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RESEARCH ARTICLE

Uniqueness of Bessel models for GSpin groups

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Abstract
We prove the uniqueness of general Bessel models for GSpin groups over a local field of characteristic zero. The proof is to reduce it to the spherical case, which has been proved by Emory and Takeda in the non-archimedean case and by Emory, Kim, and Maiti in the archimedean case.

1. Introduction

Let V be a non-degenerate quadratic space over a local field F of characteristic 0. Let $V_0 \subset V$ be a non-degenerate subspace such that the orthogonal complement $V_0^{\perp}$ is split and has dimension $2r+1$ for a non-negative integer r. Denote by $\mathrm{GSpin}(V)$ and $\mathrm{GSpin}(V_0)$ the corresponding GSpin groups. We have an embedding $\mathrm{GSpin}(V_0) \subseteq \mathrm{GSpin}(V)$, and their centers share the same subgroup $F^{\times}$. Associated with the pair (V_0, V), we define the rth Bessel subgroup of $G = \mathrm{GSpin}(V) \times \mathrm{GSpin}(V_0)$ by

$$H = N_{Q_r} \rtimes \mathrm{GSpin}(V_0),$$

where N_{Q_r} is the unipotent radical of the parabolic subgroup of $\mathrm{GSpin}(V)$ stabilizing a complete flag of isotropic subspaces determined by $V_0^{\perp}$. Let ξ be a generic character of H, as defined in Section 3.1. The goal of this article is to establish the following result on the uniqueness of local Bessel models for G.

Theorem 1.1. *Let V, V_0, H, G, ξ be as above. For any irreducible admissible representation π of $\mathrm{GSpin}(V)$ and π_0 of $\mathrm{GSpin}(V_0)$, which are of Casselman–Wallach type if F is archimedean, the following inequality holds:*

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \pi_0, \xi) \leq 1.$$

When F is archimedean, the symbol $\otimes$ stands for the completed projective tensor product, and the symbol Hom_H stands for the space of continuous H-intertwining maps.

Denote by ω_π and ω_{π_0} the central characters of π and π_0, respectively. Note that $F^{\times}$ is contained in both the centers of $\mathrm{GSpin}(V)$ and $\mathrm{GSpin}(V_0)$ (see Lemma 2.2). If $\omega_\pi|_{F^{\times}} \cdot \omega_{\pi_0}|_{F^{\times}}$ is not the trivial character of $F^{\times}$, then $\mathrm{Hom}_H(\pi \otimes \pi_0, \xi) = 0$ (see Remark 4.1). In the rest of the article, we always assume that $\omega_\pi|_{F^{\times}} \cdot \omega_{\pi_0}|_{F^{\times}}$ is the trivial character of $F^{\times}$.

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We discuss two extreme cases. When $r = 0$, the subgroup H is given by $H = \mathrm{GSpin}(V_0)$, which is diagonally embedded into $G = \mathrm{GSpin}(V) \times \mathrm{GSpin}(V_0)$, with ξ being trivial. In this case, the inequality in Theorem 1.1 recovers the uniqueness for spherical models, proved in [8] in the non-archimedean case and in [7] in the archimedean case. On the other hand, when $V_0 = 0$, we have $\mathrm{GSpin}(V_0) = F^\times$, which is precisely the center of $\mathrm{GSpin}(V)$ where V is odd-dimensional (see Lemma 2.2). In this case, the inequality in Theorem 1.1 recovers the uniqueness for Whittaker models (see [5] when F is archimedean and [17] when F is non-archimedean). Hence, the family of Bessel models generalizes both the Whittaker model ($V_0 = 0$) and the spherical model ($r = 0$).

Remark 1.2. We discuss some low-rank cases. When $\dim(V) \leq 2$, $\mathrm{GSpin}(V)$ is abelian, hence π and π_0 are both one-dimensional, and hence the inequality in Theorem 1.1 is clear. If $\dim(V) = 3$, then either $\dim(V_0) = 2$ or $\dim(V_0) = 0$. The case $(\dim(V), \dim(V_0)) = (3, 2)$ is known by the uniqueness of spherical models in [7] and [8]. The case $(\dim(V), \dim(V_0)) = (3, 0)$ is known by the uniqueness of Whittaker models in [5] and [17]. Therefore, Theorem 1.1 is known when $\dim(V) \leq 3$. In the rest of the article, we will always assume that $\dim(V) > 3$.

For classical groups, the uniqueness of Bessel models is proved in [1], [9], [21] when F is non-archimedean, and in [11], [20], when F is archimedean.

We remark that Theorem 1.1 establishes an analog of the result in [9, Section 15] and completes the first step toward the local Gan–Gross–Prasad conjecture for GSpin groups.

Another important application of Theorem 1.1 is the Euler product factorization of certain global Rankin–Selberg integrals for GSpin groups, studied in the author's work in [22]. As a further application of the Rankin–Selberg integrals, certain cases of the global Gan–Gross–Prasad conjecture for GSpin groups are proved in [22]. More recently, Bessel models are used in the construction of twisted automorphic descent from the general linear groups to odd GSpin groups in [23].

We now discuss the organization of the article as well as the idea of the proof. In Section 2, we review the structure of the GSpin group using Clifford algebra and its parabolic subgroups. In Section 3, as a preparation of the proof of Theorem 1.1, we discuss the Bessel subgroup H and the character ξ, as well as representations of GSpin groups. In Section 4.1, we introduce a local zeta integral $\mathcal{Z}_\mu$ associated with a Bessel functional $\mu \in \mathrm{Hom}_H(\pi \otimes \pi_0, \xi)$, which depends on a complex parameter s, following an approach in [11]. Proposition 4.3 asserts that if μ is nonzero, then the local zeta integral $\mathcal{Z}_\mu$ can be made nonzero. Proposition 4.4 states that $\mathcal{Z}_\mu$ converges absolutely when $\mathrm{Re}(s)$ is large enough. As a consequence, $\mathcal{Z}_\mu$ defines an element in the spherical model $\mathrm{Hom}_{\mathrm{GSpin}(V)}(\pi'_s \otimes \pi, \mathbb{C})$, where π'_s is defined in Section 3.3. In Section 4.2, we prove Proposition 4.3. Finally, in Section 4.3, we prove Theorem 1.1.

1.1. Notation

Throughout the article, we assume that F is a local field of characteristic zero. All representations in this article are complex and smooth. For a representation π, we also use π to denote its underlying space. Over archimedean fields, by an admissible representation, we mean an admissible Casselman–Wallach representation. Morphisms are continuous and induction is smooth, and $\otimes$ is the completed projective tensor product, over archimedean fields.

2. The GSpin group and its parabolic subgroups

2.1. The GSpin *group*

In this section, we recall how the GSpin group is realized via the Clifford algebra. Our main reference is [18], which introduces the Clifford algebras associated with quadratic spaces over arbitrary field of characteristic different from 2.

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Let F be a local field of characteristic zero. Let V be a finite-dimensional quadratic space over F equipped with a non-degenerate quadratic form q_V, giving rise to the groups $\mathrm{O}(V)$ and $\mathrm{SO}(V)$:

$$\mathrm{O}(V) = \{g \in \mathrm{GL}_F(V) : q_V(gv) = q_V(v) \text{ for every } v \in V\},$$
$$\mathrm{SO}(V) = \{g \in \mathrm{O}(V) : \det(g) = 1\}.$$

The Clifford algebra $C(V)$ associated with (V, q_V) is the quotient of the tensor algebra $T(V) = \bigoplus_{d=0}^{\infty} V^{\otimes d}$ by the two-sided ideal generated by all $v \otimes v - q_V(v)$, for $v \in V$. To simplify notation, we also write $v_1 v_2 \dots v_k$ for $v_1 \otimes v_2 \otimes \cdots \otimes v_k$.

The inclusion map $V \to T(V)$ induces a canonical injection $\iota : V \to C(V)$. Identifying v with $\iota(v)$ for every $v \in V$, we view V as a subspace of $C(V)$, so that $v^2 = q_V(v)$. For $v_1, v_2 \in V$, we set $\langle v_1, v_2 \rangle := q_V(v_1+v_2) - q_V(v_1) - q_V(v_2)$ the induced symmetric bilinear form, so that $\langle v, v \rangle = 2q_V(v)$. Similarly, for $x, y \in C(V)$, we denote $\langle x, y \rangle = (x+y)^2 - x^2 - y^2$. In particular, $\langle x, y \rangle$ measures if x and y anti-commute in $C(V)$ since

$$\langle x, y \rangle = (x+y)^2 - x^2 - y^2 = xy + yx \in C(V). \tag{1}$$

Denote by $C^{\pm}(V)$ the even and odd parts of $C(V)$, so that $C(V) = C^+(V) \oplus C^-(V)$. The Clifford algebra $C(V)$ has a canonical involution $* : C(V) \to C(V)$, induced by reversing the order of products in the tensor algebra, that is, $(v_1 v_2 \dots v_k)^* = v_k \dots v_2 v_1$ for $v_1, \dots, v_k \in V$. The canonical involution preserves both $C^+(V)$ and $C^-(V)$. We also define

$$\alpha : C(V) \to C(V), \quad \alpha(v_+ + v_-) = v_+ - v_-,$$

where $v_+ \in C^+(V)$ and $v_- \in C^-(V)$. So α acts as the identity on $C^+(V)$ and as multiplication by -1 on $C^-(V)$, and $\alpha(v_1 v_2 \dots v_k) = (-1)^k v_1 v_2 \dots v_k$ for $v_1, \dots, v_k \in V$. Moreover, for all $x \in C(V)$, we define the Clifford involution

$$\overline{x} = (\alpha(x))^* = \alpha(x^*)$$

and the Clifford norm

$$N : C(V) \to C(V), \quad x \mapsto x\overline{x}.$$

It is clear that $N(\lambda x) = \lambda^2 N(x)$ for all $\lambda \in F$.

The GPin group and GSpin group of (V, q_V) are defined by

$$\mathrm{GPin}(V) = \{g \in C(V)^\times : \alpha(g) V g^{-1} = V\},$$
$$\mathrm{GSpin}(V) = \mathrm{GPin}(V) \cap C^+(V).$$

These are also called the Clifford group and the even Clifford group in the literature. Since α acts as identity on $C^+(V)$, we have the inclusion $\mathrm{GSpin}(V) \subset \mathrm{GPin}(V)$. In fact, we have $[\mathrm{GPin}(V) : \mathrm{GSpin}(V)] = 2$, and $\mathrm{GSpin}(V)$ is the identity component of $\mathrm{GPin}(V)$.

There is a natural projection map

$$\mathrm{pr} : \mathrm{GPin}(V) \to \mathrm{O}(V) \tag{2}$$

sending $g \in \mathrm{GPin}(V)$ to the map $v \mapsto \alpha(g) v g^{-1}$. The kernel of this homomorphism consists of scalars, thus we get an exact sequence

$$1 \to F^\times \to \mathrm{GPin}(V) \to \mathrm{O}(V) \to 1.$$

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Moreover, $\mathrm{pr}^{-1}(\mathrm{SO}(V)) = \mathrm{GSpin}(V)$, and we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & \mathrm{GSpin}(V) & \longrightarrow & \mathrm{SO}(V) & \longrightarrow & 1 \\ & & \| & & \downarrow\subseteq & & \downarrow\subseteq & & \\ 1 & \longrightarrow & F^\times & \longrightarrow & \mathrm{GPin}(V) & \longrightarrow & \mathrm{O}(V) & \longrightarrow & 1, \end{array}$$

where the rows are exact. It is known (see, e.g., [14, Theorem 3.1]) that the elements of $\mathrm{GPin}(V)$ are all products

$$g = v_1 \dots v_k, \tag{3}$$

where $v_1, \dots, v_k \in V$ are non-isotropic. Given $g = v_1 \dots v_k \in \mathrm{GPin}(V)$, the corresponding element $\mathrm{pr}(g) \in \mathrm{O}(V)$ is a product

$$\mathrm{pr}(g) = r_{v_1} \dots r_{v_k},$$

where each $r_{v_i} \in \mathrm{O}(V)$ is the reflection in the hyperplane orthogonal to Fv_i, that is, $r_{v_i} v_i = -v_i$ and $r_{v_i} x = x$ for every $x \in (Fv_i)^\perp$. In particular, $\mathrm{GSpin}(V)$ is given by products in (3) with k even.

