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Mock theta functions and weakly holomorphic modular forms modulo 2 and 3

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Abstract

We prove that the coefficients of the mock theta functions

$$f(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$

and

$$\omega(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(1+q)^2(1+q^3)^2\cdots(1+q^{2n+1})^2}$$

possess no linear congruences modulo 3. We prove similar results for the moduli 2 and 3 for a wide class of weakly holomorphic modular forms and discuss applications. This extends work of Radu on the behavior of the ordinary partition function.

1. *Introduction*

The arithmetic properties of Fourier coefficients of modular forms are involved in many areas of number theory and they are the object of a vast amount of research. A prototypical modular form is the Dedekind eta function

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q := e^{2\pi iz},$$

whose inverse generates the partition function (the prototypical function of additive number theory) via the well-known relation

$$\frac{1}{\eta(z)} = q^{-1/24}\sum_{n=0}^{\infty} p(n)q^n. \tag{1·1}$$

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Most of what is known about the arithmetic properties of $p(n)$ stems from the interpretation of (1·1) as a weakly holomorphic modular form of weight $-1/2$. Although there are far too many results to mention individually, we mention Ono's paper [**25**] and the subsequent work [**3**] which show that Ramanujan's famous congruence

$$p(5n+4) \equiv 0 \pmod 5 \tag{1·2}$$

has an analogue for any modulus M coprime to 6 (see also the recent work of Folsom, Kent and Ono [**17**]). Treneer [**32**, **33**] has extended these results to cover any weakly holomorphic modular form. In the other direction, the first author and Boylan [**1**] showed that there are no congruences as simple as (1·2) for primes $\geqslant 13$.

The situation has been less clear for the primes which divide 24. Parkin and Shanks [**27**] conjectured that

$$\#\{n \leqslant x : p(n) \not\equiv 0 \pmod 2\} \sim \frac{x}{2},$$

and there is an analogous folklore conjecture modulo 3. The current results are far from this expectation; for example the best result for the number of odd values of $p(n)$ is due to Nicolas [**24**], who obtained the bound $\sqrt{x}(\log\log x)^K/\log x$ for any K. An old conjecture of Subbarao [**31**] states that there are no linear congruences for $p(n)$ modulo 2. In striking recent work, Radu [**30**] has proved Subbarao's conjecture, as well as its analog modulo 3, with clever and technical arguments. In this paper we will adapt the methods of Radu's paper to prove non-vanishing results for the coefficients of mock theta functions and weakly holomorphic modular forms of certain types.

The function

$$f(q) = \sum_{n=0}^{\infty} a(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$

is a prototypical mock theta function (see, for example, works of Zwegers [**35**], Bringmann and Ono [**11**], and Zagier [**34**]). Using a standard argument and [**32**], [**33**], one deduces that there are linear congruences $a(mn+t) \equiv 0 \pmod{\ell^j}$ for any prime power ℓ^j with $\ell \geqslant 5$. The function $f(q)$ coincides modulo 2 with the generating function for partitions. To see this, define the *rank* of a partition λ as $\lambda_1 - \ell(\lambda)$, where λ_1 is the largest part of λ and $\ell(\lambda)$ is the number of parts. Define $N_e(n)$ and $N_o(n)$ as the number of partitions of n with even and odd ranks, respectively. Then we have

$$f(q) := \sum_{n=0}^{\infty} a(n)q^n = 1 + \sum_{n=1}^{\infty} (N_e(n) - N_o(n))\, q^n. \tag{1·3}$$

Since $p(n) = N_e(n) + N_o(n)$, we have $a(n) \equiv p(n) \pmod 2$ for all n, and Radu's result implies that there are no linear congruences for $a(n)$ modulo 2. Here we will prove:

THEOREM 1·1. *For any positive integer m and any integer t we have*

$$\sum a(mn+t)q^n \not\equiv 0 \pmod 3.$$

The mock theta function

$$\omega(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(1+q)^2(1+q^3)^2\ldots(1+q^{2n+1})^2} = \sum_{n=0}^{\infty} c(n)q^n$$

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appears naturally with $f(q)$ as the component of a vector-valued mock modular form (see, for example, [**35**]). Andrews [**7**] gives a partition theoretic interpretation for $c(n)$ as the number of partitions of $n+1$ into nonnegative integers such that every part in the partition, with the possible exception of the largest part, appears as a pair of consecutive integers. For example, there are 6 such partitions of 5:

$$5, \quad 4+(1+0), \quad 3+(1+0)+(1+0), \quad 2+(2+1),$$
$$2+(1+0)+(1+0)+(1+0), \quad (1+0)+\cdots+(1+0).$$

This function behaves quite differently modulo 2. In fact, Andrews [**7**, Theorem 31] has shown that $c(n)$ is odd if and only if $n = 6j^2+4j$ for some integer j. For the modulus 3, we will prove

THEOREM 1·2. *For any positive integer m and any integer t we have*

$$\sum c(mn+t)q^n \not\equiv 0 \pmod 3.$$

It is natural to ask how these results extend to more general classes of modular forms. In this direction, we investigate the class

$$\mathcal{S}(B,k,N,\chi) := \left\{\eta^B(z)F(z) : F(z) \in M_k^!(\Gamma_0(N),\chi)\right\},$$

where k is an integer or half-integer and $M_k^!(\Gamma_0(N),\chi)$ is the space of weakly holomorphic modular forms of weight k and level N with character χ (see Section 2 for definitions). If $f(z) \in \mathcal{S}(B,k,N,\chi)$, then we have

$$f(z) = q^{B/24}\sum_{n\geqslant n_0} a_f(n)q^n.$$

We show that certain forms of this type do not possess linear congruences modulo 2 or 3. If m is a positive integer and B is an integer with $6 \nmid B$, then write $m = 2^r3^sm'$ with $(m',6)=1$, and define a divisor of m by

$$Q_{m,B} = \begin{cases} m' & \text{if } (B,6)=1, \\ 2^rm' & \text{if } (B,6)=2, \\ 3^sm' & \text{if } (B,6)=3. \end{cases} \tag{1·4}$$

THEOREM 1·3. *Suppose that $\ell = 2$ or $\ell = 3$. Suppose that $f \in \mathcal{S}(B,k,N,\chi)$ and that f has a pole at infinity and leading coefficient equal to 1. Suppose that $\ell \nmid BN$ and that the coefficients of f are ℓ-integral rational numbers. Then for any positive integers m and t with $(Q_{m,B},N)=1$, we have*

$$\sum a_f(mn+t)q^n \not\equiv 0 \pmod \ell.$$

Remark 1. The analogous statement will hold for forms with algebraic coefficients, where ℓ is replaced by any prime ideal over ℓ.

