolic space, and that G is defined to be of divergence type if its Poincaré series diverges at the exponent of convergence $\delta(G)$. We refer the reader to [**Bea95**, **Mas88**, **Nic89**] for details of Kleinian groups.

THEOREM 2. [**MY09**, Theorem 4.2 and Corollary 4.3] *Let N denote a non-trivial normal subgroup of a Kleinian group Γ. If N is of divergence type, then we have that $\delta(N) = \delta(\Gamma)$.*

In this subsection, we briefly explain how Theorem 2 is related to Proposition 1.4. We consider the special case in which Γ is a non-elementary Fuchsian group of Schottky type acting on the Poincaré disc model of hyperbolic 2-space $(\mathbb{D}, d)$, where d denotes the hyperbolic metric. Recall that, in this case, Γ is isomorphic to the free group $\mathbb{F}_t = \langle g_1, \ldots, g_t \rangle$ with $t \geq 2$. It is well known (see e.g. [**Ser81**]) that the limit set of Γ can be identified with the state space of the Markov shift $\Sigma_{\mathbb{F}_t}$ with alphabet $I := \{g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_t, g_t^{-1}\}$ given by

$$\Sigma_{\mathbb{F}_t} = \{\omega \in I^{\mathbb{N}} \mid \omega_i \neq \omega_{i+1}^{-1}\}.$$

Note that $\Sigma_{\mathbb{F}_t}$ is equal to the space of ends of the Cayley graph of $\mathbb{F}_t$ with respect to I.

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It is well known that there exists a Hölder continuous potential $\zeta : \Sigma_{\mathbb{F}_t} \to \mathbb{R}$ which captures the modulus of the derivatives of the generators of Γ acting on the limit set (see [**Ser81**]). This potential has the geometric property that there exists $C > 0$ such that, for all $n \in \mathbb{N}$ and $\omega \in \Sigma^n_{\mathbb{F}_t}$, we have that

$$C^{-1} e^{\sup S_n \zeta_{|[\omega]}} \leq e^{-d(0,\omega_1\omega_2\cdots\omega_n(0))} \leq C e^{\sup S_n \zeta_{|[\omega]}}. \tag{1.1}$$

For a non-trivial normal subgroup N of $\mathbb{F}_t$, let $\Psi_N : I^* \to \mathbb{F}_t/N$ be the unique semigroup homomorphism such that $\Psi_N(g_i) := Ng_i$ for each $g_i \in I$. It follows from (1.1) that the exponents of convergence $\delta(\Gamma)$ and $\delta(N)$ are characterized by the following equations, often referred to as Bowen's formula (cf. [**Bow79**]):

$$\mathcal{P}(\delta(\Gamma)\zeta, \sigma) = 0, \quad \mathcal{P}(\delta(N)(\zeta \circ \pi_1), \sigma \rtimes \Psi_N) = 0. \tag{1.2}$$

Moreover, using the fact that each $g \in \Gamma$ acts isometrically on $(\mathbb{D}, d)$, one immediately deduces from (1.1) that ζ is symmetric on average with respect to Ψ_N. Moreover, we have that N is of divergence type if and only if $\delta(N)(\zeta \circ \pi_1)$ is recurrent with respect to $\sigma \rtimes \Psi_N$. Hence, if N is of divergence type, then Proposition 1.4 yields

$$\mathcal{P}(\delta(N)\zeta, \sigma) = \mathcal{P}(\delta(N)(\zeta \circ \pi_1), \sigma \rtimes \Psi_N) = 0,$$

which, in light of (1.2), gives $\delta(N) = \delta(\Gamma)$.

1.2. *An application to the cogrowth of group presentations.* For $t \geq 2$, we consider the presentation $\langle g_1, \ldots, g_t | r_1, r_2, \ldots \rangle$ of a finitely generated group G and we let N denote the non-trivial normal subgroup of $\mathbb{F}_t$ generated by the relations $r_1, r_2, \ldots$. The *cogrowth* η of $\langle g_1, \ldots, g_t | r_1, r_2, \ldots \rangle$, which was independently introduced by Grigorchuk [**Gri80**] and Cohen [**Coh82**], is given by

$$\eta := \frac{\log \gamma}{\log(2t-1)} \quad \text{where } \gamma := \limsup_{n\to\infty} (\mathrm{card}(\{\omega \in \mathbb{F}_t \cap N \mid |\omega| = n\}))^{1/n}.$$

Here, $|\omega|$ refers to the word length of ω with respect to $\{g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_t, g_t^{-1}\}$. It is known that $\frac{1}{2} < \eta \leq 1$ and that $\eta = 1$ if and only if G is amenable [**Gri80**, **Coh82**]. Since N is a non-trivial normal subgroup of $\mathbb{F}_t$, there exist $h \in N \setminus \{\mathrm{id}\}$ and $\{ghg^{-1} \mid g \in \mathbb{F}_t\} \subset H$. A simple counting argument then shows that

$$\sum_{n\in\mathbb{N}} (\mathrm{card}\{\omega \in N, |\omega| = n\})(2t-1)^{-n/2} = \infty, \tag{1.3}$$

which immediately gives that $(2t-1)^{1/2} \leq \gamma$ and, hence, $\eta \geq \frac{1}{2}$. We aim to give an ergodic-theoretic proof of the strict inequality $\eta > \frac{1}{2}$ by using Proposition 1.4. In order to apply Proposition 1.4, we consider the constant zero function $0 : \Sigma_{\mathbb{F}_t} \to \mathbb{R}$ and the group-extended Markov system $\sigma \rtimes \Psi_N : \Sigma_{\mathbb{F}_t} \times (\mathbb{F}_t/N) \to \Sigma_{\mathbb{F}_t} \times (\mathbb{F}_t/N)$ as in §1.1. Clearly, we have $\mathcal{P}(0, \sigma) = \log(2t-1)$ and $\mathcal{P}(0, \sigma \rtimes \Psi_N) = \log \gamma$. Further, 0 is symmetric on average with respect to Ψ_N. By Proposition 1.4, we conclude that either 0 is recurrent with respect to $\sigma \rtimes \Psi_N$ and $\gamma = 2t-1$, or 0 is transient with respect to $\sigma \rtimes \Psi_N$. In the latter case, it follows from (1.3) that $\gamma \neq (2t-1)^{1/2}$. We have thus shown that, in both cases, $(2t-1)^{1/2} < \gamma$ and, hence, $\eta > \frac{1}{2}$.

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1.3. *Amenable groups.* Lifting a potential to a group extension may result in a drop of Gurevič pressure, that is, $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) < \mathcal{P}(\varphi, \sigma)$. This phenomenon is intimately linked to non-amenability of the associated group and was first observed by Kesten for random walks on countable discrete groups driven by a symmetric independent identically distributed process on a set of generators [**Kes59b**, **Kes59a**]. The cogrowth criterion due to Grigorchuk and Cohen [**Gri80**, **Coh82**], which characterizes amenability of groups in terms of cogrowth, is a similar result, in which the independent identically distributed process is replaced by a certain Markov process. Moreover, a result of Brooks [**Bro85**] for Kleinian groups certainly fits into this context, where equality of the exponents of convergence of normal subgroups of certain convex cocompact Kleinian groups is characterized in terms of amenability. Recently, amenability of groups has been studied in the framework of the thermodynamic formalism of group-extended Markov systems, where the random walk is driven by a Gibbs measure associated to a Hölder continuous potential on a Markov shift [**Jae11**, **Sta13**, **Jae14a**, **Jae14b**]. We refer the reader to §2.3 for a short review of these results.