Since $\mathrm{GPin}(V)$ consists of products in (3), the restriction of the Clifford norm N to $\mathrm{GPin}(V)$ has its image in $F^\times$, thus we have a homomorphism

$$N : \mathrm{GPin}(V) \to F^\times, \quad g \mapsto g\overline{g}.$$

For any $z \in \ker(\mathrm{pr}) = F^\times$, we have $N(z) = z^2$. Also, for any $g \in \mathrm{GPin}(V)$, we have $g^{-1} = \frac{1}{N(g)}\overline{g}$. In particular, for any $g \in \mathrm{GSpin}(V)$, we have $g^{-1} = \frac{1}{N(g)}\overline{g} = \frac{1}{gg^*}g^*$.

We now describe the centers of $\mathrm{GPin}(V)$ and $\mathrm{GSpin}(V)$. For this purpose, take an orthogonal basis $\{b_1, \dots, b_n\}$ of V and denote $\zeta = b_1 \dots b_n$. It is clear that $\zeta \in \mathrm{GSpin}(V)$ if $\dim(V)$ is even, and $\zeta \in \mathrm{GPin}(V) \setminus \mathrm{GSpin}(V)$ if $\dim(V)$ is odd.

Lemma 2.1 [18, Lemma 2.7]. *The element ζ enjoys the following properties:*

1. $\alpha(\zeta) v \zeta^{-1} = -v$ *for all* $v \in V$ *and hence* $\mathrm{pr}(\zeta) = -1$.
2. *We have* $\zeta^2 = (-1)^{\frac{n(n-1)}{2}} b_1^2 b_2^2 \dots b_n^2$ *and* $\zeta^* = (-1)^{\frac{n(n-1)}{2}} \zeta$.
3. $F\zeta$ *is independent of the choice of the orthogonal basis* $\{b_1, \dots, b_n\}$.
4. $F + F\zeta$ *is the center of* $C(V)$ *or of* $C^+(V)$ *according as* n *is odd or even, respectively.*

Lemma 2.2 [18, Theorem 3.6]. *Let* $Z_{\mathrm{GPin}(V)}$ *and* $Z_{\mathrm{GSpin}(V)}$ *denote the centers of* $\mathrm{GPin}(V)$ *and* $\mathrm{GSpin}(V)$, *respectively.*

1. *If* $\dim(V)$ *is even and* $\dim(V) > 2$, *then*

$$Z_{\mathrm{GPin}(V)} = F^\times, \quad Z_{\mathrm{GSpin}(V)} = F^\times \cup F^\times \zeta.$$

If $\dim(V) = 2$, *then*

$$Z_{\mathrm{GPin}(V)} = F^\times, \quad Z_{\mathrm{GSpin}(V)} = \mathrm{GSpin}(V).$$

2. *If* $\dim(V)$ *is odd, then*

$$Z_{\mathrm{GPin}(V)} = F^\times \cup F^\times \zeta, \quad Z_{\mathrm{GSpin}(V)} = F^\times.$$

In particular, $\ker(\mathrm{pr})$ *is the connected component of the center of* $\mathrm{GPin}(V)$, *as well as* $\mathrm{GSpin}(V)$ *when* $\dim(V) > 2$.

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Note that if $W \subset V$ is a non-degenerate subspace, then we have the inclusions of Clifford algebra $C(W) \subset C(V)$ and of even Clifford algebra $C^+(W) \subset C^+(V)$, which map $F^\times$ onto itself. These induce the inclusions $\mathrm{GPin}(W) \subset \mathrm{GPin}(V)$ and $\mathrm{GSpin}(W) \subset \mathrm{GSpin}(V)$.

Now let $V = V_1 \oplus V_2$ be an orthogonal sum decomposition, where both V_1, V_2 are non-degenerate. If $g_1 = v_1 \dots v_k \in \mathrm{GPin}(V_1)$ and $g_2 = w_1 \dots w_m \in \mathrm{GPin}(V_2)$, by using (1), we see that

$$g_1 g_2 g_1^{-1} = (-1)^{km} g_2.$$

In particular, if $g_1 \in \mathrm{GSpin}(V_1)$ and $g_2 \in \mathrm{GSpin}(V_2)$, then $g_1 g_2 = g_2 g_1$. Thus, we can form the direct product $\mathrm{GSpin}(V_1) \times \mathrm{GSpin}(V_2)$. We have a homomorphism $\mathrm{GSpin}(V_1) \times \mathrm{GSpin}(V_2) \to \mathrm{GSpin}(V)$ given by $(g_1, g_2) \mapsto g_1 g_2$ for $g_1 \in \mathrm{GSpin}(V_1), g_2 \in \mathrm{GSpin}(V_2)$, which induces an injection

$$(\mathrm{GSpin}(V_1) \times \mathrm{GSpin}(V_2))/\{(z, z^{-1}) : z \in \mathrm{GL}_1\} \to \mathrm{GSpin}(V).$$

Moreover, the diagram

$$\begin{array}{ccc} \mathrm{GSpin}(V_1) \times \mathrm{GSpin}(V_2) & \longrightarrow & \mathrm{GSpin}(V) \\ \downarrow{\scriptstyle \mathrm{pr}\times\mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathrm{SO}(V_1) \times \mathrm{SO}(V_2) & \xrightarrow{\iota_{V_1,V_2}} & \mathrm{SO}(V) \end{array}$$

is commutative, where ι_{V_1,V_2} is the block diagonal embedding.

2.2. Parabolic subgroups

In this section, we recall an explicit realization of parabolic subgroups of $\mathrm{GSpin}(V)$ in terms of Clifford algebra from [15]. Let $X \subset V$ be an isotropic subspace, and we consider the parabolic subgroup P_X of $\mathrm{GSpin}(V)$ that stabilizes X, namely,

$$P_X = \{g \in \mathrm{GSpin}(V) : gXg^{-1} = X\}.$$

The unipotent radical N_X of P_X has a two-step filtration $N_X \supset N_X' \supset 1$. Inside the Clifford algebra, its Lie algebra is given by $X^\perp \cdot X$, and the exponential map identifies this nilpotent Lie algebra with N_X. Moreover, this filtration identifies

$$N_X' \cong \wedge^2 X, \qquad N_X/N_X' \cong (X^\perp/X) \otimes X.$$

If X is one-dimensional, then $N_X' = 1$, while if $\dim(V)$ is even and $\dim(X) = \dim(V)/2$, then $X^\perp = X$ and hence $N_X = N_X'$. In all other cases, we have both N_X/N_X' and N_X' are nontrivial. The image of P_X under pr, $\mathrm{pr}(P_X)$, is the corresponding parabolic subgroup of $\mathrm{SO}(V)$.

Associated with X, one can define an increasing filtration $\mathcal{W}_\bullet^X$ on $C(V)$ as follows. First, we define an increasing filtration on V by

$$\mathcal{W}_k^X(V) := \begin{cases} 0 & \text{if } k < -1, \\ X & \text{if } k = -1, \\ X^\perp & \text{if } k = 0, \\ V & \text{if } k \geq 1. \end{cases}$$

Next, we define

$$\mathcal{W}_k^X(C(V)) := \{x \in C(V) : x = \sum x_1 x_2 \dots x_m, x_i \in \mathcal{W}_{\alpha_i}^X(V), \alpha_1 + \alpha_2 + \dots \alpha_m \leq k\}.$$

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Lemma 2.3 [15, Lemma 2.3]. *The parabolic subgroup P_X and its unipotent radical N_X are given by*

$$P_X = \mathcal{W}_0^X(C(V)) \cap \mathrm{GSpin}(V), \quad N_X = (1 + \mathcal{W}_{-1}^X(C(V))) \cap \mathrm{GSpin}(V).$$

Now suppose Y is an isotropic subspace of V dual to X, and we take an orthogonal sum decomposition $V = X \oplus W \oplus Y$. Let M_X be the Levi subgroup of the parabolic subgroup P_X of $\mathrm{GSpin}(V)$. Then the image in $\mathrm{SO}(V)$ of M_X is identified with $\mathrm{GL}_F(X) \times \mathrm{SO}(W)$. To describe the lifting of the $\mathrm{GL}_F(X)$ factor to $\mathrm{GSpin}(V)$ explicitly, we take $X = \mathrm{Span}\{e_1, \dots, e_m\}$ and $Y = \mathrm{Span}\{e_{-m}, \dots, e_{-1}\}$ such that $\langle e_i, e_{-j}\rangle = \delta_{i,j}$ for $1 \le i, j \le m$. Let $\{w_1, \dots, w_{\dim W}\}$ be a basis of W. Then we have the following basis of V:

$$\{e_1, \dots, e_m, w_1, \dots, w_{\dim W}, e_{-m}, \dots, e_{-1}\}. \tag{4}$$

Let J_m denote the $m \times m$ anti-diagonal matrix with entries 1 on the anti-diagonal and 0 everywhere else.

Lemma 2.4 [18, Proposition 4.8]. *There exists an injective algebraic group homomorphism*

$$\ell_m : \mathrm{GL}_F(X) = \mathrm{GL}_m(F) \to P_X$$

with the following properties:

1. *For any $g \in \mathrm{GL}_m(F)$, we have* $\mathrm{pr}(\ell_m(g)) = \begin{pmatrix} g & & \\ & I_W & \\ & & J_m {}^t g^{-1} J_m \end{pmatrix} \in \mathrm{SO}(V)$, *where the matrix is written with respect to the ordered basis (4).*
2. *For any $g \in \mathrm{GL}_m(F)$, we have $N(\ell_m(g)) = \det(g)$.*
3. *For any diagonal element* $\mathrm{diag}(a_1, \dots, a_m) \in \mathrm{GL}_m(F)$, *we have* $\ell_m(\mathrm{diag}(a_1, \dots, a_m)) = \prod_{i=1}^m (a_i e_i e_{-i} + e_{-i} e_i)$.
4. *If $1 \le m' < m$ and $g_0 \in \mathrm{GL}_F(\mathrm{Span}\{e_1, \dots, e_{m'}\}) = \mathrm{GL}_{m'}(F)$, let $g \in \mathrm{GL}_m(F)$ be the element such that g acts as g_0 on $\mathrm{Span}\{e_1, \dots, e_{m'}\}$ and acts trivially on $\mathrm{Span}\{e_{m'+1}, \dots, e_m\}$. Then $\ell_m(g) = \ell_{m'}(g_0)$.*
5. *The image $\ell_m(\mathrm{GL}_m(F))$ is an algebraic subgroup of P_X.*

Moreover, ℓ_m is uniquely determined, as an algebraic group homomorphism, by properties (1)–(3).

