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Mixed boundary-value problems for an elastic half-space

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1. *Introduction.* In this paper the mixed problem for an isotropic, elastic half-space is considered. Boundary conditions are prescribed interior and exterior to a circular region of unit radius, and the state of stress is assumed to be axially symmetric. Several authors have treated this problem. Mossakovskii (1) considered a punch adhering to and indenting an elastic half-space. His solution was obtained by introducing certain operators that transformed the half-space problem into a problem in plane potential theory. The method of linear relationship was used to solve this auxiliary problem and inverse operators returned the plane to the half-space. The general case of a circular, rigid punch adhering to a half-space was treated by Ufliand (2, 3) and a solution was obtained through the use of toroidal coordinates and the Mehler–Fok integral transforms.

Using integral transform theory, Abramian *et al.* (4) have reduced the axially symmetric problem, involving simultaneous pairs of dual integral equations, to a single integral equation which they were able to solve by a direct application of the results from a paper by Rapoport (5). A recent paper by Mossakovskii and Rybka (6) solved the problem of a penny-shaped crack between two dissimilar materials using the same techniques as in (1). Erdogan (7) solved several axially symmetric mixed problems by means of a reduction to a singular integral equation. He found that the solution to the dominant part of this equation very nearly approximated the behaviour of the surface stresses. The solution of the singular integral equation had to be obtained numerically.

The method of solution to be used in this paper is somewhat different although it will be shown that the results reduce to those given by Mossakovskii. The method of Green and Zerna (8) and Collins (9), as applied to axially symmetric potential theory problems, is used. This technique, similar in form to that of Sneddon (10), introduces an auxiliary function that may be determined by the solution of Abel's integral equation. For the problems considered in this paper, it will be shown that these auxiliary functions may be determined exactly from the solution of a singular integral equation. Quantities of interest may be calculated directly from these auxiliary functions.

2. *Solution to punch problems.* The indentation of an isotropic, elastic half-space by a rigid, axially symmetric punch acting along its axis of symmetry is considered. The half-space ($z \leqslant 0$) is deformed infinitesimally by the punch in the contact region $0 \leqslant \rho \leqslant 1$. Cylindrical polar coordinates (ρ, θ, z) are used, the z-axis coinciding with the axis of the punch. The displacements corresponding to these coordinates are

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u_ρ, u_z with stresses $\sigma_{z\rho}$, σ_{zz} on the surface whose normal is parallel to the z-axis. The boundary conditions are as follows:

$$u_\rho = p(\rho), \quad z = 0 \quad (0 \leqslant \rho \leqslant 1), \tag{1}$$

$$u_z = q(\rho), \quad z = 0 \quad (0 \leqslant \rho \leqslant 1), \tag{2}$$

$$\sigma_{zz} = 0, \quad z = 0 \quad (1 < \rho < \infty), \tag{3}$$

$$\sigma_{z\rho} = 0, \quad z = 0 \quad (1 < \rho < \infty). \tag{4}$$

A representation for the displacement in terms of two harmonic functions, which is convenient for the subsequent use of axially symmetric potential functions, is the following:

$$2\mu u_\rho = (1-2\nu)\frac{\partial \phi}{\partial \rho} + 2(1-\nu)\frac{\partial \chi}{\partial \rho} + z\frac{\partial^2}{\partial \rho\, \partial z}(\phi + \chi), \tag{5}$$

$$2\mu u_z = -2(1-\nu)\frac{\partial \phi}{\partial z} - (1-2\nu)\frac{\partial \chi}{\partial z} + z\frac{\partial^2}{\partial z^2}(\phi + \chi). \tag{6}$$

Equations (5) and (6) lead to the stresses

$$\sigma_{zz} = -\frac{\partial^2 \phi}{\partial z^2} + z\frac{\partial^3}{\partial z^3}(\phi + \chi), \tag{7}$$

$$\sigma_{z\rho} = \frac{\partial^2 \chi}{\partial \rho\, \partial z} + z\frac{\partial^3}{\partial \rho\, \partial z^2}(\phi + \chi), \tag{8}$$

where $\nabla^2 \phi = \nabla^2 \chi = 0$. If the representation in equations (6) through (8) is used, then equations (3) and (4) are automatically satisfied, when $\partial\phi/\partial z$ and $\partial\chi/z$ have the following forms:

$$\frac{\partial \phi}{\partial z} = \frac{1}{2}\int_{-1}^{1} f(t)\,[\rho^2 + (z+it)^2]^{-\frac{1}{2}}\,dt, \quad f(t) = f(-t), \tag{9}$$

$$\frac{\partial \chi}{\partial z} = \frac{1}{2i}\int_{-1}^{1} g(t)\,[\rho^2 + (z+it)^2]^{-\frac{1}{2}}\,dt, \quad g(t) = -g(-t). \tag{10}$$

