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DECIDABILITY OF SOME COMPLICATED STRUCTURES DEFINABLE IN $\mathbb{C}(T)$

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In memory of Thanases Pheidas, 1958 – 2023

Abstract. Several infinite unions of algebraic sets in $\mathbb{C}^n$ are definable in $\mathbb{C}(t)$, including the set CM of j-invariants of complex elliptic curves with complex multiplication. It has been suggested that one could prove the undecidability of $\mathrm{Th}(\mathbb{C}(t))$ by showing that the theory of the structure $\mathfrak{CM} := (\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{CM})$ of the field of complex numbers considered with a unary predicate picking out CM is undecidable. We show, using an effective version of the André–Oort conjecture, that to the contrary $\mathrm{Th}(\mathfrak{CM})$ is stable and decidable. We discuss some related structures on the complex numbers definable in $\mathbb{C}(t)$ and how their theories may be connected to the Zilber–Pink conjectures.

1. Introduction

It is a longstanding open problem [19] whether the first-order theory of the field $\mathbb{C}(t)$ of rational functions in the single variable t with coefficients from the field $\mathbb{C}$ of complex numbers is decidable. In various public lectures, Pheidas described a strategy, which he attributed to an anonymous reviewer of a grant proposal[1], for showing that this theory is undecidable based on interpreting a complicated structure on the complex numbers.

Using the fact that there are no nonconstant rational maps from the projective line to curves of positive genus, it is easy to see that $\mathbb{C}$ is definable in $\mathbb{C}(t)$. For example, we have

$$\mathbb{C} = \{a \in \mathbb{C}(t) : (\exists y \in \mathbb{C}(t))\ a^3 + y^3 = 1\}\ .$$

The harder observation is that the set CM of j-invariants of elliptic curves with complex multiplication is also definable in $\mathbb{C}(t)$. The set CM is a countably infinite set of algebraic integers and is known to have rich arithmetic. As such, one might expect that the structure $\mathfrak{CM} := (\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{CM})$ of the field of complex numbers considered with a unary predicate picking out CM would have a complicated and, in particular, an undecidable theory. Since $\mathfrak{CM}$ is interpretable in $\mathbb{C}(t)$, if this expectation were verified, then one would have succeeded in showing that the theory of $\mathbb{C}(t)$ is undecidable.

In this note, we observe that it follows from theorems of Zilber [29] on the model theory of special varieties, of Casanovas and Ziegler [2] on expansions of stable

[1]After I talked about this topic some years ago, Clifton Ealy wrote to me saying that he had asked Lou van den Dries whether he may have been the source of this observation. Van den Dries' answer on 30 June 2008 was that he recalled having learned about the interpretation of the endomorphism rings (as groups) of elliptic curves in $\mathbb{C}(t)$ from Pheidas and then noticed the consequence that this would give the definability of CM. This is not exactly a confirmation that he was the source of the observation, but it is also not exactly a denial.

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structures by predicates, and of Binyamini [1] on an effective version of the André–Oort conjecture for products of modular curves that the theory of $\mathfrak{CM}$ is stable and decidable.

The considerations which give the definability of CM also yield the definability of several other sets which are naturally infinite unions of algebraic sets. For example, the set

$$\mathsf{Isog} := \{(x,y) \in \mathbb{C}^2 : (\exists E)(\exists E')(\exists \psi : E \to E' \text{ an isogeny}) \ j(E) = x \wedge j(E') = y\}$$

is definable in $\mathbb{C}(t)$ where $j(E)$ denotes the j-invariant of the elliptic curve E and an isogeny between elliptic curves is a map of algebraic groups with a finite kernel. I would *guess* that the Zilber–Pink conjecture [17, 28, 6, 27] would be relevant in the analysis of such structures as $\mathfrak{CMI} := (\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{CM}, \mathsf{Isog})$. However, I have been unsuccessful in showing both that stability of $\mathfrak{CMI}$ follows from the Zilber–Pink conjecture and that decidability of the theory of $\mathfrak{CMI}$ follows from an effective form of the Zilber–Pink conjecture.

This paper is organized as follows. In Section 2 we recall some basic results and constructions involving elliptic curves. In Section 3 we explain in detail how to define CM, Isog, and related sets in $\mathbb{C}(t)$. Section 4 contains the proof of the stability and decidability of $\mathrm{Th}(\mathfrak{CM})$. We use the André–Oort conjecture proven by Pila to describe the quantifier-free induced structure on CM, a quantifier elimination due to Zilber to see that there is no more structure, a theorem of Casanovas and Ziegler to lift this analysis to a description of $\mathfrak{CM}$ yielding stability, and then an effective refinement of the André–Oort conjecture due to Binyamini to conclude decidability. Section 5 contains some questions and observations around the connections between the Zilber–Pink conjecture and the complexity of the logical theory of the structures on $\mathbb{C}$ interpreted in $\mathbb{C}(t)$.

As noted above, I learned about the strategy for proving the undecidability of $\mathrm{Th}(\mathbb{C}(t))$ from Thanases Pheidas. He and I talked about the relevance of the André–Oort conjecture to this problem on several occasions, initially at the 2007 ICMS workshop on Number Theory and Computability and most recently at the Summer 2022 MSRI program on Decidability, Definability and Computability in Number Theory. I had been hoping to resolve the kinds of problems described in Section 5 before releasing this paper, but it is clear that I have already waited too long. I had wanted to show the final version to Thanases, but it is too late for that. Instead, this paper is dedicated to his memory.

I have presented these ideas in various lectures, including at the 2023 Arizona Winter School. Further exposition of the number theoretic background is provided in my notes [20] for those lectures.

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2. Some background on elliptic curves

An elliptic curve E over a field K is an irreducible, projective, one-dimensional (necessarily commutative) algebraic group over K. For more details on elliptic curves, consult [8] or [24]. For higher-dimensional abelian varieties, see [12].

Notation 2.1. Fix a field K of characteristic not equal to 2 or 3. We shall take our elliptic curves in Weierstrass form. For $g_2 \in K$ and $g_3 \in K$ with $\Delta(g_2, g_3) := g_2^3 - 27g_3^2 \neq 0$ we write $E = E_{g_2,g_3}$ for the elliptic curve whose affine equation is $y^2 = 4x^3 - g_2x - g_3$ and which has one point at infinity. As explained in our references, there is an algebraic group structure on E in which the point at infinity is the identity and addition is given by the secant-tangent method.

In practice, we will only work with constant g_2 and g_3, that is, we take $K = \mathbb{C}$.

Notation 2.2. The algebraic j-invariant of E is

$$j^{\mathrm{alg}}(g_2, g_3) := \frac{12^3 g_2^3}{\Delta}$$

and it is known that over an algebraically closed field $E_{g_2,g_3} \cong E_{g_2',g_3'}$ as algebraic groups if and only if $j^{\mathrm{alg}}(g_2, g_3) = j^{\mathrm{alg}}(g_2', g_3')$.