Lemma 2.5. *Let ℓ_m be the map in Lemma 2.4. Then $\ell_m(\mathrm{GL}_m(F)) \cap \mathrm{GSpin}(W) = \{1\}$.*

Proof. Let $g \in \mathrm{GL}_m(F)$ and assume that $\ell_m(g) \in \mathrm{GSpin}(W)$. Since $\mathrm{GSpin}(W)$ acts on X trivially, $\mathrm{pr}(\ell_m(g))$ acts on X trivially. On the other hand, by Lemma 2.4, $\mathrm{pr}(\ell_m(g))$ acts X by g. Thus, $g = I_m$ and $\ell_m(g) = 1$. □

Let $Z_m \subset \mathrm{GL}_m(F)$ be the unipotent radical of the Borel subgroup $B_m = T_m Z_m$ of $\mathrm{GL}_m(F)$ stabilizing the flag

$$\mathrm{Span}\{e_1\} \subset \mathrm{Span}\{e_1, e_2\} \subset \cdots \subset \mathrm{Span}\{e_1, e_2, \dots, e_m\}.$$

In the next lemma, we describe the lifting of the unipotent group Z_m.

Lemma 2.6. *For $1 \le i < j \le m$, let $E_{i,j}$ denote the $m \times m$ matrix with one in the (i, j)th entry and zeros everywhere else, and let $u_{i,j}(\gamma) = I_m + \gamma E_{i,j}$ be the unipotent matrix in $\mathrm{GL}_m(F)$, where $\gamma \in F$. Then $\ell_m(u_{i,j}(\gamma)) = 1 + \gamma e_i e_{-j}$.*

Proof. Set $x = \gamma e_i e_{-j}$. Since $i \ne j$, we have $\langle e_i, e_{-j}\rangle = e_i e_{-j} + e_{-j} e_i = 0$, and hence $(e_i e_{-j})^2 = e_i(-e_i e_{-j}) e_{-j} = 0$. Therefore, $(1 + x)^{-1} = 1 - x$. By the uniqueness of ℓ_m in Lemma 2.4, it suffices to check that $1 + x$ agrees with $u_{i,j}(\gamma)$ on properties (1)–(3) of Lemma 2.4.

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We first compute the action of $1+x$ on V by conjugation. For e_j, using $e_{-j}e_j = 1 - e_je_{-j}$, we get

$$(1+x)e_j(1-x) = e_j + xe_j - e_jx - xe_jx = e_j + \gamma e_ie_{-j}e_j - \gamma e_je_ie_{-j} - \gamma^2 e_ie_{-j}e_je_ie_{-j}.$$

Now $e_ie_{-j}e_j = e_i(1-e_je_{-j}) = e_i - e_ie_je_{-j}$ and $e_je_ie_{-j} = -e_ie_je_{-j}$. Moreover, the last term is zero because $e_ie_{-j}e_je_i = -e_{-j}e_ie_je_i = e_{-j}e_je_i^2 = 0$. Hence, $(1+x)e_j(1-x) = e_j + \gamma e_i$. For $k \neq j$, a direct calculation gives $(1+x)e_k(1-x) = e_k$. Thus, the action of $1+x$ on $X = \mathrm{Span}\{e_1, \dots, e_m\}$ is given by

$$e_j \mapsto e_j + \gamma e_i, \quad e_k \mapsto e_k \quad \text{for } k \neq j.$$

Next, since every $w \in W$ is orthogonal to both e_i and e_{-j}, it commutes with the product e_ie_{-j}. Hence,

$$(1+x)w(1-x) = w.$$

On the Y-part, one checks similarly that the action of $1+x$ is given by

$$e_{-i} \mapsto e_{-i} - \gamma e_{-j}, \quad e_{-k} \mapsto e_{-k} \quad \text{for } k \neq i.$$

We conclude that the action of $1+x$ on V is given by

$$\begin{pmatrix} u_{i,j}(\gamma) & & \\ & I_W & \\ & & {}^t u_{i,j}(\gamma)^{-1} J_m \end{pmatrix}.$$

We now compute the Clifford norm of $1+x$. Since $(e_ie_{-j})^* = e_{-j}e_i = -e_ie_{-j}$, we have $(1+\gamma e_ie_{-j})^* = 1 - \gamma e_ie_{-j}$. Thus,

$$N(1+\gamma e_ie_{-j}) = (1+\gamma e_ie_{-j})(1-\gamma e_ie_{-j}) = 1 - \gamma^2(e_ie_{-j})^2 = 1.$$

This agrees with $\det(u_{i,j}(\gamma)) = 1$.

Finally, the unipotent matrix $u_{i,j}(\gamma)$ is diagonal if and only if $\gamma = 0$, reducing it to the identity matrix $I_m = \mathrm{diag}(1, \dots, 1)$, which agrees with $\prod_{i=1}^{m}(e_ie_{-i} + e_{-i}e_i) = 1$.

By the uniqueness of the lift ℓ_m in Lemma 2.4, we obtain $\ell_m(u_{i,j}(\gamma)) = 1 + \gamma e_ie_{-j}$. □

Lemma 2.7. *Let ℓ_m be as in Lemma 2.4. Then* $\mathrm{GSpin}(W)$ *commutes with* $\ell_m(\mathrm{GL}_F(X))$.

Proof. Note that every $w \in W$ anti-commutes with each of the isotropic basis vectors $e_1, \dots, e_m$, $e_{-m}, \dots, e_{-1}$, so every $w \in W$ commutes with every quadratic monomial in $e_1, \dots, e_m, e_{-m}, \dots, e_{-1}$. By Lemma 2.4(3), every $w \in W$ commutes with the lifts of diagonal matrices $\ell_m(\mathrm{diag}(a_1, \dots, a_m))$. By Lemma 2.6, every $w \in W$ commutes with the lifts of upper triangular matrices $I_m + \gamma E_{i,j}$ with $1 \leq i < j \leq m$. By a similar computation as in Lemma 2.6 for the lifts of lower triangular unipotent matrices, every $w \in W$ commutes with the lifts of lower triangular matrices $I_m + \gamma E_{i,j}$ with $1 \leq j < i \leq m$. Hence, every $w \in W$ commutes with $\ell_m(\mathrm{GL}_F(X))$. Every element in $\mathrm{GSpin}(W)$ can be written as a product $h = w_1 \dots w_k$, where $w_1, \dots, w_k \in W$ are non-isotropic and k is even (see (3)). Hence, $\mathrm{GSpin}(W)$ commutes with $\ell_m(\mathrm{GL}_F(X))$. □

3. Preparation of the proof

3.1. The Bessel subgroup H and the character ξ

Let V be a non-degenerate quadratic space over F, and assume that $\dim(V) > 3$ (see Remark 1.2). Let $V_0 \subset V$ be a non-degenerate subspace such that $V_0^{\perp}$ is split of dimension $2r+1$. Write $V_0^{\perp} = X_r \oplus Y_r \oplus Fe_0$, where X_r and Y_r are maximal isotropic subspaces of $V_0^{\perp}$ in duality, and e_0 is non-isotropic. Set $V_0' = V_0 \oplus Fe_0$. Thus, $V = X_r \oplus V_0' \oplus Y_r$.

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Let $P_r = N_{P_r} \rtimes M_{P_r}$ be the parabolic subgroup of $\mathrm{GSpin}(V)$ stabilizing X_r, namely,

$$P_r = \{g \in \mathrm{GSpin}(V) : gX_r g^{-1} = X_r\}.$$

Then the Levi subgroup $M_{P_r} = \ell_r(\mathrm{GL}_F(X_r)) \times \mathrm{GSpin}(V_0') \cong \mathrm{GL}_F(X_r) \times \mathrm{GSpin}(V_0')$, where we recall ℓ_r is the map from Lemma 2.4. The unipotent radical N_{P_r} has a two-step filtration $N_{P_r} \supset N'_{P_r} \supset 1$, where $N'_{P_r} \cong \wedge^2 X_r$ and $N_{P_r}/N'_{P_r} \cong X_r \otimes V_0'$. We have a short exact sequence

$$1 \longrightarrow N'_{P_r} \longrightarrow N_{P_r} \longrightarrow N_{P_r}/N'_{P_r} \longrightarrow 1.$$

Let $l_{X_r} : X_r \to F$ be a nonzero F-linear homomorphism, and let $l_{V_0} : V_0' \to F$ be a nonzero F-linear homomorphism which vanishes on V_0. Then l_{V_0} is supported on the line Fe_0. We obtain a linear map $l_{X_r} \otimes l_{V_0} : X_r \otimes V_0' \longrightarrow F$. Explicitly, we have

$$(l_{X_r} \otimes l_{V_0})(x \otimes (v + ce_0)) = c \cdot l_{X_r}(x) l_{V_0}(e_0) \quad \text{for } x \in X_r, v \in V_0, c \in F.$$

By composing with the natural quotient map, we define a map $l_{N_{P_r}} : N_{P_r} \longrightarrow F$ by

$$N_{P_r} \longrightarrow N_{P_r}/N'_{P_r} \xrightarrow{\sim} X_r \otimes V_0' \xrightarrow{l_{X_r} \otimes l_{V_0}} F.$$

Let Z_{X_r} be a maximal unipotent subgroup of $\mathrm{GL}_F(X_r)$ stabilizing l_{X_r}. Then the subgroup $Z_{X_r} \times \mathrm{GSpin}(V_0)$ of the standard Levi subgroup M_{P_r} fixes $l_{N_{P_r}}$. Indeed, Z_{X_r} fixes l_{X_r} by definition, and $\mathrm{GSpin}(V_0)$ acts trivially on Fe_0.

Set

$$N = N_{P_r} \rtimes Z_{X_r} \subset \mathrm{GSpin}(V).$$

We define the rth Bessel subgroup H of $\mathrm{GSpin}(V)$ by

$$H = N \rtimes \mathrm{GSpin}(V_0).$$

We extend $l_{N_{P_r}}$ to $N = N_{P_r} \rtimes Z_{X_r}$ by making it trivial on Z_{X_r}. Let ψ be a nontrivial additive unitary character of F, and let

$$\lambda_{X_r} : Z_{X_r} \to \mathbb{S}^1$$

be a generic unitary character. We define a character ξ of H by

$$\xi(nzh) = \lambda_{X_r}(z)\psi(l_{N_{P_r}}(n)),$$

for $z \in Z_{X_r}$, $n \in N_{P_r}$, and $h \in \mathrm{GSpin}(V_0)$. This is well-defined because Z_{X_r} and $\mathrm{GSpin}(V_0)$ fix $l_{N_{P_r}}$. The pair (H, ξ) is uniquely determined, up to conjugacy in $G = \mathrm{GSpin}(V) \times \mathrm{GSpin}(V_0)$, by the pair $V_0 \subset V$.

We now give a more explicit description of (H, ξ). Let $\{e_1, \dots, e_r\}$ be a basis of X_r, and let $\{e_{-r}, \dots, e_{-1}\}$ be the dual basis of Y_r, with $\langle e_i, e_{-j} \rangle = \delta_{ij}$ for $1 \le i, j \le r$. Thus, in the Clifford algebra, we have

$$e_i^2 = e_{-i}^2 = 0, \quad e_i e_{-i} + e_{-i} e_i = 1, \quad e_i e_{-j} = -e_{-j} e_i \tag{5}$$

for $1 \le i, j \le r$ with $i \ne j$.

