Bull. Aust. Math. Soc. **113** (2026), 529–542
doi:10.1017/S0004972726101026

COMPLEX DIFFERENTIAL EQUATIONS AND SYSTEMS FROM TRIGONOMETRIC FORMULAS

KAI LIU✉, YINHAO GUO and LINKUI GAO

(Received 27 August 2025; accepted 31 December 2025; first published online 20 February 2026)

Abstract

From classical trigonometric formulas, complex differential equations of various types have been formulated and widely studied. We investigate perturbed nonlinear complex differential equations and explore their corresponding complex differential systems. Some open questions on nonlinear complex differential equations and systems are proposed.

2020 *Mathematics subject classification*: primary 30D35; secondary 34A05.

Keywords and phrases: complex differential equations, complex differential systems, entire functions, trigonometric identities.

1. Complex differential equations from $2\sin z\cos z = \sin 2z$

The existence and growth of transcendental meromorphic solutions of complex differential equations have been widely studied (see Laine's monograph [10]). Interesting nonlinear complex differential equations can be derived from trigonometric identities. Observe the well-known trigonometric identity

$$2\sin z\cos z = \sin 2z \tag{1.1}$$

and the formula $(\sin z)' = \cos z$. Zhang and Yi [27, Corollary 1.7] studied the complex differential equation corresponding to (1.1) and the existence of meromorphic solutions, obtaining the following result.

THEOREM 1.1. *The nonlinear complex differential equation*

$$2f(z)f'(z) = \sin 2z \tag{1.2}$$

has only four meromorphic solutions, $f(z) = \pm\sin z$ *and* $f(z) = \pm i\cos z$.

REMARK 1.2. (1) The trigonometric identity (1.1) is also connected to functional-differential equations; Beardon [1] considered the pantograph-type equation

This work was partially supported by the NSFC (No. 12061042) and the NSF of Jiangxi (No. 20242BAB25003).

© The Author(s), 2026. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

$$f(2z) = 2f'(z)f(z)$$

and obtained all the formal power series solutions. Ludwig and Tomaschek [20] perturbed $2z$ and the constant 2 in (1.1) and considered the equation

$$f(\varphi(z)) = a(z)f'(z)f(z),$$

where $\varphi(z) = kz + \varphi_2 z^2 + \cdots$ and $a(z) = a_0 + a_1 z + \cdots$, and either $|k| > 1$ and k is a Siegel number, or $|k| < 1$.

(2) Theorem 1.1 is a corollary of [27, Theorem 1.6] dealing with a generalised complex differential equation of the form

$$[f(z)^2]^{(k)} = \sin 2z + p(z),$$

where $p(z)$ is a polynomial and k is a positive integer. The main proof idea relies on the integral form, that is,

$$f(z)^2 = \frac{1}{2^k}\sin\left(2z - \frac{k}{2}\pi\right) + P(z),$$

where $P^{(k)}(z) = p(z)$.

(3) The periodicity of $f(z)$ when $f(z)f'(z)$ is a periodic function was confirmed by Wang and Hu [23]. Theorems 1.1 and 4.1 below have inspired recent research on the periodicity of complex differential polynomials (see [18, 26]).

We will provide a new method to deal with the meromorphic solutions of (1.2) by differentiating and using the following basic lemmas of Nevanlinna theory. For the basic notations and fundamental results of Nevanlinna theory, see [8, 10].

LEMMA 1.3 (Clunie's lemma; [10, Lemma 2.4.2]). *Let f be a transcendental meromorphic solution of*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\mu(z), \mu \in I\}$, such that $m(r, a_\mu(z)) = S(r, f)$ for all $\mu \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives does not exceed n, then

$$m(r, P(z, f)) = S(r, f).$$

LEMMA 1.4 (Valiron–Mohon'ko lemma; [10, Theorem 2.2.5]). *Let f be a meromorphic function and*

$$R(z, f) = \frac{a_0 + a_1 f + \cdots + a_p f^p}{b_0 + b_1 f + \cdots + b_q f^q}$$

be irreducible and rational in $f(z)$ with meromorphic coefficients a_j, b_k of small growth $S(r, f)$. Then $T(r, R(z, f)) = \max(p, q)T(r, f) + S(r, f)$.