Using this invariance under isomorphism, we write $j(E)$ for $j^{\mathrm{alg}}(g_2, g_3)$ if E is an elliptic curve which is isomorphic to the elliptic curve E_{g_2,g_3}.

We may understand elliptic curves over $\mathbb{C}$ via complex analysis. See [24, Chapter VI] and [8, Chapter 11] for some of this theory.

Notation 2.3. Let $\mathrm{GL}_2(\mathbb{R})^+$ denote the group of 2×2 matrices with real entries and positive determinant. There is an action of $\mathrm{GL}_2(\mathbb{R})^+$ on $\mathfrak{h} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ via the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d}\ .$$

If $\tau \in \mathfrak{h}$, then the complex Lie group $T_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is isomorphic, as a complex Lie group, to (the analytification of) an elliptic curve. This identification is achieved through the use of Eisenstein series and the Weierstrass $\wp$-function. The exact formulae are not really necessary for our discussion, but for the sake of culture we recall them.

Notation 2.4. The Weierstrass $\wp$-function is a meromorphic map $\wp : \mathbb{C} \times \mathfrak{h} \to \mathbb{C}$ defined by

$$\wp(z, \tau) := \frac{1}{z^2} + \sum_{(n,m)\in\mathbb{Z}^2\smallsetminus\{(0,0)\}} \left(\frac{1}{(z - n - m\tau)^2} - \frac{1}{(n + m\tau)^2} \right)\ .$$

We write $\wp'(z, \tau)$ for the derivative of $\wp$ with respect to z.

Notation 2.5. For $k \in \mathbb{Z}_+$ the Eisenstein function $E_{2k} : \mathfrak{h} \to \mathbb{C}$ is defined by

$$E_{2k}(\tau) := \sum_{(n,m)\in\mathbb{Z}^2\smallsetminus\{(0,0)\}} \frac{1}{(n + m\tau)^{2k}}\ .$$

We set $g_2(\tau) := 60E_4(\tau)$, $g_3(\tau) := 140E_6(\tau)$, and define Klein's j-function by $j(\tau) := j^{\mathrm{alg}}(g_2(\tau), g_3(\tau))$.

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With Notation 2.4 and 2.5 in place, we may recall some key facts about these functions.

The complex torus T_τ is isomorphic to (the analytification of) the elliptic curve $E_{g_2(\tau),g_3(\tau)}$ via the map $z \mapsto (\wp(z,\tau), \wp'(z,\tau))$.

A map of complex Lie groups between two one-dimensional complex tori $\psi : T_\tau \to T_{\tau'}$ is given by multiplication by some complex number λ for which $\lambda(\mathbb{Z} + \mathbb{Z}\tau) \subseteq (\mathbb{Z} + \mathbb{Z}\tau')$. We say that the complex torus T_τ has complex multiplication if and only if there is some $\lambda \in \mathbb{C} \smallsetminus \mathbb{Z}$ for which $\lambda(\mathbb{Z} + \mathbb{Z}\tau) \subseteq (\mathbb{Z} + \mathbb{Z}\tau)$. It is an easy computation to check that T_τ has complex multiplication if and only if $\mathbb{Z}[\tau]$ is a free $\mathbb{Z}$-module of rank two. We say that an elliptic curve E over $\mathbb{C}$ has complex multiplication if it (or, really, its analytification) is a complex torus with complex multiplication. Alternatively, we may detect that the elliptic curve has complex multiplication through its endomorphism ring $\mathrm{End}(E)$, the set of maps of algebraic groups $\psi : E \to E$ with addition defined by $(\psi + \phi)(x) = (\psi(x) +_E \phi(x))$ where the second addition is in the sense of the algebraic group structure on E and multiplication defined by composition of functions. For an elliptic curve E over the complex numbers, either $\mathrm{End}(E) \cong \mathbb{Z}$ or $\mathrm{End}(E)$ is an order in a quadratic imaginary field, and in particular has rank two as an abelian group. The elliptic curve E has complex multiplication if and only if we are in this second case.

It is known that $j : \mathfrak{h} \to \mathbb{C}$ is surjective. Moreover, it follows from the properties of the algebraic j-function that $j(\tau) = j(\tau')$ if and only if $T_\tau \cong T_{\tau'}$ if and only if $E_{g_2(\tau),g_3(\tau)} \cong E_{g_2(\tau'),g_3(\tau')}$ where the first isomorphism is taken in the category of complex Lie groups and the second is in the category of algebraic groups.

This equivalence may be described via the action of $\mathrm{GL}_2(\mathbb{R})^+$ on $\mathfrak{h}$ as $j(\tau) = j(\tau')$ if and only if there is some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\tau' = \gamma \cdot \tau$.

Taking $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+ := \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{GL}_2(\mathbb{R})^+$ instead, we find that the complex tori T_τ and $T_{\gamma\cdot\tau}$ are *isogenous*, even by cyclic isogenies. That is, there is a map of complex tori $T_\tau \to T_{\gamma\cdot\tau}$ whose kernel is a finite cyclic group. This relation is expressed by the modular polynomials.

Notation 2.6. For each positive integer $\ell \in \mathbb{Z}_+$, we let $\Phi_\ell(x,y) \in \mathbb{Z}[x,y]$ be the ℓ^{th} modular polynomial [10, page 335]. This polynomial Φ_ℓ is characterized by the properties that it is monic in x and for any pair of complex numbers (j_1, j_2) we have $\Phi_\ell(j_1, j_2) = 0$ if and only if there are elliptic curves E_1 and E_2 with $j(E_1) = j_1$, $j(E_2) = j_2$, and a map of algebraic groups $\psi : E_1 \to E_2$ with kernel isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$. It is a general fact that if two elliptic curves E_1 and E_2 are isogenous, meaning that there is a map of algebraic groups $\psi : E_1 \to E_2$ with finite kernel, then E_1 has complex multiplication if and only if E_2 does [24, Exercise VI.6.9].

For $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ there is a number $\ell(\gamma) \in \mathbb{Z}_+$ so that $\Phi_{\ell(\gamma)}(j(\tau), j(\gamma \cdot \tau)) \equiv 0$. That is, the elliptic curves $E_{g_2(\tau),g_3(\tau)}$ and $E_{g_2(\gamma\cdot\tau),g_3(\gamma\cdot\tau)}$ are isogenous by maps $\psi : E_{g_2(\tau),g_3(\tau)} \to E_{g_2(\gamma\cdot\tau),g_3(\gamma\cdot\tau)}$ having kernel cyclic of order $\ell(\gamma)$. Moreover, for each ℓ there are $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ with $\ell(\gamma) = \ell$.

3. Some structures interpreted in $\mathbb{C}(t)$

In this section we explain in detail how CM, Isog, and related structures are defined in $\mathbb{C}(t)$. As noted in the introduction, these constructions are not due to me, but as they are not documented in the literature, they are included here.

We start by recording the observation (which already appears in [19]) that the field of constants is definable in $\mathbb{C}(t)$.