Let $Q_r = N_{Q_r} \rtimes M_{Q_r}$ denote the standard parabolic subgroup of $\mathrm{GSpin}(V)$ stabilizing the flag

$$0 = X_0 \subset X_1 = Fe_1 \subset X_2 = Fe_1 \oplus Fe_2 \subset \cdots \subset X_r = Fe_1 \oplus \cdots \oplus Fe_r. \tag{6}$$

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Then the Levi subgroup $M_{Q_r} = \ell_r((\mathrm{GL}_1)^r) \times \mathrm{GSpin}(V_0') \cong (\mathrm{GL}_1)^r \times \mathrm{GSpin}(V_0')$, where $(\mathrm{GL}_1)^r \subset \mathrm{GL}_F(X_r)$ is the diagonal torus with respect to the basis $\{e_1, \dots, e_r\}$. The $(\mathrm{GL}_1)^r$ factor is represented explicitly by $\prod_{i=1}^r (a_i e_i e_{-i} + e_{-i} e_i) \in \mathrm{GSpin}(V)$, where $a_i \in F^\times$. It acts by

$$\prod_{i=1}^r (a_i e_i e_{-i} + e_{-i} e_i) e_k \prod_{i=1}^r (a_i e_i e_{-i} + e_{-i} e_i)^{-1} = a_k e_k, \quad 1 \le k \le r, \tag{7}$$

and

$$\prod_{i=1}^r (a_i e_i e_{-i} + e_{-i} e_i) e_{-k} \prod_{i=1}^r (a_i e_i e_{-i} + e_{-i} e_i)^{-1} = a_k^{-1} e_{-k}, \quad 1 \le k \le r. \tag{8}$$

The torus element $\prod_{i=1}^r (a_i e_i e_{-i} + e_{-i} e_i)$ fixes the line $F e_0$. Moreover, $\mathrm{GSpin}(V_0')$ acts trivially on $X_r \oplus Y_r$. The unipotent radical of Q_r is $N_{Q_r} = Z_r \ltimes N_{P_r}$, where $Z_r = Z_{X_r}$ is the unipotent radical of the Borel subgroup $B_r = T_r Z_r$ of $\mathrm{GL}_r = \mathrm{GL}_F(X_r)$ stabilizing the flag (6). Thus, with this choice of Z_r, we have $N_{Q_r} = N$.

Define a map

$$l_r : N_{Q_r} \longrightarrow F^r$$

by

$$l_r(u) = (x_1, \dots, x_{r-1}, z),$$

where

$$x_i = x_i(u) = \langle \mathrm{pr}(u) e_{i+1}, e_{-i} \rangle, \quad 1 \le i \le r-1, \tag{9}$$

and

$$z = z(u) = \langle \mathrm{pr}(u) e_0, e_{-r} \rangle. \tag{10}$$

We first observe that l_r is fixed by $\mathrm{GSpin}(V_0)$-conjugation. Indeed, for $h \in \mathrm{GSpin}(V_0)$ and $u \in N_{Q_r}$, since $\mathrm{pr}(h)$ fixes each e_i, e_{-i}, and e_0, we have

$$\begin{aligned} \langle \mathrm{pr}(huh^{-1}) e_{i+1}, e_{-i} \rangle &= \langle \mathrm{pr}(h)\mathrm{pr}(u)\mathrm{pr}(h^{-1}) e_{i+1}, e_{-i} \rangle \\ &= \langle \mathrm{pr}(h)\mathrm{pr}(u) e_{i+1}, e_{-i} \rangle \\ &= \langle \mathrm{pr}(u) e_{i+1}, \mathrm{pr}(h^{-1}) e_{-i} \rangle \\ &= \langle \mathrm{pr}(u) e_{i+1}, e_{-i} \rangle. \end{aligned}$$

Similarly,

$$\langle \mathrm{pr}(huh^{-1}) e_0, e_{-r} \rangle = \langle \mathrm{pr}(u) e_0, e_{-r} \rangle.$$

Hence, l_r is invariant under $\mathrm{GSpin}(V_0)$-conjugation. The conjugation action of the torus $t = \prod_{i=1}^r (a_i e_i e_{-i} + e_{-i} e_i)$, where $a_i \in F^\times$, is given by

$$\begin{aligned} \langle \mathrm{pr}(tut^{-1}) e_{i+1}, e_{-i} \rangle &= \langle \mathrm{pr}(t)\mathrm{pr}(u)\mathrm{pr}(t^{-1}) e_{i+1}, e_{-i} \rangle \\ &= a_{i+1}^{-1} \langle \mathrm{pr}(t)\mathrm{pr}(u) e_{i+1}, e_{-i} \rangle \quad \text{(by (7))} \\ &= a_{i+1}^{-1} \langle \mathrm{pr}(u) e_{i+1}, \mathrm{pr}(t^{-1}) e_{-i} \rangle \\ &= a_i a_{i+1}^{-1} \langle \mathrm{pr}(u) e_{i+1}, e_{-i} \rangle \quad \text{(by (8))}. \end{aligned}$$

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Similarly, since t fixes e_0 and $t^{-1}e_{-r}t = a_r e_{-r}$, we have $\langle \mathrm{pr}(tut^{-1})e_0, e_{-r}\rangle = a_r\langle \mathrm{pr}(u)e_0, e_{-r}\rangle$. The subgroup of M_{Q_r} fixing l_r is precisely $\mathrm{GSpin}(V_0)$.

For a nontrivial additive unitary character ψ of F, define a character ξ of N_{Q_r} by

$$\xi(u) = \psi\left(\sum_{i=1}^{r-1} x_i + z\right), \tag{11}$$

where x_i and z are given by (9) and (10). Since l_r is invariant under $\mathrm{GSpin}(V_0)$-conjugation, ξ is invariant under $\mathrm{GSpin}(V_0)$-conjugation. Hence, ξ extends to a character

$$\xi : N_{Q_r} \rtimes \mathrm{GSpin}(V_0) \longrightarrow \mathbb{C}^\times$$

by $\xi(nh) = \xi(n)$ for $n \in N_{Q_r}$, $h \in \mathrm{GSpin}(V_0)$. As noted above, up to conjugacy in $G = \mathrm{GSpin}(V) \times \mathrm{GSpin}(V_0)$, the pair (H, ξ) depends only on the pair $V_0 \subset V$, and not on the choices of ψ, or the basis vectors e_i.

We now prove that the Bessel subgroup H is unimodular.

Lemma 3.1. *The group H is unimodular.*

Proof. Recall that $H = N_{Q_r} \rtimes \mathrm{GSpin}(V_0)$, $N_{Q_r} = Z_r \ltimes N_{P_r}$. The group N_{Q_r} is unipotent, and hence is unimodular. The group $\mathrm{GSpin}(V_0)$ is reductive, and hence is also unimodular. Therefore, it remains to check that the conjugation action of $\mathrm{GSpin}(V_0)$ on N_{Q_r} has determinant one.

The group $\mathrm{GSpin}(V_0)$ acts trivially on the factor Z_r, since Z_r is contained in the lifted $\mathrm{GL}_F(X_r)$-factor and $\mathrm{GSpin}(V_0)$ acts trivially on $X_r \oplus Y_r$. Thus, it suffices to analyze its action on N_{P_r}.

At the level of Lie algebras, the two-step filtration of N_{P_r} gives $\mathrm{Lie}(N_{P_r}) \simeq (X_r \otimes V_0') \oplus \wedge^2 X_r$ as a vector space. The subgroup $\mathrm{GSpin}(V_0)$ acts trivially on X_r, on $\wedge^2 X_r$, and on the line $Fe_0 \subset V_0'$. On the V_0-summand of $V_0' = V_0 \oplus Fe_0$, it acts through $\mathrm{pr} : \mathrm{GSpin}(V_0) \longrightarrow \mathrm{SO}(V_0)$. Hence, for $h \in \mathrm{GSpin}(V_0)$, the determinant of its action on $\mathrm{Lie}(N_{P_r})$ is $\det(\mathrm{pr}(h))^{\dim X_r} = 1$, because $\mathrm{pr}(h) \in \mathrm{SO}(V_0)$. Therefore, the modular character of the semidirect product $N_{Q_r} \rtimes \mathrm{GSpin}(V_0)$ is trivial. Hence, H is unimodular. □

3.2. The group $\mathbf{GSpin}(V')$

Consider $V' = V \oplus Ff_0$. We equip V' with a symmetric bilinear form $\langle \cdot, \cdot \rangle_{V'}$ so that

$$\langle v_1, v_2\rangle_{V'} = \langle v_1, v_2\rangle, \quad \langle v, f_0\rangle_{V'} = 0, \quad \langle f_0, f_0\rangle_{V'} = -\langle e_0, e_0\rangle, \quad \text{for all } v, v_1, v_2 \in V.$$

Denote $e_{r+1} = e_0 + f_0$, $e_{-(r+1)} = \frac{1}{2\langle e_0, e_0\rangle}(e_0 - f_0)$. Then V' is the orthogonal sum

$$V' = (X'_{r+1} \oplus Y'_{r+1}) \oplus V_0,$$

where

$$X'_{r+1} = X_r \oplus Fe_{r+1}, \quad Y'_{r+1} = Y_r \oplus Fe_{-(r+1)}$$

are totally isotropic subspaces.

Recall $V_0' = V_0 \oplus Fe_0$, and $P_r = N_{P_r} \rtimes M_{P_r}$ is the parabolic subgroup of $\mathrm{GSpin}(V)$ stabilizing X_r, where $M_{P_r} \cong \mathrm{GSpin}(V_0') \times \mathrm{GL}_r$, $\mathrm{GL}_r = \mathrm{GL}_F(X_r)$. Denote by $P'_{r+1} = N_{P'_{r+1}} \rtimes M_{P'_{r+1}}$ the parabolic subgroup of $\mathrm{GSpin}(V')$ stabilizing X'_{r+1}. Then we have

$$M_{P'_{r+1}} \cong \mathrm{GSpin}(V_0) \times \mathrm{GL}_{r+1}, \quad \mathrm{GL}_{r+1} := \mathrm{GL}_F(X'_{r+1}) \supset \mathrm{GL}_r\,.$$

We view GL_r as a subgroup of GL_{r+1} by acting trivially on e_{r+1}. Recall that Z_r is the unipotent radical of the Borel subgroup $B_r = T_r Z_r$ of $\mathrm{GL}_F(X_r)$ stabilizing the flag (6). Let Z_{r+1} be the unipotent radical

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of the Borel subgroup $B_{r+1} = T_{r+1}Z_{r+1}$ of $\mathrm{GL}(X'_{r+1})$ stabilizing the flag

$$X_1 \subset X_2 \subset \cdots \subset X_r \subset X'_{r+1}. \tag{12}$$

We observe that

$$H = N_{P_r} \rtimes (\mathrm{GSpin}(V_0) \times Z_r) \subset P_r = N_{P_r} \rtimes (\mathrm{GSpin}(V'_0) \times \mathrm{GL}_r) \tag{13}$$

and

$$H \subset P'_{r+1} \cap \mathrm{GSpin}(V).$$

Now we analyze the structure of the group $P'_{r+1} \cap \mathrm{GSpin}(V)$. Let $P'_r = N_{P'_r} \rtimes M_{P'_r}$ be the parabolic subgroup of $\mathrm{GSpin}(V')$ stabilizing X_r.

Lemma 3.2. *We have* $P'_{r+1} \cap \mathrm{GSpin}(V) = (N_{P'_r} \cap \mathrm{GSpin}(V)) \rtimes (\mathrm{GL}(X_r) \times \mathrm{GSpin}(V_0)) \subset P'_r \cap \mathrm{GSpin}(V)$.