As an application, we consider eta-quotients, which we express in the standard form

$$f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}. \tag{1·5}$$

Writing

$$B = B_f := \sum \delta r_\delta, \tag{1·6}$$

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we have

$$f(z) = q^{\frac{B}{24}} \sum a_f(n) q^n. \tag{1·7}$$

COROLLARY 1·4. *Suppose that $f(z)$ is an eta-quotient as in* (1·5)–(1·7) *and that f has a pole at infinity. Suppose that $\ell = 2$ or $\ell = 3$ and that $\ell \not| B$. Write $N = \ell^s N'$ with $\ell \not| N'$. Then for any positive integers m and t with $(Q_{m,B}, N') = 1$, we have*

$$\sum a_f(mn + t) q^n \not\equiv 0 \pmod{\ell}.$$

Remark 2. The hypotheses are satisfied if $\ell \not| BN$ and $(m, N) = 1$.

We give some examples involving Corollary 1·4.

Example 1. A k-multipartition of n is a k-tuple of partitions $(\pi_1, \pi_2, \ldots, \pi_k)$ such that $|\pi_1| + |\pi_2| + \cdots + |\pi_k| = n$. The generating function for k-multipartitions is

$$\sum_{n=0}^{\infty} p_k(n) q^{n-k/24} = \eta^{-k}(z).$$

Various congruences for $p_k(n)$ have been studied (see, for example, [**8**], [**18**], [**19**]). For example, Andrews [**8**] showed that for each prime $p \geqslant 5$ there are $(p + 1)/2$ values of b with $1 \leqslant b \leqslant p$ for which $p_{p-3}(pn + b) \equiv 0 \pmod{p}$. Corollary 1·4 shows that if $\ell = 2$ or $\ell = 3$ and $\ell \not| k$, then there are no linear congruences for $p_k(n)$ modulo ℓ.

Example 2. A cubic partition of n is a bi-partition (π_1, π_2) such that π_2 contains no odd part. For example, the cubic partitions of 3 are

$$(3, \varnothing), \quad (2 + 1, \varnothing), \quad (1 + 1 + 1, \varnothing), \quad (1, 2).$$

The generating function for these partitions is

$$f_{\mathrm{cu}}(z) = \sum_{n=0}^{\infty} \mathrm{cu}(n) q^{n-3/24} = \eta^{-1}(z)\eta^{-1}(2z).$$

In this case, the quantity B from (1·6) is a multiple of 3. So Corollary 1·4 does not apply for the prime $\ell = 3$. In fact, H.-C. Chan [**13**] has shown that

$$\mathrm{cu}(3n + 2) \equiv 0 \pmod{3}.$$

Corollary 1·4 implies that there is no linear congruence for $\mathrm{cu}(n)$ modulo 2.

Example 3. To explain Ramanujan's congruences for $p(n)$, the *crank* of a partition was introduced by Andrews and Garvan [**9**]. Let $M_e(n)$ and $M_o(n)$ denote the number of partitions of n with even and odd crank, respectively. As a counterpart to (1·3), we have

$$\sum_{n=0}^{\infty} (M_e(n) - M_o(n)) q^{n-1/24} = \frac{\eta^3(z)}{\eta^2(2z)}.$$

In [**14**], Choi, Lovejoy and Kang showed that

$$M_e(5n + 4) - M_o(5n + 4) \equiv 0 \pmod{5}.$$

Here $B = -1$, so Corollary 1·4 guarantees that there are no linear congruences modulo 2 or 3.

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Example 4. Andrews [**6**] introduced the generalised Frobenius symbol $c\phi_2$ and showed that

$$\sum_{n=0}^{\infty} c\phi_2(n)q^{n-1/12} = \frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)}.$$

Andrews [**6**, Corollary 10·1 and Theorem 10·2] proved that

$$c\phi_2(2n+1) \equiv 0 \pmod 2 \quad \text{and} \quad c\phi_2(5n+3) \pmod 5,$$

and many congruence properties of these symbols have since been investigated (see, for example, [**10**, **22**, **28**]). Corollary 1·4 shows that there is no linear congruence of the form $c\phi_2(mn+t)$ modulo 3 with odd m (it is likely that an adaptation of these methods can be used to remove the restriction on m in this case).

Example 5. The assumption that f has a pole at infinity is necessary. The function $\eta(z) = q^{\frac{1}{24}}\Sigma(-1)^k q^{\frac{k(3k+1)}{2}}$ provides a simple example. For another example, the generating function for the number of 4-core partitions of n is given by

$$\sum_{n=0}^{\infty} c_4(n)q^{n+15/24} = \frac{\eta^4(4z)}{\eta(z)},$$

and M. Hirschhorn and J. Sellers [**20**] have shown that

$$c_4(9n+2) \equiv 0 \pmod 2.$$

Example 6. The assumption that $(Q_{m,B}, N) = 1$ is also necessary in general. For example, if we define $a(n)$ by

$$\sum_{n=0}^{\infty} a(n)q^{n-5/24} = \eta^{-1}(5z),$$

then we have $a(5n+1) \equiv a(5n+2) \equiv \cdots \equiv a(5n+4) \equiv 0 \pmod 2$.

The first author and Boylan [**2**] proved that if ℓ is prime and $f = \Sigma a(n)q^n$ is a weakly holomorphic modular form with $f \not\equiv 0 \pmod \ell$, then then

$$\#\left\{n \leqslant x : a_f(n) \not\equiv 0 \pmod \ell\right\} \gg_{f,K} \frac{\sqrt{x}}{\log x}(\log\log x)^K \tag{1·8}$$

for any positive integer K. In each situation where the results described above imply that $\Sigma a(mn+t)q^n \not\equiv 0 \pmod \ell$, the lower bound (1·8) applies to the number of non-zero coefficients (this will be clear from the method of proof).

The proofs follow the outline given by Radu [**30**], and require a careful analysis of the integrality properties of the functions $\Sigma a(mn+t)q^n$ at various cusps. In Section 2, we give some background on modular forms and mock modular forms. In Sections 3 and 4, we prove Theorems 1·1 and 1·2. In Section 5, we prove Theorem 1·3 and its corollary.

Finally, we mention that Radu [**29**] used similar methods to study the behavior of the partition function modulo primes $\ell \geqslant 5$. Andersen [**4**] has obtained similar results in the present context.

2. *Preliminaries*

We recall the definitions of harmonic weak Maass forms and mock modular forms (see, for example, [**12**, **26** or **34**] for details). Given $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, $z = x + iy$ with $x, y \in \mathbb{R}$,

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4 | N and a Dirichlet character χ modulo N, a *harmonic weak Maass form of weight k with Nebentypus χ on $\Gamma_0(N)$* is a smooth function $F : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(i) for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and all $z \in \mathbb{H}$, we have

$$F\left(\frac{az+b}{cz+d}\right) = \left(\frac{c}{d}\right) \epsilon_d^{-2k} \chi(d)\,(cz+d)^k\, F(z),$$

where

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4; \end{cases}$$

(ii) $\Delta_k(F) = 0$, where Δ_k is the weight k hyperbolic Laplacian, given by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right);$$

(iii) the function F has at most linear exponential growth at the cusps of $\Gamma_0(N)$.