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Lemma 3.1. *The field of complex numbers $\mathbb{C}$ is a definable subset of $\mathbb{C}(t)$.*

Proof. See [7, Section IV.2] for the necessary results on the genus of algebraic curves.

It suffices to consider any two-variable polynomial $f(x, y) \in \mathbb{Z}[x, y]$ for which the equation $f(x, y) = 0$ defines the affine part of a smooth curve X of genus at least one and for which the one-variable polynomial $f(a, y)$ is nonconstant for every choice of $a \in \mathbb{C}$. Then the formula $\phi(x) := (\exists y) f(x, y) = 0$ defines $\mathbb{C}$. Indeed, if $a \in \mathbb{C}$, then because $f(a, y) \in \mathbb{C}[y]$ is a nonconstant polynomial, it has a root. On the other hand, if $a \in \mathbb{C}(t)$ and $b \in \mathbb{C}(t)$ witnesses that $\phi(a)$ holds, then (a, b) defines a rational map $\mathbb{P}^1_{\mathbb{C}} \dashrightarrow X$. As the genus of X is at least one, this map is necessarily constant, which means in particular that a is constant.

For a concrete choice of f, we may take $f(x, y) = x^3 + y^3 - 1$. □

There is a standard construction to interpret a finite algebraic extension L of the field K in K. Indeed, for any fixed positive integer d the class of all extensions of K of degree d is uniformly interpretable. Let us formalize this fact with the following proposition.

Proposition 3.2. *For each positive integer d the class of field extensions of degree d is uniformly interpretable relative to the theory of fields. That is, there is a formula $\theta(\mu)$ in the d^3 variables μ, a formula ξ in d variables, and definable functions $\otimes_\mu$ and $\oplus_\mu$ taking $2d$ arguments and returning a d-tuple, depending on the parameters μ so that for any field K and any $d \times d \times d$-array μ with entries from K such that $K \models \theta(\mu)$, we have that $L_\mu := (K^d, \oplus_\mu, \otimes_\mu)$ is a field and $\xi(K) \subseteq L_\mu$ picks out a subfield isomorphic to K for which $[L_\mu : K] = d$. Moreover, for every extension L of K of degree d there is some choice of parameters μ for which $L_\mu \cong L$ over the identification of K with $\xi(K)$.*

Proof. While this construction is standard, allow us to recall some of the details. For an extension L of the field K of degree d, we may fix some K-basis $\{e_1, \ldots, e_d\}$ of L. Making a change of basis if need be, we may assume that $e_1 = 1$.

Let μ be the $d \times d \times d$ array of coefficients for which

$$e_i e_j = \sum_{k=1}^{d} \mu_{i,j,k} e_k \ .$$

Note that the coefficients $\mu_{i,j,k}$ are uniquely determined as $\{e_1, \ldots, e_d\}$ is a K-basis. Since $e_i e_j = e_j e_i$, we could get away with recording these values only for $i \leq j$, but there is no harm in writing too much.

Let $\xi(x_1, \ldots, x_d)$ be the formula $\bigwedge_{i=2}^{d} x_i = 0$. Define

$$(x_1, \ldots, x_d) \oplus_\mu (y_1, \ldots, y_d) := (x_1 + y_1, \ldots, x_k + y_k, \ldots, x_d + y_d)$$

and

$$(x_1, \ldots, x_d) \otimes_\mu (y_1, \ldots, y_d) := \Big(\sum_{1 \leq i,j \leq d} x_i y_j \mu_{i,j,1}, \ldots, \sum_{1 \leq i,j \leq d} x_i y_j \mu_{i,j,k}, \ldots, \sum_{1 \leq i,j \leq d} x_i y_j \mu_{i,j,d} \Big) \ .$$

Then, $L \cong L_\mu = (K^d, \oplus_\mu, \otimes_\mu)$ and $\xi(K)$ identifies with the image of K in L under the usual embedding of K-algebras.

In general, the formula $\theta(\mu)$ asserts that L_μ is a field with multiplicative identity $(1, 0, \ldots, 0)$ by relativizing the finitely many field axioms to K. □

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Let us use the construction of Proposition 3.2 to encode the function fields of the elliptic curves E_{g_2,g_3} as g_2 and g_3 range through $\mathbb{C}$ with $\Delta \neq 0$.

Notation 3.3. For $g_2 \in \mathbb{C}$ and $g_3 \in \mathbb{C}$ with $\Delta(g_2, g_3) \neq 0$, let μ be the array describing multiplication on the field $L_\mu := L^{g_2,g_3} := \mathbb{C}(t)[y]/(y^2 - 4t^3 + g_2 t + g_3)$ as a quadratic extension of $K = \mathbb{C}(t)$ relative to the basis $(e_1, e_2) = (1, y)$. That is, $\mu_{1,1,1} = 1$, $\mu_{1,1,2} = 0$, $\mu_{1,2,1} = 0$, $\mu_{1,2,2} = 1$, $\mu_{2,1,1} = 0$, $\mu_{2,1,2} = 1$, $\mu_{2,2,1} = 4t^3 - g_2 t - g_3$, and $\mu_{2,2,2} = 0$.

Proposition 3.4. *For $r = 0$, 1, and 2 there is a formula $\rho_r(g_2, g_3, g_2', g_3')$ of the four variables g_2, g_3, g_2', and g_3' so that $\mathbb{C}(t) \models \rho_r(g_2, g_3, g_2', g_3')$ if and only if*

- *g_2, g_3, g_2', and g_3' all belong to $\mathbb{C}$,*
- *$\Delta(g_2, g_3) \neq 0$ and $\Delta(g_2', g_3') \neq 0$, and*
- *$\operatorname{rk} \operatorname{Hom}(E_{g_2,g_3}, E_{g_2',g_3'}) = r$.*

Proof. By Lemma 3.1 the first two conditions may be expressed with a formula. Using the fact that every rational map from one elliptic curve to another takes the form of a map of algebraic groups followed by a translation [11, Corollary 3.7], we see that the interpretable group $E_{g_2',g_3'}(L^{g_2,g_3}) = E_{g_2',g_3'}(\mathbb{C}(E_{g_2,g_3}))$ fits into the following exact sequence:

$$0 \longrightarrow E_{g_2',g_3'}(\mathbb{C}) \longrightarrow E_{g_2',g_3'}(L^{g_2,g_3}) \longrightarrow \operatorname{Hom}(E_{g_2,g_3}, E_{g_2',g_3'}) \longrightarrow 0$$

That is, as a group, we may identify $\operatorname{Hom}(E_{g_2,g_3}, E_{g_2',g_3'})$ with the interpretable group $E_{g_2',g_3'}(L^{g_2,g_3})/E_{g_2',g_3'}(\mathbb{C})$. On general grounds (see [24, Theorem VI.6.1]), this group is isomorphic to one of

- 0, if E_{g_2,g_3} and $E_{g_2',g_3'}$ are not isogenous,
- $\mathbb{Z}$, if E_{g_2,g_3} and $E_{g_2',g_3'}$ are isogenous and $E_{g_2',g_3'}$ does not have complex multiplication, or
- $\mathbb{Z}^2$, if E_{g_2,g_3} and $E_{g_2',g_3'}$ are isogenous and $E_{g_2',g_3'}$ has complex multiplication.

