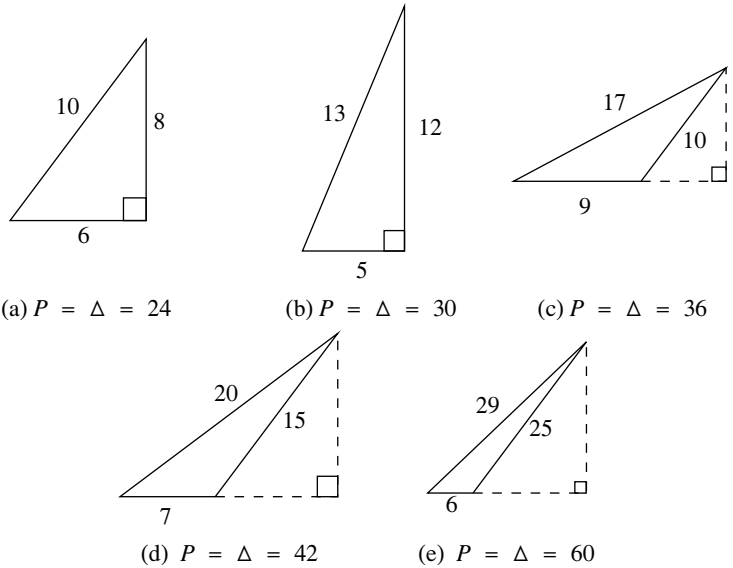
Less than equable Heronian triangles

STAN DOLAN

It is well known that there are precisely five integer-sided triangles which have equal area, Δ, and perimeter, P. These triangles are called *equable* Heronian triangles.



(a) $P = \Delta = 24$ (b) $P = \Delta = 30$ (c) $P = \Delta = 36$

(d) $P = \Delta = 42$ (e) $P = \Delta = 60$

FIGURE 1: Equable Heronian triangles

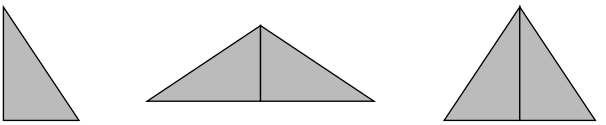
A proof of this result was given by Whitworth [1]. Since Whitworth's time, much attention has been given to triangles whose areas are integer multiples of their perimeters, for example [2, 3]. However, as this paper will show, Heronian triangles with areas less than their perimeters have some mathematical interest.

1. *What lies between* $\frac{1}{2}$ *and* 1?

The (3,4,5) triangle is an obvious example of an integer-sided triangle for which the ratio $\dfrac{\text{Area}}{\text{Perimeter}}$ is a rational number less than 1. As we shall see, it actually has the smallest such ratio.

Furthermore, it can also be used to construct two further examples, the (5, 5, 8) and (5, 5, 6) triangles.

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Area	6	12	12
Perimeter	12	18	16
Ratio	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$

This raises the issue of what ratios of $\dfrac{\text{Area}}{\text{Perimeter}}$ are possible and whether a general method can be found for determining all integer-sided triangles for which the ratio is a rational number less than 1.

The purpose of this paper is to prove that all the possible rational ratios are given by the terms of two sequences. The first few terms of these sequences are as follows and the sequences converge to $\dfrac{\sqrt{2}}{2}$ and $\dfrac{\sqrt{3}}{2}$, respectively

$$\frac{1}{2}, \frac{2}{3}, \frac{7}{10}, \frac{12}{17}, \frac{41}{58}, \frac{70}{99}, \frac{239}{338}, \cdots$$

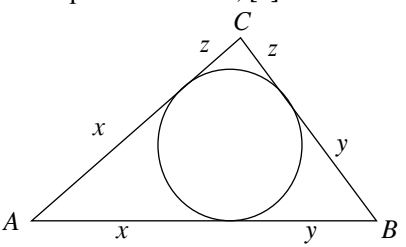
$$\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{6}{7}, \frac{19}{22}, \frac{45}{52}, \frac{71}{82}, \frac{84}{97}, \cdots$$

Alternate terms of the first series are alternate convergents of a continued fraction and the other terms are the reciprocals of the remaining convergents, multiplied by a suitable scalar so as to have the same limit. Similarly for the second series. The resultant series are then monotonic, unlike the convergents of a continued fraction.

There is a 1-1 correspondence between terms of the sequences and solution triangles other than the three given. Furthermore, it will be shown that all the new solution triangles are the result of subtracting two Pythagorean triangles.

2. *Some basic results*

We shall use a standard method for considering integer-sided triangles, based upon the incircle, [2].