Using the harmonic functions (9) and (10), it is found, by appropriate limiting procedures described in (8) and (9), that the following values for their derivatives as $z \to 0$ $(0 \leqslant \rho \leqslant 1)$ are:

$$\frac{\partial \phi}{\partial z} = \int_0^\rho f(t)\,(\rho^2 - t^2)^{-\frac{1}{2}}\,dt, \tag{11}$$

$$\frac{\partial \chi}{\partial z} = -\int_\rho^1 g(t)(t^2 - \rho^2)^{-\frac{1}{2}}\,dt, \tag{12}$$

$$\rho\frac{\partial \phi}{\partial \rho} = \int_0^1 f(t)\,dt - \int_\rho^1 t f(t)\,(t^2 - \rho^2)^{-\frac{1}{2}}\,dt, \tag{13}$$

$$\rho\frac{\partial \chi}{\partial \rho} = -\int_0^\rho t g(t)\,(t^2 - \rho^2)^{-\frac{1}{2}}\,dt. \tag{14}$$

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When the above results are used for boundary conditions (1), (2) the following set of simultaneous integral equations for $f(t)$ and $g(t)$ is obtained:

$$(1-2\nu)\int_\rho^1 tf(t)(t^2-\rho^2)^{-\frac{1}{2}}\,dt+2(1-\nu)\int_0^\rho tg(t)\,(\rho^2-t^2)^{-\frac{1}{2}}\,dt = (1-2\nu)\int_0^1 f(t)\,dt-2\mu p(\rho), \quad (15)$$

$$-2(1-\nu)\int_0^\rho f(t)\,(\rho^2-t^2)^{-\frac{1}{2}}\,dt+(1-2\nu)\int_\rho^1 g(t)\,(t^2-\rho^2)^{-\frac{1}{2}}\,dt = 2\mu q(\rho). \quad (16)$$

The purpose of this section is to determine the unknown functions, $f(t)$ and $g(t)$ from equations (15), (16). Noting that

$$-\tfrac{1}{2}\log\left|\frac{\xi-x}{\xi+x}\right| = \int_0^\xi \frac{\sigma\,d\sigma}{(\xi^2-\sigma^2)^{\frac{1}{2}}(x^2-\sigma^2)^{\frac{1}{2}}} \quad (x>\xi)$$

$$= \int_0^x \frac{\sigma\,d\sigma}{(\xi^2-\sigma^2)^{\frac{1}{2}}(x^2-\sigma^2)^{\frac{1}{2}}} \quad (\xi>x),$$

multiplying equations (15) and (16) by the operator

$$\frac{d}{ds}\int_0^s \rho(s^2-\rho^2)^{-\frac{1}{2}}\,d\rho, \quad (17)$$

and reversing the order of integration, the equations are reduced to simultaneous integral equations involving Cauchy type kernels. (It is interesting to note that the procedure followed here is nearly the reverse of Peters (11), who reduces an integral equation with a Cauchy kernel to the solution of Abel integral equations.) The result, after performing the indicated differentiation, is

$$\tfrac{1}{2}(1-2\nu)\int_{-1}^1 \frac{tf(t)}{t-s}\,dt+2(1-\nu)\,\tfrac{1}{2}\pi\,sg(s) = \frac{d}{ds}\int_0^s \rho A(\rho)\,(s^2-\rho^2)^{-\frac{1}{2}}\,d\rho, \quad (18)$$

$$-2(1-\nu)\tfrac{1}{2}\pi f(s)+\tfrac{1}{2}(1-2\nu)\int_{-1}^1 \frac{g(t)\,dt}{t-s} = 2\mu\frac{d}{ds}\int_0^s q(\rho)\,(s^2-\rho^2)^{-\frac{1}{2}}\,d\rho, \quad (19)$$

