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On a syntactically defined invariant of symbolic dynamics

WOLFGANG KRIEGER

Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, D-69120 Heidelberg, Germany

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Abstract. A partially ordered set that is invariantly associated to a subshift is constructed. A property of subshifts, also an invariant of topological conjugacy, is described. If this property is present in a subshift then the constructed partially ordered set is a partially ordered semigroup (with zero). In the description of these invariants the notion of context is instrumental.

1. *Introduction*

Let Σ be a finite alphabet, and let S_Σ be the shift on the shift space $\Sigma^{\mathbb{Z}}$:

$$S_\Sigma(x_i)_{i\in\mathbb{Z}} = (x_{i+1})_{i\in\mathbb{Z}}, \quad (x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}.$$

In symbolic dynamics one studies subshifts (C, S), C a S_Σ-invariant closed subset of $\Sigma^{\mathbb{Z}}$ and S the restriction of S_Σ to C. An introduction to the theory of subshifts is in [**8, 9**] (see also [**1, 2**]). In this paper we describe certain invariants of the topological conjugacy of subshifts. Given a subshift $C \subset \Sigma^{\mathbb{Z}}$ the context of an admissible block a of C is the set of all pairs (a^-, a^+) of blocks such that a^-aa^+ is C-admissible. Using the notion of context, in §2 we associate to a subshift C a partially ordered set $\mathcal{M}_C$ that is an invariant of topological conjugacy.

Again using the notion of context, in §3 we describe a property of subshifts that we refer to as property (A). While in describing property (A) we make reference to a presentation of the subshift, we establish that either all or no presentations of the subshift possess this property. In a sense this property is of a geometrical and dynamical nature, not only by the formulation of its defining description, but also by its invariance. We show then for a subshift C with property (A) that the partially ordered set $\mathcal{M}_C$ is a partially ordered semigroup (with zero), that is also invariantly associated to C. In §4 we give some examples.

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2. *A partially ordered set associated to a subshift*

We fix notation and terminology. Given a subshift $C \subset \Sigma^{\mathbb{Z}}$ we set

$$x_{[i,k]} = (x_j)_{i \le j \le k}, \quad x \in C, i, k \in \mathbb{Z}, i \le k,$$

and

$$C_{[i,k]} = \{x_{[i,k]} : x \in C\}.$$

We also use similar notation for blocks,

$$b_{[i',k']} = (b_j)_{i' \le j \le k'}, \quad b \in C_{[i,k]}, i \le i' \le k' \le k, i, k \in \mathbb{Z}, i \le k,$$

and also if indices range in infinite intervals. For the n-block system of a subshift $C \subset \Sigma^{\mathbb{Z}}$ we use notation like

$$x^{[1,n]} = (x_{(i,i+n]})_{i \in \mathbb{Z}}, \quad x \in C,$$
$$C^{[1,n]} = \{x^{[1,n]} : x \in C\}, \quad n \in \mathbb{N}.$$

Unstable and stable sets are denoted by $W^-(x)$, $W^+(x)$, $x \in C$,

$$W^-(x) = \bigcup_{I \in \mathbb{Z}} \{y \in C : y_i = x_i, i < I\},$$

with the time symmetrical definition for $W^+(x)$. The context of a C-admissible block $a \in C_{[i,k]}$, $i, k \in \mathbb{Z}$, $i \le k$, is denoted by $\Gamma(a)$:

$$\Gamma(a) = \{(x^-, x^+) \in C_{(-\infty,i)} \times C_{(k,\infty)} : (x^-, a, x^+) \in C\}.$$

When comparing contexts, we feel free to identify blocks with the words they carry. With the notation

$$\Gamma^-(a) = \{x^- \in C_{(-\infty,i)} : (x^-, a) \in C_{(-\infty,k]}\},$$

for the left context of a block we set

$$\omega^+(a) = \bigcup_{n \in \mathbb{N}} \bigcap_{c^- \in \Gamma^-(a)} \{b \in C_{(k,k+n]} : (c^-, a, b) \in C_{(-\infty,k+n]}\},$$
$$a \in C_{[i,k]}, i, k \in \mathbb{Z}, i \le k.$$

$\omega^-(a)$ has the time symmetrical meaning.

We recall that for every homomorphism $\varphi : C \to \overline{C}$ of subshifts $C \subset \Sigma^{\mathbb{Z}}$, $\overline{C} \subset \overline{\Sigma}^{\mathbb{Z}}$ there is for some $L \in \mathbb{Z}_+$ a block mapping

$$\Phi : C_{[-L,L]} \to \overline{\Sigma}$$

such that

$$\varphi(x) = (\Phi(x_{[i-L,i+L]}))_{i \in \mathbb{Z}}.$$

We then say that φ is given by Φ, and we write

$$\Phi(a) = (\Phi(a_{[j-L,j+L]})_{i+L \le j \le k-L}), \quad a \in C_{[i,k]}, i, k \in \mathbb{Z}, \ k - i \ge 2L,$$

and we also use similar notation if indices range in infinite intervals.

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LEMMA 2.1. *Let $\overline{C} \subset \overline{\Sigma}^{\mathbb{Z}}$, $C \subset \Sigma^{\mathbb{Z}}$ be subshifts and let $\varphi : \overline{C} \to C$ be a topological conjugacy that is given by the 1-block map $\overline{\Phi} : \overline{\Sigma} \to \Sigma$, with φ^{-1} given for some $L \in \mathbb{Z}_+$ by the block map*

$$\Phi : C_{[-L,L]} \to \overline{\Sigma}.$$

Let $\overline{x} \in \overline{C}$, $x = \varphi(\overline{x})$ and, for some $i \in \mathbb{Z}$, let

$$x_{(i,i+L+1]} \in \omega^+(x_i). \tag{1}$$

Then

$$\overline{x}_{i+1} \in \omega^+(\overline{x}_{[i-2L,i]}).$$

Proof. Let

$$\overline{y}^- \in \overline{C}_{(-\infty,i]}, \quad \overline{y}^-_{[i-2L,i]} = \overline{x}_{[i-2L,i]},$$

and let

$$y^- = \overline{\Phi}\overline{y}^-.$$

Then

$$\overline{y}^-_{[i-2L,i]} = x_{[i-2L,i]}.$$

By (1),

$$(y^-, x_{[i,i+L+1]}) \in C_{(-\infty,i+L+1]}.$$

By construction

$$(\overline{y}^-, \overline{x}_{i+1}) = \Phi(y^-, x_{(i,i+L+1]}). \qquad \square$$

For a subshift $C \subset \Sigma^{\mathbb{Z}}$ we set

$$X_n(C) = \bigcap_{i\in\mathbb{Z}} \{x \in C : x_i \in \omega^+(x_{[i-n,i)}) \cap \omega^-(x_{(i,i+n]})\}, \quad n \in \mathbb{N}.$$

$X_n(C)$, if not empty, is an n-step topological Markov shift and

$$X_n(C) \subset X_{n+1}(C), \quad n \in \mathbb{N}.$$

We set

$$X_C = \bigcup_{n\in\mathbb{N}} X_n(C).$$

We abbreviate $X_1(C)$ to $X(C)$.