We denote by $H_k(\Gamma_0(N), \chi)$ the space of harmonic weak Maass forms of weight k with Nebentypus χ on $\Gamma_0(N)$. We denote the subspace of weakly holomorphic forms (i.e., those meromorphic forms whose poles are supported at the cusps of $\Gamma_0(N)$) by $M_k^!(\Gamma_0(N), \chi)$, and the space of holomorphic forms by $M_k(\Gamma_0(N), \chi)$ (if χ is trivial, then we drop it from the notation). Each harmonic weak Maass form F decomposes uniquely as the sum of a holomorphic part F^+ and a non-holomorphic part F^-. The holomorphic part, which is known as a *mock modular form*, is a power series in q with at most finitely many negative exponents.

Now define

$$F(z) = (F_0(z), F_1(z), F_2(z))^T := \left(q^{-1/24} f(q),\ 2q^{1/3}\omega(q^{1/2}),\ 2q^{1/3}\omega(-q^{1/2})\right)^T \tag{2·1}$$

and

$$G := (g_1, g_0, -g_2)^T,$$

where the $g_i(z)$ are theta functions defined by

$$\begin{aligned} g_0(z) &= \sum_{n\in\mathbb{Z}} (-1)^n \left(n+\tfrac{1}{3}\right) q^{\frac{3}{2}\left(n+\frac{1}{3}\right)^2}, \\ g_1(z) &= -\sum_{n\in\mathbb{Z}} \left(n+\tfrac{1}{6}\right) q^{\frac{3}{2}\left(n+\frac{1}{6}\right)^2}, \\ g_2(z) &= \sum_{n\in\mathbb{Z}} \left(n+\tfrac{1}{3}\right) q^{\frac{3}{2}\left(n+\frac{1}{3}\right)^2}. \end{aligned} \tag{2·2}$$

Zwegers [**35**] showed that

$$T(z) := F(z) - 2i\sqrt{3}\int_{-\bar{z}}^{i\infty} \frac{G(\tau)}{(-i(z+\tau))^{\frac{1}{2}}}\, d\tau \tag{2·3}$$

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transforms as

$$\begin{aligned} T(z+1) &= \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} T(z), \\ T(-1/z) &= \sqrt{-iz} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} T(z). \end{aligned} \tag{2·4}$$

We also require the incomplete gamma function, given by

$$\Gamma(\alpha, x) := \int_x^\infty e^{-t} t^{\alpha-1}\, dt.$$

3. *Proof of Theorem* 1·1

We work with the function

$$M(z) = F_0(z) - 2i\sqrt{3} \int_{-\bar{z}}^{i\infty} \frac{g_1(\tau)}{(-i(z+\tau))^{\frac{1}{2}}}\, d\tau.$$

A computation using (2·1), (2·2) and (2·3) shows that $M(z)$ has the form

$$M(z) = q^{-1/24} \sum a(n) q^n + q^{-1/24} \sum b(n) \Gamma\left(\tfrac{1}{2}, 4\pi y (n + \tfrac{1}{24})\right) q^{-n},$$

where

$$b(n) = 0 \qquad \text{unless} \qquad n = \frac{k(3k+1)}{2} \qquad \text{for some integer } k. \tag{3·1}$$

For ease of notation we will write

$$M(z) = H(z) + NH(z), \tag{3·2}$$

where

$$H(z) = F_0(z) = q^{-1/24} \sum a(n) q^n, \tag{3·3}$$

and $NH(z)$ is the non-holomorphic part.

Suppose that m is a positive integer and that t is a non-negative integer. Setting $\zeta_m := e^{\frac{2\pi i}{m}}$, we define

$$M_{m,t} := \frac{1}{m} \sum_{\lambda=0}^{m-1} M_{\lambda,m,t} \tag{3·4}$$

where

$$M_{\lambda,m,t} := \zeta_m^{-\lambda(t-1/24)} M\left(\begin{pmatrix} 1 & \lambda \\ 0 & m \end{pmatrix} z\right). \tag{3·5}$$

Using notation in analogy with (3·2), we write

$$M_{m,t} = H_{m,t} + NH_{m,t}.$$

A calculation gives

$$H_{m,t} = q^{\frac{t-1/24}{m}} \sum a(mn+t) q^n \tag{3·6}$$

and

$$NH_{m,t} = q^{\frac{t-1/24}{m}} \sum b(mn-t) \Gamma\left(\tfrac{1}{2}, 4\pi y \cdot (n + \tfrac{1/24-t}{m})\right) q^{-n}. \tag{3·7}$$

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In order to prove Theorem 1·1 we must show that for any progression $t \pmod m$ we have

$$H_{m,t} \not\equiv 0 \pmod 3. \tag{3·8}$$

We call a progression $t \pmod m$ *good* if for some $p \mid m$ we have $(1-24t)/p = -1$. By (3·1) and (3·7) we see that if $t \pmod m$ is good, then $NH_{m,t} = 0$. Suppose that the progression $t \pmod m$ is not good. In this case we pick a prime $p \geqslant 5$ with $p \nmid m$ and a quadratic non-residue $x \pmod p$, and we find a solution to the system of congruences

$$\begin{aligned} T &\equiv t \pmod m \\ T &\equiv \frac{1-x}{24} \pmod p. \end{aligned}$$

After replacing $t \pmod m$ by the sub-progression $T \pmod{mp}$, we are reduced in proving (3·8) to considering progressions which are good.

The next two propositions describe the transformation properties of the forms $M_{m,t}$. Given a positive integer m, we define

$$N_m := \begin{cases} 2m & \text{if } (m,6)=1, \\ 8m & \text{if } (m,6)=2, \\ 6m & \text{if } (m,6)=3, \\ 24m & \text{if } (m,6)=6. \end{cases} \tag{3·9}$$

PROPOSITION 3·1. *Suppose that $t \pmod m$ is good. Then*

$$M_{m,t}^{24m} = H_{m,t}^{24m} \in M_{12m}^!(\Gamma_1(N_m)).$$

Now if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_m)$ has $3 \nmid a$ then we define t_A to be any integer with

$$t_A \equiv a^2 t + \frac{1-a^2}{24} \pmod m. \tag{3·10}$$

PROPOSITION 3·2. *For every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_m)$ with $3 \nmid a$ we have*

$$M_{m,t}^{24m}\big|_{12m} A = M_{m,t_A}^{24m}. \tag{3·11}$$

In each case, the matrices A as above with $3 \nmid a$ generate $\Gamma_1(N_m)$. Therefore Proposition 3·1 follows immediately from Proposition 3·2. For Proposition 3·2 we require a transformation law of Andrews [**5**] which is also recorded in work of Bringman and Ono [**11**, p. 251]. In particular, for $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ with $c > 0$, we have

$$M\left(\frac{az+b}{cz+d}\right) = w(A) \cdot (cz+d)^{\frac{1}{2}} M(z), \tag{3·12}$$

where $w(A)$ is the root of unity given by

$$w(A) := i^{-\frac{1}{2}} \cdot e^{-\pi i s(-d,c)} \cdot (-1)^{\frac{c+1+ad}{2}} \cdot e^{2\pi i\left(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}\right)} \tag{3·13}$$

and $s(d,c)$ is the Dedekind sum defined by

$$s(d,c) = \sum_{r=1}^{c-1} \left(\frac{r}{c} - \left\lfloor \frac{r}{c} \right\rfloor - \frac{1}{2}\right)\left(\frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2}\right). \tag{3·14}$$