Proof. We first show that $P'_{r+1} \cap \mathrm{GSpin}(V) \subset P'_r \cap \mathrm{GSpin}(V)$. Take any $g \in P'_{r+1} \cap \mathrm{GSpin}(V)$. It suffices to show that $\mathrm{pr}(g)$ also stabilizes X_r, that is, for any $x \in X_r$, we need to show that $\langle \mathrm{pr}(g)x, e_0 - f_0 \rangle_{V'} = 0$. We compute that

$$\begin{aligned} \langle \mathrm{pr}(g)x, e_0 - f_0 \rangle_{V'} &= \langle x, \mathrm{pr}(g)^{-1}(e_0 - f_0) \rangle_{V'} \\ &= \langle x, \mathrm{pr}(g)^{-1}(e_{r+1} - 2f_0) \rangle_{V'} \\ &= \langle x, \mathrm{pr}(g)^{-1} e_{r+1} \rangle_{V'} - 2\langle x, \mathrm{pr}(g)^{-1} f_0 \rangle_{V'}. \end{aligned}$$

Since $g^{-1} \in \mathrm{GSpin}(V)$, g^{-1} acts trivially on f_0, that is, $\mathrm{pr}(g^{-1})f_0 = f_0$. Hence,

$$\langle x, \mathrm{pr}(g)^{-1} f_0 \rangle_{V'} = \langle x, f_0 \rangle_{V'} = 0.$$

Since $g^{-1} \in P'_{r+1}$, it stabilizes X'_{r+1} and $\mathrm{pr}(g^{-1})e_{r+1} \in X'_{r+1}$. Since X'_{r+1} is totally isotropic, we have

$$\langle x, \mathrm{pr}(g)^{-1} e_{r+1} \rangle_{V'} = 0.$$

Thus,

$$\langle \mathrm{pr}(g)x, e_0 - f_0 \rangle_{V'} = 0.$$

This proves $P'_{r+1} \cap \mathrm{GSpin}(V) \subset P'_r \cap \mathrm{GSpin}(V)$.

Next, we show that as a subgroup of $P'_r \cap \mathrm{GSpin}(V)$,

$$P'_{r+1} \cap \mathrm{GSpin}(V) = (N_{P'_r} \cap \mathrm{GSpin}(V)) \rtimes (\mathrm{GL}(X_r) \times \mathrm{GSpin}(V_0)).$$

Let h be an element of $P'_r \cap \mathrm{GSpin}(V)$, so that $\mathrm{pr}(h)x \in X_r$ for all $x \in X_r$ and $\mathrm{pr}(h)f_0 = f_0$. Note that

$$\begin{aligned} h \in P'_{r+1} \cap \mathrm{GSpin}(V) &\iff \mathrm{pr}(h)e_{r+1} \in X'_{r+1} \\ &\iff \mathrm{pr}(h)e_0 + f_0 \in X'_{r+1} \\ &\iff \mathrm{pr}(h)e_0 - e_0 \in X'_{r+1}. \end{aligned}$$

Also, if $x \in X_r$, then $\mathrm{pr}(h)^{-1}x \in X_r$ and hence

$$\langle \mathrm{pr}(h)e_0, x \rangle_{V'} = \langle e_0, \mathrm{pr}(h)^{-1}x \rangle_{V'} = 0.$$

Thus, we may write $\mathrm{pr}(h)e_0 = \lambda e_0 + w + x$ for some $\lambda \in F, w \in V_0, x \in X_r$. The condition $\mathrm{pr}(h)e_0 - e_0 \in X'_{r+1}$ if and only if $\lambda = 1$ and $w = 0$, implying that $\mathrm{pr}(h)$ fixes e_0 modulo X_r and hence stabilizes

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V_0 modulo X_r, consequently, $\mathrm{pr}(h) \in (N_{P'_r} \cap \mathrm{GSpin}(V)) \rtimes (\mathrm{GL}(X_r) \times \mathrm{GSpin}(V_0))$, as desired. This completes the proof. □

Let $\alpha : P'_{r+1} \twoheadrightarrow \ell_{r+1}(\mathrm{GL}_{r+1}) \times \mathrm{GSpin}(V_0) \cong \mathrm{GL}_{r+1} \times \mathrm{GSpin}(V_0)$ be the natural projection, whose kernel is the unipotent radical $N_{P'_{r+1}}$.

Lemma 3.3. *The natural projection map $\alpha : P'_{r+1} \twoheadrightarrow \mathrm{GSpin}(V_0) \times \mathrm{GL}_{r+1}$ induces the following commutative diagram with exact rows, where the vertical arrows are inclusions:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N_{P'_{r+1}} & \longrightarrow & P'_{r+1} & \longrightarrow & \mathrm{GL}_{r+1} \times \mathrm{GSpin}(V_0) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & N_{P'_{r+1}} \cap \mathrm{GSpin}(V) & \longrightarrow & P'_{r+1} \cap \mathrm{GSpin}(V) & \longrightarrow & R \times \mathrm{GSpin}(V_0) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & N_{P'_{r+1}} \cap \mathrm{GSpin}(V) & \longrightarrow & N_{P'_r} \cap \mathrm{GSpin}(V) & \longrightarrow & \mathrm{Hom}(Fe_{r+1}, X_r) & \longrightarrow & 1.
\end{array}$$

Here, $R \subset \mathrm{GL}(X'_{r+1})$ is the mirabolic subgroup which stabilizes the co-dimension one subspace $X_r \subset X'_{r+1}$ and fixes e_{r+1} modulo X_r, and $\mathrm{Hom}(Fe_{r+1}, X_r)$ is the unipotent radical of R.

Proof. We consider the restriction of α to the subgroup $P'_{r+1} \cap \mathrm{GSpin}(V) = N_{P_r} \rtimes (\mathrm{GL}(X_r) \times \mathrm{GSpin}(V_0))$. Observe that the subgroup $\mathrm{GL}_r \times \mathrm{GSpin}(V_0)$ is mapped isomorphically to its image in $\mathrm{GL}_{r+1} \times \mathrm{GSpin}(V_0)$, and its image is precisely $\mathrm{GL}_r \times \mathrm{GSpin}(V_0)$, which is the Levi subgroup of $R \times \mathrm{GSpin}(V_0)$.

Next, we analyze the kernel of α restricted to $P'_{r+1} \cap \mathrm{GSpin}(V)$, which is given by

$$\ker(\alpha|_{P'_{r+1} \cap \mathrm{GSpin}(V)}) = N_{P'_{r+1}} \cap \mathrm{GSpin}(V).$$

We claim that $\ker(\alpha|_{P'_{r+1} \cap \mathrm{GSpin}(V)})$ is contained in $N_{P'_r} \cap \mathrm{GSpin}(V)$. To prove this, take $h \in N_{P'_{r+1}} \cap \mathrm{GSpin}(V)$, so that $\mathrm{pr}(h) f_0 = f_0$. Our goal is to show that $\mathrm{pr}(h)$ acts trivially on X_r and acts trivially on V'_0 modulo X_r. Since $h \in N_{P'_{r+1}}$, $\mathrm{pr}(h)$ acts trivially on X'_{r+1} and acts trivially on V_0 modulo X'_{r+1}. Since $X_r \subset X'_{r+1}$, $\mathrm{pr}(h)$ acts trivially on X_r. Also, for $w \in V_0$, we have $\mathrm{pr}(h)w - w \in X'_{r+1} \cap V = X_r$. It remains to show that $\mathrm{pr}(h)e_0 - e_0 \in X_r = X'_{r+1} \cap V$. Since $\mathrm{pr}(h)e_0 - e_0 \in V$, it suffices to show that $\mathrm{pr}(h)e_0 - e_0 \in X'_{r+1}$. We have

$$\mathrm{pr}(h)e_0 - e_0 = \mathrm{pr}(h)(e_0 + f_0) - (e_0 + f_0) = \mathrm{pr}(h)e_{r+1} - e_{r+1} \in X'_{r+1},$$

as desired. This proves the claim.

Now we show that the map α induces an isomorphism

$$(N_{P'_r} \cap \mathrm{GSpin}(V))/(N_{P'_{r+1}} \cap \mathrm{GSpin}(V)) \cong \mathrm{Hom}(Fe_{r+1}, X_r).$$

If $h \in N_{P'_r} \cap \mathrm{GSpin}(V)$, then $\mathrm{pr}(h)$ acts trivially on X_r, and $\mathrm{pr}(h)$ acts trivially on V'_0 modulo X_r, and $\mathrm{pr}(h) f_0 = f_0$. Then

$$\mathrm{pr}(h)e_{r+1} - e_{r+1} = \mathrm{pr}(h)(e_0 + f_0) - (e_0 + f_0) = \mathrm{pr}(h)e_0 - e_0 \in X_r.$$

It follows that $\alpha(h)$ lies in the unipotent radical $\mathrm{Hom}(Fe_{r+1}, X_r)$ of R. It remains to prove that $\alpha|_{P'_{r+1} \cap \mathrm{GSpin}(V)}$ is surjective onto $\mathrm{Hom}(Fe_{r+1}, X_r)$. For any $x \in X_r$, set $h_x = 1 + \frac{1}{\langle e_0, e_0 \rangle} x e_0$. Then $h_x \in N_{P'_r} \cap \mathrm{GSpin}(V)$, fixes X_r and V_0, and satisfies $\mathrm{pr}(h_x)e_0 = e_0 + x$. Then $\mathrm{pr}(\alpha(h_x))(e_{r+1}) = x$, as desired. Thus, we have the commutative diagram. □

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By Lemma 3.3, we see that the preimage of $Z_{r+1}\times\mathrm{GSpin}(V_0)$ in $P'_{r+1}\cap\mathrm{GSpin}(V)$ under the restriction of α is precisely the Bessel subgroup

$$H = N_{P_r} \rtimes (\mathrm{GSpin}(V_0)\times Z_r).$$

Thus, the restriction of α gives a surjective homomorphism

$$\alpha|_H : H \twoheadrightarrow Z_{r+1}\times\mathrm{GSpin}(V_0).$$

Moreover, the character $\xi : H \to \mathbb{C}^\times$ is trivial on the kernel of $\alpha|_H$. Indeed, the kernel of $\alpha|_H$ is $N_{P'_{r+1}}\cap\mathrm{GSpin}(V)$, and this subgroup is contained in the kernel of ξ. Hence, ξ descends to a character on $Z_{r+1}\times\mathrm{GSpin}(V_0)$. Equivalently, there exists a character χ of $Z_{r+1}\times\mathrm{GSpin}(V_0)$ such that

$$\xi = \chi\circ\alpha|_H. \tag{14}$$

Here, χ is a unitary generic character on Z_{r+1}, extended to a character on $Z_{r+1}\times\mathrm{GSpin}(V_0)$ by $\chi(uh)=\chi(u)$ for $u\in Z_{r+1}, h\in\mathrm{GSpin}(V_0)$.

Remark 3.4. The character χ is compatible with the character ξ defined in Section 3.1 in the following sense. Recall that on $N_{Q_r} = Z_r \ltimes N_{P_r}$, ξ was defined as $\xi(u)=\psi(x_1+\cdots+x_{r-1}+z)$, where x_i and z are given by (9) and (10). Under the quotient map $\alpha|_H : H \twoheadrightarrow Z_{r+1}\times\mathrm{GSpin}(V_0)$, the coordinates $x_1,\dots,x_{r-1}$ correspond to the simple root coordinates of the subgroup $Z_r\subset Z_{r+1}$, while the coordinate z corresponds to the additional simple root subgroup $\mathrm{Hom}(Fe_{r+1},X_r)$ appearing in the unipotent radical of the mirabolic subgroup $R\subset\mathrm{GL}_{r+1}=\mathrm{GL}(X'_{r+1})$. Consequently, up to conjugation, the character χ of Z_{r+1} is the generic character obtained from the same linear functional $(x_1,\dots,x_{r-1},z)\longmapsto x_1+\cdots+x_{r-1}+z$.