We may recognize the first case that $\operatorname{Hom}(E_{g_2,g_3}, E_{g_2',g_3'}) = 0$, with the formula expressing that $E_{g_2',g_3'}(L^{g_2,g_3}) = E_{g_2',g_3'}(\mathbb{C})$, giving the formula $\rho_0(g_2, g_3, g_2', g_3')$. To recognize the second case we use the formula expressing that there is some $P \in E_{g_2',g_3'}(L^{g_2,g_3}) \smallsetminus E_{g_2',g_3'}(\mathbb{C})$ and that for all $Q \in E_{g_2',g_3'}(L^{g_2,g_3})$ there is some $R \in E_{g_2',g_3'}(L^{g_2,g_3})$ so that either $[2]R = Q$ or $[2]R = Q + P$, where multiplication by two and addition are taken in the sense of the group law on $E_{g_2',g_3'}$. This gives the formula $\rho_1(g_2, g_3, g_2', g_3')$. Finally, $\rho_2(g_2, g_3, g_2', g_3')$ may be taken to be the formula asserting that there are P and Q in $E_{g_2',g_3'}(L^{g_2,g_3})$ so that for all R none of the equations $[2]R = P$, $[2]R = Q$, or $[2]R = P + Q$ hold. □

Corollary 3.5. *Each of the sets* CM *and* Isog *is definable in $\mathbb{C}(t)$. Hence, each of the structures $\mathfrak{CM} = (\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{CM})$ and $\mathfrak{CMI} = (\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{CM}, \mathsf{Isog})$ is interpretable in $\mathbb{C}(t)$.*

Proof. The set CM is defined by the formula

$$\exists u \exists v (\rho_2(u, v, u, v) \wedge x = j^{\mathrm{alg}}(u, v))$$

and Isog is defined by

$$\exists u \exists v \exists w \exists z (x = j^{\mathrm{alg}}(u, v) \wedge y = j^{\mathrm{alg}}(w, z) \wedge (\rho_1(u, v, w, z) \vee \rho_2(u, v, w, z))) \ .$$

□

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4. Stability and decidability of $\mathfrak{CM}$

The structure $\mathfrak{CM}$ is not definable in $\mathbb{C}$ considered as a field. Indeed, the algebraically closed field $\mathbb{C}$ is strongly minimal but the set CM is a countably infinite set of algebraic integers, so that both it and its complement are infinite. One might expect that some of the arithmetic of the algebraic integers could be recovered in $\mathfrak{CM}$. As the theory of the ring of all algebraic integers is itself decidable [26], this would not yield undecidability of $\mathfrak{CM}$, but it would imply instability. In this section, we observe that it follows from the André–Oort conjecture for products of modular curves [13] and a theorem of Casanovas and Ziegler [2] on expansions of stable theories by predicates that $\mathfrak{CM}$ is stable and then using another part of the work of Casanovas and Ziegler and an effective version of the André–Oort conjecture due to Binyamini [1], that the theory of $\mathfrak{CM}$ is decidable.

Notation 4.1. Recall that for any $\mathcal{L}$-structure $\mathfrak{M}$ and a nonempty subset $A \subseteq M^n$ of the n^{th} Cartesian power of the universe M of $\mathfrak{M}$ (for some natural number n), the induced structure, A^{ind}, on A is the structure having universe A and an m-ary relation R_ϕ for each $\mathcal{L}$-formula ϕ in the variables $x_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ to be interpreted as $R_\phi^{A^{\text{ind}}} = A^m \cap \phi(\mathfrak{M})$. We refer to this associated relational language as $\mathcal{L}^{\text{ind}}$. Strictly speaking, we should indicate the arity n in the notation for the name of this language, but we omit it.

Notation 4.2. Let $\mathcal{L}^{\text{mod}}$ be the relational language having an n-ary relation symbol R_X for each irreducible component X of an embedded affine algebraic subvariety of affine n-space $\mathbb{A}^n_{\mathbb{C}}$ defined by a system of equations of the form $\Phi_{\ell_k}(x_{i_k}, x_{j_k}) = 0$ for some sequence $(i_1, j_1, \ell_1), \ldots, (i_m, j_m, \ell_m)$ where $1 \leq i_k \leq j_k \leq n$ and $\ell_k \in \mathbb{Z}_+$ for $1 \leq k \leq m$. We call such X *special varieties*. A component of a variety for which we further allow equations of the form $x_i = c$ for $c \in \mathbb{C}$ is called *weakly special*.

Let $\mathbb{C}^{\text{mod}}$ be the $\mathcal{L}^{\text{mod}}$-structure with universe $\mathbb{C}$ in which for each special variety X we interpret $R_X^{\mathbb{C}^{\text{mod}}}$ as $X(\mathbb{C})$. Let CM^{mod} be the substructure of $\mathbb{C}^{\text{mod}}$ with universe CM.

Remark 4.3. An equation of the form $\Phi_\ell(x, x) = 0$ with $\ell > 1$ defines a finite set of CM-points and more generally a pair of equations $\Phi_\ell(x, y) = 0$ and $\Phi_{\ell'}(x, y) = 0$ with $\ell \neq \ell'$ will define a finite set of pairs of CM-points. It follows from these observations that if $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ is a special variety, then for each projection $\rho_j : \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ given by $\rho_j(x_1, \ldots, x_n) = x_j$, either the image of X under ρ_j is infinite, or $\rho_j(X) = \{\xi\}$ for some CM-point ξ.

Proposition 4.4. *The theory of the structure* CM^{mod} *has quantifier elimination, is strongly minimal, and, hence, a fortiori, stable, and is decidable.*

In Section 3 of the original version of the present paper, available as a preprint at [21], a more hands-on proof of Proposition 4.4 is given. As most of this proposition is essentially proven in [29], we follow this route. However, we will need to explain how to correct some minor misstatements and to extract the present proposition from the more abstract results shown there.