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For any $A \in \mathrm{SL}_2(\mathbb{Z})$, we have the following (see, e.g., [**21**, p. 247]):

$$12s(-d, c) + \frac{a+d}{c} \in \mathbb{Z}, \tag{3·15}$$

and so

$$w(A)^{24} = 1. \tag{3·16}$$

We also require a transformation law which follows from Lewis [**21**, Lemma 2] (we have corrected a sign error in the statement of that lemma).

LEMMA 3·1. *Let m be a positive integer. Then for every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_m)$ with $c > 0$ and $3 \nmid a$ we have*

$$s(d + c, mc) = s(d, mc) + \frac{1 - a^2}{12m} + \textit{even integer}.$$

Proof of Proposition 3·2. Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_m)$ has $3 \nmid a$. We may suppose that $c > 0$. For each $\lambda \pmod m$, (3·5) gives

$$M_{\lambda,m,t}(Az) = \zeta_m^{-\lambda(t-1/24)} M\left(\begin{pmatrix} 1 & \lambda \\ 0 & m \end{pmatrix} Az\right) = \zeta_m^{-\lambda(t-1/24)} M\left(A_\lambda \begin{pmatrix} 1 & \lambda' \\ 0 & m \end{pmatrix} z\right), \tag{3·17}$$

where

$$A_\lambda = \begin{pmatrix} a + c\lambda & \frac{-\lambda' c\lambda - \lambda' a + b + d\lambda}{m} \\ mc & d - c\lambda' \end{pmatrix}, \tag{3·18}$$

and $\lambda' \in \{0, 1, \ldots, m-1\}$ is chosen with

$$a\lambda' \equiv b + d\lambda \pmod m. \tag{3·19}$$

Note that λ' travels the residue classes mod m with λ.

Using (3·17) and recalling the definitions (3·5) and (3·10), we obtain

$$M_{\lambda,m,t}(Az) = (cz + d)^{\frac{1}{2}} w(A_\lambda) \zeta_m^{-\lambda(t-1/24)} \zeta_m^{\lambda'(t_A - 1/24)} M_{\lambda',m,t_A}(z). \tag{3·20}$$

We find that

$$\begin{aligned} w(A_\lambda) &\zeta_m^{-\lambda(t-1/24)} \zeta_m^{\lambda'(t_A-1/24)} \\ &= i^{-\frac{1}{2}} e^{-\pi i s(-d+c\lambda', mc)} \cdot (-1)^{\frac{mc+1+(a+c\lambda)(d-c\lambda')}{2}} \cdot e^{2\pi i(-\frac{a+d}{24mc} - \frac{a+c\lambda}{4} + \frac{3mc(d-c\lambda')}{8})} \zeta_m^{-\lambda t + \lambda' t_A} \\ &= \zeta_1(A, m) e^{-\pi i s(-d+c\lambda', mc)} \cdot (-1)^{\frac{-ac\lambda' + cd\lambda}{2}} \cdot e^{2\pi i(-\frac{a+d}{24mc} - \frac{c\lambda}{4} - \frac{3mc^2\lambda'}{8})} \zeta_m^{-\lambda t + \lambda' t_A}, \end{aligned}$$

where $\zeta_1(A, m)$ is a 24th root of unity which depends only on A and m (and not on λ). We see that

$$(-1)^{\frac{-ac\lambda' + cd\lambda}{2}} \cdot e^{2\pi i(-\frac{c\lambda}{4} - \frac{3mc^2\lambda'}{8})} = 1 \tag{3·21}$$

by writing $c = 2c_0$ and separating cases according to the parity of c_0 (note that if c_0 is odd then m is odd). Therefore we have

$$w(A_\lambda) \zeta_m^{-\lambda(t-1/24)} \zeta_m^{\lambda'(t_A-1/24)} = \zeta_1(A, m) e^{-\pi i s(-d+c\lambda', mc)} \cdot e^{2\pi i(-\frac{a+d}{24mc})} \zeta_m^{-\lambda t + \lambda' t_A}.$$

We apply Lemma 3·1 iteratively to the matrices

$$\begin{pmatrix} -a & b - ja \\ c & -d + jc \end{pmatrix} \in \Gamma_0(N_m), \quad 1 \leqslant j \leqslant \lambda' - 1$$

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to find that

$$s(-d+\lambda' c, mc) = s(-d, mc) + \lambda' \frac{1-a^2}{12m} + \text{an even integer}.$$

Also, since $\begin{pmatrix} a(1-bc) & -b^2c/m \\ mc & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we see from (3·15) that

$$12s(-d, mc) + \frac{a+d}{mc} - \frac{ab}{m} \in \mathbb{Z}.$$

It follows from the last three equations that

$$w(A_\lambda)\zeta_m^{-\lambda(t-1/24)}\zeta_m^{\lambda'(t_A-1/24)} = \zeta_2(A,m)e^{-2\pi i\lambda'\frac{1-a^2}{24m}}\zeta_m^{-\lambda t+\lambda' t_A}, \tag{3·22}$$

where $\zeta_2(A,m)$ is a $24m$th root of unity which depends only on A and m.

Recalling (3·10) and the fact that $\lambda \equiv a^2\lambda' - ab \pmod m$, we find that

$$\zeta_m^{-\lambda t+\lambda' t_A} = \zeta_m^{abt}\zeta_m^{\lambda'\frac{1-a^2}{24}}. \tag{3·23}$$

Combining (3·22) and (3·23), we conclude that

$$w(A_\lambda)\zeta_m^{-\lambda(t-1/24)}\zeta_m^{\lambda'(t_A-1/24)} = \zeta_3(A,m), \tag{3·24}$$

where $\zeta_3(A,m)$ is a $24m$th root of unity which depends only on A and m. Proposition 3·2 follows immediately from (3·24), (3·20) and (3·4).

LEMMA 3·2. *Suppose that $t \pmod m$ is good. Write $m = 2^s3^rQ$ with $(Q,6)=1$. If*

$$\sum a(mn+t)q^n \equiv 0 \pmod 3$$

then

$$\sum a(Qn+t)q^n \equiv 0 \pmod 3.$$

Proof. The arithmetic progression $t \pmod Q$ is the disjoint union of the arithmetic progressions

$$t + \ell Q \pmod m, \quad 0 \leqslant \ell < 2^s3^r. \tag{3·25}$$

By the argument in [**30**, Theorem 4·2], we see that as a ranges over integers with $(a, 6m) = 1$, the quantity

$$t_A \equiv ta^2 + \frac{1-a^2}{24} \pmod m$$

covers each of the progressions in (3·25).