Let us now explain how these results are related to our results given in this paper. Let $(\Sigma \times G, \sigma \rtimes \Psi)$ be an irreducible group-extended Markov system and let $\varphi : \Sigma \to \mathbb{R}$ be Hölder continuous. Corollary 1.1(1) and Proposition 1.4 prove that, for a recurrent potential $\varphi \circ \pi_1$, the associated group is amenable, and that the skew product pressure $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi)$ and the base pressure $\mathcal{P}(\varphi, \sigma)$ coincide if and only if φ is symmetric on average with respect to Ψ. If φ is asymptotically symmetric with respect to Ψ (see Definition 2.11) or weakly symmetric (cf. [**Sta13**]) and the group is amenable, then $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$ by Theorem 6 (cf. [**Sta13**, Theorem 4.1]). Hence, a recurrent and asymptotically symmetric potential is symmetric on average. For an amenable group and a potential which is not necessarily recurrent, a necessary condition for the base pressure and skew product pressure to coincide is not known. A sufficient condition is that the potential is asymptotically symmetric or weakly symmetric. It seems that the proofs given in [**Jae14b**] or [**Sta13**, Theorem 4.1], which make use of Kesten's classical criterion for amenability, cannot be generalized to potentials which are symmetric on average.

1.4. *Recurrent groups.* A classical result by Pólya [**Pól21**] states that the simple random walk on $\mathbb{Z}^d$ is recurrent if and only if $d \leq 2$. Dudley [**Dud62**] has shown that a countable abelian group is recurrent if it has rank at most two, where a *recurrent group* is a group which carries an irreducible recurrent random walk. Kesten [**Kes67**] motivated the interesting task of classifying recurrent groups and it became known as Kesten's conjecture that finitely generated recurrent groups are finite or contain $\mathbb{Z}$ or $\mathbb{Z}^2$ as a finite-index subgroup. Kesten's conjecture was proved by Varopoulos [**Var86**] using a famous result of Gromov [**Gro81**]. An extension for random walks on locally finite graphs with quasitransitive automorphism groups is due to Woess (see [**Woe94a**, Corollary 4.7] and [**Woe94b**, Theorem 4.1]). Another related result is due to Rees [**Ree81a**, **Ree81b**], which shows the following for a non-trivial normal subgroup N of a geometrically finite Fuchsian group Γ such that $\Gamma/N \simeq \mathbb{Z}^d$ for some $d \in \mathbb{N}$: firstly, the exponents of convergence $\delta(N)$ and $\delta(\Gamma)$ coincide. Further, if Γ contains no parabolic elements, then the Fuchsian group

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N is of divergence type if and only if $d \leq 2$. If Γ contains parabolic elements, then the situation is more complicated; however, the implication that a divergence type subgroup N satisfies $d \leq 2$ remains true. In fact, the results in [**Ree81a**] are deduced for general Gibbs measures satisfying a certain symmetry condition. However, we remark that, in contrast to the result of Matsuzaki and Yabuki and to Proposition 1.4, it is *a priori* assumed in the work of Rees that $\Gamma/N \simeq \mathbb{Z}^d$ for some $d \in \mathbb{N}$.

In our framework, the classification of recurrent groups implies the following. Let $\Psi : I^* \to G$ denote a surjective semigroup homomorphism and let $\varphi : I^{\mathbb{N}} \to \mathbb{R}$ be a potential depending only on the first coordinate. If $\varphi \circ \pi_1$ is recurrent, then G is a recurrent group. In particular, since each recurrent group is amenable, we obtain from Kesten's classical theorem that if φ is symmetric on average with respect to Ψ, then $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$. Regarding Example 1.6, we observe that the potential $\varphi \circ \pi_1$ is recurrent, $\mathbb{Z}$ is a recurrent group and $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) < \mathcal{P}(\varphi, \sigma)$ whenever the potential is not symmetric on average with respect to Ψ.

If we replace the full shift $I^{\mathbb{N}}$ by an irreducible Markov shift Σ with finite alphabet I, and if we consider $\varphi : \Sigma \to \mathbb{R}$ depending only on the first coordinate, then we obtain the following by investigating the group of graph automorphisms using a result of Woess [**Woe94b**, Theorem 4.1].

PROPOSITION 1.7. *Let* $(\Sigma \times G, \sigma \rtimes \Psi)$ *be an irreducible group-extended Markov system with* $\operatorname{card}(I) < \infty$. *Suppose that* $\varphi : \Sigma \to \mathbb{R}$ *depends only on the first coordinate. If* $\varphi \circ \pi_1$ *is recurrent, then* G *is a recurrent group.*

Conjecture 1.8. Let $(\Sigma \times G, \sigma \rtimes \Psi)$ be an irreducible group-extended Markov system with $\operatorname{card}(I) < \infty$. Suppose that $\varphi : \Sigma \to \mathbb{R}$ is Hölder continuous. If $\varphi \circ \pi_1$ is recurrent, then G is a recurrent group.

The assertion of the conjecture was proved for a large class of Hölder continuous potentials on $\Sigma_{\mathbb{F}_t}$ with $t \geq 2$ under the additional assumption that G is a free abelian group in [**Ree81a**, Theorem 4.7]. It also worth noting that, if the conjecture is true, then it provides a significant strengthening of Corollary 1.1(1), stating that recurrence of $\varphi \circ \pi_1$ implies that G is amenable.

2. *Preliminaries*

2.1. *Symbolic dynamics.* Throughout, the underlying state space for the symbolic thermodynamic formalism will be a *Markov shift* Σ, given by

$$\Sigma := \{\omega := (\omega_1, \omega_2, \ldots) \in I^{\mathbb{N}} \mid a(\omega_i, \omega_{i+1}) = 1 \ \forall i \in \mathbb{N}\},$$

where $I \subset \mathbb{N}$ denotes a finite or countable infinite *alphabet*, the matrix $A = (a(i, j)) \in \{0, 1\}^{I \times I}$ is the *incidence matrix* and the *left shift map* $\sigma : \Sigma \to \Sigma$ is defined by $\sigma((\omega_1, \omega_2, \ldots)) := (\omega_2, \omega_3, \ldots)$. The set of *A-admissible words* of length $n \in \mathbb{N}$ is given by

$$\Sigma^n := \{\omega \in I^n \mid a(\omega_i, \omega_{i+1}) = 1, \forall i \in \{1, \ldots, n-1\}\}.$$

The set of A-admissible words of arbitrary length is denoted by $\Sigma^* := \bigcup_{n \in \mathbb{N}} \Sigma^n$. Let us also define the *word length function* $|\cdot| : \Sigma^* \cup \Sigma \to \mathbb{N} \cup \{\infty\}$, where for $\omega \in \Sigma^*$ we set

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$|\omega|$ to be the unique $n \in \mathbb{N}$ such that $\omega \in \Sigma^n$ and, for $\omega \in \Sigma$, we set $|\omega| := \infty$. For each $\omega \in \Sigma^* \cup \Sigma$ and $n \in \mathbb{N}$ with $n \leq |\omega|$, we define $\omega_{|n} := (\omega_1, \ldots, \omega_n)$. For $\omega, \tau \in \Sigma$, we set $\omega \wedge \tau$ to be the longest common initial block of ω and τ, that is, $\omega \wedge \tau := \omega_{|l}$, where $l := \sup\{n \in \mathbb{N} \mid \omega_{|n} = \tau_{|n}\}$. For $n \in \mathbb{N}$ and $\omega \in \Sigma^n$, we let $[\omega] := \{\tau \in \Sigma \mid \tau_{|n} = \omega\}$ denote the *cylindrical set* given by ω.