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

PROOF OF THEOREM 1.1. By taking the first derivative of (1.2),

$$f'^2 + ff'' = \cos 2z$$

and, by taking the derivative of this equation,

$$3f'f'' + ff''' = -2\sin 2z = -4ff'.$$

Thus,

$$3\frac{f''}{f} + \frac{f'''}{f'} = -4. \tag{1.3}$$

Let $f''/f = \alpha$. We will affirm that α is a constant. Otherwise, if α is not a constant, from (1.3),

$$3\alpha + \frac{\alpha' f + \alpha f'}{f'} = -4,$$

so we conclude that

$$\frac{f'}{f} = -\frac{1}{4}\frac{\alpha'}{\alpha + 1}.$$

Taking the derivative of this equation,

$$-\frac{1}{4}\frac{\alpha''(\alpha+1) - \alpha'^2}{(\alpha+1)^2} = \alpha - \frac{\alpha'^2}{16(\alpha+1)^2},$$

that is,

$$4\alpha''(\alpha+1) + 16\alpha(\alpha+1)^2 = 5\alpha'^2, \tag{1.4}$$

which we rewrite as

$$16\alpha^3 = 5\alpha'^2 - 4\alpha''(\alpha+1) - 32\alpha^2 - 16\alpha. \tag{1.5}$$

If α is a nonconstant entire function, then $m(r,\alpha) = S(r,\alpha)$ by Lemma 1.3, which is impossible.

If α is a properly meromorphic function and z_0 is a pole of α, then z_0 must be the simple pole, since z_0 is a simple zero of $f(z)$ by (1.2). Computing the multiplicity of the pole at z_0 from (1.5) gives a contradiction.

Hence, α is a constant and it follows from (1.4) that $\alpha = -1$. Therefore, $f(z) = c_1 e^{iz} + c_2 e^{-iz}$. Substituting back into (1.2) gives

$$i(c_1^2 e^{2iz} - c_2^2 e^{-2iz}) = \frac{e^{2iz} - e^{-2iz}}{4i},$$

that is,

$$\left(ic_1^2 - \frac{1}{4i}\right)e^{4iz} + \left(\frac{1}{4i} - ic_2^2\right) = 0.$$

To avoid a contradiction with Lemma 1.4, we obtain $c_1^2 = -\frac{1}{4}$ and $c_2^2 = -\frac{1}{4}$. So $f(z) = \pm\sin z$ or $f(z) = \pm i\cos z$. Theorem 1.1 is thus proved. □

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

Next, we consider a polynomial perturbation of (1.2). Mao and Liu [19, Theorem 1.3] also considered the general perturbed equation $f(z)f'(z) = p(z)\sin(\alpha(z))$, where $p(z)$ is a polynomial and $\alpha(z)$ is an entire function, using a different method.

THEOREM 1.5. *If the complex differential equation*

$$f(z)f'(z) = p(z)\sin 2z$$

admits a transcendental meromorphic solution $f(z)$, then $p(z)$ must be a nonzero constant $\frac{1}{2}\lambda^2$ and $f(z) = \pm\lambda\sin z$ or $f(z) = \pm\lambda i\cos z$.

Before giving the proof of Theorem 1.5, we need the following lemma.

LEMMA 1.6 ([19, Lemma 2.2]). *Let k be a positive integer, λ a nonzero constant and let Q_j $(j = 1, 2, 3)$ be small functions of e^{z^k} such that $Q_1Q_2 \not\equiv 0$. If the equation*

$$f(z)^2 = Q_1(z)e^{\lambda z^k} + Q_2(z)e^{-\lambda z^k} + Q_3(z)$$

admits a meromorphic solution $f(z)$, then $4Q_1(z)Q_2(z) = Q_3(z)$. Furthermore, $f(z) = \beta_1(z)e^{\lambda z^k/2} + \beta_2(z)e^{-\lambda z^k/2}$, where $\beta_1(z)$ and $\beta_2(z)$ are small functions of f satisfying $\beta_j^2 = Q_j$ and $2\beta_1\beta_2 = Q_3$.

PROOF OF THEOREM 1.5. Assume that $p(z)$ is a nonconstant polynomial of degree m. From the hypothesis of the theorem, $f(z)f'(z) = \frac{1}{2}(f(z)^2)' = p(z)\sin 2z$. Integrating,

$$\tfrac{1}{2}f(z)^2 = Q(z)\cos 2z + R(z)\sin 2z + c, \tag{1.6}$$

where

$$Q(z) = -\frac{1}{2}p(z) + \frac{1}{2^3}p''(z) - \frac{1}{2^5}p^{(4)}(z) + \cdots + (-1)^{[m+1/2]}\frac{1}{2^{2m+1}}p^{(2m)}(z)$$

and

$$R(z) = \frac{1}{2^2}p'(z) - \frac{1}{2^4}p'''(z) + \frac{1}{2^6}p^{(5)}(z) + \cdots + (-1)^{[m/2]}\frac{1}{2^{2m}}p^{(2m-1)}(z).$$

Now,

$$Q(z)\cos 2z + R(z)\sin 2z + c = \left(\frac{Q(z)}{2} + \frac{R(z)}{2i}\right)e^{2iz} + \left(\frac{Q(z)}{2} - \frac{R(z)}{2i}\right)e^{-2iz} + c.$$