3.3. Induced representations of $\mathbf{GSpin}(V')$

Let π_0 be an irreducible admissible representation of $\mathrm{GSpin}(V_0)$ and let σ be an irreducible admissible χ^{-1}-generic representation of GL_{r+1}. The existence of such σ is well known (see, e.g., [24, Theorem 9.7] when F is non-archimedean and [5, Theorem 9.1] when F is archimedean). Put

$$\rho := \pi_0\otimes\sigma,$$

which is an admissible representation of the Levi subgroup $M_{P'_{r+1}}\cong\mathrm{GSpin}(V_0)\times\mathrm{GL}_{r+1}$. For $s\in\mathbb{C}$, define $\rho_s(hm)=|\det(m)|^s\rho(hm)$, for $h\in\mathrm{GSpin}(V_0)$, $m\in\mathrm{GL}_{r+1}$. *Throughout, we identify* GL_{r+1} *with its lift* $\ell_{r+1}(\mathrm{GL}_{r+1})$ *in* $\mathrm{GSpin}(V')$, *where* ℓ_{r+1} *is given in Lemma 2.4.*

Let $\pi'_s = \mathrm{Ind}_{P'_{r+1}}^{\mathrm{GSpin}(V')}(\rho_s)$ denote the (unnormalized) induced representation of $\mathrm{GSpin}(V')$, consisting of all smooth functions $f:\mathrm{GSpin}(V')\to\rho$ satisfying

$$f(ugmx) = |\det(m)|^s\rho(gm)(f(x)) \tag{15}$$

for all $u\in N_{P'_{r+1}}$, $g\in\mathrm{GSpin}(V_0)$, $m\in\mathrm{GL}_{r+1}$, and $x\in\mathrm{GSpin}(V')$. Note that for $z\in F^\times\subset\mathrm{GSpin}(V_0)$, we have

$$f(zx)=\rho(z)f(x)=\omega_{\pi_0}(z)f(x),\quad \forall x\in\mathrm{GSpin}(V'). \tag{16}$$

We have the following lemma, which will be used in Section 4.3.

Lemma 3.5. *The representation* π'_s *is irreducible except for a measure zero subset of* $\mathbb{C}$.

Proof. When F is archimedean, this is a consequence of [19, Theorem 1.1]. When F is non-archimedean, this follows from Bernstein's generic irreducible theorem for parabolic induction (see [16, Proposition VI.8.4]). □

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4. Proof of Theorem 1.1

4.1. A local integral

Recall that σ is χ^{-1}-generic. We fix a nonzero χ^{-1}-Whittaker functional λ of σ, satisfying

$$\lambda(\sigma(u)v_\sigma) = \chi^{-1}(u)\lambda(v_\sigma), \qquad \text{for all } u \in Z_{r+1}, v_\sigma \in \sigma. \tag{17}$$

Then we define a linear map $\Lambda : \rho \to \pi_0$ by

$$\begin{aligned} \Lambda : \rho = \pi_0 \otimes \sigma &\to \pi_0 \\ v_0 \otimes v_\sigma &\mapsto \lambda(v_\sigma)v_0. \end{aligned}$$

Let π, π_0 be irreducible admissible representations of $\mathrm{GSpin}(V)$ and $\mathrm{GSpin}(V_0)$, respectively, and let

$$\mu : \pi \otimes \pi_0 \to \mathbb{C}$$

be a Bessel functional, that is, a map in the space $\mathrm{Hom}_H(\pi \otimes \pi_0, \xi)$.

Remark 4.1. Let $\omega_\pi, \omega_{\pi_0}$ denote the central characters of π and π_0, respectively. Note that $F^\times$ lies in both the centers of $\mathrm{GSpin}(V)$ and $\mathrm{GSpin}(V_0)$ (see Lemma 2.2), and $\xi|_{F^\times}$ is trivial. If $\omega_\pi|_{F^\times}\omega_{\pi_0}|_{F^\times}$ is not the trivial character of $F^\times$, then $\mathrm{Hom}_H(\pi \otimes \pi_0, \xi) = 0$. Thus, we may assume that $\omega_\pi|_{F^\times}\omega_{\pi_0}|_{F^\times}$ is the trivial character of $F^\times$.

We have the following result.

Lemma 4.2. *For every $s \in \mathbb{C}$, $v \in \pi$, and $f \in \pi'_s$, the smooth function on* $\mathrm{GSpin}(V)$ *defined by*

$$g \mapsto \mu(\pi(g)v, \Lambda(f(g)))$$

is left invariant under H.

Proof. Let $g \in \mathrm{GSpin}(V)$ and $b \in H \subset P'_{r+1}$. We may write $b = nhm$, where $n \in N_{P'_{r+1}}$, $h \in \mathrm{GSpin}(V_0)$, and $m \in Z_{r+1}$. Recall that we have identified Z_{r+1} with its lift $\ell_{r+1}(Z_{r+1}) \subset \mathrm{GSpin}(V')$. Then

$$\Lambda(f(bg)) = \Lambda(\rho(hm)f(g)) = \chi^{-1}(m)\pi_0(h)(\Lambda(f(g))) = \xi^{-1}(b)\pi_0(h)(\Lambda(f(g)))$$

and hence

$$\begin{aligned} \mu(\pi(bg)v, \Lambda(f(bg))) &= \xi^{-1}(b)\mu(\pi(bg)v, \pi_0(h)(\Lambda(f(g)))) \\ &= \mu(\pi(b^{-1})\pi(bg)v, \pi_0(b^{-1})\pi_0(h)(\Lambda(f(g)))) \\ &= \mu(\pi(g)v, \Lambda(f(g))), \end{aligned}$$

where the last equality holds because b maps to h under the quotient map $H \to \mathrm{GSpin}(V_0)$, and π_0 is viewed as a representation of H via inflation. □

Since $\mathrm{GSpin}(V)$ and H are unimodular, there exists a right $\mathrm{GSpin}(V)$-invariant positive measure on $H\backslash \mathrm{GSpin}(V)$ unique up to scalar, and we write dg for such a measure. The following local zeta integral plays an important role in the proof of Theorem 1.1:

$$\mathcal{Z}_\mu(f, v) := \int_{H\backslash \mathrm{GSpin}(V)} \mu(\pi(g)v, \Lambda(f(g)))dg, \quad f \in \pi'_s, v \in \pi. \tag{18}$$

Whenever the integral converges absolutely, the right $\mathrm{GSpin}(V)$-invariance of dg implies that, for every $g_0 \in \mathrm{GSpin}(V)$, we have

$$\mathcal{Z}_\mu(\pi'_s(g_0)f, \pi(g_0)v) = \mathcal{Z}_\mu(f, v). \tag{19}$$

Indeed, this follows from the change of variable $gg_0 \mapsto g$.

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We postpone the proof of the following proposition to Section 4.2.

Proposition 4.3. *Let $\mu \in \mathrm{Hom}_H(\pi \otimes \pi_0, \xi)$ be nonzero. There exist an element $f_\rho \in \pi'_s$ and a vector $v \in \pi$ such that $\mathcal{Z}_\mu(f_\rho, v)$ converges absolutely and yields a nonzero number.*

We also have the following result.

Proposition 4.4. *There exists a number $c_\mu \in \mathbb{R}$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c_\mu$, the integral $\mathcal{Z}_\mu(f, v)$ converges absolutely for all $f \in \pi'_s$ and all $v \in \pi$. In the archimedean case, in the domain of absolute convergence, $\mathcal{Z}_\mu(f, v)$ is continuous in the input data.*

Proof. We prove this result by reducing from $\mathrm{GSpin}(V)$ to $\mathrm{SO}(V)$. A similar approach has been used in [2, Theorem 7.1(i)] and [13, Proposition 3.1].

Recall that the kernel of the projection map $\mathrm{pr} : \mathrm{GSpin}(V) \to \mathrm{SO}(V)$ is $\ker(\mathrm{pr}) = F^\times$. Since $F^\times \subset H$, the projection induces a homeomorphism

$$H \backslash \mathrm{GSpin}(V) \cong \mathrm{pr}(H) \backslash \mathrm{SO}(V). \tag{20}$$

Let $v \in \pi$ and $f \in \pi'_s$, and denote

$$\mathcal{G}_{\mu,f,v}(g) := \mu(\pi(g)v, \Lambda(f(g))), \qquad g \in \mathrm{GSpin}(V).$$

By Lemma 4.2, $\mathcal{G}_{\mu,f,v}$ is left invariant under H. Note that $F^\times$ lies in the centers of both $\mathrm{GSpin}(V)$ and $\mathrm{GSpin}(V_0)$. It follows that $\mathcal{G}_{\mu,f,v}$ is invariant under multiplication by $F^\times$. Indeed, for any $z \in F^\times$, we have

$$\begin{aligned}
\mathcal{G}_{\mu,f,v}(gz) &= \mathcal{G}_{\mu,f,v}(zg) \\
&= \mu(\pi(zg)v, \Lambda(f(zg))) \\
&= \mu(\omega_\pi(z)\pi(g)v, \omega_{\pi_0}(z)\Lambda(f(g))) \quad \text{(by (16))} \\
&= \omega_\pi(z)\omega_{\pi_0}(z)\mu(\pi(g)v, \Lambda(f(g))) \\
&= \mu(\pi(g)v, \Lambda(f(g))) \\
&= \mathcal{G}_{\mu,f,v}(g),
\end{aligned} \tag{21}$$

where the second to the last equality follows from the assumption that $\omega_\pi|_{F^\times} \cdot \omega_{\pi_0}|_{F^\times}$ is the trivial character of $F^\times$ (see Remark 4.1). Thus, the function $\mathcal{G}_{\mu,f,v}$ descends to a function on $\mathrm{SO}(V)$ as follows. For any $\overline{g} \in \mathrm{SO}(V)$, we set

$$\overline{\mathcal{G}}_{\mu,f,v}(\overline{g}) := \mathcal{G}_{\mu,f,v}(g),$$

where $g \in \mathrm{GSpin}(V)$ is any lift of $\overline{g}$. This is well-defined by (21). Moreover, $\overline{\mathcal{G}}_{\mu,f,v}$ is left invariant under $\mathrm{pr}(H)$. Indeed, let $\overline{h} \in \mathrm{pr}(H)$, and choose any lift $h \in H$. Then we have

$$\overline{\mathcal{G}}_{\mu,f,v}(\overline{h}\overline{g}) = \overline{\mathcal{G}}_{\mu,f,v}(\overline{hg}) = \mathcal{G}_{\mu,f,v}(hg) = \mathcal{G}_{\mu,f,v}(g) = \overline{\mathcal{G}}_{\mu,f,v}(\overline{g}).$$

Hence, $\overline{\mathcal{G}}_{\mu,f,v}$ is left invariant under $\mathrm{pr}(H)$.

Using the homeomorphism (20), we may rewrite the local zeta integral $\mathcal{Z}_\mu(f, v)$ as

$$\begin{aligned}
\mathcal{Z}_\mu(f, v) &= \int_{H \backslash \mathrm{GSpin}(V)} \mathcal{G}_{\mu,f,v}(g)\, dg \\
&= \int_{\mathrm{pr}(H) \backslash \mathrm{SO}(V)} \overline{\mathcal{G}}_{\mu,f,v}(\overline{g})\, d\overline{g},
\end{aligned}$$

where $d\overline{g}$ denotes the quotient measure on $\mathrm{pr}(H) \backslash \mathrm{SO}(V)$ transported from the quotient measure dg on $H \backslash \mathrm{GSpin}(V)$ through the homeomorphism (20). The absolute convergence of the integral

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$\int_{\mathrm{pr}(H)\backslash \mathrm{SO}(V)} \overline{\mathcal{G}}_{\mu,f,v}(\overline{g})d\overline{g}$ and hence of $\mathcal{Z}_\mu(f,v)$ now follows from the absolute convergence of the analogous local integral for special orthogonal groups in [12, Lemma 4.1]. Finally, in the archimedean case, the same reduction together with [11, Proposition 3.4] shows that $\mathcal{Z}_\mu(f,v)$ is continuous on the input data in the domain of absolute convergence. □

4.2. *Proof of Proposition 4.3*

The goal of this section is to prove Proposition 4.3, and hence we assume that $\mu \in \mathrm{Hom}_H(\pi \otimes \pi_0, \xi)$ is nonzero. By twisting the representation σ by a character if necessary, we may assume $s = 0$.