If Σ is the Markov shift with alphabet I whose incidence matrix consists entirely of 1s, then we have that $\Sigma = I^{\mathbb{N}}$ and $\Sigma^n = I^n$ for all $n \in \mathbb{N}$. Then we set $I^* := \Sigma^*$. For $\omega, \tau \in I^*$, we denote by $\omega\tau \in I^*$ the *concatenation* of ω and τ, which is defined by $\omega\tau := (\omega_1, \ldots, \omega_{|\omega|}, \tau_1, \ldots, \tau_{|\tau|})$ for $\omega, \tau \in I^*$. Note that I^* forms a semigroup (without identity element) with respect to the concatenation operation. The semigroup I^* is the free semigroup on the set I and satisfies the following universal property: for each countable semigroup S and for every map $u : I \to S$, there exists a unique semigroup homomorphism $\widehat{u} : I^* \to S$ such that $\widehat{u}(i) = u(i)$ for all $i \in I$ (see [**Ber98**, §3.10]).

We equip $I^{\mathbb{N}}$ with the product topology of the discrete topology on I. The Markov shift $\Sigma \subset I^{\mathbb{N}}$ is equipped with the subspace topology. A countable basis of this topology on Σ is given by the cylindrical sets $\{[\omega] \mid \omega \in \Sigma^*\}$. We will make use of the following metric generating the topology on Σ. For $\alpha > 0$, we define the metric d_α on Σ given by

$$d_\alpha(\omega, \tau) := e^{-\alpha|\omega\wedge\tau|} \quad \text{for all } \omega, \tau \in \Sigma.$$

For a function $f : \Sigma \to \mathbb{R}$ and $n \in \mathbb{N}$, we use the notation $S_n f : \Sigma \to \mathbb{R}$ to denote the *ergodic sum* of f with respect to the left shift σ, in other words, $S_n f := \sum_{i=0}^{n-1} f \circ \sigma^i$. Also, we set $S_0 f := 0$.

We say that $f : \Sigma \to \mathbb{R}$ is *α-Hölder continuous*, for some $\alpha > 0$, if

$$V_\alpha(f) := \sup_{n\geq 1}\{V_{\alpha,n}(f)\} < \infty, \tag{2.1}$$

where, for each $n \in \mathbb{N}$, we let

$$V_{\alpha,n}(f) := \sup\left\{\frac{|f(\omega) - f(\tau)|}{d_\alpha(\omega, \tau)} \mid \omega, \tau \in \Sigma, |\omega \wedge \tau| \geq n\right\}.$$

We say that f is *Hölder continuous* if there exists $\alpha > 0$ such that f is α-Hölder continuous.

The following lemma is well known (see e.g. [**MU03**, Lemma 2.3.1]).

LEMMA 2.1. (Bounded distortion property) *Let Σ be a Markov shift. If $f : \Sigma \to \mathbb{R}$ is Hölder continuous, then there exists a constant $C_f \geq 1$ such that, for all $\omega \in \Sigma^*$ and $\tau, \tau' \in [\omega]$, we have*

$$|S_{|\omega|} f(\tau) - S_{|\omega|} f(\tau')| \leq \log C_f.$$

We will make use of the following topological mixing properties for Markov shifts.

Definition 2.2. Let Σ be a Markov shift with alphabet $I \subset \mathbb{N}$.

- Σ is *irreducible* if, for all $i, j \in I$, there exists $\omega \in \Sigma^*$ such that $i\omega j \in \Sigma^*$.
- Σ is *topologically mixing* if, for all $i, j \in I$, there exists $n_0 \in \mathbb{N}$ with the property that, for all $n \geq n_0$, there exists $\omega \in \Sigma^n$ such that $i\omega j \in \Sigma^*$.
- Σ is *finitely primitive* if there exist $l \in \mathbb{N}$ and a finite set $\Lambda \subset \Sigma^l$ with the property that, for all $i, j \in I$, there exists $\omega \in \Lambda$ such that $i\omega j \in \Sigma^*$.

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Remark. Σ is finitely primitive if and only if Σ is topologically mixing and if Σ satisfies the big images and preimages property (see [**Sar03**]).

The following definition of the Gurevič pressure was introduced by Sarig [**Sar99**, Definition 1] and extends the notion of the Gurevič entropy [**Gur69**, **Gur70**] corresponding to $\varphi = 0$.

Definition 2.3. Let Σ be a Markov shift with alphabet I and left shift $\sigma : \Sigma \to \Sigma$. Let $f : \Sigma \to \mathbb{R}$ be Hölder continuous. For each $a \in I$ and $n \in \mathbb{N}$, we set

$$Z_n(f, a, \sigma) := \sum_{x \in \Sigma, x_1 = a, \sigma^n(x) = x} e^{S_n f(x)}$$

and

$$Z_n^*(f, a, \sigma) := \sum_{x \in \Sigma, x_1 = a, \sigma^n(x) = x, x_2, \ldots, x_n \neq a} e^{S_n f(x)}.$$

If Σ is irreducible, then the *Gurevič pressure* of f with respect to σ is, for each $a \in I$, given by

$$\mathcal{P}(f, \sigma) := \limsup_{n \to \infty} \frac{1}{n} \log Z_n(f, a, \sigma).$$

If Σ is reducible, then we define

$$\mathcal{P}(f, \sigma) := \sup_{V \in \mathcal{V}} \mathcal{P}(f|_V, \sigma|_V),$$

where $\mathcal{V}$ denotes the set of irreducible components of Σ.

Remark 2.4. [**JKL14**, Fact 3.1] Let Σ be a Markov shift with alphabet I and left shift $\sigma : \Sigma \to \Sigma$. If Σ is irreducible and f is Hölder continuous, then $\mathcal{P}(f, \sigma)$ is independent of $a \in I$.

The next definition goes back to the work of Ruelle and Bowen (cf. [**Rue69**, **Bow75**]).

Definition 2.5. (Gibbs measure) Let $f : \Sigma \to \mathbb{R}$ be Hölder continuous with $\mathcal{P}(f, \sigma) < \infty$. We say that a Borel probability measure μ is a *Gibbs measure for* f if there exists a constant $C > 0$ such that

$$C^{-1} \leq \frac{\mu([\omega])}{e^{S_{|\omega|} f(\tau) - |\omega| \mathcal{P}(f, \sigma)}} \leq C \quad \text{for all } \omega \in \Sigma^* \text{ and } \tau \in [\omega]. \tag{2.2}$$

The following necessary and sufficient condition for the existence of Gibbs measures is taken from [**Sar03**, Theorem 1]. The uniqueness part also follows from [**MU03**, Theorem 2.2.4]. The existence of a σ-invariant Gibbs measure on a finitely primitive Markov shift follows from [**MU03**].

THEOREM 3. (Existence of Gibbs measures) *Let Σ be topologically mixing and let $f : \Sigma \to \mathbb{R}$ be Hölder continuous. Then there exists a σ-invariant Gibbs measure for f if and only if Σ is finitely primitive* and $\mathcal{P}(f, \sigma) < \infty$. *If a σ-invariant Gibbs measure for f exists, then it is unique.*

2.2. *Recurrent potentials.* In this subsection, we recall the definition of a recurrent potential, which was introduced by Sarig for Hölder continuous potentials on a topologically mixing countable state Markov shift [**Sar01**, Definition 1]. This notion generalizes the notion of recurrence for Markov chains and the definition of R-recurrence

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for positive matrices by Vere-Jones [**VJ62**]. In this paper, we adapt the definition of recurrence to Hölder continuous potentials on an irreducible Markov shift, which is not necessarily topologically mixing. One can easily verify that also in this case, the notion of recurrence is independent of the choice of $a \in I$.