Since $p(z)$ is a nonconstant polynomial, $Q(z)/2 + R(z)/2i = \frac{1}{2}(Q(z) - iR(z))$ and $Q(z)/2 - R(z)/2i = \frac{1}{2}(Q(z) + iR(z))$ are nonconstant polynomials. If $c = 0$, then

$$Q(z)\cos 2z + R(z)\sin 2z = \frac{\frac{1}{2}(Q(z) - iR(z))e^{4iz} + \frac{1}{2}(Q(z) + iR(z))}{e^{2iz}} = \frac{G(z)}{e^{2iz}} \quad \text{(say)}.$$

From the second main theorem of Nevanlinna theory for three small functions [8, Theorem 2.5] and (1.6),

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

$$\begin{aligned} T(r, e^{4iz}) &\le \overline{N}(r, e^{4iz}) + \overline{N}\Big(r, \frac{1}{e^{4iz}}\Big) + \overline{N}\Big(r, \frac{1}{G(z)}\Big) + S(r, e^{4iz}) \\ &\le \overline{N}\Big(r, \frac{1}{G(z)}\Big) + S(r, e^{4iz}) \le \frac{1}{2}N\Big(r, \frac{1}{G(z)}\Big) + S(r, e^{4iz}) \\ &\le \frac{1}{2}T(r, e^{4iz}) + S(r, e^{4iz}), \end{aligned}$$

which is impossible.

If $c \neq 0$, by Lemma 1.6,

$$\Big(\frac{Q(z)}{2} + \frac{R(z)}{2i}\Big)\Big(\frac{Q(z)}{2} - \frac{R(z)}{2i}\Big) = \frac{c}{8},$$

which is impossible. In both cases, (1.6) leads to a contradiction. Hence, $p(z)$ must be a constant and $p(z) = \lambda^2/2$. The expressions for $f(z)$ are given by Theorem 1.1. □

We next consider a generalisation of the complex differential equation (1.2):

$$f(z)f^{(4k+1)}(z) = \tfrac{1}{2}\sin 2z, \tag{1.7}$$

where k is a positive integer. Obviously, (1.7) also has four meromorphic solutions $f(z) = \pm\sin z$ and $f(z) = \pm i\cos z$ and all meromorphic solutions of (1.7) are entire functions. However, the uniqueness of the meromorphic solutions to this equation remains an open question. So far, neither the integral method in Zhang and Yi [27] nor the differential method in the paper has yielded an answer. The following question is a natural one to ask.

QUESTION 1.7. Let $f(z)$ be a transcendental meromorphic solution of

$$f(z)f^{(k)}(z) = p(z)\sin 2z. \tag{1.8}$$

Does it follow that $k = 4k_1 + 1$, $p(z) = \frac{1}{2}\lambda^2$ is a nonzero constant and $f(z) = \pm\lambda\sin z$ or $f(z) = \pm\lambda i\cos z$, where k_1 is a nonnegative integer and λ is a nonzero constant?

REMARK 1.8. Question 1.7 has a connection with Hayman's conjecture [7]. Equation (1.8) can be written as

$$\begin{aligned} g(z)g^{(k)}(z) &= -p\Big(z - \frac{\pi}{4}\Big)\cos 2z = -p\Big(z - \frac{\pi}{4}\Big) + 2p\Big(z - \frac{\pi}{4}\Big)\sin^2 z \\ &= p\Big(z - \frac{\pi}{4}\Big) - p\Big(z - \frac{\pi}{4}\Big)\cos^2 z \end{aligned}$$

by letting $g(z) = f(z - \pi/4)$. The two complex differential polynomials

$$g(z)g^{(k)}(z) \pm p\Big(z - \frac{\pi}{4}\Big)$$

have infinitely many zeros which are double except for the finitely many zeros of $p(z - \pi/4)$. Hayman's conjecture states that if $f(z)$ is a transcendental meromorphic function, then $f(z)^n f'(z) - a$ has infinitely many zeros, where a is a nonzero constant and n is a positive integer. Hayman's conjecture has been confirmed, but we do

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

not know if $g(z)g^{(k)}(z) - p(z)$ has infinitely many zeros, where $p(z)$ is a nonzero polynomial. The case in which $k = 2$ and $p(z)$ is a nonzero constant can be seen in Langley [11].

2. Complex differential systems from $2\sin z\cos z = \sin 2z$

Complex differential systems have been studied recently by Gao [3, 5, 17]. Results on systems of complex delay-differential equations [16, Ch. 11], Malmquist- or Riccati-type differential systems [4, 21, 22] and Fermat-type differential-difference systems [3, 14] have also been derived from the corresponding differential equations. We will use the results in Section 1 to investigate the meromorphic solutions of the corresponding complex differential systems, where the coefficients are not necessarily small functions.