where

$$A = (1-2\nu)\int_0^1 f(t)\,dt-2\mu p(\rho).$$

Equation (18) may be written in a more useful form by the use of the properties of integrals with Cauchy type kernels. Thus, equation (18) is prepared for subsequent operations by writing it as

$$\tfrac{1}{2}(1-2\nu)\int_{-1}^1 \frac{f(t)\,dt}{t-s}+2(1-\nu)\,\tfrac{1}{2}\pi g(s) = -2\mu\frac{d}{ds}\int_0^s \rho p(\rho)\,(s^2-\rho^2)^{-\frac{1}{2}}\,d\rho. \quad (20)$$

If equation (20) is multiplied by $-i$ and equation (19) subtracted from it, then the following singular integral equation is obtained

$$2(1-\nu)\,\psi(s)+\frac{(1-2\nu)}{\pi i}\int_{-1}^1 \frac{\psi(t)}{t-s} = l(s) \quad (|s|<1), \quad (21)$$

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where $\psi = f - ig$ and

$$l(s) = \frac{4\mu}{\pi}\frac{d}{ds}\int_0^s \rho(ip-q)\,(s^2-\rho^2)^{-\frac{1}{2}}\,d\rho.$$

Equation (21) has been discussed in detail by Muskhelishvili (12), and its solution may be written immediately. Defining

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{\psi\,dt}{t-z} \tag{22}$$

and using the Plemelj formulae

$$\psi = \Phi^+ - \Phi^-, \quad \frac{1}{\pi i}\int_{-1}^{1}\frac{\psi\,dt}{t-z} = \Phi^+ + \Phi^-, \tag{23}$$

it is easily seen that the following Hilbert problem must be solved

$$(3-4\nu)\,\Phi^+ - \Phi^- = l(s) \quad (|s| < 1). \tag{24}$$

The solution to the Hilbert problem, equation (24), may be immediately written as

$$(3-4\nu)\,\Phi(z) = \frac{X}{2\pi i}\int_{-1}^{1}\frac{l(t)\,dt}{X(t-z)} + P(z)\,X, \tag{25}$$

where $X = X^+ = X^-/(3-4\nu)$, $P(z)$ is an arbitrary polynomial, and

$$X(z) = \left(\frac{z-1}{z+1}\right)^{i\beta}, \quad \beta = \frac{1}{2\pi}\log(3-4\nu). \tag{26}$$

It is noted that the function X, having an exponent that is pure imaginary, must be of the form given in (26) and is bounded at the end-points. For $\Phi(z)$ vanishing as $z \to \infty$, $P(z)$ is taken to be zero.

The normal and tangential surface stresses in the region $z = 0\,(0 \leqslant \rho \leqslant 1)$ are given by

$$\sigma_{zz} = -\frac{\partial^2\phi}{\partial z^2}\bigg|_{z=0} = -\frac{1}{\rho}\frac{d}{d\rho}\int_\rho^1 tf(t)\,(t^2-\rho^2)^{-\frac{1}{2}}\,dt, \tag{27}$$

$$\sigma_{z\rho} = \frac{\partial^2\chi}{\partial\rho\,\partial z}\bigg|_{z=0} = -\frac{d}{d\rho}\int_\rho^1 g(t)\,(t^2-\rho^2)^{-\frac{1}{2}}\,dt, \tag{28}$$

or in terms of Φ, they may be written as

$$\sigma_{zz} = -\frac{1}{\rho}\frac{d}{d\rho}\int_0^\rho t(\rho^2-t^2)^{-\frac{1}{2}}\,\Im\{\Phi^+ + \Phi^-\}\,dt, \tag{29}$$

$$\sigma_{z\rho} = -\frac{d}{d\rho}\int_0^\rho (\rho^2-t^2)^{-\frac{1}{2}}\,\Re\{\Phi^+ + \Phi^-\}\,dt. \tag{30}$$

Equations (29) and (30) are obtained from (27) and (28) after appropriate use has been made of the operator (17). In these equations Φ is evaluated on $0 \leqslant t \leqslant 1$.