PROPOSITION 2.2. *Let $C \subset \Sigma^{\mathbb{Z}}$, $\overline{C} \subset \overline{\Sigma}^{\mathbb{Z}}$ be subshifts and let $\varphi : C \to \overline{C}$ be a topological conjugacy. Then*

$$\varphi(X_C) = X_{\overline{C}}.$$

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Proof. We consider the situation where the topological conjugacy $\varphi^{-1} : \overline{C} \to C$ is given by the 1-block map $\overline{\Phi} : \overline{\Sigma} \to \Sigma$ with φ given for some $L \in \mathbb{Z}_+$ by the block map

$$\Phi : C_{[-L,L]} \to \overline{\Sigma}.$$

Set

$$\widehat{\overline{C}} = \overline{C}^{[-L,L]}.$$

By Lemma 2.1

$$\varphi(X(C))^{[-L,L]} \subset X(\widehat{\overline{C}}).$$

One can conclude that

$$\varphi(X_n(C)) \subset X_{n+2L+1}(\overline{C}), \quad n \in \mathbb{N},$$

and therefore that

$$\varphi(X_C) = X_{\overline{C}}. \qquad \square$$

We denote by $P_n(C)$ the set of periodic points in $X_n(C)$, $n \in \mathbb{N}$, and we set

$$P_C = \bigcup_{n\in\mathbb{N}} P_n(C).$$

We abbreviate $P_1(C)$ to $P(C)$. We introduce a reflexive and transitive relation $\leq (C)$ into the set P_C (actually, into the set of orbits in P_C). For $u, u' \in P_C$, we mean by $u \geq (C)u'$ that there exists a point in X_C that is negatively asymptotic to the orbit of u and positively asymptotic to the orbit of u'. We denote the resulting equivalence relation by $= (C)$ and the set of $= (C)$-equivalence classes by $\mathcal{P}_C$. The partial order that is induced by $\leq (C)$ on $\mathcal{P}_C$ we also denote by $\leq (C)$. The structure $(\mathcal{P}_C, \leq (C))$ is invariantly associated to C. Here it is an auxiliary structure.

We denote by $Y_n(C)$ the set of points in C that are negatively asymptotic to a point in $P_n(C)$ and also positively asymptotic to a point in $P_n(C)$, $n \in \mathbb{N}$, and we set

$$Y_C = \bigcup_{n\in\mathbb{N}} Y_n(C).$$

We abbreviate $Y_1(C)$ to $Y(C)$.

Now consider a subshift $C \subset \Sigma^{\mathbb{Z}}$. Let $H \in \mathbb{N}$. We introduce a reflexive and transitive relation $\lesssim (C, H)$ into the set $Y(C)$ (actually into the set of orbits in $Y(C)$). For

$$u \in W^-(q^-) \cap W^+(q^+), \quad v \in W^-(r^-) \cap W^+(r^+), \quad q^-, q^+, r^-, r^+ \in P(C),$$

we define

$$u \lesssim (C, H)v$$

to mean:

$$q^- \geq (C)r^-, \quad q^+ \leq (C)r^+,$$

and if

$$I_-, I_+, J_-, J_+ \in \mathbb{Z}, \quad I_- < I_+, \quad J_- < J_+,$$

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and

$$u_{(-\infty,I_-]} = q^-_{(-\infty,I_-]}, \quad u_{[I_+,\infty)} = q^+_{[I_+,\infty)},$$
$$v_{(-\infty,J_-]} = r_{(-\infty,J_-]}, \quad v_{[J_+,\infty)} = r^+_{[J_+,\infty)},$$

and if $i_-, i_+ \in \mathbb{Z}$,

$$i_- < I_- - 2H, \quad i_+ > I_+ + 2H,$$
$$j_- < J_- - 2H, \quad j_+ > J_+ + 2H,$$

and

$$a \in C_{[i_-,i_+]}, \quad b \in C_{[j_-,j_+]},$$

are such that

$$a_{[I_- - H, I_+ + H]} = u_{[I_- - H, I_+ + H]},$$
$$b_{[J_- - H, J_+ + H]} = v_{[J_- - H, J_+ + H]},$$
$$a_{i_- + h} = b_{j_- + h}, \quad a_{i_+ - h} = b_{j_+ - h}, \quad 0 \le h \le H,$$

and

$$a_{[i_-, I_- - H]} \in X(C)_{[i_-, I_- - H]}, \quad a_{[I_+ + H, i_+]} \in X(C)_{[I_+ + H, i_+]},$$
$$b_{[j_-, J_- - H]} \in X(C)_{[j_-, J_- - H]}, \quad b_{[J_+ + H, j_+]} \in X(C)_{[J_+ + H, j_+]},$$

then the context of a is contained in the context of b. We introduce a reflexive and transitive relation $\lesssim (C)$ into Y_C (actually, into the set of orbits in Y_C). For $u, v \in Y_C$ we mean by

$$u \lesssim (C) v$$

that there are $n_0 \in \mathbb{N}$ and $H_n \in \mathbb{N}$, $n \ge n_0$, such that

$$u, v \in Y_{n_0}(C)$$

and

$$u^{[1,n]} \lesssim (C^{[1,n]}, H_n) v^{[1,n]}, \quad n \ge n_0.$$

We denote the resulting equivalence relation by $\approx (C)$.

PROPOSITION 2.3. *Let $C \subset \Sigma^{\mathbb{Z}}$, $\overline{C} \subset \overline{\Sigma}^{\mathbb{Z}}$ be subshifts and let $\varphi : C \to \overline{C}$ be a topological conjugacy. Let $u, v \in Y_C$, $u \lesssim (C) v$. Then*

$$\varphi(u) \lesssim (\overline{C}) \varphi(v).$$

Proof. We consider the situation where the conjugacy φ^{-1} is given by a 1-block map

$$\overline{\Phi} : \overline{\Sigma} \to \Sigma$$

with φ given for some $L \in \mathbb{Z}_+$ by a block map

$$\Phi : C_{[-L,L]} \to \overline{\Sigma},$$

and we set

$$\widehat{\overline{C}} = \overline{C}^{[-L,L]}.$$

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Also let $u, v \in Y(C)$,

$$u \in W^-(q^-) \cap W^+(q^+), \quad v \in W^-(r^-) \cap W^+(r^+), \quad q^-, q^+, r^-, r^+ \in P(C),$$

and set

$$\widehat{\overline{u}} = \varphi(u)^{[-L,L]}, \quad \widehat{\overline{v}} = \varphi(v)^{[-L,L]},$$
$$\widehat{\overline{q}}^- = \varphi(q^-)^{[-L,L]}, \quad \widehat{\overline{q}}^+ = \varphi(q^+)^{[-L,L]}, \quad \widehat{\overline{r}}^- = \varphi(r^-)^{[-L,L]}, \quad \widehat{\overline{r}}^+ = \varphi(r^+)^{[-L,L]}.$$

By Lemma 2.1

$$\widehat{\overline{q}}^-, \widehat{\overline{q}}^+, \widehat{\overline{r}}^-, \widehat{\overline{r}}^+ \in X(\widehat{\overline{C}}), \quad \widehat{\overline{u}}, \widehat{\overline{v}} \in Y(\widehat{\overline{C}}).$$

Further, we set

$$\widehat{C} = C^{[-4L,4L]}, \quad \widehat{u} = u^{[-4L,4L]}, \quad \widehat{v} = v^{[-4L,4L]}.$$

Let $\widehat{H} \in \mathbb{N}$, and assume that

$$\widehat{u} \lesssim (\widehat{C}, \widehat{H})\widehat{v}. \tag{2}$$

We set

$$\widehat{\overline{H}} = \widehat{H} + 2L$$

and we establish the proposition by showing that

$$\widehat{\overline{u}} \lesssim (\widehat{\overline{C}}, \widehat{\overline{H}})\widehat{\overline{v}}.$$