THEOREM 2.1. *Let $p_1(z)$ and $p_2(z)$ be nonzero polynomials and n be a positive integer, and consider the nonlinear differential system*

$$\begin{cases} f(z)^n g'(z) = p_1(z)\sin 2z, \\ g(z)^n f'(z) = p_2(z)\sin 2z. \end{cases} \tag{2.1}$$

(1) *If $n \geq 2$, then (2.1) has no meromorphic solutions.*
(2) *Assume that $n = 1$, $f(z) = \alpha(z)\sin z$ and $g(z) = \beta(z)\sin z$, where $\alpha(z)$ and $\beta(z)$ are small functions with respect to $\sin z$. Then $p_1(z) = p_2(z) = p$ where p is a nonzero constant. Thus, $f(z) = \gamma g(z)$, and $f(z) = \tau\sin z$ or $f(z) = i\tau\cos z$ where $2p = \tau^2/\gamma$.*

PROOF. (1) Assume that $n \geq 2$ and $f(z)$ and $g(z)$ are properly transcendental meromorphic functions with infinitely many poles. Assume that z_0 is a pole of $f(z)$ with multiplicity p which is not a zero of $p_1(z)$ and $p_2(z)$. Then z_0 is a zero of $g'(z)$ with multiplicity at least np and at most $np + 1$ by the first equation of (2.1), and z_0 is a zero of $g(z)$ with multiplicity at least $(p + 1)/n$ and at most $(p + 2)/n$ by the second equation of (2.1). Therefore, z_0 is a zero of $g(z)$ and $g'(z)$. This gives the four possibilities listed below, all of which are impossible because they imply that $p \leq 0$ when $n \geq 2$.

Case 1. If $(p + 2)/n - 1 = np$, then $p = (n - 2)/(1 - n^2)$.

Case 2. If $(p + 2)/n - 1 = np + 1$, then $p = (2n - 2)(1 - n^2)$.

Case 3. If $(p + 1)/n - 1 = np$, then $p = (n - 1)/(1 - n^2)$.

Case 4. If $(p + 1)/n - 1 = np + 1$, then $p = 2n - 1/(1 - n^2)$.

Thus, if $n \geq 2$, both $f(z)$ and $g(z)$ are transcendental entire functions or properly transcendental meromorphic functions with finitely many poles. From the first and the second equation of (2.1),

$$nT(r, f) \leq T(r, g') + T(r, \sin 2z) + O(\log r) \leq T(r, g) + T(r, \sin 2z) + O(\log r)$$

and

$$nT(r, g) \leq T(r, f') + T(r, \sin 2z) + O(\log r) \leq T(r, f) + T(r, \sin 2z) + O(\log r).$$

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

Hence,

$$(n-1)(T(r,f)+T(r,g)) \le 2T(r,\sin 2z)+O(\log r).$$

Thus, $f(z)$ and $g(z)$ must be transcendental entire functions or properly transcendental meromorphic functions having finitely many poles with order at most 1. Since $n \ge 2$, $f(z)$ and $g(z)$ have finitely many zeros. Therefore, we can assume that $f(z)=t_1(z)e^{s_1z}$ and $g(z)=t_2(z)e^{s_2z}$, where $t_1(z)$ and $t_2(z)$ are rational functions. Substituting $f(z)$ and $g(z)$ into (2.1),

$$t_1(z)^n(t_2'(z)+t_2(z)s_2)e^{ns_1z+s_2z}=p_1(z)\sin 2z,$$

and checking the numbers of zeros on the two sides of this equation gives a contradiction.

Thus, we have proved that $n=1$. Consider

$$\begin{cases} f(z)g'(z)=p_1(z)\sin 2z, \\ g(z)f'(z)=p_2(z)\sin 2z, \end{cases} \tag{2.2}$$

and let $f(z)=\alpha(z)\sin z$ and $g(z)=\beta(z)\sin z$.

Case 1. If $\alpha(z)$ and $\beta(z)$ are nonzero constants, the conclusion of the theorem is obviously true.

Case 2. If one of $\alpha(z)$ and $\beta(z)$ is a nonconstant small function with respect to $\sin z$, then (2.2) reduces to

$$\begin{cases} \alpha(z)(\beta'(z)\sin z+\beta(z)\cos z)=2p_1(z)\cos z, \\ \beta(z)(\alpha'(z)\sin z+\alpha(z)\cos z)=2p_2(z)\cos z. \end{cases}$$

Hence,

$$\sin z(\alpha(z)\beta'(z)-\beta(z)\alpha'(z))=2\cos z(p_1(z)-p_2(z)). \tag{2.3}$$

If $\alpha(z)\beta'(z)-\beta(z)\alpha'(z)\not\equiv 0$, since $\alpha(z)$ and $\beta(z)$ are small functions with respect to $\sin z$ and $\cos z$, (2.3) contradicts Lemma 1.4.