Denote $\overline{P}_r = \{g \in \mathrm{GSpin}(V) : gY_r g^{-1} = Y_r\}$. This is the parabolic subgroup of $\mathrm{GSpin}(V)$ opposite to P_r with respect to the decomposition $V = X_r \oplus V_0' \oplus Y_r$. It has a Levi decomposition $\overline{P}_r = (\mathrm{GL}_r \times \mathrm{GSpin}(V_0')) \ltimes \overline{N}_{P_r}$, where the GL_r-factor is embedded in $\mathrm{GSpin}(V)$ through the lift ℓ_r described in Lemma 2.4, and $\overline{N}_{P_r}$ is the unipotent radical, which is given by

$$\overline{N}_{P_r} = \{g \in \mathrm{GSpin}(V) : \mathrm{pr}(g)y = y \text{ for all } y \in Y_r, \mathrm{pr}(g)w \equiv w (\mathrm{mod}\ Y_r) \text{ for all } w \in V_0'\}.$$

By [4, Theorem 14.20], the multiplication map $P_r \times \overline{N}_{P_r} \to \mathrm{GSpin}(V)$ given by $(p, \overline{n}) \mapsto p\overline{n}$ is an F-morphic open immersion, and hence the product

$$P_r\overline{N}_{P_r} = \left(N_{P_r} \rtimes (\mathrm{GSpin}(V_0') \times \mathrm{GL}_r)\right)\overline{N}_{P_r} \tag{22}$$

is the open Bruhat cell of $\mathrm{GSpin}(V)$. In particular, it is an open dense subset of $\mathrm{GSpin}(V)$, and its image in $H\backslash \mathrm{GSpin}(V)$ has full measure in $H\backslash \mathrm{GSpin}(V)$. Therefore, the integral defining $\mathcal{Z}_\mu(f,v)$ may be computed over $H\backslash P_r\overline{N}_{P_r}$.

Recall that

$$H = N_{P_r} \rtimes (\mathrm{GSpin}(V_0) \times Z_r) \subset P_r = N_{P_r} \rtimes (\mathrm{GSpin}(V_0') \times \mathrm{GL}_r).$$

It follows that multiplication induces an identification

$$H\backslash P_r\overline{N}_{P_r} \cong (Z_r\backslash \mathrm{GL}_r) \times (\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')) \times \overline{N}_{P_r},$$

given by $(Z_r m, \mathrm{GSpin}(V_0)g_0', \overline{n}) \mapsto H\ell_r(m)g_0'\overline{n}$. Thus, we have

$$\mathcal{Z}_\mu(f,v) = \int_{(Z_r\backslash \mathrm{GL}_r)\times(\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0'))\times\overline{N}_{P_r}} \mu(\pi(mg_0'\overline{n})v, \Lambda(f(mg_0'\overline{n})))\delta_{P_r}^{-1}(m)dm dg_0' d\overline{n}, \tag{23}$$

where δ_{P_r} is the modular character of P_r defined by $p \mapsto |\det(\mathrm{Ad}_p)|$. We now briefly explain the Jacobian factor in (23). Since $\overline{N}_{P_r}$ is unipotent, the $\overline{N}_{P_r}$ part does not contribute to the Jacobian. Moreover, δ_{P_r} is trivial on the $\mathrm{GSpin}(V_0')$ part, since $\mathrm{GSpin}(V_0')$ acts on $\mathrm{Lie}(N_{P_r})$ through the projection to the orthogonal action which has determinant 1. Therefore, the factor $\delta_{P_r}^{-1}(m)$ in the right-hand side of (23) accounts for the full Jacobian. *Here and in what follows, we identify* GL_r *with its image under* ℓ_r *when no confusion can arise.*

We have the following non-vanishing lemma on $\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')$.

Lemma 4.5. *There exist a vector* $v \in \pi$ *and a smooth function* $f_{\pi_0} : \mathrm{GSpin}(V_0') \to \pi_0$*, compactly supported modulo* $\mathrm{GSpin}(V_0)$*, such that*

$$f_{\pi_0}(gg_0') = \pi_0(g)f_{\pi_0}(g_0'), \quad \textit{for all } g \in \mathrm{GSpin}(V_0),\ g_0' \in \mathrm{GSpin}(V_0'),$$

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and

$$\int_{\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')} \mu(\pi(g_0')v, f_{\pi_0}(g_0'))dg_0' \neq 0. \tag{24}$$

Proof. Since $\mu \neq 0$, we can find $v \in \pi$ and $v_0 \in \pi_0$ such that $\mu(v, v_0) = 1$. Consider the function $\phi(g_0') := \mu(\pi(g_0')v, v_0)$, $g_0' \in \mathrm{GSpin}(V_0')$. This is continuous, and $\phi(1) = 1$. Hence, there exists a compact neighborhood K' of the identity in $\mathrm{GSpin}(V_0')$ such that $\mathrm{Re}(\phi(h)) > 0$ for all $h \in K'$. Let $\beta : \mathrm{GSpin}(V_0') \to \mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')$ be the quotient map. Shrinking K' if necessary, we may assume that $\beta|_{K'}$ is injective, and that there exists an open neighborhood Ω of the base point $\beta(1)$ in $\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')$ together with a smooth section $\widetilde{\beta} : \Omega \to \mathrm{GSpin}(V_0')$ such that $\widetilde{\beta}(\Omega) \subset K'$.

Let $\phi_1 \in C_c^\infty(\Omega)$ be a non-negative nonzero smooth function with compact support contained in Ω. Define

$$f_{\pi_0}(g_0') = \begin{cases} \phi_1(x)\,\pi_0(g)v_0, & \text{if } g_0' = g\widetilde{\beta}(x) \text{ with } g \in \mathrm{GSpin}(V_0),\ x \in \Omega, \\ 0, & \text{if } g_0' \notin \mathrm{GSpin}(V_0)\widetilde{\beta}(\Omega). \end{cases}$$

Since $\widetilde{\beta}$ is a section of the quotient map, the decomposition $g_0' = g\widetilde{\beta}(x)$ is unique whenever $g_0' \in \mathrm{GSpin}(V_0)\widetilde{\beta}(\Omega)$, so f_{π_0} is well-defined. By construction, we have

$$f_{\pi_0}(gg_0') = \pi_0(g)f_{\pi_0}(g_0'), \qquad \text{for all } g \in \mathrm{GSpin}(V_0),\ g_0' \in \mathrm{GSpin}(V_0').$$

Moreover, f_{π_0} is smooth. Indeed, on the open subset $\mathrm{GSpin}(V_0)\widetilde{\beta}(\Omega) \subset \mathrm{GSpin}(V_0')$, the map

$$\mathrm{GSpin}(V_0) \times \Omega \longrightarrow \mathrm{GSpin}(V_0)\widetilde{\beta}(\Omega), \qquad (g, x) \longmapsto g\widetilde{\beta}(x)$$

is a diffeomorphism in the archimedean case and a homeomorphism onto an open subset in the non-archimedean case. Since ϕ_1 is smooth and $g \mapsto \pi_0(g)v_0$ is smooth, the function f_{π_0} is smooth on the open subset $\mathrm{GSpin}(V_0)\widetilde{\beta}(\Omega)$. Outside $\mathrm{GSpin}(V_0)\widetilde{\beta}(\Omega)$, the function f_{π_0} is identically zero. Since $\mathrm{supp}(\phi_1) \subset \Omega$ is compact, the support of f_{π_0} is contained in $\mathrm{GSpin}(V_0)\widetilde{\beta}(\mathrm{supp}(\phi_1))$, which is compact modulo $\mathrm{GSpin}(V_0)$, and f_{π_0} vanishes near the boundary of $\mathrm{GSpin}(V_0)\widetilde{\beta}(\Omega)$. Hence, the extension by zero is smooth on all of $\mathrm{GSpin}(V_0')$.

Finally, we have

$$\int_{\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')} \mu(\pi(g_0')v, f_{\pi_0}(g_0'))dg_0' = \int_\Omega \phi_1(x)\,\mu(\pi(\widetilde{\beta}(x))v, v_0)dx.$$

By taking the real parts, we obtain

$$\mathrm{Re}\left(\int_{\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')} \mu(\pi(g_0')v, f_{\pi_0}(g_0'))dg_0'\right) = \int_\Omega \phi_1(x)\mathrm{Re}(\phi(\widetilde{\beta}(x)))dx > 0.$$

Thus, we have (24). This finishes the proof of the lemma. □

Take $v \in \pi$ and f_{π_0} as in Lemma 4.5. For $m \in \mathrm{GL}_r$ and $\overline{n} \in \overline{N}_{P_r}$, put

$$\Phi(m, \overline{n}) := \int_{\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')} \mu(\pi(mg_0'\overline{n})v, f_{\pi_0}(g_0'))dg_0',$$

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which is a smooth function on $\mathrm{GL}_r \times \overline{N}_{P_r}$. By Lemma 4.5, $\Phi(1,1) \neq 0$, and hence $\Phi \neq 0$. Note that for any $z \in Z_r$, $m \in \mathrm{GL}_r$, and $\overline{n} \in \overline{N}_{P_r}$, we have

$$\begin{aligned}\Phi(zm,\overline{n}) &= \int_{\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')} \mu(\pi(zmg_0'\overline{n})v, f_{\pi_0}(g_0'))dg_0' \\ &= \int_{\mathrm{GSpin}(V_0)\backslash \mathrm{GSpin}(V_0')} \mu(\pi(zmg_0'\overline{n})v, \pi_0(z)f_{\pi_0}(g_0'))dg_0' \\ &= \chi(z)\Phi(m,\overline{n}),\end{aligned}$$

where the second equality follows from the fact that the representation π_0 of H has trivial restriction to Z_r, and the third equality follows from the fact that $\chi|_{Z_r} = \xi|_{Z_r}$ (see (14) and Remark 3.4).

Let W_r be a smooth function on GL_r with compact support modulo Z_r such that

$$W_r(zm) = \chi^{-1}(z)W_r(m), \quad \text{for all } z \in Z_r, m \in \mathrm{GL}_r .$$

Then for each $\overline{n} \in \overline{N}_{P_r}$, the function on GL_r given by $m \mapsto \Phi(m,\overline{n})W_r(m)\delta_{P_r}^{-1}(m)$ is left-invariant under Z_r. We remind the reader that $\mathrm{GL}_r = \mathrm{GL}_F(X_r)$ is viewed as a subgroup of $\mathrm{GL}_{r+1} = \mathrm{GL}_F(X_{r+1}')$ by acting trivially on e_{r+1}, and that Z_r (resp. Z_{r+1}) is the unipotent radical of the Borel subgroup of GL_r (resp. GL_{r+1}) stabilizing the flag (6) (resp. the flag (12)), as in Section 3.2.