Definition 2.6. Let Σ be an irreducible Markov shift with alphabet I and left shift $\sigma : \Sigma \to \Sigma$. Let $f : \Sigma \to \mathbb{R}$ be Hölder continuous with $\mathcal{P}(f, \sigma) < \infty$. We say that

$$f \text{ is } \textit{recurrent} \text{ if } \sum_{n\in\mathbb{N}} e^{-n\mathcal{P}(f,\sigma)} Z_n(f, a, \sigma) = \infty \quad \text{for some (hence all) } a \in I,$$

and f is *transient*, otherwise. Moreover, if f is recurrent, then we say that

$$f \text{ is } \textit{positive recurrent} \text{ if } \sum_{n\in\mathbb{N}} n e^{-n\mathcal{P}(f,\sigma)} Z_n^*(f, a, \sigma) < \infty \quad \text{for some (hence all) } a \in I,$$

$$f \text{ is } \textit{null recurrent} \text{ if } \sum_{n\in\mathbb{N}} n e^{-n\mathcal{P}(f,\sigma)} Z_n^*(f, a, \sigma) = \infty \quad \text{for some (hence all) } a \in I.$$

If an irreducible Markov shift Σ has period $p \in \mathbb{N}$, then it is convenient to study the pth iterate of the dynamics. The next remark shows how this is reflected in pressure and recurrence.

Remark 2.7. Let Σ be an irreducible Markov shift with alphabet I, incidence matrix $A = (a(i, j))$ and left shift $\sigma : \Sigma \to \Sigma$. If Σ has period $p \in \mathbb{N}$, then we consider $\sigma^p : \Sigma \to \Sigma$, which is conjugated to the left shift $\sigma^{(p)} : \Sigma^{(p)} \to \Sigma^{(p)}$ on the Markov shift $\Sigma^{(p)}$ via the canonical bijection $\iota : \Sigma^{(p)} \to \Sigma$, where $\Sigma^{(p)}$ is the Markov shift with the alphabet Σ^p, for which $\omega, \tau \in \Sigma^p$ are admissible if $a(\omega_p, \tau_1) = 1$. Let $f^{(p)} : \Sigma^{(p)} \to \mathbb{R}$ be given by $f^{(p)} := S_p f \circ \iota$ and denote the irreducible components of $\Sigma^{(p)}$ by $\mathcal{V}$. One easily verifies that, for each $V \in \mathcal{V}$,

$$\mathcal{P}(f^{(p)}, \sigma^{(p)}) = \mathcal{P}(f^{(p)}|_V, \sigma^{(p)}|_V) = p\mathcal{P}(f, \sigma)$$

and that $f^{(p)}|_V$ is (positive) recurrent if and only if f is (positive) recurrent.

The following was proved by Sarig [**Sar01**, Theorem 1, Remark 2, Proposition 1, Proposition 3].

THEOREM 4. *Let Σ be a topologically mixing Markov shift and let $f : \Sigma \to \mathbb{R}$ be Hölder continuous with $\mathcal{P}(f, \sigma) < \infty$. Then f is recurrent if and only if there exist $\rho > 0$ and a conservative measure ν on Σ, positive and finite on cylindrical sets, which satisfies $\mathcal{L}_f^*(\nu) = \rho\nu$. In this case,* $\log \rho = \mathcal{P}(f, \sigma)$ *and there exists a continuous function $h : \Sigma \to \mathbb{R}^+$ such that $\mathcal{L}_f(h) = \rho h$ and such that* $\log h$ *and* $\log h \circ \sigma$ *are Hölder continuous. Moreover, ν is the unique measure (up to a constant multiple), which is fixed by $e^{-\mathcal{P}(f,\sigma)}\mathcal{L}_f^*$ and which is positive and finite on cylindrical sets, and h is the unique positive continuous function (up to a constant multiple), which is fixed by $e^{-\mathcal{P}(f,\sigma)}\mathcal{L}_f$ and which is bounded on cylindrical sets. Furthermore, we have that f is positive recurrent if and only if $\int h \, d\nu < \infty$.*

Remark 2.8. The uniqueness of h for a recurrent potential f, as stated in Theorem 4, follows because h defines the invariant and σ-finite measure $h \, d\nu$, which is absolutely continuous with respect to the conservative ergodic measure ν. Therefore, h is unique

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ν-almost everywhere (see, for example, [**Aar97**, Theorem 1.5.6]). Since ν is positive on cylindrical sets and h is continuous, uniqueness of h follows.

2.3. *Group extensions and amenability.* In this section, we consider a group-extended Markov system $(\Sigma \times G, \sigma \rtimes \Psi)$ for a Markov shift Σ with alphabet $I \subset \mathbb{N}$, a countable group G and a semigroup homomorphism $\Psi : I^* \to G$. Ergodic properties of group extensions given by locally compact abelian groups have also been studied in [**AD00**, **AD02**].

Remark 2.9. $(\Sigma \times G, \sigma \rtimes \Psi)$ is conjugated to the Markov shift with state space

$$\{((\omega_j, g_j)) \in (I \times G)^{\mathbb{N}} \mid \omega \in \Sigma \text{ and } g_j \Psi(\omega_j) = g_{j+1},\ j \in \mathbb{N}\}.$$

We do not distinguish between this Markov shift and $(\Sigma \times G, \sigma \rtimes \Psi)$.

The following definition is due to von Neumann [**Neu29**].

Definition 2.10. A discrete group G is *amenable* if there exists a finitely additive probability measure ν on the power set of G such that $\nu(A) = \nu(g(A))$ for all $g \in G$ and $A \subset G$.

Let us now state some recent results on amenability for group extensions of Markov shifts. We will refer to these results in §1.3 in order to clarify the context of the results obtained in this paper. The next theorem is due to Stadlbauer [**Sta13**, Theorem 5.4]. For locally constant potentials on a finite state Markov shift Σ, a similar statement has been proved in [**Jae14a**, Theorem 1.1] using different methods.

THEOREM 5. *Let Σ be finitely primitive and let $(\Sigma \times G, \sigma \rtimes \Psi)$ be an irreducible group-extended Markov system. Let $\varphi : \Sigma \to \mathbb{R}$ be Hölder continuous with $\mathcal{P}(\varphi, \sigma) < \infty$. If $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$, then G is amenable.*

The next theorem, which provides a converse of the previous theorem, is taken from [**Jae14b**, Corollary 1.6]. Under slightly different assumptions, the theorem was proved in [**Jae11**, Theorem 5.3.11] and, independently, by Stadlbauer in [**Sta13**, Theorem 4.1].

Definition 2.11. We say that φ is *asymptotically symmetric with respect to* Ψ [**Jae14b**, Definition 1.3] if there exist $n_0 \in \mathbb{N}$ and sequences $(c_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ with the property that $\lim_n (c_n)^{1/n} = 1$, $\lim_n n^{-1} N_n = 0$ and such that, for each $g \in G$ and $n \geq n_0$, we have

$$\sum_{\omega\in\Sigma^n:\Psi(\omega)=g} e^{\sup S_n\varphi_{|[\omega]}} \leq c_n \sum_{\omega\in\Sigma^*:\Psi(\omega)=g^{-1},n-N_n\leq|\omega|\leq n+N_n} e^{\sup S_{|\omega|}\varphi_{|[\omega]}}.$$

THEOREM 6. *Let Σ be finitely primitive and let $(\Sigma \times G, \sigma \rtimes \Psi)$ be an irreducible group-extended Markov system. Let $\varphi : \Sigma \to \mathbb{R}$ be Hölder continuous with $\mathcal{P}(\varphi, \sigma) < \infty$. If φ is asymptotically symmetric with respect to Ψ and if G is amenable, then $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi)$.*

3. *Proof of the main results*

3.1. *Proof of Theorem 1.* The following lemma, which shows that, for a recurrent potential $\varphi \circ \pi_1$, the associated eigenfunction of $\mathcal{L}_{\varphi\circ\pi_1}$ has product structure, is crucial for the proof of the main theorem.