Semi-perimeter

$$s = x + y + z$$

Sides $a = s - x$,

$b = s - y$, $c = s - z$

Area $\Delta = \sqrt{xyz(x + y + z)}$

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We are interested in triangles such that $\dfrac{\text{Area}}{\text{Perimeter}}$ is a rational number less than 1 and it will prove convenient to consider this as two separate conditions:

- $xyz < 4(x + y + z)$
- $(x + y + z)xyz$ is the square of a rational number.

It would considerably simplify our analysis if the perimeter of the triangle were even, since then x, y and z would all be integers. Fortunately this can be proved to be the case, as follows.

Suppose first that the perimeter $2s$ is odd. In that case, $X = 2x, Y = 2y$ and $Z = 2z$ are all odd integers. Then $(X + Y + Z)XYZ$ is both the square of a rational number and also an odd integer. It is therefore an integer square and so is 1(modulo 4).

If $X \equiv Y \equiv Z$ (modulo 4), then $(X + Y + Z)XYZ \equiv 3Z^4 \equiv 3$ (modulo 4), a contradiction. Without loss of generality we can suppose $X \not\equiv Y$ (modulo 4). Then $X + Y \equiv 0$ (modulo 4) and $(X + Y + Z)XYZ \equiv 3Z^4 \equiv 3$ (modulo 4), another contradiction.

We conclude that $2s$ must be an even integer and therefore x, y and z are positive integers.

3. *Eliminating cases*

We can suppose without loss of generality that $x \geqslant y \geqslant z$.

First suppose that $yz > 4$. Then the inequality $xyz < 4(x + y + z)$ can be used to limit the sizes of x, y and z and can be rewritten as

$$\begin{aligned} &(xz - 4)(yz - 4) < 4z^2 + 16 \\ \Rightarrow \quad &yz - 4 < \sqrt{4z^2 + 16} \\ \Rightarrow \quad &y < \frac{4}{z} + \sqrt{4 + \frac{16}{z^2}}. \end{aligned}$$

Running through the possibilities for z we see that:
If $z = 1$, then $4 < y < 4 + \sqrt{20}$ and so $y = 5,\ 6,\ 7$ or 8.
If $z = 2$, then $2 < y < 2 + \sqrt{8}$ and so $y = 3$ or 4.
If $z \geqslant 3$, then $z \leqslant y < \frac{1}{3}(4 + \sqrt{52})$ and so $y = z = 3$.

The following table shows the values of x determined by these cases and the expression which we require to be a square number.

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z	y		Square?
1	5	$5x < 4(x+6) \Rightarrow 5 \leqslant x \leqslant 23$	$5x(x+6)$
1	6	$6x < 4(x+7) \Rightarrow 6 \leqslant x \leqslant 13$	$6x(x+7)$
1	7	$7x < 4(x+8) \Rightarrow 7 \leqslant x \leqslant 10$	$7x(x+8)$
1	8	$8x < 4(x+9) \Rightarrow x = 8$	$8x(x+9)$
2	3	$6x < 4(x+5) \Rightarrow 3 \leqslant x \leqslant 9$	$6x(x+5)$
2	4	$8x < 4(x+6) \Rightarrow 4 \leqslant x \leqslant 5$	$8x(x+6)$
3	3	$9x < 4(x+6) \Rightarrow 3 \leqslant x \leqslant 4$	$9x(x+6)$

The various cases can be quickly checked by computer. There is just one solution, $x = y = 3$, $z = 2$. This gives the $(5, 5, 6)$ triangle that has already been noted.

[If an interested reader wishes to check through these cases by hand it is useful to remember the result of Section 2 that not all of x, y and z can be odd. Simple congruences can also be used to eliminate cases with little effort.]

Now suppose that $yz \leqslant 4$. The only condition that needs to be satisfied is that $(x + y + z)xyz$ is a square. The possibilities are listed in the following table.

z	y	$(x+y+z)xyz$
1	1	$x(x+2)$
1	2	$2x(x+3)$
1	3	$3x(x+4)$
1	4	$4x(x+5)$
2	2	$4x(x+4)$

Three of these possibilities, $(y, z) = (1, 1), (4, 1)$ and $(2, 2)$, require the solution in positive integers of a Diophantine equation of the form

$$x(x + n) = a^2, \text{ for } n = 2,\ 4 \text{ and } 5.$$

Then $a = x + b$, where $b < \frac{1}{2}n$ and so $b = 1$ or 2. The only solution is $n = 5$, $b = 2$, $x = 4$ which gives the $(5, 5, 8)$ triangle that has already been noted.

The two remaining possibilities, $(y, z) = (2, 1)$ and $(3, 1)$, require a more detailed treatment. We shall make use of the known solutions of Pell-type equations. An excellent and simple treatment of these is given in [4].