If $\alpha(z)\beta'(z)-\beta(z)\alpha'(z)\equiv 0$, then $p_1(z)-p_2(z)\equiv 0$. We have $f(z)=\gamma g(z)$, where γ is a nonzero constant. By Theorem 1.5, $p_1(z)$ and $p_2(z)$ must be constants and $f(z)=\tau\sin z$ or $f(z)=i\tau\cos z$, where $2p=\tau^2/\gamma$. □

REMARK 2.2. We have assumed that $f(z)=\alpha(z)\sin z$ and $g(z)=\beta(z)\sin z$ in Theorem 2.1, where $\alpha(z)$ and $\beta(z)$ are small functions with respect to $\sin z$. We believe that the condition on $\alpha(z)$ and $\beta(z)$ can be removed, but we have not yet been able to show this.

3. Complex differential equations from $\cos 2z=\cos^2 z-\sin^2 z$

Gundersen *et al.* [6] derived another nonlinear complex differential equation from the trigonometric identity $\cos 2z=\cos^2 z-\sin^2 z$.

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

THEOREM 3.1 [6]. *The complex differential equation*

$$f(z)^2 - f'(z)^2 = \cos 2z \tag{3.1}$$

has only four meromorphic solutions $f(z) = \pm\cos z$ *and* $f(z) = \pm i\sin z$.

Gao and Gao [2] extended (3.1) by proving the following result.

THEOREM 3.2 [2, Theorem 7]. *The complex differential equation*

$$f(z)^4 - f'(z)^4 = \cos 2z$$

has only eight meromorphic solutions

$$f_j(z) = e^{2j\pi i/4}\cos z, \quad \tilde{f}_k(z) = e^{(2k+1)\pi i/4}\sin z,$$

for $j = 0, 1, 2, 3$ *and* $k = 0, 1, 2, 3$.

THEOREM 3.3. *Let* $p(z) \not\equiv 0$ *be a polynomial. If the complex differential equation*

$$f(z)^4 - f'(z)^4 = p(z)\cos 2z \tag{3.2}$$

admits meromorphic solutions $f(z)$*, then* $p(z)$ *must be a nonzero constant* p*. Furthermore, all meromorphic solutions of (3.2) are*

$$f_j(z) = p^{1/4}e^{2j\pi i/4}\cos z, \quad \tilde{f}_k(z) = p^{1/4}e^{(2k+1)\pi i/4}\sin z,$$

for $j = 0, 1, 2, 3$ *and* $k = 0, 1, 2, 3$.

For the proof of Theorem 3.3, we need the following lemma.

LEMMA 3.4. *If* $p(z)$ *is a nonzero polynomial and* $f(z)$ *is a meromorphic solution of (3.2), then* $N(r, 1/f) \neq S(r, f)$ *and* $N(r, 1/f') \neq S(r, f)$.

PROOF. Obviously, the meromorphic solution $f(z)$ must be a transcendental entire function. Define $\phi(z) = 1 - (f'/f)^4$. If $N(r, 1/f) = S(r, f)$, then

$$N\left(r, 1 - \left(\frac{f'}{f}\right)^4\right) = 4\overline{N}\left(r, \frac{1}{f}\right) = S(r, f).$$

We also have

$$m\left(r, 1 - \left(\frac{f'}{f}\right)^4\right) = S(r, f)$$

by the lemma of the logarithmic derivative. Thus, $T(r, \phi(z)) = S(r, f)$. Rewrite (3.2) as

$$\phi(z)f(z)^4 + p(z) = 2p(z)\cos^2(z).$$

By the second main theorem of Nevanlinna theory for three small functions [8, Theorem 2.5],

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

$$\begin{aligned} T(r,f^4) &\le \overline{N}(r,f^4)+\overline{N}\Big(r,\frac{1}{f^4}\Big)+\overline{N}\Big(r,\frac{1}{f^4+p(z)/\phi}\Big)+S(r,f^4)\\ &\le \overline{N}\Big(r,\frac{1}{f}\Big)+\frac{1}{2}N\Big(r,\frac{1}{f^4+p(z)/\phi}\Big)+S(r,f^4)\\ &\le 3T(r,f)+S(r,f) \end{aligned}$$

which is impossible. The same reasoning can be used to show $N(r,1/f')\neq S(r,f)$. □

PROOF OF THEOREM 3.3. Taking the first derivative of (3.2),

$$4f^3f'-4(f')^3f''=p'\cos 2z-2p\sin 2z. \tag{3.3}$$

Computing $(p'-2ip)$ times (3.2) minus p times (3.3) gives

$$(p'-2ip)f^4-(p'-2ip)(f')^4-4pf^3f'+4p(f')^3f''=-2ip^2e^{2iz}. \tag{3.4}$$

Taking the derivative of (3.4) and eliminating e^{2iz},

$$\begin{aligned} &A_1f^4+A_2f^3f'+A_3f^3f''+A_4f^2(f')^2\\ &\quad -A_1(f')^4-A_2(f')^3f''-A_3(f')^3f'''-A_4(f')^2(f'')^2=0, \end{aligned} \tag{3.5}$$

where A_i $(i=1,2,3,4)$ are the polynomials

$$\begin{cases} A_1=(2p'+2ip)(p'-2ip)-p(p''-2ip'),\\ A_2=-8pp',\\ A_3=4p^2,\\ A_4=12p^2. \end{cases}$$