As an example case the problem given in (1) will be solved. Taking

$$p = 0, \quad q = \epsilon \tag{31}$$

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and using the results of Muskhelishvili (13) for the evaluation of the integral in (25), it is easily found that

$$\Phi = \frac{2\mu\epsilon}{\pi(1-2\nu)}[1-X], \tag{32}$$

$$\psi = \frac{4\mu\epsilon}{\pi}X^{+}.$$

The pressure under the punch is thus given by (29) as

$$\sigma_{zz} = -\frac{8\mu\epsilon(1-\nu)}{\pi(1-2\nu)(3-4\nu)^{\frac{1}{2}}}\frac{1}{\rho}\frac{d}{d\rho}\int_0^\rho \frac{t\,dt}{(\rho^2-t^2)^{\frac{1}{2}}}\sin\left[\beta\log\left|\frac{1-t}{1+t}\right|\right] \tag{33}$$

in agreement with Mossakovskii.

3. *Solution to crack problem.* The adhesion of an elastic half-space to a rigid foundation is investigated. The half-space ($z \leqslant 0$) is bonded perfectly to the foundation everywhere except in a circular region $0 \leqslant \rho \leqslant 1$, where the contact is smooth. It is assumed that infinitesimal surface stresses applied in this region tend to open the crack. The coordinate system and notation are the same as in section 2. The boundary conditions are as follows:

$$u_\rho = 0, \quad z = 0 \quad (1 < \rho < \infty), \tag{34}$$

$$u_z = 0, \quad z = 0 \quad (1 < \rho < \infty), \tag{35}$$

$$\sigma_{zz} = p, \quad z = 0 \quad (0 \leqslant \rho \leqslant 1), \tag{36}$$

$$\sigma_{z\rho} = q, \quad z = 0 \quad (0 \leqslant \rho \leqslant 1). \tag{37}$$

A representation for the displacement in terms of two harmonic functions, similar to that given by equations (5) through (8), will be used in this section. The displacements and stresses are given as

$$2\mu u_\rho = (3-4\nu)\frac{\partial\gamma}{\partial\rho} + z\frac{\partial^2}{\partial\rho\,\partial z}(\gamma-\eta), \tag{38}$$

$$2\mu u_z = (3-4\nu)\frac{\partial\eta}{\partial z} + z\frac{\partial^2}{\partial z^2}(\gamma-\eta), \tag{39}$$

$$\sigma_{zz} = (1-2\nu)\frac{\partial^2\gamma}{\partial z^2} + 2(1-\nu)\frac{\partial^2\eta}{\partial z^2} + z\frac{\partial^3}{\partial z^3}(\gamma-\eta), \tag{40}$$

$$\sigma_{z\rho} = 2(1-\nu)\frac{\partial^2\gamma}{\partial\rho\,\partial z} + (1-2\nu)\frac{\partial^2\eta}{\partial\rho\,\partial z} + z\frac{\partial^3}{\partial\rho\,\partial z^2}(\gamma-\eta), \tag{41}$$

where equations (40) and (41) are stresses on the surface whose normal is parallel to the z-axis and $\nabla^2\gamma = \nabla^2\eta = 0$. In this form it is easily seen that the functions

$$\frac{\partial\eta}{\partial z} = \frac{1}{2i}\int_{-1}^{1} h(t)\,[\rho^2+(z+it)^2]^{-\frac{1}{2}}\,dt, \quad h(t) = -h(-t), \tag{42}$$

$$\frac{\partial\gamma}{\partial z} = \frac{1}{2}\int_{-1}^{1} j(t)\,[\rho^2+(z+it)^2]^{-\frac{1}{2}}\,dt, \quad j(t) = j(-t), \tag{43}$$

where

$$\int_0^1 j(t)\,dt = 0, \tag{44}$$

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will automatically satisfy equations (34) and (35), provided that condition (44) is met. In the sequel this condition is used to determine the value of an arbitrary constant of integration. With the use of appropriate limiting procedures as $z \to 0$, equations (40) and (41) may be written in the following form:

$$\sigma_{zz} = 2(1-\nu)\frac{1}{\rho}\frac{d}{d\rho}\int_0^\rho th(t)\,(\rho^2-t^2)^{-\frac{1}{2}}\,dt + (1-2\nu)\,\frac{1}{\rho}\frac{d}{d\rho}\int_\rho^1 tj(t)\,(t^2-\rho^2)^{-\frac{1}{2}}\,dt, \tag{45}$$

$$\sigma_{z\rho} = -(1-2\nu)\frac{d}{d\rho}\int_\rho^1 h(t)\,(t^2-\rho^2)^{-\frac{1}{2}}\,dt + 2(1-\nu)\,\frac{d}{d\rho}\int_0^\rho j(t)\,(\rho^2-t^2)^{-\frac{1}{2}}\,dt. \tag{46}$$