For this, let

$$\widehat{\overline{I}}_-, \widehat{\overline{I}}_+, \widehat{\overline{J}}_-, \widehat{\overline{J}}_+ \in \mathbb{Z}, \quad \widehat{\overline{I}}_- < \widehat{\overline{I}}_+, \quad \widehat{\overline{J}}_- < \widehat{\overline{J}}_+,$$

and

$$\widehat{\overline{u}}_{(-\infty,\widehat{\overline{I}}_-]} = \widehat{\overline{q}}^-_{(-\infty,\widehat{\overline{I}}_-]}, \quad \widehat{\overline{u}}_{[\widehat{\overline{I}}_+,\infty)} = \widehat{\overline{q}}^+_{[\widehat{\overline{I}}_+,\infty)},$$
$$\widehat{\overline{v}}_{(-\infty,\widehat{\overline{J}}_-]} = \widehat{\overline{r}}^-_{(-\infty,\widehat{\overline{J}}_-]}, \quad \widehat{\overline{v}}_{[\widehat{\overline{J}}_+,\infty)} = \widehat{\overline{q}}^+_{[\widehat{\overline{J}}_+,\infty)},$$

and let $\widehat{\overline{i}}_-, \widehat{\overline{i}}_+ \in \mathbb{Z}$,

$$\widehat{\overline{i}}_- < \widehat{\overline{I}}_- - 2\widehat{\overline{H}}, \quad \widehat{\overline{i}}_+ > \widehat{\overline{I}}_+ + 2\widehat{\overline{H}},$$
$$\widehat{\overline{j}}_- < \widehat{\overline{J}}_+ - 2\widehat{\overline{H}}, \quad \widehat{\overline{j}}_+ > \widehat{\overline{J}}_+ + 2\widehat{\overline{H}},$$

and

$$\widehat{\overline{a}} \in \widehat{\overline{C}}_{[\widehat{\overline{i}}_-,\widehat{\overline{i}}_+]}, \quad \widehat{\overline{b}} \in \widehat{\overline{C}}_{[\widehat{\overline{j}}_-,\widehat{\overline{j}}_+]},$$

be such that

$$\widehat{\overline{a}}_{[\widehat{\overline{I}}_- - \widehat{\overline{H}}, \widehat{\overline{I}}_+ + \widehat{\overline{H}}]} = \widehat{\overline{u}}_{[\widehat{\overline{I}}_- - \widehat{\overline{H}}, \widehat{\overline{I}}_+ + \widehat{\overline{H}}]},$$
$$\widehat{\overline{b}}_{[\widehat{\overline{J}}_- - \widehat{\overline{H}}, \widehat{\overline{J}}_+ + \widehat{\overline{H}}]} = \widehat{\overline{v}}_{[\widehat{\overline{J}}_- - \widehat{\overline{H}}, \widehat{\overline{J}}_+ + \widehat{\overline{H}}]},$$
$$\widehat{\overline{a}}_{\widehat{\overline{i}}_- + h} = \widehat{\overline{b}}_{\widehat{\overline{j}}_- + h}, \quad \widehat{\overline{a}}_{\widehat{\overline{i}}_+ - h} = \widehat{\overline{b}}_{\widehat{\overline{j}}_+ - h}, \quad 0 \le h \le \widehat{\overline{H}},$$

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and

$$\widehat{\overline{a}}_{[\widehat{\overline{i}}_-,\widehat{\overline{I}}_- -\widehat{\overline{H}}]} \in X(\widehat{\overline{C}})_{[\widehat{\overline{i}}_-,\widehat{\overline{I}}_- -\widehat{\overline{H}}]}, \quad \widehat{\overline{a}}_{[\widehat{\overline{I}}_+ +\widehat{\overline{H}},\widehat{\overline{i}}_+]} \in X(\widehat{\overline{C}})_{[\widehat{\overline{I}}_+ +\widehat{\overline{H}},\widehat{\overline{i}}_+]},$$
$$\widehat{\overline{b}}_{[\widehat{\overline{j}}_-,\widehat{\overline{J}}_- -\widehat{\overline{H}}]} \in X(\widehat{\overline{C}}_{[\widehat{\overline{j}}_-,\widehat{\overline{J}}_- -\widehat{\overline{H}}]}, \quad \widehat{\overline{b}}_{[\widehat{\overline{J}}_+ +\widehat{\overline{H}},\widehat{\overline{j}}_+]} \in X(\widehat{\overline{C}})_{[\widehat{\overline{J}}_+ +\widehat{\overline{H}},\widehat{\overline{j}}_+]}.$$

We have to show that the context of $\widehat{\overline{a}}$ is contained in the context of $\widehat{\overline{b}}$. Denote the block map that implements the conjugacy of $\widehat{\overline{C}}$ onto $\widehat{C}$ by $\widehat{\overline{\Phi}}$,

$$\widehat{\overline{\Phi}}((\overline{c}_{[i-L,i+L]})_{-3L\le i\le 3L}) = (\overline{\Phi}(\overline{c}_j))_{-4L\le j\le 4L}, \quad \overline{c} \in \overline{C}_{[-4L,4L]},$$

and set

$$\widehat{a} = \widehat{\overline{\Phi}}(\widehat{\overline{a}}), \quad \widehat{b} = \widehat{\overline{\Phi}}(\widehat{\overline{b}}).$$

By (2) the context of $\widehat{a}$ is contained in the context of $\widehat{b}$. Denote the 1-block map that implements the conjugacy of $\widehat{C}$ onto $\widehat{\overline{C}}$ by $\widehat{\Phi}$:

$$\widehat{\Phi}(c) = \Phi(c_{[-2L,2L]}), \quad c \in C_{[-4L,4L]}.$$

With

$$\widehat{i}_- = \widehat{\overline{i}}_- + 2L, \quad \widehat{i}_+ = \widehat{\overline{j}}_+ - 2L,$$

one has

$$\Gamma(\widehat{\overline{a}}) = \Big\{(\widehat{\Phi}(\widehat{x}^-_{(-\infty,\widehat{\overline{i}}_-)}), \widehat{\Phi}(\widehat{x}^+_{(\widehat{\overline{i}}_+,\infty)})) : (\widehat{x}^-,\widehat{x}^+) \in \Gamma(\widehat{a}),$$
$$\widehat{\Phi}(\widehat{x}^-_{[\widehat{\overline{i}}_-,\widehat{i}_-)}) = \widehat{\overline{a}}_{[\widehat{\overline{i}}_-,\widehat{i}_-)}, \widehat{\Phi}(\widehat{x}^+_{(\widehat{i}_+,\widehat{\overline{i}}_+]}) = \widehat{\overline{a}}_{(\widehat{i}_+,\widehat{\overline{i}}_+]}\Big\},$$
$$\Gamma(\widehat{\overline{b}}) = \Big\{(\widehat{\Phi}(\widehat{x}^-_{(-\infty,\widehat{\overline{i}}_-)}), \widehat{\Phi}(\widehat{x}^+_{(\widehat{\overline{i}}_+,\infty)})) : (\widehat{x}^-,\widehat{x}^+) \in \Gamma(\widehat{b}),$$
$$\widehat{\Phi}(\widehat{x}^-_{[\widehat{\overline{i}}_-,\widehat{i}_-)}) = \widehat{\overline{b}}_{[\widehat{\overline{i}}_-,\widehat{i}_-)}, \widehat{\Phi}(\widehat{x}^+_{(\widehat{i}_+,\widehat{\overline{i}}_+]}) = \widehat{\overline{b}}_{(\widehat{i}_+,\widehat{\overline{i}}_+]}\Big\},$$

and it is shown that the context of $\widehat{\overline{a}}$ is contained in that of $\widehat{\overline{b}}$. □

We denote the set of $\approx$ (C)-equivalence classes by $\mathcal{M}_C$. The partial order that is induced by $\lesssim$ (C) on $\mathcal{M}_C$ we also denote by $\lesssim$ (C). The structure $(\mathcal{M}_C, \lesssim (C))$ is by Proposition 2.3 invariantly associated to C.