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4. *The equation* $2x(x + 3) = a^2$

First note that

$$2x^2 \equiv a^2 \text{ (modulo 3)}.$$

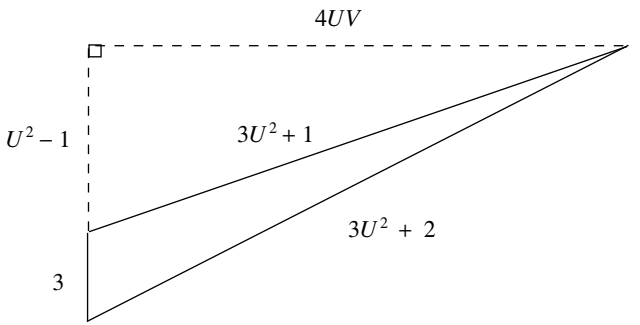
Squares are 0 or 1 (modulo 3) and so x and a must both be divisible by 3. Furthermore, a is even. Let $x = 3X$ and $a = 6A$, then $X(X + 1) = 2A^2$.

As X and $X + 1$ can have no common factor, one is a square and the other is twice a square. Therefore X is the smaller of U^2 and $2V^2$, where U and V are positive integers such that $U^2 - 2V^2 = \pm 1$. Then U and V are the numerators and denominators, respectively, of the well-known sequence of convergents to $\sqrt{2}$,

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \cdots .$$

The case $X = U^2$

Here $U^2 = 2V^2 - 1$. The sides of the triangle are obtained from $x = 3U^2, y = 2, z = 1$ and are as shown.



The construction of the right-angled triangle can be checked by Pythagoras:

$$(3U^2 + 2)^2 - (U^2 + 2)^2 = 2U^2(4U^2 + 4) = 16U^2V^2,$$

$$(3U^2 + 1)^2 - (U^2 - 1)^2 = 4U^2(2U^2 + 2) = 16U^2V^2.$$

The values of U are from the odd terms of the sequence of convergents to $\sqrt{2}$ and are

$$U = 1, 7, 41, 239, \ldots \qquad U_{n+2} = 6U_{n+1} - U_n.$$

Then

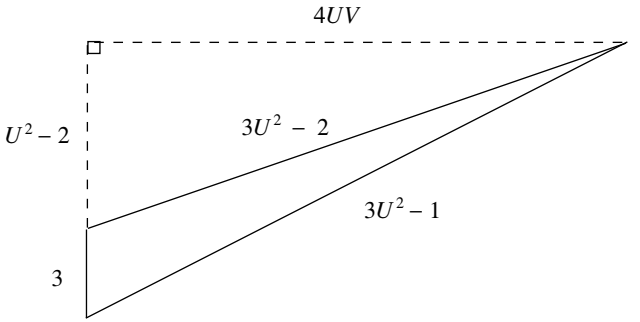
$$\frac{\text{Area}}{\text{Perimeter}} = \frac{6UV}{12V^2} = \frac{U}{2V} = \frac{1}{2}, \frac{7}{10}, \frac{41}{58}, \frac{239}{338}, \cdots \to \frac{\sqrt{2}}{2}.$$

Note that $U = 1$ gives the (3, 4, 5) triangle.

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The case $X = 2V^2$

Here $U^2 = 2V^2 + 1$. The sides of the triangle are obtained from $x = 6V^2 = 3U^2 - 3, y = 2, z = 1$ and are as shown.



Checking with Pythagoras:

$$(3U^2 - 1)^2 - (U^2 + 1)^2 = 4U^2(2U^2 - 2) = 16U^2V^2,$$
$$(3U^2 - 2)^2 - (U^2 - 2)^2 = 2U^2(4U^2 - 4) = 16U^2V^2.$$

The values of U are from the even terms of the sequence of convergents to $\sqrt{2}$ and are

$$U = 3,\ 17,\ 99,\ \ldots \qquad U_{n+2} = 6U_{n+1} - U_n.$$

Then

$$\frac{\text{Area}}{\text{Perimeter}} = \frac{6UV}{6U^2} = \frac{V}{U} = \frac{2}{3}, \frac{12}{17}, \frac{70}{99}, \ldots \to \frac{\sqrt{2}}{2}.$$

Note that we have produced a second triangle with ratio $\frac{2}{3}$.

5. *The equation* $3x(x + 4) = a^2$

First note that

$$3x^2 \equiv a^2 \text{ (modulo 4)}.$$

Squares are 0 or 1 (modulo 4) and so x and a must both be divisible by 2. Furthermore, a is divisible by 3. Let $x = 2X$ and $a = 6A$, then $X(X + 2) = 3A^2$.