Dividing the two sides of (3.5) by f^3f' gives

$$m\Big(r,\frac{f}{f'}\Big)=S(r,f)$$

by the lemma of the logarithmic derivative. Let z_0 be a zero of f' but not a zero of $p(z)$. From (3.2), z_0 is not a zero of f because all zeros of $\cos 2z$ are simple. By (3.5), z_0 is a zero of A_1f+A_3f''. Define $\varphi(z):=(A_1f+A_3f'')/f'$. Then $T(r,\varphi(z))=S(r,f)$. From the definition of $\varphi(z)$,

$$A_3f''=\varphi f'-A_1f. \tag{3.6}$$

Differentiating (3.6),

$$A_3f'''=(\varphi-A_3')f''+(\varphi'-A_1)\varphi f'-A_1'f. \tag{3.7}$$

Combining (3.6) and (3.7),

$$A_3f'''=\Big(\frac{\varphi(\varphi-A_3')}{A_3}+(\varphi'-A_1)\varphi\Big)f'-\Big(\frac{A_1(\varphi-A_3')}{A_3}+A_1'\Big)f. \tag{3.8}$$

Substituting (3.6) and (3.8) into (3.5),

$$H_1f^3f'+H_2f^2(f')^2+H_3f(f')^3+H_4(f')^4=0, \tag{3.9}$$

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

where

$$\begin{cases} H_1 = A_2 + \varphi, \\ H_2 = A_4 - \dfrac{A_4(A_1)^2}{(A_3)^2}, \\ H_3 = \dfrac{A_1A_2}{A_3} + \dfrac{A_1}{A_3}(\varphi - A_3') + A_1' + \dfrac{2A_1A_4\varphi}{(A_3)^2}, \\ H_4 = -A_1 - \dfrac{A_2\varphi}{A_3} - \dfrac{\varphi(\varphi - A_3')}{A_3} - (\varphi' - A_1)\varphi - \dfrac{A_4(\varphi)^2}{(A_3)^2}. \end{cases}$$

By Lemma 3.4, noting that the zeros of $f(z)$ and $f'(z)$ are different except for the zeros of $p(z)$, we have $N(r, 1/f) = S(r, f)$ unless $H_i \not\equiv 0$. But $H_2 \equiv 0$ gives $A_3 = \pm A_1$, so that

$$(2p' + 2ip)(p' - 2ip) - p(p'' - 2ip') = \pm 4p^2,$$

that is,

$$4p^2 + 2p'^2 - pp'' = \pm 4p^2.$$

It follows that $p(z)$ is a constant and $A_3 = A_1$. We now rewrite (3.2) as

$$(F(z))^4 - (F'(z))^4 = \cos 2z,$$

where $F(z) = p^{-1/4} f(z)$. From Theorem 3.2, all solutions of (3.2) are

$$f_j(z) = p^{1/4} e^{2j\pi i/4} \cos z, \quad \tilde{f}_k(z) = p^{1/4} e^{(2k+1)\pi i/4} \sin z,$$

for $j = 0, 1, 2, 3$ and $k = 0, 1, 2, 3$. □

There are many other ways to perturb (3.1). For example, Gundersen *et al.* [6, Theorem 7] asked for all entire solutions of

$$(z + 1)f(z)^2 + (z - 1)f'(z)^2 = z + \cos 2z. \tag{3.10}$$

They stated that $f(z) = \pm \cos z$ and $f(z) = \pm i \sin z$ are the entire solutions of the above equation. However, $f(z) = \pm i \sin z$ are not solutions. The remaining question is to prove the uniqueness of $f(z) = \pm \cos z$ for (3.10). We make some observations which may contribute to the uniqueness problem. Let $f(z) = \cos z + F(z)$ so that

$$(z + 1)(\cos z + F(z))^2 + (z - 1)(F'(z) - \sin z)^2 = z + \cos 2z.$$

By elementary calculation,

$$(z + 1)(2 \cos z + F(z))F(z) = (z - 1)(2 \sin z - F'(z))F'(z).$$

Let $G(z) = 2 \cos z + F(z)$. Then

$$(z + 1)G(z)F(z) = (1 - z)G'(z)F'(z),$$

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

where $G(z)$ and $F(z)$ are entire functions. If $G(z) \equiv 0$, then $f(z) = -\cos z$; if $F(z) \equiv 0$, then $f(z) = \cos z$. If $G(z) \not\equiv 0$ and $F(z) \not\equiv 0$,

$$\frac{G'(z)}{G(z)}\frac{F'(z)}{F(z)} = \frac{z+1}{1-z}.$$

We conjecture that this equation has no entire solutions satisfying $G(z) = 2\cos z + F(z)$.