Proof. Zilber works in a more general context where $\mathbb{X}$ is a complex algebraic variety admitting an analytic covering $p : \mathbb{U} \to \mathbb{X}(\mathbb{C})$ where $\mathbb{U} \subseteq \mathbb{C}^n$ is a semi-algebraic set in $\mathbb{C}^n$ for some $n \in \mathbb{Z}_+$. He defines the class of weakly special varieties by a bi-algebraicity condition: $Y \subseteq \mathbb{X}^m$ is weakly special if Y is an irreducible subvariety

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for which there is an irreducible complex analytic set $\mathbb{Y} \subseteq \mathbb{U}^m$ which is also semi-algebraic for which $Y(\mathbb{C}) = p(\mathbb{Y})$. Here, we write p also for the function defined on m-tuples by the coordinatewise action of p. He considers the structure $\mathbb{X}_w$ having universe $\mathbb{X}$ and basic m-ary predicates for each weakly special subvariety of $\mathbb{X}^m$. Theorem 4.11(b) of [29] says that the theory of $\mathbb{X}_w$ has quantifier elimination. It is known in the case we are considering, where $\mathbb{X} = \mathbb{A}^1_{\mathbb{C}}$, $\mathbb{U} = \mathfrak{h}$, and $p = j$, that the weakly special varieties as defined by Zilber are the same as what we call weakly special varieties in Notation 4.2. For instance, this follows from the modular Ax-Lindemann-Weierstrass theorem of Pila [13] and is part of the general theory of bi-algebraicity [9].

In [29, Section 6] Zilber defines a reduct $\mathbb{X}_s$ in which we have distinguished predicates only for *special* subvarieties.

The conditions as given in [29, Definition 6.1(iii)] for the special varieties are not quite right: instead, the correct definition is that $S \subseteq \mathbb{X}^n$ is *special* if S is weakly special, defined over $\mathbb{Q}^{\mathrm{alg}}$, and for some weakly co-special $\widetilde{S}$ corresponding to S the Zariski closure $\widetilde{S}^{\mathrm{Zar}}$ is defined over $\mathbb{Q}^{\mathrm{alg}}$ (whereas the original definition required $\widetilde{S}^{\mathrm{Zar}}$ to be defined over $\mathbb{Q}$). That we need to correct the definition may be seen from [29, Example 6.4] where it is asserted that the special subvarieties of Shimura varieties as usually defined match the new definition via bi-algebraicity, which is incorrect. For example, regarding the affine line as the coarse moduli space of elliptic curves via $j : \mathfrak{h} \to \mathbb{A}^1(\mathbb{C})$, the special points, i.e., the CM-points, are the images of quadratic imaginary points, not rational points.

As defined on the first page of [25], a special subvariety of a Shimura variety is a weakly special variety containing a special point. From [25, Theorem 1.2], we see that the weakly special subvarieties of Shimura varieties are precisely the weakly special subvarieties in Zilber's sense. As expressed in [9, Theorem 4.18], the bi-algebraic points in Shimura varieties of abelian type, which include products of modular curves, are exactly the special points in the Shimura theoretic sense of having a torus as a Mumford-Tate group. In our case, this is a celebrated theorem of Schneider [22] and in general is due to Cohen, Shiga, and Wolfart [3, 23]. That the special subvarieties of Shimura varieties of abelian type in our sense are the same as what appears in the literature may be seen from [25, Definition 2.1].

Zilber offers two conjectures, Conjecture A saying that every weakly special variety may be obtained as a fiber of a special variety and Conjecture B saying that for every special variety $Y \subseteq \mathbb{X}^m$ the set $S^m \cap Y(\mathbb{Q}^{\mathrm{alg}})$ is Zariski dense in Y. He notes that if Conjectures A and B hold for a given triple $(\mathbb{X}, \mathbb{U}, p)$, then one may deduce Conjecture AB that $S^{\mathrm{ind}} \preceq \mathbb{X}_s$. (Here, "$S^{\mathrm{ind}}$" is the structure induced on S from $\mathbb{X}_s$. By the André–Oort conjecture [15], this is the same as the structure induced on S from the field structure.) Zilber notes that the main theorem of [25] shows that Conjecture A holds for Shimura varieties and that Conjecture B is also known for Shimura varieties. His observations are correct if "special point" is understood in the usual Shimura theoretic sense, which in the case of Shimura varieties of abelian type, including in particular products of modular curves, agrees with our corrected definition of special point. Thus, with the corrected definition of *special variety*, these observations are correct and Shimura varieties of abelian type satisfy Conjecture AB.

Let us expand on Zilber's observations. The key to quantifier elimination for $\mathbb{X}$ in the language where every weakly special variety is named by a predicate

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is [29, Proposition 4.8]. The proof of that proposition goes through with "special" replacing the word "weakly special" as the relevant Boolean operations preserve bi-algebraicity and the fiber of a special variety over a special point is itself special. The conclusion that $S^{\mathrm{ind}} \preceq \mathbb{X}_s$ follows from this quantifier elimination result and the fact that every special variety contains a Zariski dense set of special points: by quantifier elimination it suffices to show that if $Y \subseteq \mathbb{X}^{n+1}$ is a special variety and $Z_1, \ldots, Z_m \subsetneq Y$ are special subvarieties of Y and $a \in \mathbb{X}^n$ is a special point for which $(Y \smallsetminus (Z_1 \cup \cdots \cup Z_m))_a = Y_a \smallsetminus ((Z_1)_a \cup \cdots \cup (Z_m)_a) \neq \varnothing$, then there is a special point in $Y_a \smallsetminus ((Z_1)_a \cup \cdots \cup (Z_m)_a)$. Since the fiber Y_a is itself a special variety (possibly reducible), the special points are Zariski dense in it. Thus, any nonempty component of $Y_a \smallsetminus ((Z_1)_a \cup \cdots \cup (Z_m)_a)$ contains a special point.

In our case, the set of special points S is CM by Schneider's theorem [22]. Thus, using Pila's characterization of the weakly special varieties we see that the special varieties as defined by Zilber's corrected strong bi-algebraicity condition are the same as our special varieties. Thus, $\mathsf{CM}^{\mathrm{mod}} \preceq (\mathbb{A}^1)_s$, which is strongly minimal (being a reduct of the complex field), has quantifier elimination, and is decidable (because the theory of the complex field is decidable and $\mathcal{L}^{\mathrm{mod}}$ is a computable sublanguage). □

With the next result we see that $\mathsf{CM}^{\mathrm{mod}}$ is bi-definable with the structure induced on CM from the language of rings.

Proposition 4.5. *The structures* $\mathsf{CM}^{\mathrm{mod}}$ *and* $\mathsf{CM}^{\mathrm{ind}}$*, where here we induce the structure from* $\mathbb{C}$ *in the language of rings, are bi-definable.*

Proof. This is the content of the André–Oort conjecture for products of modular curves [13, Theorem 1.1*].

Indeed, by quantifier elimination for algebraically closed fields, for each formula ϕ in the language of rings with free variables amongst $\{x_1, \ldots, x_n\}$ we may write

$$\phi(\mathbb{C}) = \bigcup_{X \in \mathcal{X}} \left(X(\mathbb{C}) \smallsetminus \bigcup_{Y \in \mathcal{Y}_X} Y(\mathbb{C}) \right)$$

where $\mathcal{X}$ is a finite set of irreducible varieties and for each $X \in \mathcal{X}$ the set $\mathcal{Y}_X$ is a finite set of proper closed irreducible subvarieties of X.