3. *A property of subshifts and an invariant of topological conjugacy*

We say that a subshift $C \subset \Sigma^{\mathbb{Z}}$, such that $X(C)$ is not empty, has property (a, K), $K \in \mathbb{Z}_+$, if the following holds: if $i_-, i_+, j_-, j_+ \in \mathbb{Z}$,

$$i_- < i_+ - 2K, \quad j_- < j_+ - 2K,$$

and if

$$a \in X(C)_{[i_-,i_+]}, \quad b \in X(C)_{[j_-,j_+]},$$
$$a_{i_-+k} = b_{j_-+k}, \quad a_{i_+-k} = b_{j_+-k}, \quad 0 \le k \le K,$$

then the context of a is equal to the context of b. We say that a subshift $C \subset \Sigma^{\mathbb{Z}}$ has property (a) if there is a $K \in \mathbb{Z}_+$ such that C has property (a, K). We say that a subshift $C \subset \Sigma^{\mathbb{Z}}$ has property (A) if there is an $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, $C^{[1,n]}$ has property (a).

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PROPOSITION 3.1. *Let the subshift* $\overline{C} \subset \overline{\Sigma}^{\mathbb{Z}}$ *have property* (A). *Then all subshifts* $C \subset \Sigma^{\mathbb{Z}}$ *that are topologically conjugate to* $\overline{C}$ *also have property* (A).

Proof. We consider the situation where a topological conjugacy $\varphi : \overline{C} \to C$ is given by the 1-block map $\overline{\Phi} : \overline{\Sigma} \to \Sigma$ with φ^{-1} given for some $L \in \mathbb{N}$ by the block map

$$\Phi : C_{[-L,L]} \to \overline{\Sigma}.$$

We denote

$$\widehat{\overline{C}} = \overline{C}^{[-L,L]},$$

and have the toplogical conjugacy $\widehat{\varphi} : \widehat{\overline{C}} \to C$ that is given by the 1-block map $\widehat{\overline{\Phi}} : \overline{C}_{[-L,L]} \to \Sigma$,

$$\widehat{\overline{\Phi}}(\overline{a}) = \overline{\Phi}(\overline{a}_0), \quad \overline{a} \in \overline{C}_{[-L,L]},$$

with $\widehat{\varphi}^{-1}$ given by the block map $\widehat{\Phi} : C_{[-2L,2L]} \to \overline{C}_{[-L,L]}$,

$$\widehat{\Phi}(a) = (\Phi(a_{[l-L,l+L]}))_{-L \le l \le L}, \quad a \in C_{[-2L,2L]}.$$

Let $\widehat{\overline{K}} \in \mathbb{Z}_+$, and assume that $\widehat{\overline{C}}$ has property $(a, \widehat{\overline{K}})$. We set

$$K = \widehat{\overline{K}} + 4L,$$

and we establish the proposition by showing that C has property $(a,\ K)$. For this let i_-, i_+, j_-, $j_+ \in \mathbb{Z}$,

$$i_- < i_+ - 2K, \quad j_- < j_+ - 2K,$$

and let

$$a \in X(C)_{[i_-,i_+]}, \quad b \in X(C)_{[j_-,j_+]},$$

be such that

$$a_{i_-+k} = b_{j_-+k}, \quad a_{i_+-k} = b_{j_+-k}, \quad 0 \le k \le K. \tag{3}$$

We have to show that the context of a is equal to the context of b. For this let

$$(x^-, x^+) \in \Gamma(a),$$

and let

$$y^- \in X(C)_{(-\infty,j_-)}, \quad y^+ \in X(C)_{(j_+,\infty)},$$

be given by

$$y^-_{j_--h} = x^-_{i_--h}, \quad y^+_{j_++h} = x^+_{i_++h}, \quad h \in \mathbb{N}.$$

We have to show that

$$(y^-, y^+) \in \Gamma(b). \tag{4}$$

Let

$$\widehat{\overline{a}} = \widehat{\Phi}(a), \quad \widehat{\overline{b}} = \widehat{\Phi}(b).$$

By (3)

$$\widehat{\overline{a}}_{i_-+2L+k} = \widehat{\overline{b}}_{j_-+2L+k}, \quad \widehat{\overline{a}}_{i_+-2L-k} = \widehat{\overline{b}}_{j_+-2L-k}, \quad 0 \le k \le \widehat{\overline{K}},$$

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and by Lemma 2.1

$$\widehat{\widehat{a}} \in X(\widehat{\overline{C}})_{[i_-+2L,i_+-2L]}, \quad \widehat{\widehat{b}} \in X(\widehat{\overline{C}})_{[j_-+2L,j_+-2L]},$$

and by property (a), $\widehat{\widehat{a}}$ and $\widehat{\widehat{b}}$ have the same context. Let

$$\widehat{\widehat{x}} = \varphi^{-1}(x^-, a, x^+)^{[-L,L]},$$

and define

$$\widehat{\widehat{y}}^- \in \widehat{\overline{C}}_{(-\infty,j_-+2L)}, \quad \widehat{\widehat{y}}^+ \in \widehat{\overline{C}}_{(j_+-2L,\infty)}$$

by

$$\widehat{\widehat{y}}^-_{j_-+2L-h} = \widehat{\widehat{x}}_{i_-+2L-h}, \quad \widehat{\widehat{y}}^+_{j_+-2L+h} = \widehat{\widehat{x}}_{i_+-2L+h}, \quad h \in \mathbb{N}.$$

Then

$$(\widehat{\widehat{y}}^-, \widehat{\widehat{y}}^+) \in \Gamma(\widehat{\widehat{b}})$$

and

$$\widehat{\widehat{\Phi}}(\widehat{\widehat{y}}^-) = y^-, \quad \widehat{\widehat{\Phi}}(\widehat{\widehat{b}}) = b, \quad \widehat{\widehat{\Phi}}(\widehat{\widehat{y}}^+) = y^+,$$

and one sees that (4) holds. □

We now consider a subshift C with property (A). We make preparations to introduce the structure of a semigroup (with zero) into $\mathcal{M}_C$ that is compatible with the partial order of $\mathcal{M}_C$. For this, first let C be a subshift with property (a). Consider

$$u \in Y(C) \cap W^+(q^+), \quad v \in Y(C) \cap W^-(r^-), \quad q^+, r^- \in P(C).$$

We say that $x \in C$ is a 1-defining point for $[u]_{\approx C}[v]_{\approx C}$ if the following structure is present: C has property $(a,\ K)$, $K \in \mathbb{Z}_+$, and there are I, $J \in \mathbb{Z}$ such that

$$u_{[I-\ K,\infty)} = q^+_{[I-\ K,\infty)}, \quad v_{(-\infty,J+K]} = r^-_{(-\infty,J+\ K]},$$

and there are I_0, $J_0 \in \mathbb{Z}$, $J_0 - I_0 \geq |\Sigma|$, such that

$$x_{[I_0,J_0]} \in X(C)_{[I_0,J_0]},$$

and

$$x_{I_0-k} = u_{I-k}, \quad x_{J_0+k} = v_{J+k}, \quad k \in \mathbb{Z}_+.$$

For a subshift $C \subset \Sigma^{\mathbb{Z}}$, and $n \in \mathbb{N}$, such that $C^{[1,n]}$ has property (a), we say that $x \in C$ is an n-defining point for $[u]_{\approx(C)}[v]_{\approx(C)}$ if $x^{[1,n]}$ is a 1-defining point in $C^{[1,n]}$ for $[u^{[1,n]}]_{\approx(C^{[1,n]})}[v^{[1,n]}]_{\approx(C^{[1,n]})}$.

