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105.24 Conditional 2×2 matrices with three prime elements and given determinant

This paper was inspired by my attempt to write a question for a Calculus III course at the University of Zanjan. The problem was to calculate the volume of a parallelepiped, a three-dimensional solid formed by six parallelograms. The volume is given by the 3 by 3 determinant whose rows are the normal vectors to the three planes. I wished this value of the determinant to be a positive integer. When I tried to generate the relevant 3 by 3 matrix, by writing a Maple code, it took about 15 minutes. I then tried to impose extra constraints on the elements of the matrix, and I finally obtained the following result for 2×2 matrices.

Theorem: Given positive integers d and M there exist infinitely many matrices $A = [a_{ij}]_{1 \leqslant i,j \leqslant 2}$ with at least three distinct prime elements satisfying $a_{ij} \geqslant M$ and $\det A = d$.

Our proof of the above result is based on the theory of linear Diophantine equations with two variables. The proof can be applied to generate matrices with large prime elements and small determinant.

Proof of Theorem: We take two distinct prime numbers p and q, both larger than the given M not being factors of the given d. Since $\gcd(p, q) = 1$, there exist integers x' and y' such that $px' - qy' = 1$. If $\gcd(x', q) \neq 1$ then $q \mid x'$ so $x' = kq$ for some integer $k \neq 0$. Thus, $pkq - qy' = 1$ and $q \mid 1$, which is not possible. Hence, $\gcd(x', q) = 1$, and similarly $\gcd(y', p) = 1$.

Let $A = [a_{ij}]_{1 \leqslant i,j \leqslant 2}$ with $a_{11} = p$, $a_{12} = q$, $a_{21} = y$ and $a_{22} = x$. The equality $\det A = d$ gives the Diophantine equation $px - qy = d$, which has infinitely many solutions

$$x = x_0 + qt \qquad \text{and} \qquad y = y_0 + pt \qquad (t \in \mathbb{Z}),$$

and the particular solutions $x_0 = dx'$ and $y_0 = dy'$, satisfying $\gcd(x_0, q) = 1$ and $\gcd(y_0, p) = 1$. Therefore, from Dirichlet's theorem both of the arithmetic progressions $(x_0 + qt)_{t \geqslant 0}$ and $(y_0 + pt)_{t \geqslant 0}$ contain infinitely many primes. By taking prime elements larger than M in one of these progressions, we conclude the proof.

As an example, we produce a 2×2 matrix $A = [a_{ij}]_{1 \leqslant i,j \leqslant 2}$ with prime elements $a_{ij} \geqslant 2020$ and $\det A = 1$. Let $p = 2027$ and $q = 2029$, which

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are twin primes. By using the identity $(2n-1)n-(2n+1)(n-1)=1$ with $2n-1=p$ we find that $x'=n=1014$ and $y'=n-1=1013$ satisfy $px'-qy'=1$. Hence, $x_0=1014$ and $y_0=1013$, too. For $t=5$ the number $y_0+qt=11159$ is prime. Also, for $t=10$ the number $y_0+pt=21283$ is prime. Thus, we obtain

$$\det\begin{bmatrix} 2027 & 2029 \\ 11148 & 11159 \end{bmatrix}=1, \text{ and } \qquad \det\begin{bmatrix} 2027 & 2029 \\ 21283 & 21304 \end{bmatrix}=1.$$

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105.25 Families of curves orthogonal to the lines $y=mx-2m-m^3$

Orthogonal pairs of family of curves represent a co-ordinate system. They may also represent an electrostatic system comprising of lines of force and equi-potentials. Textbooks [1, 2] discuss the method and exercises of finding orthogonal family of curves to a given one-parameter family of curves. First, one sets up a first order ordinary differential equation (ODE) by eliminating the free parameter of the given family of curves, by differentiation. Next, in this ODE(c) one changes y' to $-\frac{1}{y'}$ to set up a new ODE(o) to solve and get the required family of their orthogonal curves. Here c denotes the curve and o denotes the orthogonal curve/trajectory.

The orthogonal trajectories to the curves $x^2-y^2=C_1$ is found by differentiating this equation and setting up the ODE(c) as $y'=\frac{x}{y}$, the ODE(o) is obtained by changing y' to $-1/y'$, we get ODE(o) as $y'=-\frac{y}{x}$, solving this, we get $xy=C_2$, where C_2 is the constant/parameter of integration. The example just discussed represents a hyperbolic co-ordinate system and an electrostatic quadrupole as well. More involved orthogonal pairs of curves are discussed in the textbooks [1, 2].

Let us find orthogonal trajectories to the family of lines $y=mx$, where m is a real parameter. Differentiating $\frac{y}{x}=m$, we eliminate m and get the ODE(c) as $y'=\frac{y}{x}$. Next we change y' to $-1/y'$ to get ODE(o) as $y'=-\frac{x}{y}$,

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