The greatest common divisor of X and $X + 2$ is either 1 or 2 and so there are positive integers U and V such that

$$\{X, X + 2\} = \{U^2,\ 3V^2\} \text{ or } \{2U^2,\ 6V^2\}.$$

Squares are congruent to 0 or 1 (modulo 3) and therefore

$$X = U^2,\ U^2 = 3V^2 - 2 \text{ or } X = 6V^2,\ U^2 = 3V^2 + 1.$$

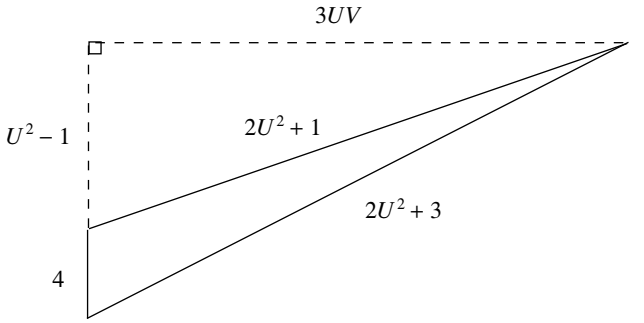
Then U and V are the numerators and denominators, respectively, of the sequence of convergents to $\sqrt{3}$ obtained from Pell's equation:

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \ldots .$$

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The case $X = U^2$

Here $U^2 = 3V^2 - 2$. The sides of the triangle are obtained from $x = 2U^2, y = 3, z = 1$ and are as shown.



The construction of the right-angled triangle can be checked by Pythagoras:

$$(2U^2 + 3)^2 - (U^2 + 3)^2 = U^2(3U^2 + 6) = 9U^2V^2,$$

$$(2U^2 + 1)^2 - (U^2 - 1)^2 = 3U^2(U^2 + 2) = 9U^2V^2.$$

The values of U are from the odd terms of the sequence of convergents to $\sqrt{3}$ and are

$$U = 1,\ 5,\ 19,\ 71,\ \ldots \qquad U_{n+2} = 4U_{n+1} - U_{n.}$$

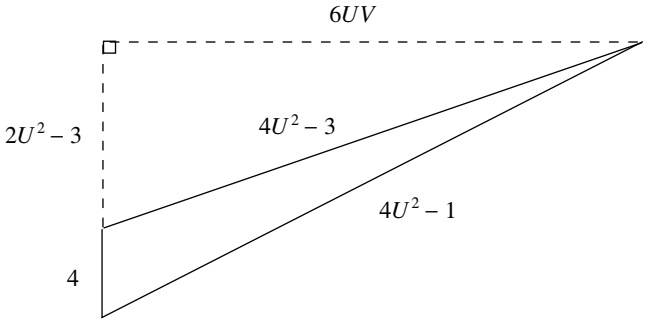
Then

$$\frac{\text{Area}}{\text{Perimeter}} = \frac{6UV}{12V^2} = \frac{U}{2V} = \frac{1}{2}, \frac{5}{6}, \frac{19}{22}, \frac{71}{82}, \ldots \to \frac{\sqrt{3}}{2}.$$

Note that, once again, $U = 1$ gives the (3, 4, 5) triangle.

The case $X = 6V^2$

Here $U^2 = 3V^2 + 1$. The sides of the triangle are obtained from $x = 12V^2 = 4U^2 - 4, y = 3, z = 1$ and are as shown.



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Checking with Pythagoras:

$$(4U^2 - 1)^2 - (2U^2 + 1)^2 = 6U^2(2U^2 - 2) = 36U^2V^2,$$

$$(4U^2 - 3)^2 - (2U^2 - 3)^2 = 2U^2(6U^2 - 6) = 36U^2V^2.$$

The values of U are from the even terms of the sequence of convergents to $\sqrt{3}$ and are

$$U = 2,\ 7,\ 26,\ 97,\ \ldots \qquad U_{n+2} = 4U_{n+1} - U_n.$$

Then

$$\frac{\text{Area}}{\text{Perimeter}} = \frac{12UV}{8U^2} = \frac{3V}{2U} = \frac{3}{4}, \frac{6}{7}, \frac{45}{52}, \frac{84}{97}, \ldots \to \frac{\sqrt{3}}{2}.$$

Note that we have produced a second triangle with ratio $\frac{3}{4}$.

6. *Applications*

The original motivation for classifying all 'less than equable' Heronian triangles came from a realisation that such triangles can form a basis for other equable shapes. For example, consider the problem of finding all integer-sided equable trapezia that are not parallelograms. Any such trapezium can be split into a triangle and a parallelogram. Furthermore, the triangle is an integer-sided triangle which has area less than its perimeter and so is one of the triangles listed in this article.

A nice exercise for an interested reader is then to determine precisely which triangles produce equable trapezia in this fashion.

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