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LEMMA 3.1. *Let $(\Sigma \times G, \sigma \rtimes \Psi)$ be a topologically mixing group-extended Markov system. Let $\varphi : \Sigma \to \mathbb{R}$ be Hölder continuous such that $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) < \infty$. Suppose that $\varphi \circ \pi_1$ is recurrent and let $h : \Sigma \times G \to \mathbb{R}^+$ be the unique continuous function (up to a constant multiple), which is fixed by $e^{-\mathcal{P}(\varphi\circ\pi_1, \sigma\rtimes\Psi)}\mathcal{L}_{\varphi\circ\pi_1}$ and which is bounded on cylindrical sets. Then there exist a unique homomorphism $c : G \to (\mathbb{R}^+, \cdot)$ and a continuous function $h_1 : \Sigma \to \mathbb{R}^+$, bounded on cylindrical sets, such that $h = (h_1 \circ \pi_1)(c \circ \pi_2)$.*

Proof. For each $g \in G$, let $g^*h : \Sigma \times G \to \mathbb{R}$ be given by $(g^*h)(\omega, \gamma) := h(\omega, g\gamma)$ for each $(\omega, \gamma) \in \Sigma \times G$. A short calculation shows that, for all $(\omega, \gamma) \in \Sigma \times G$,

$$\begin{aligned}\mathcal{L}_{\varphi\circ\pi_1}(g^*h)(\omega, \gamma) &= \sum_{i\in I: i\omega\in\Sigma} e^{\varphi\circ\pi_1(i\omega, \gamma(\Psi(i))^{-1})} h(i\omega, g\gamma(\Psi(i))^{-1}) \\ &= \sum_{i\in I: i\omega\in\Sigma} e^{\varphi(i\omega)} h(i\omega, g\gamma(\Psi(i))^{-1}) \\ &= \mathcal{L}_{\varphi\circ\pi_1}(h)(\omega, g\gamma) = e^{\mathcal{P}(\varphi\circ\pi_1, \sigma\rtimes\Psi)} h(\omega, g\gamma).\end{aligned}$$

We have thus shown that $\mathcal{L}_{\varphi\circ\pi_1}(g^*h) = e^{\mathcal{P}(\varphi\circ\pi_1, \sigma\rtimes\Psi)} g^*h$ for each $g \in G$. By Theorem 4, there exists a constant $c(g) \in \mathbb{R}^+$ such that $g^*h = c(g)h$. This defines a homomorphism $c : G \to (\mathbb{R}^+, \cdot)$, since we have $c(g_1g_2)h = (g_1g_2)^*(h) = g_2^*(g_1^*(h)) = c(g_1)c(g_2)h$ for all $g_1, g_2 \in G$. We conclude that there exists a continuous function $h_1 : \Sigma \to \mathbb{R}^+$, bounded on cylindrical sets, such that $h = (h_1 \circ \pi_1)(c \circ \pi_2)$. □

We are now in the position to prove the main result.

Proof of Theorem 1. First, suppose that $\varphi \circ \pi_1$ is recurrent. Let $h : \Sigma \times G \to \mathbb{R}^+$ be the unique continuous function (up to a constant multiple), which is fixed by $e^{-\mathcal{P}(\varphi\circ\pi_1, \sigma\rtimes\Psi)}\mathcal{L}_{\varphi\circ\pi_1}$ and which is bounded on cylindrical sets. By Lemma 3.1, there exist a unique homomorphism $c : G \to (\mathbb{R}^+, \cdot)$ and a continuous function $h_1 : \Sigma \to \mathbb{R}^+$, bounded on cylindrical sets, such that $h = (h_1 \circ \pi_1)(c \circ \pi_2)$. Hence, the coboundary $\log h - \log h \circ (\sigma \rtimes \Psi)$ is given by

$$\begin{aligned}\log h - \log h \circ (\sigma \rtimes \Psi) &= \log(h_1 \circ \pi_1) - \log(h_1 \circ \pi_1) \circ (\sigma \rtimes \Psi) \\ &\quad + \log(c \circ \pi_2) - \log(c \circ \pi_2) \circ (\sigma \rtimes \Psi).\end{aligned}$$

A short calculation shows that, for each $(x, g) \in \Sigma \times G$,

$$\log(c \circ \pi_2)(x, g) - \log(c \circ \pi_2)(\sigma x, g\Psi(x_1)) = -\log c(\Psi(x_1)). \tag{3.1}$$

Therefore, the coboundary $\log h - \log h \circ (\sigma \rtimes \Psi)$ is independent of the second coordinate. Let $\varphi_c : \Sigma \to \mathbb{R}$ be given by $\varphi_c(x) := \varphi(x) - \log c(\Psi(x_1))$ for each $x \in \Sigma$. Clearly, φ_c is Hölder continuous. For ease of notation, let us also define $\tilde{\varphi} : \Sigma \to \mathbb{R}^+$ given by $\tilde{\varphi} := \varphi_c + \log h_1 - \log h_1 \circ \sigma$. We have

$$\tilde{\varphi} \circ \pi_1 = \varphi \circ \pi_1 + \log h - \log h \circ (\sigma \rtimes \Psi). \tag{3.2}$$

Since $\log h$ and $\log h \circ (\sigma \rtimes \Psi)$ are Hölder continuous by Theorem 4, so are $\log h_1$, $\log h_1 \circ \sigma$ and $\tilde{\varphi}$. As a consequence, we have

$$\mathcal{P}(\tilde{\varphi} \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) \quad \text{and} \quad \mathcal{P}(\tilde{\varphi}, \sigma) = \mathcal{P}(\varphi_c, \sigma). \tag{3.3}$$

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Since h satisfies $\mathcal{L}_{\varphi\circ\pi_1}(h) = e^{\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}h$, we have $\mathcal{L}_{\tilde{\varphi}\circ\pi_1}(\mathbb{1}\circ\pi_1) = e^{\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}(\mathbb{1}\circ\pi_1)$ by (3.2). Combining this with the identity $\mathcal{L}_{\tilde{\varphi}\circ\pi_1}(\mathbb{1}\circ\pi_1) = (\mathcal{L}_{\tilde{\varphi}}(\mathbb{1}))\circ\pi_1$, we conclude that

$$\mathcal{L}_{\tilde{\varphi}}(\mathbb{1}) = e^{\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}\mathbb{1}. \tag{3.4}$$

In particular, we have $\mathcal{P}(\tilde{\varphi},\sigma) \le \mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi) < \infty$. Since Σ is finitely primitive and $\tilde{\varphi}$ is Hölder continuous with $\mathcal{P}(\tilde{\varphi},\sigma) < \infty$, we have that $e^{-n\mathcal{P}(\tilde{\varphi},\sigma)}\mathcal{L}^n_{\tilde{\varphi}}(\mathbb{1})$ converges uniformly to a bounded Hölder continuous function (see [**MU03**, Theorem 2.4.6]). Since $e^{-n\mathcal{P}(\tilde{\varphi},\sigma)}\mathcal{L}^n_{\tilde{\varphi}}(\mathbb{1}) = e^{-n(\mathcal{P}(\tilde{\varphi},\sigma)-\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi))}\mathbb{1}$ by (3.4), we conclude that $\mathcal{P}(\tilde{\varphi},\sigma) = \mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)$. Combining this with (3.3), we have thus shown that

$$\mathcal{P}(\tilde{\varphi}\circ\pi_1,\sigma\rtimes\Psi) = \mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi) = \mathcal{P}(\tilde{\varphi},\sigma) = \mathcal{P}(\varphi_c,\sigma). \tag{3.5}$$