For $X \in \mathcal{X}$ and $Y \in \mathcal{Y}_X$ define $\widetilde{X} := \overline{X(\mathbb{C}) \cap \mathsf{CM}^n}^{\mathrm{Zariski}}$ and $\widetilde{Y} := \overline{Y(\mathbb{C}) \cap \mathsf{CM}^n}^{\mathrm{Zariski}}$.

Decompose these as $\widetilde{X} = \bigcup_{Z \in \mathcal{Z}_X} Z$ and then for each $Z \in \mathcal{Z}_X$ write $Z \cap \bigcup_{Y \in \mathcal{Y}_X} \widetilde{Y}$ as $\bigcup_{W \in \mathcal{W}_Z} W$ where each W is irreducible. By the André–Oort conjecture, each Z and W is a special variety. That is, R_Z and R_W are predicates in $\mathcal{L}^{\mathrm{mod}}$. Using the fact that $\widetilde{X} \subseteq X$ and $\widetilde{Y} \subseteq Y$, we see that $\widetilde{X}(\mathbb{C}) \cap \mathsf{CM}^n = X(\mathbb{C}) \cap \mathsf{CM}^n$, and likewise for $\widetilde{Y}$ and Y in place of $\widetilde{X}$ and X. Putting this all together we have

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$$\begin{aligned}
R_\phi(\mathsf{CM}^{\mathrm{ind}}) &= \phi(\mathbb{C}) \cap \mathsf{CM}^n \\
&= \bigcup_{X \in \mathcal{X}} \left(X(\mathbb{C}) \smallsetminus \bigcup_{Y \in \mathcal{Y}_X} Y(\mathbb{C}) \right) \cap \mathsf{CM}^n \\
&= \bigcup_{X \in \mathcal{X}} \left(\widetilde{X}(\mathbb{C}) \smallsetminus \bigcup_{Y \in \mathcal{Y}_X} \widetilde{Y}(\mathbb{C}) \right) \cap \mathsf{CM}^n \\
&= \bigcup_{X \in \mathcal{X}} \bigcup_{Z \in \mathcal{Z}_X} \left(Z(\mathbb{C}) \smallsetminus \bigcup_{W \in \mathcal{W}_Z} W(\mathbb{C}) \right) \cap \mathsf{CM}^n \\
&= \bigcup_{X \in \mathcal{X}} \bigcup_{Z \in \mathcal{Z}_X} \left(Z(\mathbb{C}) \cap \mathsf{CM}^n \smallsetminus \bigcup_{W \in \mathcal{W}_Z} (W(\mathbb{C}) \cap \mathsf{CM}^n) \right) \\
&= \widehat{\phi}(\mathsf{CM}^{\mathrm{mod}})
\end{aligned}$$

where

$$\widehat{\phi} = \bigvee_{X \in \mathcal{X}} \bigvee_{Z \in \mathcal{Z}_X} \left(R_Z \wedge \neg \bigvee_{W \in \mathcal{W}_Z} R_W \right)$$

is a quantifier-free formula in $\mathcal{L}^{\mathrm{mod}}$.

Thus, every basic predicate in $\mathcal{L}^{\mathrm{ind}}$ is interpreted in $\mathsf{CM}^{\mathrm{ind}}$ by a quantifier-free $\mathcal{L}^{\mathrm{mod}}$-formula. Hence, every $\mathcal{L}^{\mathrm{ind}}$-definable set in $\mathsf{CM}^{\mathrm{ind}}$ is already $\mathcal{L}^{\mathrm{mod}}$-definable. Since $\mathsf{CM}^{\mathrm{mod}}$ is a reduct of $\mathsf{CM}^{\mathrm{ind}}$, we see that $\mathsf{CM}^{\mathrm{mod}}$ and $\mathsf{CM}^{\mathrm{ind}}$ are bi-definable. □

Strong minimality of $\mathsf{CM}^{\mathrm{ind}}$ follows from Proposition 4.4 and Proposition 4.5.

Corollary 4.6. *The theory of* $\mathsf{CM}^{\mathrm{ind}}$ *is strongly minimal. A fortiori this theory is stable without the finite cover property.*

In fact, by the effective André–Oort theorem of Binyamini, the bi-definability of $\mathsf{CM}^{\mathrm{mod}}$ and $\mathsf{CM}^{\mathrm{ind}}$ is itself effective.

Proposition 4.7. *There is a computable function* $\phi \mapsto \widehat{\phi}$ *from the language of rings to* $\mathcal{L}^{\mathrm{mod}}$ *so that for every formula* ϕ *of the language of rings, we have* $\mathsf{CM}^{\mathrm{ind}} \models R_\phi \leftrightarrow \widehat{\phi}$.

Proof. The theory of algebraically closed fields itself has an effective quantifier elimination algorithm. So to compute $\widehat{\phi}$ we may take ϕ to be quantifier-free. Our proof of Proposition 4.5 shows that to compute $\widehat{\phi}$ we need only have a way to compute from an algebraic variety $X \subseteq \mathbb{A}^n_{\overline{\mathbb{Q}}}$ the (possibly reducible) variety $\overline{X(\mathbb{C}) \cap \mathsf{CM}^n}^{\mathrm{Zariski}}$.

This may be accomplished using the degree bounds of [1, Theorem 1]; as described in the footnote on [1, page 18] we may carry out such a computation.

Let us explain some of the details. Using just the language of rings and our construction in the proof of Proposition 4.5, we need to give an effective method to transform the presentation of an algebraic variety $X \subseteq \mathbb{A}^n_{\mathbb{Q}}$ into a presentation of $\overline{X(\mathbb{C}) \cap \mathsf{CM}^n}^{\mathrm{Zariski}}$. We can effectively compute the irreducible components over $X_{\overline{\mathbb{Q}}}$, that is, the components of X regarded as an algebraic variety over the algebraic numbers. Computing $\overline{Y(\mathbb{C}) \cap \mathsf{CM}^n}^{\mathrm{Zariski}}$ for each such component gives

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a way to compute $\overline{X(\mathbb{C}) \cap \mathsf{CM}^n}^{\text{Zariski}}$ itself by taking a finite union. So, we will pass to the problem where X is an absolutely irreducible variety defined over some number field K. From [1, Theorem 1], we have a bound on the degree of any maximal special subvariety of X computed just from the degree of X and $[K : \mathbb{Q}]$. From this we obtain a finite list of possible equations of the form $\Phi_\ell(x_i, x_j) = 0$ to test. The difficulty is determining which of the equations $x_j = \xi$ with $\xi \in \mathsf{CM}$ to test. The third bound in [1, Theorem 1] says that the discriminant of such a ξ is bounded by a function of $\deg(X)$, $[K : \mathbb{Q}]$, and a quantity Δ^* computed as the discriminant of the ring of integers of a certain exceptional number field K^* associated with a theorem of Siegel and Tatuzawa. Such a bound constrains ξ to a finite set, allowing us to complete our computation. However, the reader might worry that we do not really know the bound as it depends on Δ^*. Indeed, the computation of Δ^* is an unresolved problem. Nevertheless, the number Δ^* *exists* and thus so too does our algorithm: for any number D we have an algorithm which we might call A_D in which we search for possible special subvarieties with constant coordinates $\xi \in \mathsf{CM}$ with discriminant bounded by $c(D, \deg(X), [K : \mathbb{Q}])$, which is a computable function of D, $\deg(X)$, and $[K : \mathbb{Q}]$. Provided that $D \geq \Delta^*$, algorithm A_D correctly produces all of the coordinates ξ we need to consider. □

Using a theorem of Casanovas and Ziegler on expansions of stable structures, we conclude that the theory of $\mathfrak{CM}$ is tame.