By application of boundary condition (36) and an integration by parts, equation (45) becomes

$$-(1-2\nu)j(1)\,(1-\rho^2)^{-\frac{1}{2}} + 2(1-\nu)\int_0^\rho j'(t)\,(\rho^2-t^2)^{-\frac{1}{2}}\,dt + (1-2\nu)\int_\rho^1 j'(t)\,(t^2-\rho^2)^{-\frac{1}{2}}\,dt = p. \tag{47}$$

Multiplying equation (47) by the operator

$$\frac{d}{ds}\int_0^s \rho(s^2-\rho^2)^{-\frac{1}{2}}\,d\rho, \tag{48}$$

reversing the order of integration, and integrating, the following equation is obtained

$$2(1-\nu)\,\tfrac{1}{2}\pi\,h(s) + \tfrac{1}{2}(1-2\nu)\int_{-1}^1 \frac{j(t)\,dt}{t-s} = \int_0^s \rho p(\rho)\,(s^2-\rho^2)^{-\frac{1}{2}}\,\rho\,d\rho. \tag{49}$$

Similarly, equation (46) is integrated and multiplied by the operator, equation (48). After changing the order of integration and differentiating the result is

$$-\tfrac{1}{2}(1-2\nu)\int_{-1}^1 \frac{h(t)\,dt}{t-s} + 2(1-\nu)\,\tfrac{1}{2}\pi\,j(s) = s\int_0^s q(\rho)\,(s^2-\rho^2)^{-\frac{1}{2}}\,d\rho + C, \tag{50}$$

where C is an arbitrary constant of integration to be determined by condition (44). If equation (50) is multiplied by i and added to equation (49), then

$$2(1-\nu)\,\psi(s) + \frac{(1-2\nu)}{\pi i}\int_{-1}^1 \frac{\psi(t)\,dt}{t-s} = r(s) \quad (|s| \leqslant 1), \tag{51}$$

where $\psi = h + ij$ and

$$r(s) = \frac{2}{\pi}\int_0^s (\rho p + isq)\,(s^2-\rho^2)^{-\frac{1}{2}}\,d\rho.$$

Condition (44) now becomes

$$\Im\left\{\int_{-1}^1 \psi\,ds\right\} = 0. \tag{52}$$

Equation (51) is identical in form to equation (21), and its solution will be treated in the same way. Defining

$$\Phi(z) = \frac{1}{2\pi i}\int_{-1}^1 \frac{\psi\,ds}{t-z} \tag{53}$$

and using the Plemelj formulae

$$\psi = \Phi^+ - \Phi^-, \quad \frac{1}{\pi i}\int_{-1}^1 \frac{\psi\,dt}{t-z} = \Phi^+ + \Phi^- \tag{54}$$

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equation (51) reduces to the following Hilbert problem:

$$(3-4\nu)\,\Phi^+ - \Phi^- = r(s) \quad (|s| \leqslant 1). \tag{55}$$

The solution is

$$(3-4\nu)\,\Phi(z) = \frac{X}{2\pi i}\int_{-1}^{1} \frac{r(t)\,dt}{X(t-z)} + P(z)\,X, \tag{56}$$

where $X = X^+ = X^-/(3-4\nu)$, $P(z)$ is an arbitrary polynomial, and

$$X = \left(\frac{z-1}{z+1}\right)^{i\beta}, \quad \beta = \frac{1}{2\pi}\log(3-4\nu). \tag{57}$$

The solution $\Phi(z)$ is taken to vanish as $z \to \infty$ and hence $P(z) = 0$.