Further, by the definition of $\tilde{\varphi}$ and (3.5), the equality in (3.4) implies

$$\mathcal{L}_{\varphi_c}(h_1) = e^{\mathcal{P}(\varphi_c,\sigma)}h_1. \tag{3.6}$$

Since Σ is finitely primitive and φ_c is Hölder continuous with $\mathcal{P}(\varphi_c,\sigma) < \infty$, we have that φ_c is positive recurrent by [**Sar03**, Corollary 2]. Since h_1 is continuous and bounded on cylindrical sets, it follows from (3.6) and [**Sar03**, Corollary 2 and 3] that h_1 is Hölder continuous and bounded away from zero and infinity. In particular, $\tilde{\varphi}$ is cohomologous to φ_c in the cohomology class of bounded continuous functions, which implies that $\mu_{\tilde{\varphi}} = \mu_{\varphi_c}$ (see [**MU03**, Theorem 2.2.7]). We now show that $\mu_{\tilde{\varphi}}\times\lambda$ is conservative. By (3.4) and (3.5), we have $\mathcal{L}^*_{\tilde{\varphi}}(\mu_{\tilde{\varphi}}) = e^{\mathcal{P}(\tilde{\varphi}\circ\pi_1,\sigma\rtimes\Psi)}\mu_{\tilde{\varphi}}$, which gives

$$\mathcal{L}^*_{\tilde{\varphi}\circ\pi_1}(\mu_{\tilde{\varphi}}\times\lambda) = e^{\mathcal{P}(\tilde{\varphi}\circ\pi_1,\sigma\rtimes\Psi)}(\mu_{\tilde{\varphi}}\times\lambda). \tag{3.7}$$

From (3.2) and since $\varphi\circ\pi_1$ is recurrent, we have that $\tilde{\varphi}\circ\pi_1$ is recurrent. Hence, (3.7) implies that $\mu_{\tilde{\varphi}}\times\lambda$ is conservative by Theorem 4.

We now turn to the proof of the converse implication and the uniqueness of the homomorphism c in Theorem 1. To prove this, suppose that $\mathcal{P}(\varphi_c,\sigma) < \infty$ and that $\mu_{\varphi_c}\times\lambda$ is conservative for some homomorphism $c : G \to (\mathbb{R}^+,\cdot)$. We show that $\varphi\circ\pi_1$ is recurrent and that c is unique. Since Σ is finitely primitive, there exists a Hölder continuous function $h_1 : \Sigma^+ \to \mathbb{R}$, bounded away from zero and infinity, such that $\mathcal{L}_{\varphi_c}(h_1) = e^{\mathcal{P}(\varphi_c,\sigma)}h_1$. Setting $\tilde{\varphi} := \varphi_c + \log h_1 - \log h_1\circ\sigma$ as above, we have $\mathcal{L}^*_{\tilde{\varphi}\circ\pi_1}(\mu_{\varphi_c}\times\lambda) = e^{\mathcal{P}(\varphi_c,\sigma)}(\mu_{\varphi_c}\times\lambda)$. Since $\mu_{\varphi_c}\times\lambda$ is conservative, it follows from Theorem 4 that $\tilde{\varphi}\circ\pi_1$ is recurrent and that $\mathcal{P}(\varphi_c,\sigma) = \mathcal{P}(\tilde{\varphi}\circ\pi_1,\sigma\rtimes\Psi)$. Since h_1 is Hölder continuous, we conclude that $\varphi_c\circ\pi_1$ is recurrent and $\mathcal{P}(\tilde{\varphi}\circ\pi_1,\sigma\rtimes\Psi) = \mathcal{P}(\varphi_c\circ\pi_1,\sigma\rtimes\Psi)$. Finally, the following relation, which is deduced from (3.1),

$$\varphi_c\circ\pi_1 = \varphi\circ\pi_1 + \log(c\circ\pi_2) - \log(c\circ\pi_2)\circ(\sigma\rtimes\Psi), \tag{3.8}$$

gives that $\varphi\circ\pi_1$ is recurrent. To prove uniqueness of c, observe that we have $\mathcal{P}(\varphi_c\circ\pi_1,\sigma\rtimes\Psi) = \mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)$ by (3.8). We have thus shown that $\mathcal{P}(\varphi_c,\sigma) = \mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)$. Therefore, we have $\mathcal{L}_{\varphi_c\circ\pi_1}(h_1\circ\pi_1) = e^{\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}(h_1\circ\pi_1)$. Hence, $\mathcal{L}_{\varphi\circ\pi_1}((c\circ\pi_2)(h_1\circ\pi_1)) = e^{\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}(c\circ\pi_2)(h_1\circ\pi_1)$ by (3.8). Since $(c\circ\pi_2)(h_1\circ\pi_1)$ is continuous and bounded on cylindrical sets, recurrence of $\varphi\circ\pi_1$ implies that c is unique by Theorem 4.

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To finish the proof, we are left to prove Theorem 1(1), (2), (3) and (4). The assertion in (1) follows from (3.5). Claim (2) follows from Lemma 3.1 and (3.6). To prove (3), let $f : \Sigma \to \mathbb{R}^+$. By (3.8), we have

$$\mathcal{L}_{\varphi\circ\pi_1}(f) = (c\circ\pi_2)\mathcal{L}_{\varphi_c\circ\pi_1}\left(\frac{f}{c\circ\pi_2}\right). \tag{3.9}$$

Let ν_1 denote the unique Borel probability measure such that $\mathcal{L}^*_{\varphi_c}(\nu_1) = e^{\mathcal{P}(\varphi_c,\sigma)}\nu_1$. By (3.5), we have

$$\mathcal{L}^*_{\varphi_c\circ\pi_1}(\nu_1\times\lambda) = e^{\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}(\nu_1\times\lambda). \tag{3.10}$$

We verify that $\mathcal{L}^*_{\varphi\circ\pi_1}((c\circ\pi_2)^{-1}\,d(\nu_1\times\lambda)) = e^{\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}(c\circ\pi_2)^{-1}\,d(\nu_1\times\lambda)$. By using (3.9) and (3.10), we have

$$\begin{aligned}\mathcal{L}^*_{\varphi\circ\pi_1}\left(\frac{d(\nu_1\times\lambda)}{c\circ\pi_2}\right)(f) &= \frac{d(\nu_1\times\lambda)}{c\circ\pi_2}\left((c\circ\pi_2)\mathcal{L}_{\varphi_c\circ\pi_1}\left(\frac{f}{c\circ\pi_2}\right)\right)\\ &= e^{\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}(\nu_1\times\lambda)\left(\frac{f}{c\circ\pi_2}\right).\end{aligned}$$

This finishes the proof of (3). Clearly, the assertion in (4) follows by combining (2), (3) and the fact that $\mu_{\varphi_c} = h_1\,d\nu_1$. The proof is complete. □

3.2. *Proof of Proposition 1.4 and Remark 1.5.*

Proof of Proposition 1.4. Let $\varphi\circ\pi_1$ be recurrent and let $c : G \to (\mathbb{R}^+, \cdot)$ denote the homomorphism given by Theorem 1. By Theorem 1(4), we have that $\mu_{\varphi_c}\times\lambda$ is the Ruelle–Perron–Frobenius measure of $\varphi\circ\pi_1$. In particular, $(\Sigma\times G, \mu_{\varphi_c}\times\lambda, \sigma\rtimes\Psi)$ is a conservative ergodic measure preserving dynamical system with transfer operator $e^{-\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}h^{-1}\mathcal{L}_{\varphi\circ\pi_1}(h\cdot)$, where $h = (h_1\circ\pi_1)(c\circ\pi_2)$, for the Hölder continuous function $h_1 : \Sigma \to \mathbb{R}^+$, bounded away from zero and infinity, given by Theorem 1(2). By the ratio ergodic theorem (see e.g. [**Aar97**, Theorem 2.2.1]), we conclude that, for each $g \in G$, we have $(\mu_{\varphi_c}\times\lambda)$-almost everywhere