LEMMA 3.2. *Let $C \subset \Sigma^{\mathbb{Z}}$ be a subshift and let $n \in \mathbb{N}$ be such that $C^{[1,n]}$ has property (a). Let $u, v, \widetilde{u}, \widetilde{v} \in Y_n(C)$, be such that $u \lesssim (C)\widetilde{u}$, $v \lesssim (C)\widetilde{v}$. Let x be an n-defining point for $[u]_{\approx(C)}[v]_{\approx(C)}$ and let $\widetilde{x}$ be an n-defining point for $[\widetilde{u}]_{\approx(C)}[\widetilde{v}]_{\approx(C)}$. Then $x \lesssim (C)\widetilde{x}$.*

Proof. We need to formulate the proof only for the case $n = 1$. Let $m_0 \in \mathbb{N}$, and let $H_m \in \mathbb{N}$, $m \geq m_0$, be such that

$$u^{[1,m]} \lesssim (C^{[1,m]}, H_m)\widetilde{u}, \quad v^{[1,m]} \lesssim (C^{[1,m]}, H_m)\widetilde{v}, \quad m \geq m_0. \tag{5}$$

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We proceed to prove that also

$$x \lesssim (C^{[1,m]}, H_m)\widetilde{x}, \quad m \geq m_0. \tag{6}$$

Consider first the case that v equals $\widetilde{v}$, and let

$$u \in Y(C) \cap W^+(q^+), \quad v \in Y(C) \cap W^-(r^-), \quad \widetilde{u} \in Y(C) \cap W^+(\widetilde{q}^+),$$
$$q^+, r^-, \widetilde{q}^+ \in P(C), \quad q^+ \geq (C)r^-, \quad \widetilde{q}^+ \geq (C)r^-.$$

Using property (a), replace, if necessary, x by a 1-defining point x' for $[u]_{\approx C}[v]_{\approx(C)}$,

$$x'^{[1,m]} \approx (C^{[1,m]}, H_m)x^{[1,m]},$$

such that one has $I, J, I_0, J_0 \in \mathbb{Z}$, $J_0 - I_0 \geq |\Sigma|$, such that

$$u_{[I-2H_m-m,\infty)} = q^+_{[I-2H_m-m,\infty)}, \quad v_{(-\infty,J+2H_m+m]} = r^-_{(-\infty,J+2H_m+m]},$$
$$x_{[I_0,J_0]} \in X(C)_{[I_0,J_0]}, \tag{7}$$
$$x_{I_0-k} = u_{I-k}, \quad x_{J_0+k} = v_{J+k}, \quad k \in \mathbb{Z}_+,$$

and replace, if necessary $\widetilde{x}$ by a 1-defining point $\widetilde{x}'$ for $[\widetilde{u}]_{\approx C}[v]_{\approx(C)}$,

$$x'^{[1,m]} \approx (C^{[1,m]}, H_m)\widetilde{x}'^{[1,m]},$$

such that one has $\widetilde{I}, \widetilde{J}, \widetilde{I}_0, \widetilde{J}_0 \in \mathbb{Z}$, $\widetilde{J}_0 - \widetilde{I}_0 \geq |\Sigma|$, such that

$$\widetilde{u}_{[\widetilde{I}-2H_m-m,\infty)} = \widetilde{q}^+_{[I-2H_m-m,\infty)}, \quad v_{(-\infty,\widetilde{J}+2H_m+m]} = r^-_{[-\infty,\widetilde{J}+2H_m+m)},$$
$$\widetilde{x}'_{[\widetilde{I}_0,\widetilde{J}_0]} \in X(C)_{[\widetilde{I}_0,\widetilde{J}_0]}, \tag{8}$$
$$\widetilde{x}'_{I_0-k} = \widetilde{u}_{I-k}, \quad \widetilde{x}'_{\widetilde{J}_0+k} = \widetilde{v}_{J+k}, \quad k \in \mathbb{Z}_+.$$

It follows then from (5), (7) and (8) that

$$x'^{[1,m]} \lesssim (C^{[1,m]}, H_m)\widetilde{x}'^{[1,m]}.$$

Repeating the same argument, now replacing v by $\widetilde{v}$, yields (6). □

LEMMA 3.3. *Let $C \subset \Sigma^{\mathbb{Z}}$ be a subshift and let $n, n' \in \mathbb{N}$, $n \leq n'$, be such that $C^{[1,n]}$ and $C^{[1,n']}$ have property (a). Let $u, v \in Y_n(C)$. Let x be an n-defining point for $[u]_{\approx(C)}[v]_{\approx(C)}$, and let x' be an n'-defining point for $[u]_{\approx(C)}[v]_{\approx(C)}$. Then $x \approx (C)x'$.*

Proof. We need to formulate the proof only for the case $n = 1$. Let $u, v \in Y_C$,

$$u \in W^+(q^+), \quad v \in W^-(r^-), \quad q^+, r^- \in P(C).$$

In view of Lemma 3.2 we establish the lemma by showing that if there is a 1-defining point at all for $[u]_{\approx(C)}[v]_{\approx(C)}$ then there is also a 1-defining point x for $[u]_{\approx(C)}[v]_{\approx(C)}$ such that $x^{[1,n']}$ is also an n'-defining point for $[u]_{\approx(C)}[v]_{\approx(C)}$. Let $K' \in \mathbb{Z}_+$ be such that $C^{[1,n']}$ has property (a, K'). Observe that if there is a 1-defining point for $[u]_{\approx(C)}[v]_{\approx(C)}$ then there is a 1-defining point x for $[u]_{\approx(C)}[v]_{\approx(C)}$ such that one has $I, J, I_0, J_0 \in \mathbb{Z}$, $J_0 - I_0 \geq |\Sigma|$, such that

$$u_{[I-n'-K',\infty)} = q^+_{[I-n'-K',\infty)}, \quad v_{(-\infty,J+n'+K']} = r^-_{[-\infty,J+n'+K']},$$
$$x_{[I_0,J_0]} \in X(C)_{[I_0,J_0]}, \tag{9}$$
$$x_{I_0-k} = u_{I-k}, \quad x_{J_0+k} = v_{j+k}, \quad k \in \mathbb{Z}_+.$$

From (9) one sees that x is also an n'-defining point for $[u]_{\approx(C)}[v]_{\approx(C)}$. □

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We continue to consider a subshift with property (A). Lemma 3.2 says that a binary operation, in fact an associative binary operation, is defined on $\mathcal{M}_C$ by setting for $u, v \in Y_C$, $[u]_{\approx(C)}[v]_{\approx(C)}$ equal to $[x]_{\approx(C)}$, x any defining point for $[u]_{\approx(C)}[v]_{\approx C}$, whenever such a defining point exists, and by setting $[u]_{\approx(C)}[v]_{\approx(C)}$ equal to zero otherwise. Moreover, by Lemma 3.2 this binary operation is compatible with the partial order $\lesssim (C)$. It is also invariantly associated to C, which is the contents of the next proposition.