To close this section, we pose the following question for further study.

QUESTION 3.5. Let $p_1(z)$ and $p_2(z)$ be nonzero polynomials and n be a positive integer. How can we describe the transcendental meromorphic solutions of the system

$$\begin{cases} f(z)^n - g'(z)^n = p_1(z)\cos 2z, \\ g(z)^n - f'(z)^n = p_2(z)\cos 2z? \end{cases}$$

4. Discussion

Observe the trigonometric identity

$$4\sin^3 z - 3\sin z = -\sin 3z.$$

The corresponding nonlinear complex differential equation

$$4f(z)^3 + 3f''(z) = -\sin 3z = \frac{e^{-3iz} - e^{3iz}}{2i}, \tag{4.1}$$

was studied by Yang [24], Heittokangas *et al.* [9] and Yang and Li [25]. They pointed out that $f_1(z) = \sin z$, $f_2(z) = -\frac{1}{2}\sqrt{3}\cos z - \frac{1}{2}\sin z$ and $f_3(z) = \frac{1}{2}\sqrt{3}\cos z - \frac{1}{2}\sin z$ are the only entire solutions of (4.1). Equation (4.1) can also be viewed as a differential equation of Tumura–Clunie type,

$$f(z)^n + P(z, f(z)) = g(z),$$

where $P(z, f(z))$ is a differential polynomial in $f(z)$ and $g(z)$ is a meromorphic function (see the related results in Liao *et al.* [15], and Mao and Liu [19]).

From the trigonometric identity

$$\cos^3 z - 3\sin^2 z\cos z = \cos 3z,$$

Gao *et al.* [5] showed that the only entire solutions of the complex differential equations

$$f(z)^3 - 3f(z)f'(z)^2 = \cos 3z \tag{4.2}$$

and

$$f(z)^3 + 3f''(z)f'(z)^2 = \cos 3z \tag{4.3}$$

are $f(z) = c_1e^z + c_2e^{-z}$, where $c_1^3 = \frac{1}{8}$ and $c_2^3 = \frac{1}{8}$. Furthermore, they considered

$$p(z)f(z)^3 + q(z)f(z)f'(z)^2 = \cos(az + b)$$

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

and showed that if this equation has entire solutions, then $p(z)$ and $q(z)$ are constants. Very recently, Li *et al.* [12] found corresponding results when $\cos 3z$ is replaced by $p_1 e^{\lambda_1(z)} + p_2 e^{\lambda_2(z)}$ in (4.2) and (4.3).

We can consider the related complex differential system

$$\begin{cases} 4f(z)^3 - 3g'(z) = \cos 3z, \\ 4g(z)^3 + 3f'(z) = -\sin 3z. \end{cases} \tag{4.4}$$

Obviously, system (4.4) has the entire solution $f(z) = \cos z$ and $g(z) = \sin z$, but we do not know if this entire solution is unique.

Finally, consider the trigonometric identity

$$\sin z \sin'' z = -\sin^2 z.$$

Li *et al.* [13, Theorem 1] obtained the following result by Laguerre's theorem on the separation of zeros.

THEOREM 4.1. *If $f(z)$ is an entire solution of*

$$f(z)f''(z) = p(z)\sin^2 z,$$

where $p(z)$ is a nonzero polynomial with real coefficients and real zeros, then $p(z)$ must be a nonzero constant $p = -a^2$ and $f(z) = a \sin z$.

We pose the following question on a related nonlinear complex differential system for further investigation.

QUESTION 4.2. Let n, m, k_1, k_2, t_1, t_2 be positive integers, and $p_1(z)$ and $p_2(z)$ be nonzero small functions with respect to $\sin z$. Suppose that $f(z)$ and $g(z)$ solve the complex differential system

$$\begin{cases} f(z)^n g^{(k_1)}(z) = p_1(z)\sin^{t_1} z, \\ g(z)^m f^{(k_2)}(z) = p_2(z)\sin^{t_2} z. \end{cases} \tag{4.5}$$

Can we describe the properties of transcendental meromorphic solutions of (4.5)?

Acknowledgement

The authors are very grateful to the reviewer and Professor Feng Lü for useful suggestions and valuable comments on the present version of the paper.

References

[1] A. F. Beardon, 'Entire solutions of $f(kz) = kf(z)f'(z)$', *Comput. Methods Funct. Theory* **12** (2012), 273–278.

[2] L. K. Gao and J. Y. Gao, 'Trigonometric identities and entire solutions of non-linear binomial differential equations', *Mediterr. J. Math.* **20** (2023), Article no. 167.

[3] L. Y. Gao, 'Entire solutions of two types of systems of complex differential-difference equations', *Acta Math. Sinica (Chin. Ser.)* **59**(5) (2016), 677–684.