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n e^{-k\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}h^{-1}\mathcal{L}^k_{\varphi\circ\pi_1}(h\mathbb{1}_{\Sigma\times\{g\}})}{\sum_{k=1}^n e^{-k\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}h^{-1}\mathcal{L}^k_{\varphi\circ\pi_1}(h\mathbb{1}_{\Sigma\times\{g^{-1}\}})} = \frac{(\mu_{\varphi_c}\times\lambda)(\mathbb{1}_{\Sigma\times\{g\}})}{(\mu_{\varphi_c}\times\lambda)(\mathbb{1}_{\Sigma\times\{g^{-1}\}})} = 1.$$

Combining this with the elementary estimate that, for each $k \in \mathbb{N}$ and $g \in G$,

$$\begin{aligned}(\sup h_1)^{-1}c(g)^{-1}\mathcal{L}^k_{\varphi\circ\pi_1}(h\mathbb{1}_{\Sigma\times\{g\}}) &\le \mathcal{L}^k_{\varphi\circ\pi_1}(\mathbb{1}_{\Sigma\times\{g\}})\\ &\le (\inf h_1)^{-1}c(g)^{-1}\mathcal{L}^k_{\varphi\circ\pi_1}(h\mathbb{1}_{\Sigma\times\{g\}}),\end{aligned}$$

we obtain that $(\mu_{\varphi_c}\times\lambda)$-almost everywhere

$$c(g)^{-2}\frac{\inf h_1}{\sup h_1} \le \limsup_{n\to\infty}\frac{\sum_{k=1}^n e^{-k\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}\mathcal{L}^k_{\varphi\circ\pi_1}(\mathbb{1}_{\Sigma\times\{g\}})}{\sum_{k=1}^n e^{-k\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}\mathcal{L}^k_{\varphi\circ\pi_1}(\mathbb{1}_{\Sigma\times\{g^{-1}\}})} \le c(g)^{-2}\frac{\sup h_1}{\inf h_1}. \tag{3.11}$$

Since Σ is finitely primitive, there exists a finite set $F \subset I$ such that, for each $a \in I$, there exists $i \in F$ with $ai \in \Sigma^2$. Fix F and choose a point $x(i) \in [i]$ for each $i \in F$. Because $(\mu_{\varphi_c}\times\lambda)$ is positive on cylindrical sets and the Hölder continuous function φ has the bounded distortion property by Lemma 2.1, it follows from (3.11) that there exists $C_\varphi \ge 1$

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such that, for each $g \in G$,

$$c(g)^{-2}\frac{\inf h_1}{\sup h_1}C_\varphi^{-2} \le \limsup_{n\to\infty} \frac{\sum_{k=1}^n e^{-k\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}\sum_{i\in F}\mathcal{L}^k_{\varphi\circ\pi_1}(\mathbb{1}_{\Sigma\times\{g\}})(x(i),\mathrm{id})}{\sum_{k=1}^n e^{-k\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}\sum_{i\in F}\mathcal{L}^k_{\varphi\circ\pi_1}(\mathbb{1}_{\Sigma\times\{g^{-1}\}})(x(i),\mathrm{id})} \le c(g)^{-2}\frac{\sup h_1}{\inf h_1}C_\varphi^2. \tag{3.12}$$

A homomorphism $c : G \to (\mathbb{R}^+, \cdot)$ is bounded away from zero (bounded away from infinity, respectively) if and only if $c = 1$. Hence, by (3.12), we obtain that $c = 1$ if and only if

$$\sup_{g\in G}\limsup_{n\to\infty} \frac{\sum_{k=1}^n e^{-k\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}\sum_{i\in F}\mathcal{L}^k_{\varphi\circ\pi_1}(\mathbb{1}_{\Sigma\times\{g\}})(x(i),\mathrm{id})}{\sum_{k=1}^n e^{-k\mathcal{P}(\varphi\circ\pi_1,\sigma\rtimes\Psi)}\sum_{i\in F}\mathcal{L}^k_{\varphi\circ\pi_1}(\mathbb{1}_{\Sigma\times\{g^{-1}\}})(x(i),\mathrm{id})} < \infty. \tag{3.13}$$

To finish the proof, we show that (3.13) holds if and only if φ is symmetric on average with respect to Ψ. Let $k \in \mathbb{N}$ and $g \in G$. Since $\mathrm{card}(F) < \infty$, we have that

$$\sum_{i\in F}\sum_{\omega\in\Sigma^k:\Psi(\omega)=g^{-1},\omega x(i)\in\Sigma} e^{S_k\varphi(\omega x(i))} \le \mathrm{card}(F)\sum_{\omega\in\Sigma^k:\Psi(\omega)=g^{-1}} e^{\sup S_k\varphi_{|[\omega]}}. \tag{3.14}$$

Because for each $\omega \in \Sigma^*$ there exists $i \in F$, such that $\omega x(i) \in \Sigma$, and since φ has the bounded distortion property, we have

$$\sum_{\omega\in\Sigma^k:\Psi(\omega)=g^{-1}} e^{\sup S_k\varphi_{|[\omega]}} \le C_\varphi\sum_{i\in F}\sum_{\omega\in\Sigma^k:\Psi(\omega)=g^{-1},\omega x(i)\in\Sigma} e^{S_k\varphi(\omega x(i))}. \tag{3.15}$$

Moreover, observe that, for each $x \in \Sigma$, we have

$$\mathcal{L}^k_{\varphi\circ\pi_1}(\mathbb{1}_{\Sigma\times\{g\}})(x,\mathrm{id}) = \sum_{\omega\in\Sigma^k:\Psi(\omega)=g^{-1},\omega x\in\Sigma} e^{S_k\varphi(\omega x)}. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we obtain that

$$C_\varphi^{-1}\sum_{\omega\in\Sigma^k:\Psi(\omega)=g^{-1}} e^{\sup S_k\varphi_{|[\omega]}} \le \sum_{i\in F}\mathcal{L}^k_{\varphi\circ\pi_1}(\mathbb{1}_{\Sigma\times\{g\}})(x(i),\mathrm{id}) \le \mathrm{card}(F)\sum_{\omega\in\Sigma^k:\Psi(\omega)=g^{-1}} e^{\sup S_k\varphi_{|[\omega]}}.$$

Hence, (3.13) holds if and only if φ is symmetric on average with respect to Ψ. □

Proof of Remark 1.5. Let $p > 1$ denote the period of the irreducible Markov shift $(\Sigma \times G, \sigma \rtimes \Psi)$. We consider the left shift $\sigma^{(p)} : \Sigma^{(p)} \to \Sigma^{(p)}$ on the Markov shift $(\Sigma^{(p)}, \sigma^{(p)})$, which is conjugated to (Σ, σ^p) via $\iota : \Sigma^{(p)} \to \Sigma$ as in Remark 2.7. Let $\Psi^{(p)} : (\Sigma^{(p)})^* \to G$ denote the unique semigroup homomorphism such that $\Psi^{(p)}(\omega) = \Psi(\omega)$ for each $\omega \in \Sigma^p$, and observe that $(\sigma \rtimes \Psi)^p : \Sigma \times G \to \Sigma \times G$ is conjugated to the aperiodic Markov shift $\sigma^{(p)} \rtimes \Psi^{(p)} : \Sigma^{(p)} \times G \to \Sigma^{(p)} \times G$. Denote the set of irreducible components of $(\Sigma^{(p)} \times G, \sigma^{(p)} \rtimes \Psi^{(p)})$ by $\mathcal{V}$. Using the facts that Σ is finitely primitive and $\Sigma \times G$ is irreducible, one verifies that there exists $l \in \mathbb{N}$ such that

$$\text{For all } i,\ j \in I,\ \text{there exists } \tau \in \Sigma^{pl} : i\tau j \in \Sigma^{pl+2} \quad \text{and} \quad \Psi(\tau) = \mathrm{id}. \tag{3.17}$$