PROPOSITION 3.4. *Let $C \subset \Sigma^{\mathbb{Z}}$, $\overline{C} \subset \overline{\Sigma}^{Z}$ be subshifts with property (A) and let $\varphi : C \to \overline{C}$ be a topological conjugacy. Then*

$$\varphi([u]_{\approx(C)})\varphi([v]_{\approx(C)}) = \varphi([u]_{\approx(C)}[v]_{\approx(C)}), \quad u, v \in Y_C.$$

Proof. Assume that C has property (a) and that $\overline{C}^{[1,n]}$ has property (a), $n \in \mathbb{N}$. We also consider the situation where for some $L \in \mathbb{Z}_+$, φ is given by a block map

$$\Phi : C_{[-L,L]} \to \overline{\Sigma}$$

and φ^{-1} is given by a 1-block map $\overline{\Phi} : \overline{\Sigma} \to \Sigma$. We again set

$$\widehat{\overline{C}} = \overline{C}^{[-L,L]}.$$

Let $u, v \in Y_C$,

$$u \in W^+(q^+), \quad v \in W^-(r^-), \quad q^+, r^- \in P(C).$$

In view of Lemma 3.3 we establish the proposition by showing that if there is a 1-defining point at all for $[u]_{\approx(C)}[v]_{\approx(C)}$ then there is also a 1-defining point x for $[u]_{\approx(C)}[v]_{\approx(C)}$ such that $\varphi(x)$ is a 1-defining point for $[\varphi(u)]_{\approx(C)}[\varphi(v)]_{\approx(C)}$. Let $\widehat{\overline{K}} \in \mathbb{N}$ be such that $\widehat{\overline{C}}$ has property $(a, \widehat{\overline{K}})$. Observe that if there is a 1-defining point for $[u]_{\approx(C)}[v]_{\approx(C)}$ then there is a 1-defining point x for $[u]_{\approx(C)}[v]_{\approx(C)}$ such that one has I, J, I_0, $J_0 \in \mathbb{Z}$, $J_0 - I_0 \geq |\Sigma|$, such that

$$\begin{gathered} u_{[I-2L-\widehat{\overline{K}},\infty)} = q^+_{[I-2L-\widehat{\overline{K}},\infty)}, \quad v_{(-\infty,J+2L+\widehat{\overline{K}}]} = r^-_{(-\infty,J+2L+\widehat{\overline{K}}]}, \\ x_{[I_0,J_0]} \in X(C)_{[I_0,J_0]}, \\ x_{I_0-k} = u_{I-k}, \quad x_{J_0+k} = v_{J+k}, \quad k \in \mathbb{Z}_+. \end{gathered} \tag{10}$$

From (10) and from Lemma 2.1 it follows that $\varphi(x)$ is a 1-defining point of $[\varphi(u)]_{\approx(C)}[\varphi(v)]_{\approx(C)}$. □

We continue to consider a subshift C with property (A). As a consequence of this property, all $u \in X_C$ such that $u \in W^-(q^-) \cap W^+(r^+)$, $q^- \in \mathfrak{q}$, $r^+ \in \mathfrak{r}$, $\mathfrak{q}, \mathfrak{r} \in \mathcal{P}_C$ are $\approx (C)$-equivalent. We denote this $\approx (C)$-equivalence class by $\varepsilon_{\mathfrak{q},\mathfrak{r}}$. $(\mathcal{P}_C, \leq (C))$ maps into $(\mathcal{M}_C, \lesssim (C))$ by

$$\mathfrak{p} \to \varepsilon_{\mathfrak{p},\mathfrak{p}} (p \in \mathcal{P}_C),$$

and the following relation holds:

$$\varepsilon_{\mathfrak{q},\mathfrak{r}}\varepsilon_{\mathfrak{r},\mathfrak{s}} = \varepsilon_{\mathfrak{q},\mathfrak{s}}, \quad \mathfrak{q}, \mathfrak{r}, \mathfrak{s} \in \mathcal{P}_C.$$

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For a subshift $C \subset \Sigma^{\mathbb{Z}}$ set

$$X_n^+(C) = \bigcap_{i\in\mathbb{Z}} \{x \in C : x_i \in \omega^+(x_{[i-n,i)})\}, \quad n \in \mathbb{N}.$$

$X_n^+(C)$, if not empty, is an n-step topological Markovshift, and

$$X_n^+(C) \subset X_{n+1}^+(C), \quad n \in \mathbb{N}.$$

We set

$$X_C^+ = \bigcup_{n\in\mathbb{N}} X_n(C).$$

X_C^- is the time symmetrically defined object. For subshifts $C \subset \Sigma^{\mathbb{Z}}$, $\overline{C} \subset \overline{\Sigma}^{\mathbb{Z}}$, $\overline{C} \subset \overline{\Sigma}^{Z}$ and a topological conjugacy $\varphi : C \to \overline{C}$ one has as a consequence of Lemma 2.1 that

$$\varphi(X_C^+) = X_{\overline{C}}^+, \quad \varphi(X_C^-) = X_{\overline{C}}^-.$$

For a subshift C with property (A) denote the image of $X_C^+ \cap Y_C$ in $\mathcal{M}_C$ by $\mathcal{M}_C^+$ and let $\mathcal{M}_C^-$ be the time symmetrically defined object. The structure $(\mathcal{M}_C, \mathcal{M}_C^+, \mathcal{M}_C^-)$ is invariantly associated to a subshift C with property (A).

4. *Examples*

We first give some prototype examples of subshifts C with property (A) and $(\mathcal{P}_C, \leq (C))$ trivial. We consider here a monoid $\mathcal{M}$ (with zero) that has generators $\alpha_{i(n)}$, $1 \leq i(n) \leq I_n$, and $\beta_{j(n)}$, $1 \leq j(n) \leq J_n$, $1 \leq n \leq N$, $N \in \mathbb{N}$. We define the monoid M by imposing relations on these generators. We require here that

$$\alpha_{i_n}(n)\beta_{j(n')}(n') = \beta_{j_{n'}}(n')\alpha_{i_n}(n), \quad 1 \leq n, n' \leq N,\ n \neq n',$$

and that

$$\alpha_{i_n}(n)\beta_{j_n}(n) \in \{0, \mathbf{1}\} \cup \{\alpha_{i_n(0)}(n) : 1 \leq i_n(0) \leq I_n\} \cup \{\beta_{j_n(0)}(n) : 1 \leq j_n(0) \leq J_n\},$$
$$1 \leq i_n \leq I_n, \quad 1 \leq j_n \leq J_n, \quad 1 \leq n \leq N.$$

Further, we require that these relations are such that for every i_n, $1 \leq i_n \leq I_n$, there are some $j_n(k)$, $1 \leq j_n(k) \leq J_n$, $1 \leq k \leq K$, $K \in \mathbb{N}$, such that

$$\alpha_{i_n}(n) \prod_{k=1}^{K} \beta_{j_n(k)}(n) = \mathbf{1},$$

and also some $j_n(k)$, $1 \leq j_n(k) \leq J_n$, $1 \leq k \leq K$, $K \in \mathbb{N}$, such that

$$\alpha_{i_n}(n) \prod_{k=1}^{K} \beta_{j_n(k)}(n) = 0,$$

and we require that these relations are such that for all i_n, i_n', $1 \leq i_n < i_n' \leq I_n$, there are some $j_n(k)$, $1 \leq j_n(k) \leq J_n$, $1 \leq k \leq K$, $K \in \mathbb{N}$, such that

$$\alpha_{i_n}(n) \prod_{k=1}^{K} \beta_{j_n(k)}(n) \in \{\mathbf{1}\} \cup \{\alpha_{i_n(0)}(n) : 1 \leq i_n(0) \leq I_n\}, \quad \alpha_{i_n'}(n) \prod_{k=1}^{K} \beta_{j_n(k)}(n) = 0,$$

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or

$$\alpha_{i_n'}(n)\prod_{k=1}^{K}\beta_{j_n(k)}(n)\in\{\mathbf{1}\}\cup\{\alpha_{i_n(0)}(n):1\le i_n(0)\le I_n\},\quad \alpha_{i_n}(n)\prod_{k=1}^{K}\beta_{j_n(k)}(n)=0,$$
$$1\le n\le N.$$