The displacements in the region, $z = 0$ $(0 \leqslant \rho \leqslant 1)$ are of interest, and from equations (38), (39), (40), and (42), they may be shown to have the values

$$2\mu u_z = -(3-4\nu)\int_0^1 h(t)\,(t^2-\rho^2)^{-\frac{1}{2}}\,dt, \tag{58}$$

$$2\mu u_\rho = -(3-4\nu)\,\rho^{-1}\int_\rho^1 j(t)\,(t^2-\rho^2)^{-\frac{1}{2}}\,dt. \tag{59}$$

Equations (58) and (59) may be put into the following form, dependent upon Φ:

$$2\mu u_z = -(3-4\nu)\int_0^\rho (\rho^2-s^2)^{-\frac{1}{2}}\,ds\,\Im\{\Phi^+ + \Phi^-\}, \tag{60}$$

$$2\mu u_\rho = -(3-4\nu)\int_0^\rho s(\rho^2-s^2)^{-\frac{1}{2}}\,ds\,\Re\{\Phi^+ + \Phi^-\} \tag{61}$$

by appropriate applications of (48).

As an example the case of a crack opened by uniform normal pressure is considered. Taking

$$p = p_0, \quad q = 0 \tag{62}$$

it is easily shown that (56) becomes

$$2(1-2\nu)\,\Phi(z) = (2/\pi)\,[p_0 z + iC - X(p_0 z + iC + 2i\beta p_0)] \tag{63}$$

and using the Plemelj formulae

$$\psi = (2/\pi)\,(p_0 z + iC + 2i\beta p_0)\,X^+. \tag{64}$$

From the application of equation (52) and with the aid of (13)

$$C = -\beta p_0 \tag{65}$$

and

$$\left.\begin{aligned} 2(1-2\nu)\,\Phi(z) &= (2/\pi)\,p_0[z - i\beta - X(z+i\beta)], \\ \psi(z) &= (2/\pi)\,p_0(z+i\beta)\,X^+. \end{aligned}\right\} \tag{66}$$

From equations (60), (61), and (66) the surface displacements are

$$2\mu u_z = \frac{(3-4\nu)}{(1-2\nu)}p_0\int_0^\rho (\rho^2-s^2)^{-\frac{1}{2}}\left\{\beta + 2(1-\nu)\,(3-4\nu)^{-\frac{1}{2}}\left[\beta\cos\left(\beta\log\left|\frac{1-s}{1+s}\right|\right)\right.\right.$$
$$\left.\left. + s\sin\left(\beta\log\left|\frac{1-s}{1+s}\right|\right)\right]\right\} ds, \tag{67}$$

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$$2\mu u_\rho = -\frac{(3-4\nu)}{(1-2\nu)} p_0 \int_0^\rho s(\rho^2-s^2)^{-\frac{1}{2}} \left\{ s - 2(1-\nu)(3-4\nu)^{-\frac{1}{2}} \left[s\cos\left(\beta \log\left|\frac{1-s}{1+s}\right|\right) - \beta \sin\left(\beta \log\left|\frac{1-s}{1+s}\right|\right)\right]\right\} ds. \quad (68)$$

The strain energy may be computed by the following expression

$$W = \tfrac{1}{2} 2\pi p_0 \int_0^1 \rho u_z d\rho. \quad (69)$$

Inserting equation (58) into (69) and reversing the order of integration leads to

$$W = \frac{(3-4\nu)}{2\mu} p_0^2 \int_{-1}^1 t(t+i\beta) X^+ dt. \quad (70)$$

The integral in (70) may be evaluated by the methods in (13) to give the result:

$$W = 2\pi(1+\nu)(3-4\nu)\beta(1+\beta^2) p_0^2/3(1-2\nu)E \quad (71)$$

which is in agreement with equation (4·11) in Mossakovskii and Rybka if $E_1 = \infty$ and $a = 1$.

4. *Discussion.* The method of Green and Zerna and Collins was used to solve two mixed boundary-value problems for an elastic half-space. If the displacement potentials were written as Hankel transforms, then a method similar to that of Sneddon (10) could be used to develop the Abel type equations in (15), (16) and (45), (46). Solution would follow in exactly the same way. Problems of internal and external cracks between two dissimilar, bonded materials lead to equations identical in form to (15), (16) and (45), (46), and aside from changes in various constants the solutions will be of the same form.

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