From (3.17), it follows that there exists a partition $(G_V)_{V\in\mathcal{V}}$ of G such that $V = \Sigma^{(p)} \times G_V$ for each $V \in \mathcal{V}$. We choose the component V_{id} containing $\Sigma^{(p)} \times \{\mathrm{id}\}$ and note that

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$V_{\mathrm{id}} = \Sigma^{(p)} \times G_0$, where $G_0 := \bigcup_{n\in\mathbb{N}} \Psi(\Sigma^{np})$. Using (3.17), one obtains that G_0 is a subsemigroup of G. To prove that G_0 is a group, let $g \in G_0$, $n \in \mathbb{N}$ and $\omega \in \Sigma^{np}$ be such that $g = \Psi(\omega)$. Since $\Sigma \times G$ is irreducible, there exist $r \in \mathbb{N}$ and $\gamma \in \Sigma^r$ such that $\overline{\omega\gamma} := (\omega\gamma\omega\gamma\dots) \in \Sigma$ and $\Psi(\omega\gamma) = \mathrm{id}$. Since $(\sigma \rtimes \Psi)^{np+r}(\overline{\omega\gamma}, \mathrm{id}) = (\overline{\omega\gamma}, \mathrm{id})$ and $(\Sigma \times G, \sigma \rtimes \Psi)$ has period p, there exists $m \in \mathbb{N}$ such that $r = mp$. Hence, we have $g^{-1} = \Psi(\gamma) \in G_0$. Let us also verify that the index of G_0 in G is at most p. To prove this, fix some $\alpha \in \Sigma^p$ and let $g \in G$. Since $(\Sigma \times G, \sigma \rtimes \Psi)$ is irreducible, there exists $\tau \in \Sigma^*$ such that $g = \Psi(\alpha)\Psi(\tau)$. Then there exists $u \in \{0, \dots, p-1\}$ such that $g \in \Psi(\alpha_1 \dots \alpha_u)G_0$, which proves that the index is at most p.

Next, define the potential $\varphi^{(p)} : \Sigma^{(p)} \to \mathbb{R}$, given by $\varphi^{(p)} := S_p\varphi \circ \iota$. As observed in Remark 2.7, we have

$$\mathcal{P}(\varphi^{(p)} \circ \pi_1|_{V_{\mathrm{id}}}, \sigma^{(p)} \rtimes \Psi^{(p)}|_{V_{\mathrm{id}}}) = p\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi), \quad \mathcal{P}(\varphi^{(p)}, \sigma^{(p)}) = p\mathcal{P}(\varphi, \sigma). \tag{3.18}$$

Further, since $\varphi \circ \pi_1$ is recurrent, so is $\varphi^{(p)} \circ \pi_1|_{V_{\mathrm{id}}}$. Moreover, $\Sigma^{(p)}$ is finitely primitive and $\sigma^{(p)} \rtimes \Psi^{(p)} : \Sigma^{(p)} \times G_0 \to \Sigma^{(p)} \times G_0$ is topologically mixing. Hence, the results of Corollary 1.1 hold with G replaced by G_0. Since the index of G_0 in G is finite, it is well known that G is amenable if and only if G_0 is amenable. Also, G is finite if and only if G_0 is finite. Hence, the results of Corollary 1.1 hold for G under the assumption that $\Sigma \times G$ is irreducible.

Next, we prove that the equality $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$ holds if and only if

$$\sup_{g\in G_0} \limsup_{n\to\infty} \frac{\sum_{k=1}^n e^{-kp\mathcal{P}(\varphi\circ\pi_1, \sigma\rtimes\Psi)} \sum_{\omega\in\Sigma^{kp}:\Psi(\omega)=g} e^{\sup S_{kp}\varphi_{|[\omega]}}}{\sum_{k=1}^n e^{-kp\mathcal{P}(\varphi\circ\pi_1, \sigma\rtimes\Psi)} \sum_{\omega\in\Sigma^{kp}:\Psi(\omega)=g^{-1}} e^{\sup S_{kp}\varphi_{|[\omega]}}} < \infty. \tag{3.19}$$

It follows from (3.18) and Proposition 1.4, applied to the group-extended Markov system $(\Sigma^{(p)} \times G_0, \sigma^{(p)} \rtimes \Psi^{(p)})$ and the potential $\varphi^{(p)}$, that $\mathcal{P}(\varphi \circ \pi_1, \sigma \rtimes \Psi) = \mathcal{P}(\varphi, \sigma)$ holds if and only if $\varphi^{(p)}$ is symmetric on average with respect to $\Psi^{(p)}$. It is easy to see that $\varphi^{(p)}$ is symmetric on average with respect to $\Psi^{(p)}$ if (3.19) holds. The proof is complete. □

3.3. *Proof of Proposition 1.7.* Without loss of generality, we may assume that $\mathrm{card}(G) = \infty$. Using the fact that $\varphi \circ \pi_1$ is recurrent, similarly as in [**Jae14a**, Proof of Theorem 1.2], we deduce that there exists a recurrent random walk P on the undirected graph X with vertex set $V := I \times G$, in which $(i, g), (j, h) \in I \times G$ are connected by an edge if

$$(\sigma \rtimes \Psi)^{-1}([i] \times \{g\}) \cap ([j] \times \{h\}) \neq \varnothing \quad \text{or} \quad (\sigma \rtimes \Psi)^{-1}([j] \times \{h\}) \cap ([i] \times \{g\}) \neq \varnothing.$$

Since $(\Sigma \times G, \sigma \rtimes \Psi)$ is irreducible, we have that X is connected. Let d denote the associated graph metric of X. Further, let $\mathrm{Aut}(X, P)$ denote the group of all invertible self-isometries γ of the metric space (X, d) which satisfy $p(x, y) = p(\gamma x, \gamma y)$ for all $x, y \in V$. Following [**Jae14a**, Proof of Theorem 1.2], each element $g \in G$ gives rise to a self-isometry γ_g on (X, d) given by $\gamma_g(i, h) := (i, gh)$ for all $(i, h) \in I \times G$. Since $\varphi \circ \pi_1$ does not depend on the second coordinate, one immediately verifies that $\gamma_g \in \mathrm{Aut}(X, P)$ for each $g \in G$. Note that $\iota : G \to \mathrm{Aut}(X, P)$ given by $g \mapsto \gamma_g$ defines a monomorphism of groups and we may thus consider G as a subgroup of $\mathrm{Aut}(X, P)$. We equip $\mathrm{Aut}(X, P)$

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with the topology of pointwise convergence and note that $\iota(G)$ is a discrete subgroup of $\mathrm{Aut}(X, P)$. We do not distinguish between G and $\iota(G)$. Since $\mathrm{card}(I) < \infty$, the action of G is quasitransitive, that is, G acts with finitely many orbits, namely the orbits are $\{i\} \times G$, $i \in I$. Since the random walk P is recurrent, it follows from [**Woe00**, Theorem 5.13] that X is roughly isometric with the one- or two-dimensional grid. In particular, the growth function of X is polynomial of degree one or two. We can then apply [**Woe94b**, Theorem 4.1] to the discrete subgroup G of $\mathrm{Aut}(X, P)$. The theorem states that there exists a compact normal subgroup N of G such that G/N contains $\mathbb{Z}$ or $\mathbb{Z}^2$ as a finite-index subgroup. Because N is compact and discrete, we have that N is finite and, hence, G is a recurrent group. □

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