Symmetrically, we require that these relations are such that for every j_n, $1\le j_n\le J_n$, there are some $i_n(\mathfrak{l})$, $1\le j_n(\mathfrak{l})\le J_n$, $1\le \mathfrak{l}\le L$, $L\in\mathbb{N}$, such that

$$\Bigg(\prod_{\mathfrak{l}=1}^{L}\alpha_{i_n(\mathfrak{l})}(n)\Bigg)\beta_{j_n}(n)=\mathbf{1},$$

and some $i_n(\mathfrak{l})$, $1\le j_n(\mathfrak{l})\le J_n$, $1\le \mathfrak{l}\le L$, $L\in\mathbb{N}$, such that

$$\Bigg(\prod_{\mathfrak{l}=1}^{L}\alpha_{i_n(\mathfrak{l})}(n)\Bigg)\beta_{j_n}(n)=0,$$

and we require that these relations are such that for all j_n, j_n', $1\le j_n<j_n'\le J_n$, there are some $i_n(\mathfrak{l})$, $1\le i_n(\mathfrak{l})\le I_n$, $1\le \mathfrak{l}\le L$, $L\in\mathbb{N}$, such that

$$\Bigg(\prod_{\mathfrak{l}=1}^{L}\alpha_{i_n(\mathfrak{l})}(n)\Bigg)\beta_{j_n}(n)\in\{\mathbf{1}\}\cup\{\beta_{j_n(0)}(n):1\le j_n(0)\le J_n\},\quad \Bigg(\prod_{\mathfrak{l}=1}^{L}\alpha_{i_n(\mathfrak{l})}(n)\Bigg)\beta_{j_n'}(n)=0,$$

or

$$\Bigg(\prod_{\mathfrak{l}=1}^{L}\alpha_{i_n(\mathfrak{l})}(n)\Bigg)\beta_{j_n'}(n)\in\{\mathbf{1}\}\cup\{\beta_{j_n(0)}(n):1\le j_n(0)\le J_n\},\quad \Bigg(\prod_{\mathfrak{l}=1}^{L}\alpha_{i_n(\mathfrak{l})}(n)\Bigg)\beta_{j_n}(n)=0,$$
$$1\le n\le N.$$

If here N is one, then these relations give rise to a finite monadic string rewriting system that is confluent. There are decision procedures for the properties that we require of these relations. (For the theory of monadic string rewriting systems see [**6**, Chapter 4]).

We consider the biprefix code $\mathcal{C}_{\mathcal{M}}$ that contains all

$$d=\prod_{r=1}^{R}{}^{*}\delta_r\in\Bigg(\bigcup_{1\le n\le\mathbb{N}}(\{\alpha_{i(n)}:1\le i(n)\le I_n\}\cup\{\beta_{j(n)}:1\le j(n)\le J_n\})\Bigg)^{*}$$

such that

$$\prod_{r=1}^{R}\delta_r=\mathbf{1}.$$

The conditions on the defining relations of $\mathcal{M}$ ensure that the surjection of $\mathcal{M}$ onto the syntactic monoid (see, for example, [**3**]) of $\mathcal{C}_{\mathcal{M}}$ is actually an isomorphism. It follows that $\mathcal{C}_{\mathcal{M}}$ determines a coded system C [**4**] that has property (A) and that is such that $\mathcal{M}_C$ is equal to $\mathcal{M}$. Direct products can also be considered here.

Next, we give examples of subshifts with property (A) and arbitrary finite $(\mathcal{P},\le)$. We will generate a partially ordered semigroup (with zero) $\mathcal{M}$ from a Markov code [**7**] with

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a finite alphabet Σ by means of the operation of concatenation with one-symbol overlap. Given words $u, v \in \Sigma^+$,

$$u = \alpha a \beta, \quad v = \gamma b \delta, \quad \alpha, \beta, \gamma, \delta \in \Sigma,$$

with one or both of the words a and b possibly empty, the concatenation with one-symbol overlap of u and v is defined precisely if β equals γ, and is then given by $\alpha a \beta b \delta$. Certain Markov codes with one-symbol overlap were already apparent in §2: given a subshift $C \subset \Sigma^{\mathbb{Z}}$ the words of the blocks $a \in C_{[0,I]}$, $I \in \mathbb{N}$, such that $a_I \in \omega^+(a_{[0,I)})$, $a_0 \in \omega^-(a_{[1,I]})$, and that cannot be written as concatenations with one-symbol overlap of such words, form a Markov code, and, in a certain sense, these Markov codes generalize the first return codes [**5**].

Let a finite partially ordered set $(\mathcal{P}, \leq)$ be given. The alphabet Σ of the Markov code to be constructed will contain the symbols $\varepsilon_{\mathfrak{p}}$, $\mathfrak{p} \in \mathcal{P}$, and every word in the code will begin and end with one of these symbols. Let $K \geq 2$, and let relations

$$\mathcal{L}_k \subset \mathcal{P} \times \mathcal{P}, \quad \mathcal{R}_k \subset \mathcal{P} \times \mathcal{P}, \quad 1 \leq k \leq K,$$

be given such that the following conditions are satisfied: for every $\mathfrak{r} \in \mathcal{P}$ there is an $\mathfrak{s} \in \mathcal{P}$ such that

$$(\mathfrak{r}, \mathfrak{s}) \in \bigcup_{1 \leq k \leq K} \mathcal{L}_k,$$

and a $\mathfrak{q} \in \mathcal{P}$ such that

$$(\mathfrak{q}, \mathfrak{r}) \in \bigcup_{1 \leq k \leq K} \mathcal{R}_k.$$

For $\mathfrak{q}, \mathfrak{q}', \mathfrak{r}, \mathfrak{r}' \in \mathcal{P}$, $\mathfrak{q}' \geq \mathfrak{q}$, $\mathfrak{r} \geq \mathfrak{r}'$, if

$$(\mathfrak{q}, \mathfrak{r}) \in \mathcal{L}_k,$$

then also

$$(\mathfrak{q}', \mathfrak{r}') \in \mathcal{L}_k,$$

and for $\mathfrak{r}, \mathfrak{r}', \mathfrak{q}, \mathfrak{q}' \in \mathcal{P}$, $\mathfrak{r}' \geq \mathfrak{r}$, $\mathfrak{q} \geq \mathfrak{q}'$, if

$$(\mathfrak{r}, \mathfrak{q}) \in \mathfrak{R}_k,$$

then also

$$(\mathfrak{r}', \mathfrak{q}') \in \mathfrak{R}_k, \quad 1 \leq k \leq K.$$

For all $(\mathfrak{q}, \mathfrak{r}) \in \mathcal{L}_k$, there is a $\mathfrak{q}' \in \mathcal{P}$, $\mathfrak{q} \geq \mathfrak{q}'$, such that

$$(\mathfrak{r}, \mathfrak{q}') \in \mathfrak{R}_k,$$

and for all $(\mathfrak{r}, \mathfrak{q}) \in \mathcal{R}_k$ there is a $\mathfrak{r}' \in \mathcal{P}$, $\mathfrak{r}' \geq r$, such that

$$(\mathfrak{r}', \mathfrak{q}) \in \mathcal{L}_k, \quad 1 \leq k \leq K.$$

We complete the alphabet of the code by adding symbols

$$\lambda_k, \rho_k, \quad 1 \leq k \leq K.$$

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The words in the Markov code are then the words

$$\varepsilon_{\mathfrak{q}}\varepsilon_{\mathfrak{r}}, \quad \mathfrak{q}, \mathfrak{r} \in \mathcal{P}, \mathfrak{q} \geq \mathfrak{r},$$

and the words

$$\varepsilon_{\mathfrak{q}}\lambda_k\varepsilon_{\mathfrak{r}}, \quad (\mathfrak{q}, \mathfrak{r}) \in \mathcal{L}_k, \quad \varepsilon_{\mathfrak{r}}\rho_k\varepsilon_{\mathfrak{q}}, \quad (\mathfrak{r}, \mathfrak{q}) \in \mathfrak{R}_k, \quad 1 \leq k \leq K.$$