Let Δ be the usual normalized cusp form of level one and weight 12. By Proposition 3·1 there is a positive integer j such that

$$M_{m,t}^{24m}\Delta^j \in M_{12(m+j)}(\Gamma_1(N_m)).$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_m)$ has $3 \nmid a$ then by Proposition 3·2 we have

$$M_{m,t}^{24m}\Delta^j\Big|_{12(m+j)} A = M_{m,t_A}^{24m}\Delta^j. \tag{3·26}$$

We require a fact proved by Deligne and Rapoport (see [**15**, VII, Corollary 3·12] or [**30**, Corollary 5·3]): If $f \in M_k(\Gamma_1(N))$ has coefficients in $\mathbb{Z}[\zeta_N]$ then the same is true of $f|_k\gamma$ for each $\gamma \in \Gamma_0(N)$.

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It follows from (3·26) that if $M_{m,t} \equiv 0 \pmod 3$, then for each t_A with $(a, 6m) = 1$ we have $M_{m,t_A} \equiv 0 \pmod 3$. By (3·25) we conclude that $M_{Q,t} \equiv 0 \pmod 3$, as desired.

After Lemma 3·2 we are reduced to proving that if Q is a positive integer with $(Q, 6) = 1$ and $t \pmod Q$ is good, then

$$M_{Q,t} \not\equiv 0 \pmod 3. \tag{3·27}$$

By Proposition 3·1 we see that for sufficiently large j, we have

$$h := M_{Q,t}^{24Q} \Delta^j \in M_{12(Q+j)}(\Gamma_1(2Q)). \tag{3·28}$$

We compute the first term in the expansion of h at the cusp $1/2$.

PROPOSITION 3·3. *Let h be the modular form defined in* (3·28). *Then we have*

$$h\Big|_{12(Q+j)} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = Q^{-12Q} q^{-Q^2+j} + \cdots .$$

Proof. By (3·4) and (3·5) we have

$$M_{Q,t}\left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} z\right) = \frac{1}{Q} \sum_{\lambda=0}^{Q-1} \zeta_Q^{-\lambda(t-1/24)} M\left(\begin{pmatrix} 1 & \lambda \\ 0 & Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} z\right). \tag{3·29}$$

We find that

$$\begin{pmatrix} 1 & \lambda \\ 0 & Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = C_\lambda \begin{pmatrix} 1 & \lambda' \\ 0 & Q/d_\lambda \end{pmatrix} \begin{pmatrix} d_\lambda & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$d_\lambda := (1 + 2\lambda, Q),$$

$\lambda' \in \{0, 1, 2, \ldots, Q/d_\lambda - 1\}$ is uniquely defined by the congruence

$$\frac{1 + 2\lambda}{d_\lambda} \lambda' \equiv \lambda \pmod{Q/d_\lambda}$$

and

$$C_\lambda := \begin{pmatrix} \frac{1+2\lambda}{d_\lambda} & \frac{-\frac{1+2\lambda}{d_\lambda}\lambda' + \lambda}{Q/d_\lambda} \\ 2Q/d_\lambda & -2\lambda' + d_\lambda \end{pmatrix} \in \Gamma_0(2).$$

By (3·12), we obtain

$$M\left(C_\lambda \frac{d_\lambda z + \lambda'}{Q/d_\lambda}\right) = w(C_\lambda) \cdot (d_\lambda(2z + 1))^{1/2} M\left(\frac{d_\lambda z + \lambda'}{Q/d_\lambda}\right). \tag{3·30}$$

Since $M = q^{-1/24} + \cdots$, we see from (3·29) and (3·30) that the leading term in the expansion of

$$(2z + 1)^{-1/2} M_{Q,t}\left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} z\right) \tag{3·31}$$

arises from those λ with $d_\lambda = Q$. The only such λ is $\lambda_0 = (Q-1)/2$, in which case we have $\lambda_0' = 0$ and $C_{\lambda_0} := \begin{pmatrix} 1 & (Q-1)/2 \\ 2 & Q \end{pmatrix}$. Therefore the leading term in (3·31) is

$$\frac{1}{\sqrt{Q}} w(C_{\lambda_0}) \zeta_Q^{(\frac{1-Q}{2})(t-1/24)} \cdot q^{\frac{-Q}{24}} + \cdots$$

Proposition 3·3 follows immediately from (3·28) and (3·16).

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Theorem 1·1 now follows.

Proof of Theorem 1·1. Deligne and Rapoport ([**15**, Corollary 3·13 and section 4·8] or [**16**, section 12·3·5]) proved that if $f \in M_k(\Gamma_1(N))$ has coefficients in $\mathbb{Z}[1/N, \zeta_N]$, then the same is true of its expansion at each cusp.

Suppose by way of contradiction that (3·27) is false. Then the modular form $3^{-24Q}h \in M_{12(Q+j)}(\Gamma_1(2Q))$ has integral coefficients. It follows that the coefficients of

$$3^{-24Q}h\Big|_{12(Q+j)}\begin{pmatrix}1&0\\2&1\end{pmatrix}$$

lie in $\mathbb{Z}[1/2Q, \zeta_{2Q}]$. This contradicts Proposition 3·3, and Theorem 1·1 is proved.

4. *Proof of Theorem* 1·2

The flow of the argument is similar to that of the last section. We now work with the function

$$\Omega(z) = 2q^{2/3}\omega(q) - 2i\sqrt{3}\int_{-2\overline{z}}^{i\infty} \frac{g_0(\tau)}{(-i(2z+\tau))^{\frac{1}{2}}}\,d\tau.$$

A computation shows that

$$\Omega(z) = 2q^{2/3}\omega(q) + q^{-1/3}\sum d(n)\Gamma\left(\tfrac{1}{2}, 4\pi(n+\tfrac{1}{3})y\right)q^{-n},$$

where

$$d(n) = 0 \qquad \text{unless} \qquad n = 3k^2+2k \qquad \text{for some integer } k.$$

To isolate arithmetic progressions, we define

$$\Omega_{m,t} := \frac{1}{m}\sum_{\lambda=0}^{m-1}\zeta_m^{-\lambda(t+2/3)}\Omega\left(\begin{pmatrix}1&\lambda\\0&m\end{pmatrix}z\right).$$

A calculation gives

$$\Omega_{m,t} = q^{\frac{t+2/3}{m}}\sum c(mn+t)q^n + q^{\frac{t+2/3}{m}}\sum d(mn-t-1)\Gamma\left(\frac{1}{2}, 4\pi\left(n-\frac{t+2/3}{m}\right)y\right)q^{-n}.$$