Theorem 4.8. *The theory of $\mathfrak{CM}$ is stable and decidable.*

Proof. The theory of the complex numbers in the language of rings is stable, the countable set CM of CM-points is small (in the sense of [2]) in $\mathbb{C}$, and, by Corollary 4.6, its induced structure is strongly minimal, and thus stable without the finite cover property. Hence, by [2, Theorem A], $\mathfrak{CM} = (\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{CM})$ has a stable theory. From [2, Corollary 2.2] we see that $\mathrm{Th}(\mathfrak{CM})$ is axiomatized by $\mathrm{Th}(\mathbb{C}, +, -, \cdot, 0, 1) = \mathrm{ACF}_0$ and $\mathrm{Th}(\mathsf{CM}^{\mathrm{ind}})$. Combining Proposition 4.4 and Proposition 4.7, we see that $\mathrm{Th}(\mathsf{CM}^{\mathrm{ind}})$ is decidable. Hence, $\mathrm{Th}(\mathfrak{CM})$ is also decidable. □

Remark 4.9. While the results of [16] are stated in a special case, namely, where the predicate by which the algebraically closed field is expanded names a finite-rank subgroup of a semiabelian variety, the proofs apply more generally and would suffice for our application in Theorem 4.8.

5. Questions on other structures interpretable in $\mathbb{C}(t)$

While Theorem 4.8 shows that the approach to proving the undecidability of the theory of $\mathbb{C}(t)$ by observing that the structure $\mathfrak{CM}$ is interpretable in $\mathbb{C}(t)$ will not succeed, it does not resolve the question of whether the other structures interpreted using similar methods are decidable. In this section, we highlight some of those questions and discuss what we would need to know in order to resolve them.

Let us recall that

$$\mathsf{Isog} := \{(x, y) \in \mathbb{C}^2 : (\exists E)(\exists E')(\exists \psi : E \to E' \text{ an isogeny}) \; j(E) = x \wedge j(E') = y\}$$

is definable in $\mathbb{C}(t)$. Using $a \in \mathbb{C}$ as a parameter, Isog_a, the set of j-invariants of elliptic curves isogenous to an elliptic curve with j-invariant a, is thus also definable in $\mathbb{C}(t)$. Using the André-Pink-Zannier Conjecture as proven by Richard

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and Yafaev [18, Theorem 1.3] in place of the André–Oort conjecture, we obtain an analogue of Proposition 4.5.

Proposition 5.1. *For any complex number $a \in \mathbb{C}$ the structures $(\mathsf{Isog}_a^{\mathrm{ind}})_{\mathsf{Isog}_a}$ and $(\mathsf{Isog}_a^{\mathrm{mod}})_{\mathsf{Isog}_a}$ are bi-definable.*

Remark 5.2. Note that in Proposition 5.1 we need to allow for parameters from Isog_a. Implicitly, we were already doing this for $\mathsf{CM}^{\mathrm{mod}}$ in Proposition 4.5 as every element of CM is already definable by a predicate there.

Unlike for $\mathsf{CM}^{\mathrm{mod}}$, $\mathsf{Isog}_a^{\mathrm{mod}}$ is not an elementary substructure of $\mathbb{C}^{\mathrm{mod}}$. However, we may still deduce quantifier elimination for $(\mathsf{Isog}_a^{\mathrm{mod}})_{\mathsf{Isog}_a}$ from what we know for $\mathbb{C}^{\mathrm{mod}}$.

Proposition 5.3. *For any complex number $a \in \mathbb{C}$ the theory of $(\mathsf{Isog}_a^{\mathrm{mod}})_{\mathsf{Isog}_a}$ admits quantifier elimination and is therefore strongly minimal.*

Proof. Working by induction and the usual reductions using irredundant representations of constructible sets, it suffices to show that if ϕ is a quantifier-free formula in the free variables $x_1, \dots, x_{n+1}$ of the form $\phi = R_X \wedge \neg \bigvee_{Y \in \mathcal{Y}} R_Y$ where X is a weakly special variety whose constant coordinates come from Isog_a and each Y is a proper weakly special subvariety of X, then $\exists x_{n+1} \phi$ is equivalent to a quantifier-free formula. By quantifier elimination for $\mathbb{C}^{\mathrm{mod}}$ there is a quantifier-free formula θ so that $\mathbb{C}^{\mathrm{mod}} \models \theta \leftrightarrow \exists x_{n+1} \phi$. Express θ in an irredundant disjunctive normal form as $\theta = \bigvee_{Z \in \mathcal{Z}} \left(R_Z \wedge \neg \bigvee_{W \in \mathcal{W}_Z} R_W \right)$ where $\mathcal{Z}$ is a finite set of weakly special varieties and for each $Z \in \mathcal{Z}$, $\mathcal{W}_Z$ is a finite set of proper weakly special subvarieties of Z. Let $\mathcal{Z}' := \{Z \in \mathcal{Z} : \text{ no constant coordinate of } Z \text{ lies outside of } \mathsf{Isog}_a\}$. Likewise, for $Z \in \mathcal{Z}'$ let $\mathcal{W}'_Z$ consist of those $W \in \mathcal{W}_Z$ not having any constant coordinates outside of Isog_a. Let $\theta' := \bigvee_{Z \in \mathcal{Z}'} \left(R_Z \wedge \neg \bigvee_{W \in \mathcal{W}'_Z} R_W \right)$. Noting that if $V \subseteq \mathbb{A}^n_{\mathbb{C}}$ has a constant coordinate outside of Isog_a, then $V(\mathbb{C}) \cap \mathsf{Isog}_a^n = \varnothing$, we see that $\mathsf{Isog}_a \models \theta \leftrightarrow \theta'$.