We turn the Markov code into a semigroup (with zero) $\mathcal{M}$ by setting the product of two elements equal to zero if their concatenation with one symbol overlap is not defined, and by imposing the relations

$$\begin{aligned}
\varepsilon_{\mathfrak{r}}\varepsilon_{\mathfrak{s}}\varepsilon_{\mathfrak{t}} &= \varepsilon_{\mathfrak{r}}\varepsilon_{\mathfrak{t}}, \quad && \mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in \mathcal{P}, \mathfrak{r} \geq \mathfrak{s} \geq \mathfrak{t},\\
\varepsilon_{\mathfrak{r}}\lambda_k\varepsilon_{\mathfrak{s}}\varepsilon_{\mathfrak{t}} &= \varepsilon_{\mathfrak{r}}\lambda_k\varepsilon_{\mathfrak{t}}, \quad && (\mathfrak{r}, \mathfrak{s}) \in \mathcal{L}_k, \mathfrak{t} \in \mathcal{P}, \mathfrak{s} \geq \mathfrak{t},\\
\varepsilon_{\mathfrak{r}}\varepsilon_{\mathfrak{s}}\lambda_k\varepsilon_{\mathfrak{t}} &= \varepsilon_{\mathfrak{r}}\lambda_k\varepsilon_{\mathfrak{t}}, \quad && (\mathfrak{s}, \mathfrak{t}) \in \mathcal{L}_k, \mathfrak{r} \in \mathcal{P}, \mathfrak{r} \geq \mathfrak{s},\\
\varepsilon_{\mathfrak{r}}\varepsilon_{\mathfrak{s}}\rho_k\varepsilon_{\mathfrak{t}} &= \varepsilon_{\mathfrak{r}}\rho_k\varepsilon_{\mathfrak{t}}, \quad && (\mathfrak{s}, \mathfrak{t}) \in \mathfrak{R}_k, \mathfrak{r} \in \mathcal{P}, \mathfrak{r} \geq \mathfrak{s},\\
\varepsilon_{\mathfrak{r}}\rho_k\varepsilon_{\mathfrak{s}}\varepsilon_{\mathfrak{t}} &= \varepsilon_{\mathfrak{r}}\rho_k\varepsilon_{\mathfrak{t}}, \quad && (\mathfrak{r}, \mathfrak{s}) \in \mathfrak{R}_k, \mathfrak{t} \in \mathcal{P}, \mathfrak{s} \geq \mathfrak{t},
\end{aligned}$$

and

$$\varepsilon_{\mathfrak{q}}\lambda_k\varepsilon_{\mathfrak{s}}\rho_k\varepsilon_{\mathfrak{r}} = \varepsilon_{\mathfrak{q}}\varepsilon_{\mathfrak{r}}, \quad (\mathfrak{q}, \mathfrak{s}) \in \mathcal{L}_k, (\mathfrak{s}, \mathfrak{r}) \in \mathfrak{R}_k, \mathfrak{q} \geq \mathfrak{r}, 1 \leq k \leq K,$$

and by setting $\varepsilon_{\mathfrak{q}}\lambda_k\varepsilon_{\mathfrak{s}}\rho_{k'}\varepsilon_{\mathfrak{r}}$, $(\mathfrak{q}, \mathfrak{s}) \in \mathcal{L}_k$, $(\mathfrak{s}, \mathfrak{r}) \in \mathfrak{R}_{k'}$, $1 \leq k, k' \leq K$, equal to zero in all other cases. We complete the definition of $\mathcal{M}$ by defining on $\mathcal{M}$ a partial order $\lesssim$ that is compatible with the algebraic structure by imposing the conditions

$$\varepsilon_{\mathfrak{q}}\varepsilon_{\mathfrak{r}} \gtrsim \varepsilon_{\mathfrak{q}'}\varepsilon_{\mathfrak{r}'}, \quad \mathfrak{q}, \mathfrak{q}', \mathfrak{r}, \mathfrak{r}' \in \mathcal{P}, \mathfrak{q}' \geq q, \mathfrak{r}' \leq \mathfrak{r},$$

and

$$\begin{aligned}
\varepsilon_{\mathfrak{q}}\lambda_k\varepsilon_{\mathfrak{r}} &\gtrsim \varepsilon_{\mathfrak{q}'}\varepsilon_{\mathfrak{r}'}, \quad (\mathfrak{q}, \mathfrak{r}) \in \mathcal{L}_k,\\
\varepsilon_{\mathfrak{q}}\rho_k\varepsilon_{\mathfrak{r}} &\gtrsim \varepsilon_{\mathfrak{q}'}\varepsilon_{\mathfrak{r}'}, \quad (\mathfrak{q}, \mathfrak{r}) \in \mathfrak{R}_k,\\
\mathfrak{q}', \mathfrak{r}' \in \mathcal{P}, &\quad \mathfrak{q}' \geq \mathfrak{q}, \quad \mathfrak{r}' \leq \mathfrak{r}, \quad 1 \leq k \leq K.
\end{aligned}$$

We now describe a subshift C with property (A). The admissible words of this subshift are the subwords of any of the languages $\mathcal{T}_{\mathfrak{q},\mathfrak{r}}$, $\mathfrak{q}, \mathfrak{r} \in \mathcal{P}$, $\mathfrak{q} \geq \mathfrak{r}$, that are produced by the grammar with terminal variables $\varepsilon_{\mathfrak{p}}$, $\mathfrak{p} \in \mathcal{P}$, ρ_k, λ_k, $1 \leq k \leq K$, and non-terminal productions

$$\begin{aligned}
&\mathcal{T}_{\mathfrak{r},\mathfrak{s}} \to \varepsilon_{\mathfrak{r}}\mathcal{T}_{\mathfrak{r},\mathfrak{s}}, \mathcal{T}_{\mathfrak{r},\mathfrak{s}}\varepsilon_{\mathfrak{s}}, \quad \mathfrak{r}, \mathfrak{s} \in \mathcal{P}, \mathfrak{r} \geq \mathfrak{s},\\
&\mathcal{T}_{\mathfrak{q},\mathfrak{t}} \to \varepsilon_{\mathfrak{q}}\lambda_k\varepsilon_{\mathfrak{r}}\mathcal{T}_{\mathfrak{r},\mathfrak{s}}\rho_k\varepsilon_{\mathfrak{t}}, \varepsilon_{\mathfrak{q}}\lambda_k\mathcal{T}_{\mathfrak{r},\mathfrak{s}}\varepsilon_{\mathfrak{s}}\rho_k\varepsilon_{\mathfrak{t}}, \quad \mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in \mathcal{P}, \quad \mathfrak{q} \geq \mathfrak{t}, \quad \mathfrak{r} \geq \mathfrak{s}, \quad 1 \leq k \leq K,
\end{aligned}$$

and with the terminal productions that delete the non-terminals. This subshift C is not necessarily a coded system. It is a coded system if and only if it is topologically transitive, and a test for this to be the case can be given. The grammar that defines the subshift C is structured such that the invariant $\mathcal{M}_C$ is identical to $\mathcal{M}$ as constructed.

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References

[1] M.-P. Béal. *Codage Symbolique.* Masson, 1993.

[2] M.-P. Béal and D. Perrin. *Symbolic Dynamics and Finite Automata (Handbook of Formal Languages, 2)*. Eds G. Rozenberg and A. Salomaa. Springer, 1997, pp. 463–506.

[3] J. Berstel and D. Perrin. *Theory of Codes*. Academic Press, 1985.

[4] F. Blanchard and G. Hansel. Systèmes codés. *Theoretical Computer Science* **44** (1986), 17–49.

[5] F. Blanchard. Codes engendrant certains systèmes sofiques. *Theoretical Computer Science* **68** (1989), 131–141.

[6] R. V. Book and F. Otto. *String-rewriting Systems*. Springer, 1993.

[7] G. Keller. Circular codes, loop counting, and zeta-functions. *J. Combinatorial Theory*, Series A, **56** (1991), 75–83.

[8] B. P. Kitchens. *Symbolic Dynamics*. Springer, 1998.

[9] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995.

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