In this case, we call the arithmetic progression $t \pmod m$ *good* if

$$\left(\frac{-3t-2}{p}\right) = -1 \qquad \text{for some prime } p \mid m.$$

If $t \pmod m$ is good, then $\Omega_{m,t}$ is weakly holomorphic. As in the last section, it suffices to prove that when $t \pmod m$ is good we have

$$\Omega_{m,t} \not\equiv 0 \pmod 3.$$

The transformation properties of $\Omega(z)$ are described in work of Andrews [**5**, Theorems 2·1 and 2·4]. Using these results with (2·4), we find that for $A = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$, we have

$$\Omega(Az) = \begin{cases} w_1(A)(cz+d)^{1/2}\Omega(z) & \text{if } c \text{ is even,}\\ w_2(A)\cdot 2^{-1/2}(cz+d)^{1/2}M(z/2) & \text{if } d \text{ is even,}\end{cases} \tag{4·1}$$

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where $w_1(A)$ and $w_2(A)$ are the roots of unity defined by

$$\begin{aligned} w_1(A) &:= (-i)^{1/2}(-1)^{(a-1)/2}e^{-\pi i s(-d,c/2)}e^{2\pi i\left(\frac{3ab}{4}-\frac{a+d}{12c}\right)}, \\ w_2(A) &:= i^{1/2}(-1)^{(32a-d)/24c}e^{-\pi i s(-d/2,c)}e^{-\frac{\pi i}{2}(2a+b-3-3ab+3a/c)}. \end{aligned} \tag{4·2}$$

Note that we have fixed a sign error in [**5**, Theorem 2·1].

PROPOSITION 4·1. *Let N_m be as defined in* (3·9). *For every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N_m)$ with $3 \nmid a$ we have*

$$\Omega_{m,t}^{24m}\big|_{12m} A = \Omega_{m,t_A}^{24m}, \tag{4·3}$$

where

$$t_A := ta^2 + \tfrac{2}{3}(a^2-1).$$

Proof of Proposition 4·1. Write

$$\Omega_{\lambda,m,t}(z) = \zeta_m^{-\lambda(t+2/3)}\Omega\left(\begin{pmatrix} 1 & \lambda \\ 0 & m \end{pmatrix} z\right).$$

Suppose that $A \in \Gamma_0(2N_m)$ with $3 \nmid a$. Then

$$\Omega_{\lambda,m,t}(Az) = \zeta_m^{-\lambda(t+2/3)}\Omega\left(A_\lambda \begin{pmatrix} 1 & \lambda' \\ 0 & m \end{pmatrix} z\right),$$

where A_λ and λ' are defined as in (3·18) and (3·19). Using (4·1), we find that

$$\Omega_{\lambda,m,t}(Az) = (cz+d)^{\frac{1}{2}} w_1(A_\lambda)\zeta_m^{-\lambda(t+\frac{2}{3})}\zeta_m^{\lambda'(t_A+\frac{2}{3})}\Omega_{\lambda',m,t_A}(z) \tag{4·4}$$

Note that $4m \mid c$ and that $ad \equiv 1 \pmod{4m}$. Moreover, from Lemma 3·1 and (3·15) we have

$$s(-d+\lambda' c, mc/2) = s(-d, mc/2) + \lambda'\frac{1-a^2}{6m} + \text{an even integer}$$

and

$$s(-d, mc/2) + \frac{a+d}{6mc} - \frac{ab}{6m} \in \mathbb{Z}/12.$$

Therefore,

$$\begin{aligned} w_1(A_\lambda) &= (-i)^{1/2}(-1)^{(a+c\lambda-1)/2}e^{-\pi i s(-d+c\lambda', mc/2)}e^{2\pi i\left(\frac{3(a+c\lambda)(-\lambda' c\lambda-\lambda' a+b+d\lambda)}{4m}-\frac{a+d}{12mc}+\frac{\lambda'-\lambda}{12m}\right)} \\ &= \omega_1 \cdot e^{-\pi i s(-d+c\lambda', mc/2)}e^{2\pi i\left(\frac{3(-\lambda' a^2+\lambda)}{4m}-\frac{a+d}{12mc}+\frac{\lambda'-\lambda}{12m}\right)} \\ &= \omega_2 \cdot e^{2\pi i\left(\frac{3(-\lambda' a^2+\lambda)}{4m}+\lambda'\frac{a^2-1}{12m}+\frac{\lambda'-\lambda}{12m}\right)} \end{aligned}$$

where ω_1, etc. denote $24m$th roots of unity which depend only on A, m, and t.

Using this together with the fact that $\lambda \equiv a^2\lambda' - ab \pmod{m}$, we obtain

$$\begin{aligned} w_1(A_\lambda)\zeta_m^{-\lambda(t+\frac{2}{3})}\zeta_m^{\lambda'(t_A+\frac{2}{3})} &= \omega_2 \cdot \zeta_m^{-\lambda t+\lambda'(t_A-\frac{2}{3}(a^2-1))} \\ &= \omega_3 \cdot \zeta_m^{\lambda'(t_A-a^2 t-\frac{2}{3}(a^2-1))} \\ &= \omega_3. \end{aligned}$$

Proposition 4·1 follows immediately from this together with (4·4).

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In this case we cannot pull out powers of 2 from the arithmetic progressions in question. We have

LEMMA 4·1. *Suppose that* $t \pmod m$ *is good. Write* $m = 3^r Q$ *with* $(Q, 3) = 1$. *If*

$$\sum c(mn+t)q^n \equiv 0 \pmod 3$$

then

$$\sum c(Qn+t)q^n \equiv 0 \pmod 3.$$

Proof. The arithmetic progression $t \pmod Q$ is the disjoint union of the arithmetic progressions

$$t + \ell Q \pmod m, \quad 0 \leqslant \ell < 3^r. \tag{4·5}$$

By the argument in [**30**, Theorem 4·2], we see that as a ranges over integers with $3 \nmid a$, the quantity

$$t_A \equiv ta^2 + \frac{2}{3}(1-a^2) \pmod m$$

covers each of the progressions in (4·5).

By Proposition 4·1 there is a positive integer j such that

$$\Omega_{m,t}^{24m}\Delta^j \in S_{12(m+j)}(\Gamma_1(2N_m)).$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N_m)$ has $3 \nmid a$ then by Proposition 4·1 we have

$$\Omega_{m,t}^{24m}\Delta^j\Big|_{12(m+j)} A = \Omega_{m,t_A}^{24m}\Delta^j.$$

If $\Omega_{m,t} \equiv 0 \pmod 3$, then the fact recorded after (3·26) shows that for each t_A with $3 \nmid a$ we have $\Omega_{m,t_A} \equiv 0 \pmod 3$. By (4·5) we conclude that $\Omega_{Q,t} \equiv 0 \pmod 3$, as desired.