Suppose that for some $b \in \mathsf{Isog}_a^n$ we have $\mathsf{Isog}_a^{\mathrm{mod}} \models \exists x_{n+1} \phi(b, x_{n+1})$. Then because Isog_a is a substructure of $\mathbb{C}^{\mathrm{mod}}$, we have $\mathbb{C}^{\mathrm{mod}} \models \exists x_{n+1} \phi(b, x_{n+1})$ as well. From the characteristic property of θ, we then have that $\mathbb{C}^{\mathrm{mod}} \models \theta(b)$, which again gives because Isog_a is a substructure that $\mathsf{Isog}_a^{\mathrm{mod}} \models \theta(b)$, which means that $\mathsf{Isog}_a \models \theta'(b)$ as θ and θ' are equivalent in Isog_a. In the other direction, if $\mathsf{Isog}_a \models \theta'(b)$, then $\mathbb{C}^{\mathrm{mod}} \models \exists x_{n+1} \phi(b, x_{n+1})$. From the equations defining X we see that either $X_b = \mathbb{A}^1_{\mathbb{C}}$ (in which case the non-emptiness of $X_b(\mathbb{C}) \smallsetminus \bigcup_{Y \in \mathcal{Y}} Y_b(\mathbb{C})$ implies that all but finitely many elements of Isog_a could serve as witnesses of $\exists x_{n+1} \phi$), the equation $x_{n+1} = c$ holds on X for some $c \in \mathsf{Isog}_a$, so that $X_b = \{c\} \subseteq \mathsf{Isog}_a$, or $\Phi_\ell(b_i, x) = 0$ holds on X_b for some $1 \leq i \leq n$ and $\ell \in \mathbb{Z}_+$ so that $X_b(\mathbb{C})$ is contained in the isogeny class of b_i, which is Isog_a. In every case, we see that $\mathsf{Isog}_a \models \exists x_{n+1} \phi$. □

Corollary 5.4. *For any $a \in \mathbb{C}$ the theory of $(\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{Isog}_a)$ is stable.*

Proof. Apply [2, Theorem A]. □

To obtain decidability for the theory of $(\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{Isog}_a)$ we would need to know that the function which takes a variety $X \subseteq \mathbb{A}^n_{\mathbb{Q}(a)^{\mathrm{alg}}}$ to $\overline{\mathsf{Isog}_a^n \cap X(\mathbb{C})}$ is computable. It seems plausible that the methods of [1] could be adapted to produce such an algorithm. In the case that a is transcendental, this can be deduced from [5].

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The argument for Proposition 5.3 can be applied to finitely many isogeny orbits and the set of CM j-invariants at one time. From this it follows that the structure $(\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{CM}, (\mathsf{Isog}_a)_{a\in\mathbb{C}})$ is stable and then if the expected analogues of Proposition 4.7 hold for Isog_a, then for any computable algebraically closed subfield $K_0 \subseteq \mathbb{C}$, the structure $(\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{CM}, (\mathsf{Isog}_a)_{a\in K_0})$ would have a decidable theory.

We have seen that the relation Isog itself is definable in $\mathbb{C}(t)$. The set Isog is not small relative to $\mathbb{C}$. Thus, the main theorems of [2] do not immediately apply. However, because the ambient theory of the complex numbers as a field is strongly minimal, the implications from stability without the finite cover property of the theory of the induced structure on Isog to stability of $(\mathbb{C}, +, -, \cdot, 0, 1, \mathsf{Isog})$ and likewise from decidability of the induced theory to decidability of the theory with a predicate still hold. Thus, we reduce to the following questions.

Question 5.5. *Is the theory of* $\mathsf{Isog}^{\mathrm{ind}}$ *stable? Is its theory decidable?*

It would seem that the Zilber–Pink conjecture should be relevant for resolving Question 5.5. However, even assuming that conjecture, the description of even the quantifier-free induced structure on Isog is much more complicated than for $\mathsf{Isog}^{\mathrm{mod}}$. In [14] a differential algebraic analogue Ξ of Isog is considered, namely the Kolchin closure of this set with the constant points removed. The set Ξ has decidable, stable induced structure because it is a reduct of something definable relative to the theory of differentially closed fields of characteristic zero. Moreover, Ξ satisfies a version of the Zilber–Pink conjecture. While $\mathsf{Isog}^{\mathrm{ind}}$ is not even a substructure of Ξ^{ind} (because Ξ is missing algebraic points), one might hope that their theories are close enough to permit a description of the sets defined with quantifiers in $\mathsf{Isog}^{\mathrm{ind}}$ through a comparison to the corresponding sets in Ξ^{ind}.

There are some other sets definable in $\mathbb{C}(t)$ using variants of the method used to prove Corollary 3.5. For general abelian varieties A and B defined over $\mathbb{C}$, it is still the case that $\mathrm{Hom}(A, B) = B(\mathbb{C}(A))/B(\mathbb{C})$. Thus, if we have a uniformly definable family $B \to S$ of abelian varieties and a uniformly definable family $\{\mathbb{C}(A)\}_{A\in F}$ of function fields of abelian varieties, there are first-order formulas specifying that the rank of $\mathrm{Hom}(A, B)$ is some fixed value.

We do not know how to interpret function fields of higher-dimensional varieties in $\mathbb{C}(t)$. If we could, then by a theorem of Eisenträger [4], we would already know that $\mathbb{C}(t)$ is undecidable. Instead, if X is a smooth projective curve of genus $g \geq 1$ over $\mathbb{C}$ and we fix some point $P \in X(\mathbb{C})$, then there is a map of algebraic varieties $f : X \to J_X$ from the curve X to its Jacobian, an abelian variety of dimension g, having the property that for any rational map $h : X \dashrightarrow A$ from X to an abelian variety A with $h(P) = 0_A$, there is a unique map $\overline{h} : J_X \to A$ with $\overline{h} \circ f = h$. Let us observe that if $h : J_X \to A$ is a map of algebraic varieties from the Jacobian of X to some abelian variety A, then $h \circ f : X \to A$ is a map of algebraic varieties from X to A. As we have noted before, every rational map from an algebraic group to an abelian variety takes the form of a map of algebraic groups followed by translation and every rational map from a smooth curve to an abelian variety is regular. Putting these observations together, we see that for any abelian variety A we have the identification $\mathrm{Hom}(J_X, A) = A(\mathbb{C}(X))/A(\mathbb{C})$. Hence, we can uniformly interpret the groups $\mathrm{Hom}(J_X, A)$ of maps from Jacobians to abelian varieties. I do not have useful conclusions about what we might do with these groups to shed light on the (un)decidability of $\mathbb{C}(t)$. Some natural questions about them follow.

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Question 5.6. *Let $A \to S$ be a family of abelian varieties over the complex numbers. Is the set $\{a \in S(\mathbb{C}) : A_a$ has complex multiplication $\}$ definable in $\mathbb{C}(t)$?*

I would guess that the answer to Question 5.6 is positive and that the definition could be obtained by working with a suitable resolution of the abelian varieties in this family by Jacobians, but to my knowledge, it remains an open problem whether such resolutions always exist.

Question 5.7. *Let $A \to S$ be a family of abelian varieties over the complex numbers and let $X \to T$ be a family of smooth projective curves of some fixed genus over $\mathbb{C}$. Fix a natural number r. Let $H_r := \{(t,s) \in (T \times S)(\mathbb{C}) : \operatorname{rk} \operatorname{Hom}(J_{X_t}, A_s) = r\}$. Is the induced structure on H_r stable? Is it decidable?*

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