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

[4] Y. C. Gao, K. Liu and X. G. Qi, 'Crossed differential systems of equations and Clunie lemma', *Electron. J. Differential Equations* **2024** (2024), Article no. 52.
[5] Z. G. Gao, M. L. Liu and L. Y. Gao, 'On entire solutions of certain types of nonlinear differential equation', *ScienceAsia* **49** (2023), 685–692.
[6] G. G. Gundersen, W. R. Lü, T. W. Ng and C. C. Yang, 'Entire solutions of differential equations that are related to trigonometric identities', *J. Math. Anal. Appl.* **507**(1) (2022), Article no. 125788.
[7] W. K. Hayman, 'Picard values of meromorphic functions and their derivatives', *Ann. of Math. (2)* **70** (1959), 9–42.
[8] W. K. Hayman, *Meromorphic Functions* (Clarendon Press, Oxford, 1964).
[9] J. Heittokangas, R. Korhonen and I. Laine, 'On meromorphic solutions of certain nonlinear differential equations', *Bull. Aust. Math. Soc.* **66** (2002), 331–343.
[10] I. Laine, *Nevanlinna Theory and Complex Differential Equations* (Walter de Gruyter, Berlin and New York, 1993).
[11] J. K. Langley, 'The zeros of $ff'' - b$', *Results Math.* **44** (2003), 130–140.
[12] N. Li, L. L. Wu and H. Y. Xu, 'Entire solutions of non-linear binomial differential equations involving trigonometric identities', *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **119** (2025), Article no. 109.
[13] P. Li, W. R. Lü and C. C. Yang, 'Entire solutions of certain types of nonlinear differential equations', *Houston J. Math.* **45**(2) (2019), 431–437.
[14] Y. X. Li and K. Liu, 'Meromorphic solutions of nonlinear systems of Fermat type', *Bull. Malays. Math. Sci. Soc.* **46** (2023), Article no. 196.
[15] L. W. Liao, C. C. Yang and J. J. Zhang, 'On meromorphic solutions of certain type of non-linear differential equations', *Ann. Acad. Sci. Fenn. Math.* **38** (2013), 581–593.
[16] K. Liu, I. Laine and L. Z. Yang, *Complex Delay-Differential Equations* (De Gruyter, Berlin and Boston, 2021).
[17] M. L. Liu, L. Y. Gao and Q. Y. Wang, 'Some results of certain nonlinear delay differential equation', *Complex Var. Elliptic Equ.* **65**(9) (2020), 1501–1514.
[18] X. L. Liu, K. Liu, R. Korhonen and G. Filipuk, 'Quasiperiodicity of transcendental meromorphic functions', *Acta Math. Sci. Ser. B (Engl. Ed.)* **44**(1) (2024), 161–172.
[19] Z. Q. Mao and H. F. Liu, 'Entire solutions of some non-linear homogeneous differential equations', *Bull. Malays. Math. Sci. Soc.* **47** (2024), Article no. 183.
[20] L. Reich and J. Tomaschek, 'On a functional-differential equation of A. F. Beardon and functional-differential equations of Briot–Bouquet type', *Comput. Methods Funct. Theory* **13** (2013), 383–395.
[21] Z. H. Tu and X. Z. Xiao, 'On the meromorphic solutions of system of higher order algebraic differential equations', *Complex Variables Theory Appl.* **15** (1990), 197–209.
[22] F. N. Wang and K. Liu, 'Crossing Malmquist systems with certain types', *J. Contemp. Math. Anal.* **58** (2023), 459–469.
[23] Q. Wang and P. C. Hu, 'On zeros and periodicity of entire functions', *Acta Math. Sci. Ser. A (Chinese Ed.)* **38**(2) (2018), 209–214.
[24] C. C. Yang, 'On entire solutions of a certain type of nonlinear differential equations', *Bull. Aust. Math. Soc.* **64** (2001), 377–380.
[25] C. C. Yang and P. Li, 'On the transcendental solutions of a certain type of nonlinear differential equations', *Arch. Math.* **82** (2004), 442–448.
[26] M. A. Zemirni, I. Laine and Z. Latreuch, 'New findings on the periodicity of entire functions and their differential polynomials', *Mediterr. J. Math.* **20** (2023), Article no. 136.
[27] X. B. Zhang and H. X. Yi, 'Entire solutions of a certain type of functional-differential equations', *Appl. Math. J. Chinese. Univ.* **28** (2013), 138–146.

KAI LIU, Department of Mathematics,
Nanchang University, Nanchang 330031, China
e-mail: liukai@ncu.edu.cn

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press

YINHAO GUO, Department of Mathematics,
Nanchang University, Nanchang 330031, China
e-mail: yinhaoguo@email.ncu.edu.cn

LINKUI GAO, School of Mathematics Sciences,
Jiangxi Science and Technology Normal University,
Nanchang 330038, China
e-mail: gaolinkui@126.com

https://doi.org/10.1017/S0004972726101026 Published online by Cambridge University Press