By Proposition 4·1 we see that for sufficiently large j, we have

$$h_\Omega := \Omega_{Q,t}^{24Q}\Delta^j \in M_{12(Q+j)}(\Gamma_1(4Q)). \tag{4·6}$$

In this case, we compute the first term in the expansion of $h_\Omega|\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

PROPOSITION 4·2. *Let* h_Ω *be the modular form defined in* (4·6). *Then we have*

$$h_\Omega\Big|_{12(Q+j)}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (-1)^Q \cdot (2Q)^{-12Q} q^{-\frac{Q^2}{2}+j} + \cdots.$$

Proof. Recall that

$$\Omega_{Q,t}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} z\right) = \frac{1}{Q}\sum_{\lambda=0}^{Q-1} \zeta_Q^{-\lambda(t+2/3)}\Omega\left(\begin{pmatrix} 1 & \lambda \\ 0 & Q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} z\right). \tag{4·7}$$

We find that

$$\begin{pmatrix} 1 & \lambda \\ 0 & Q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = D_\lambda \begin{pmatrix} 1 & \lambda^* \\ 0 & Q/d_\lambda \end{pmatrix}\begin{pmatrix} d_\lambda & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$d_\lambda := (1+\lambda, Q),$$

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λ^* is chosen to satisfy the congruence

$$\frac{1+\lambda}{d_\lambda}\lambda^* \equiv \lambda \pmod{Q/d_\lambda}$$

and

$$D_\lambda := \begin{pmatrix} \frac{1+\lambda}{d_\lambda} & \frac{-\frac{1+\lambda}{d_\lambda}\lambda^*+\lambda}{Q/d_\lambda} \\ Q/d_\lambda & -\lambda^*+d_\lambda \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Moreover, we may choose the values of λ^* in such a way that

$$-\lambda^* + d_\lambda \text{ is even whenever } Q/d_\lambda \text{ is odd.}$$

By (4·1), we obtain

$$\Omega\left(D_\lambda \frac{d_\lambda z+\lambda^*}{Q/d_\lambda}\right) = \begin{cases} w_1(D_\lambda)\cdot(d_\lambda(z+1))^{1/2}\,\Omega\left(\frac{d_\lambda z+\lambda^*}{Q/d_\lambda}\right) & \text{if } Q/d_\lambda \text{ is even,} \\ w_2(D_\lambda)\cdot\left(\frac{d_\lambda}{2}(z+1)\right)^{1/2} M\left(\frac{d_\lambda z+\lambda^*}{2Q/d_\lambda}\right) & \text{if } Q/d_\lambda \text{ is odd.} \end{cases} \tag{4·8}$$

Since $M = q^{-1/24} + \cdots$ and $\Omega = q^{2/3} + \cdots$, we see from (4·7) and (4·8) that the leading term in the expansion of

$$(z+1)^{-1/2}\Omega_{Q,t}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} z\right) \tag{4·9}$$

arises from those λ with $d_\lambda = Q$.

The only such λ is $\lambda_0 = Q-1$. We may choose $\lambda_0^* = Q$, so that $D_{\lambda_0} := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore the leading term in (4·9) is

$$\frac{1}{\sqrt{2Q}} w_2(D_{\lambda_0}) \zeta_Q^{(1-Q)(t+2/3)} e^{\frac{-2\pi i Q}{48}} \cdot q^{\frac{-Q}{48}} + \cdots,$$

and Proposition 4·2 follows from (4·2).

Theorem 1·2 follows by the argument used to prove Theorem 1·1 in the last section.

5. *Proof of Theorem* 1·3 *and Corollary* 1·4

Recall that

$$\mathcal{S}(B,k,N,\chi) := \left\{\eta^B(z)F(z) : F(z) \in M_k^!(\Gamma_0(N),\chi)\right\}.$$

Suppose that $\ell = 2$ or $\ell = 3$ and that

$$f(z) = q^{B/24}\sum_{n=n_0}^{\infty} a_f(n)q^n = q^{n_0+B/24} + \cdots \in \mathcal{S}(B,k,N,\chi) \tag{5·1}$$

has rational, ℓ-integral coefficients and a pole at infinity. Given m and t we define

$$f_{m,t} := \frac{1}{m}\sum_{\lambda=0}^{m-1} f_{\lambda,m,t} := \frac{1}{m}\sum_{\lambda=0}^{m-1} \zeta_m^{-\lambda(t+B/24)} f\left(\begin{pmatrix} 1 & \lambda \\ 0 & m \end{pmatrix} z\right).$$

A calculation as in (3·6) reveals that

$$f_{m,t} = q^{\frac{t+B/24}{m}} \sum a_f(mn+t)q^n.$$

We recall the transformation formula of the eta function (see. e.g. [**21**]).

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LEMMA 5·1. *Suppose that* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(1)$ *has* $c > 0$. *Then*

$$\eta(Az) = \exp\left(\frac{\pi i}{12}\left(\frac{a+d}{c} - 12s\,(d,c)\right)\right)\sqrt{-i(cz+d)}\cdot\eta(z),$$

where $s(d,c)$ *is the Dedekind sum defined in* (3·14).

Define

$$N_m := \begin{cases} m & \text{if } (m,6) = 1, \\ 8m & \text{if } (m,6) = 2, \\ 3m & \text{if } (m,6) = 3, \\ 24m & \text{if } (m,6) = 6 \end{cases} \tag{5·2}$$

(this differs slightly from the definition of N_m in Section 3). Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(NN_m)$ has $(a,6) = 1$, and let A_λ and λ' be as defined in (3·18) and (3·19).

Suppose that k is an integer. Then we have

$$\begin{aligned} f_{\lambda,m,t}(Az) &= \zeta_m^{-\lambda(t+B/24)} f\left(\begin{pmatrix} 1 & \lambda \\ 0 & m \end{pmatrix} Az\right) \\ &= \zeta_m^{-\lambda(t+B/24)} f\left(A_\lambda \begin{pmatrix} 1 & \lambda' \\ 0 & m \end{pmatrix} z\right) \\ &= \sqrt{-i(cz+d)}^{B}(cz+d)^k \chi(d-c\lambda') e^{\frac{B\pi i}{12}\left(\frac{a+d}{mc} + \frac{\lambda-\lambda'}{m} - 12s(d-c\lambda',mc)\right)} \\ &\qquad \cdot \zeta_m^{-\lambda(t+B/24)} f\left(\begin{pmatrix} 1 & \lambda' \\ 0 & m \end{pmatrix} z\right) \\ &= \sqrt{-i(cz+d)}^{B}(cz+d)^k \chi(d) e^{\frac{B\pi i}{12}\left(\frac{a+d}{mc} - 12s(d-c\lambda',mc)\right)} \zeta_m^{-\lambda t + \lambda' t_A} f_{\lambda',m,t_A}, \end{aligned} \tag{5·3}$$