Lemma 4.6. *For every W_r as above, there exists a vector $v_\sigma \in \sigma$ such that*

$$W_r(m) = \lambda(\sigma(m)v_\sigma), \quad m \in \mathrm{GL}_r .$$

Proof. This is due to Bernstein and Zelevinsky [3] when F is non-archimedean, and Jacquet and Shalika [10] when F is archimedean (see also [6, Lecture 4]). Note that for any $z \in Z_r$ and $m \in \mathrm{GL}_r$, we have $\lambda(\sigma(zm)v_\sigma) = \chi^{-1}(z)\lambda(\sigma(m)v_\sigma)$ by (17). □

Let ϕ_2 be a smooth and compactly supported function on $\overline{N}_{P_r}$. Since $\Phi \neq 0$, we can pick W_r and ϕ_2 such that

$$\int_{(Z_r\backslash \mathrm{GL}_r)\times \overline{N}_{P_r}} \Phi(m,\overline{n})W_r(m)\phi_2(\overline{n})\delta_{P_r}^{-1}(m)dmd\overline{n} \neq 0. \tag{25}$$

We now define a section $f_\rho \in \pi_s'$. Note that

$$P_{r+1}' \cap \mathrm{GSpin}(V) = N_{P_r} \rtimes (\mathrm{GL}_r \times \mathrm{GSpin}(V_0)),$$

and the multiplication map

$$\begin{aligned}\iota : (N_{P_{r+1}'} \rtimes (\mathrm{GL}_{r+1} \times \mathrm{GSpin}(V_0')) \ltimes \overline{N}_{P_r}) &\to \mathrm{GSpin}(V') \\ (u, m, g_0', \overline{n}) &\mapsto umg_0'\overline{n}\end{aligned}$$

is an open embedding. Put

$$f_\rho(x) := \begin{cases} \phi_2(\overline{n})f_{\pi_0}(g_0') \otimes (\sigma(m)v_\sigma) & \text{if } x = \iota(u,m,g_0',\overline{n}), \\ 0 & \text{if } x \text{ is not in the image of } \iota, \end{cases}$$

where $v_\sigma \in \sigma$ is as in Lemma 4.6. By construction, f_ρ is smooth and compactly supported modulo P_{r+1}'. Moreover, it satisfies the equivariance property (15) as we explain below. Let $x = \iota(u,m,g_0',\overline{n})$ be in the image of ι. For any $u_0 \in N_{P_{r+1}'}$, we have

$$f_\rho(u_0x) = f_\rho(\iota(u_0u,m,g_0',\overline{n})) = \phi_2(\overline{n})f_{\pi_0}(g_0') \otimes (\sigma(m)v_\sigma) = f_\rho(x).$$

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For any $g_0 \in \mathrm{GSpin}(V_0)$, we have

$$\begin{aligned} f_\rho(g_0 x) &= f_\rho(g_0 u m g_0' \overline{n}) \\ &= f_\rho((g_0 u g_0^{-1}) g_0 m g_0' \overline{n}) \\ &= f_\rho((g_0 u g_0^{-1}) m g_0 g_0' \overline{n}) \quad \text{(by Lemma 2.7)} \\ &= f_\rho(\iota(g_0 u g_0^{-1}, m, g_0 g_0', \overline{n})) \\ &= \rho(g_0) f_\rho(x). \end{aligned}$$

For any $m_0 \in \mathrm{GL}_{r+1}$, we have

$$f_\rho(m_0 x) = f_\rho(m_0 u m g_0' \overline{n}) = f_\rho(\iota(m_0 u m_0^{-1}, m_0 m, g_0', \overline{n})) = \rho(m_0) f_\rho(x).$$

On the other hand, if x is not in the image of ι, then for any $u_0 \in N_{P'_{r+1}}$, $g_0 \in \mathrm{GSpin}(V_0)$, and $m_0 \in \mathrm{GL}_{r+1}$, we have that $u_0 g_0 m_0 x$ is not in the image of ι and hence

$$f_\rho(u_0 g_0 m_0 x) = f_\rho(x) = 0.$$

Thus, $f_\rho \in \pi_s'$ (recall that s is assumed to be zero in this section). For any $m \in \mathrm{GL}_r$, $g_0' \in \mathrm{GSpin}(V_0')$, $\overline{n} \in \overline{N}_{P_r}$, we have

$$\Lambda(f_\rho(m g_0' \overline{n})) = \Lambda(f_\rho(\iota(1, m, g_0', \overline{n}))) = \Lambda(\phi_2(\overline{n}) f_{\pi_0}(g_0') \otimes (\sigma(m) v_\sigma)) = \phi_2(\overline{n}) \lambda(\sigma(m) v_\sigma) f_{\pi_0}(g_0').$$

We are ready to finish the proof of Proposition 4.3. Starting from (23), we compute that

$$\begin{aligned} \mathcal{Z}_\mu(f_\rho, v) &= \int_{(Z_r \backslash \mathrm{GL}_r) \times (\mathrm{GSpin}(V_0) \backslash \mathrm{GSpin}(V_0')) \times \overline{N}_{P_r}} \mu(\pi(m g_0' \overline{n}) v, \Lambda(f_\rho(m g_0' \overline{n}))) \delta_{P_r}^{-1}(m) dm dg_0' d\overline{n} \\ &= \int_{(Z_r \backslash \mathrm{GL}_r) \times (\mathrm{GSpin}(V_0) \backslash \mathrm{GSpin}(V_0')) \times \overline{N}_{P_r}} \mu(\pi(m g_0' \overline{n}) v, \phi_2(\overline{n}) \lambda(\sigma(m) v_\sigma) f_{\pi_0}(g_0')) \delta_{P_r}^{-1}(m) dm dg_0' d\overline{n} \\ &= \int_{(Z_r \backslash \mathrm{GL}_r) \times \overline{N}_{P_r}} \Phi(m, \overline{n}) \phi_2(\overline{n}) W_r(m) \delta_{P_r}^{-1}(m) dm d\overline{n}, \end{aligned}$$

which converges to a nonzero number by (25). This concludes the proof of Proposition 4.3.

4.3. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We are given π, π_0 and a generic character ξ of H as in (14) (and hence a unitary generic character χ of Z_{r+1}). We take σ to be an irreducible χ^{-1}-generic representation of GL_{r+1}. For each $\mu \in \mathrm{Hom}_H(\pi \otimes \pi_0, \xi)$, we may define the integral $\mathcal{Z}_\mu$ as in (18).

Let $\mathcal{F}$ be a finite-dimensional subspace of $\mathrm{Hom}_H(\pi \otimes \pi_0, \xi)$. By Proposition 4.4, there exists a number $c_{\mathcal{F}} \in \mathbb{R}$ such that for all $\mu \in \mathcal{F}$ and all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c_{\mathcal{F}}$, the integral $\mathcal{Z}_\mu(f, v)$ converges absolutely for all $f \in \pi_s'$ and all $v \in \pi$, and defines a linear functional on $\pi_s' \otimes \pi$.

By Lemma 3.5, we may choose one s with $\mathrm{Re}(s) > c_{\mathcal{F}}$ such that π_s' is irreducible. By (19) and Proposition 4.3, we obtain a linear embedding

$$\mathcal{F} \hookrightarrow \mathrm{Hom}_{\mathrm{GSpin}(V)}(\pi_s' \otimes \pi, \mathbb{C}), \quad \mu \mapsto \mathcal{Z}_\mu.$$

The injectivity follows from Proposition 4.3. The $\mathrm{GSpin}(V)$-equivariance of $\mathcal{Z}_\mu$ follows from (19). The linearity of the map $\mu \mapsto \mathcal{Z}_\mu$ follows from the definition of the integral $\mathcal{Z}_\mu$ and Proposition 4.4. The space $\mathrm{Hom}_{\mathrm{GSpin}(V)}(\pi_s' \otimes \pi, \mathbb{C})$ is at most one-dimensional (see [8] when F is non-archimedean and [7]

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when F is archimedean). Therefore, $\mathcal{F}$ is at most one-dimensional. Since $\mathcal{F}$ was arbitrary, we conclude that $\mathrm{Hom}_H(\pi \otimes \pi_0, \xi)$ is at most one-dimensional. □

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References

[1] A. Aizenbud, D. Gourevitch, S. Rallis and G. Schiffmann, *Multiplicity one theorems*, Ann. Math. (2) 172 (2010), no. 2, 1407–1434.
[2] M. Asgari, J. W. Cogdell and F. Shahidi. *Rankin-Selberg L-functions for $GSpin \times GL$ groups*, preprint, 2024. https://arxiv.org/abs/2409.17323
[3] J. Bernstein and A. Zelevinsky, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk 31 (1976), no. 3(189), 5–70.
[4] A. Borel, *Linear algebraic groups*, 2nd ed., volume 126 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1991.
[5] W. Casselman, H. Hecht and D. Miličić, "Bruhat filtrations and Whittaker vectors for real groups" in Robert S. Doran and V. S. Varadarajan (eds.), *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998)*, volume 68 of Proceedings of Symposia in Pure Mathematics, American Mathematical Society, Providence, RI, 2000, pp. 151–190.
[6] J. W. Cogdell, *Lectures on L-functions, converse theorems, and functoriality for GL_n. in lectures on automorphic L-functions*, volume 20 of Fields Institute Monographs, American Mathematical Society, Providence, RI, 2004, pp. 1–96.
[7] M. Emory, Y. Kim and A. Maiti, *Multiplicity one theorem for general spin groups: The Archimedean case*, Can. Math. Bull. (2025). Published online. DOI: https://doi.org/10.4153/S0008439525101483
[8] M. Emory and S. Takeda, *Contragredients and a multiplicity one theorem for general spin groups*, Math. Z. 303 (2023), no. 3, Article no. 70, 54 pp.
[9] W. T. Gan, B. H. Gross and D. Prasad, *Symplectic local root numbers, central critical Lvalues, and restriction problems in the representation theory of classical groups*, Astérisque 346 (2012), 1–109. Sur les conjectures de Gross et Prasad. I.
[10] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations I*, Amer. J. Math. 103 (1981), no. 3, 499–558.
[11] D. Jiang, B. Sun and C.-B. Zhu, *Uniqueness of Bessel models: The Archimedean case*, Geom. Funct. Anal. 20 (2010), no. 3, 690–709.
[12] D. Jiang and L. Zhang, *A product of tensor product L-functions of quasi-split classical groups of Hermitian type*, Geom. Funct. Anal. 24 (2014), no. 2, 552–609.
[13] E. Kaplan, J. F. Lau and B. Liu, *Local descent to quasi-split even general spin groups*, Math. Z. 303 (2023), no. 3, Article no. 69, 29 pp.
[14] E. Meinrenken, *Clifford algebras and Lie theory*, volume 58 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], Springer, Heidelberg, 2013.
[15] A. Pollack, *Unramified Godement-Jacquet theory for the spin similitude group*, J. Ramanujan Math. Soc. 33 (2018), no. 3, 249–282.
[16] D. Renard, *Représentations des groupes réductifs p-adiques*, volume 17 of Cours Spécialisés [Specialized Courses], Société Mathématique de France, Paris, 2010.
[17] J. A. Shalika, *The multiplicity one theorem forGL_n*, Ann. of Math. (2) 100 (1974), 171–193.
[18] G. Shimura, *Arithmetic and analytic theories of quadratic forms and Clifford groups*, volume 109 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2004.
[19] B. Speh and D. A. Vogan, Jr., *Reducibility of generalized principal series representations*, Acta Math. 145 (1980), nos. 3–4, 227–299.
[20] B. Sun and C.-B. Zhu, *Multiplicity one theorems: The Archimedean case*, Ann. of Math. (2) 175 (2012), no. 1, 23–44.
[21] J.-L. Waldspurger, *Une variante d'un résultat de Aizenbud, Gourevitch, Rallis et Schiffmann*, Astérisque 346 (2012), 313–318. Sur les conjectures de Gross et Prasad. I.
[22] P. Yan, *Rankin-Selberg integrals for GSpin groups with application to the global Gan-Gross-Prasad conjecture*, preprint, 2025. https://arxiv.org/abs/2508.09066
[23] P. Yan, *Twisted automorphic descent to odd GSpin groups and applications*, preprint, 2026. https://arxiv.org/abs/2607.25127
[24] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of GL(n)*, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 2, 165–210.

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