where

$$t_A \equiv ta^2 - \frac{B(1-a^2)}{24} \pmod m. \tag{5·4}$$

From [**21**, Lemma 2] (correcting the sign error), we find that

$$s(d - c\lambda', mc) = s(d, mc) - \lambda'\frac{1-a^2}{12m} + \text{an even integer}$$

and that

$$12s(d,mc) - \frac{a+d}{mc} + \frac{ab}{m} \in \mathbb{Z}.$$

Since $\lambda \equiv a^2\lambda' - ab \pmod m$, (5·3) reduces to

$$f_{\lambda,m,t}(Az) = \zeta_{24m}^{\Phi(A,B,m,t)}\chi(d)\sqrt{-i(cz+d)}^{B}(cz+d)^k f_{\lambda',m,t_A},$$

where $\Phi(A,B,m,t)$ is an integer depending only on A, B, m, and t. We conclude that for $A \in \Gamma_0(NN_m)$ with $(a,6) = 1$ we have

$$(f_{m,t}(Az))^{24mN} = (cz+d)^{24mN(k+B/2)}(f_{m,t_A}(z))^{24mN}. \tag{5·5}$$

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If k is not an integer, then the factor $\chi(d)(cz+d)^k$ in (5·3) is replaced by

$$\left(\frac{mc}{d-c\lambda'}\right)\epsilon_{d-c\lambda'}^{-2k}\chi(d)(cz+d)^k.$$

We have $\epsilon_{d-c\lambda'}=\epsilon_d$ since c is a multiple of 4. To show that $mc/(d-c\lambda')$ is independent of λ', write $mc=2^e p_1\dots p_t$ with odd primes p_i. For each i we have $p_i/(d-c\lambda')=(-1)^{\frac{p_i^2-1}{2}\frac{d^2-1}{2}}(d/p_i)$. If $8\mid c$ then $2^e/(d-c\lambda')=2^e/d$, while if $8\nmid c$ then m is odd, so that $e=2$. Thus, we can conclude (5·5) in this case as well.

By (5·5), there is a positive integer j such that

$$f_{m,t}^{24mN}\Delta^j\in S_{24mN(k+B/2)+12j}(\Gamma_1(NN_m))$$

and if $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(NN_m)$ has $(a,6)=1$ then

$$f_{m,t}^{24mN}\Delta^j\Big|A=f_{m,t_A}^{24mN}\Delta^j.$$

Now recall the definition (1·4) of $Q_{m,b}$, and write $Q=Q_{m,B}$ for simplicity. The argument in [**30**, Theorem 4·2] shows that as A ranges over elements of $\Gamma_0(NN_m)$ with $(a,6)=1$, the quantity t_A in (5·4) covers each of the progressions

$$t+jQ\pmod m,\quad 0\leqslant j<m/Q.$$

So if $f_{m,t}\equiv 0\pmod\ell$ we may conclude as before that $f_{Q,t}\equiv 0\pmod\ell$, where $(Q,N)=1$.

To obtain a contradiction we calculate the expansion of $f_{Q,t}$ at the cusp $1/N$. We have

$$\begin{aligned}f_{Q,t}\left(\begin{pmatrix}1&0\\N&1\end{pmatrix}z\right)&=\frac{1}{Q}\sum_{\lambda=0}^{Q-1}\zeta_Q^{-\lambda(t+B/24)}f\left(\begin{pmatrix}1&\lambda\\0&Q\end{pmatrix}\begin{pmatrix}1&0\\N&1\end{pmatrix}z\right)\\&=\frac{1}{Q}\sum_{\lambda=0}^{Q-1}\zeta_Q^{-\lambda(t+B/24)}f\left(C'\begin{pmatrix}1&\lambda'\\0&Q/d_\lambda\end{pmatrix}\begin{pmatrix}d_\lambda&0\\0&1\end{pmatrix}z\right),\end{aligned}$$

where $d_\lambda=\gcd(1+\lambda N,Q)$, λ' is a solution of $(1+\lambda N)\lambda'/d_\lambda\equiv\lambda(\mathrm{mod}(Q/d_\lambda)$ and

$$C'=\begin{pmatrix}\frac{1+\lambda N}{d_\lambda}&\frac{-\left(\frac{1+\lambda N}{d_\lambda}\right)\lambda'+\lambda}{Q/d_\lambda}\\NQ/d_\lambda&-N\lambda'+d_\lambda\end{pmatrix}\in\Gamma_0(N).$$

Since f has a pole at infinity, we see that the leading term of

$$(Nz+1)^{-k-B/2}f_{Q,t}$$

arises from the unique λ_0 with $d_{\lambda_0}=Q$. Since $\lambda'=0$, the coefficient of this term is

$$\xi Q^{B/2+k-1},$$

where ξ is a $24QN$th root of unity. We have $f_{Q,t}^{24QN}\Delta^j\in S_j(\Gamma_1(NN_Q))$. Note that if $\ell\nmid B$ then $\ell\nmid N_Q$ by (1·4) and (5·2). Therefore $\ell\nmid NN_Q$, so we obtain a contradiction as before, and Theorem 1·3 is proved.

We finish by treating the case of eta-quotients.

Proof of Corollary 1·4. Suppose that f is the η-quotient

$$f(z)=\prod_{\delta\mid N}\eta(\delta z)^{r_\delta}$$

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and recall that

$$B = \sum_{\delta | N} \delta r_\delta. \tag{5·6}$$

If $\ell \mid \delta$ for some δ, then, writing $\delta = \ell^s \delta'$, we replace the factor $\eta(\delta z)^{r_\delta}$ by the factor $\eta(\delta' z)^{\ell^s r_\delta}$. These factors are congruent modulo ℓ, and the value of B is not affected by this replacement. After this discussion, we may assume that $\ell \not\mid N$.

Write

$$f(z) = \eta^B(z) \frac{f(z)}{\eta^B(z)}$$

and set

$$k = \frac{1}{2}\left(\sum_{\delta | N} r_\delta - B\right). \tag{5·7}$$

Suppose first that $\ell = 2$. In this case N is odd, so B and $\Sigma_{\delta | N} r_\delta$ have the same parity, and k is an integer. We have

$$N^2 \left(\sum_{\delta | N} \frac{r_\delta}{\delta} - B\right) \equiv 0 \pmod{24} \tag{5·8}$$

(to see this, consider the cases $3 \not\mid N$ and $3 \mid N$ separately). In view of (5·6) and (5·8), a standard criterion [**23**] applies to show that $f(z)/\eta^B(z) \in M_k^!(\Gamma_0(N^2), \chi)$ for some character χ. If $\ell = 3$ and k is an integer, then a similar argument shows that $f(z)/\eta^B(z) \in M_k^!(\Gamma_0(N^4), \chi)$ for some χ.

Finally, suppose that k is not an integer. Then (5·7) and (5·6) show that N must be even. So we again have $f(z)/\eta^B(z) \in M_k^!(\Gamma_0(N^4), \chi)$ for some χ. In each of the cases, Corollary 1·4 follows from Theorem 1·3.

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