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Codimension and analytic spread

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In this paper, I shall establish the sufficiency of certain conditions on an ideal A of a local ring Q, and on a set $\{g_1, \ldots, g_k\}$ of elements of Q generating a proper ideal G, for the ideals A and G to be analytically disjoint. Hence I shall establish an upper bound for the analytic spread of A.

The maximal ideal of Q will be denoted throughout by M, and it will be assumed that the field Q/M is infinite.

The concept of the *analytic spread* $l(A)$ of an ideal A was defined and developed by Northcott and Rees in (1). I begin by recalling some of their definitions and results.

If we define $\chi_A(r)$ to be the number of elements in a set minimally generating A^r, then, for large r, $\chi_A(r)$ is a polynomial in r; $l(A)$ is defined by taking the degree of this polynomial to be $l(A)-1$ (adopting the convention that the zero polynomial is of degree -1). Ideals A and B of Q are said to be *analytically disjoint* if

$$l(A)+l(B) = l(A+B).$$

A set $\{a_1, \ldots, a_m\}$ of elements of an ideal A is *analytically independent in* A if, whenever $V(X_1, \ldots, X_m)$ is a homogeneous polynomial of arbitrary degree r over Q such that $V(a_1, \ldots, a_m)$ is contained in MA^r, then every coefficient of V is contained in M; $\{a_1, \ldots, a_m\}$ is *analytically independent* if it is analytically independent in the ideal it generates. The ideal generated by an analytically independent set $\{a_1, \ldots, a_m\}$ is of analytic spread m; such ideals are called *basic* ideals.

An element c of Q is *analytically dependent* on A if, for some r, c^{r+1} is contained in $A((c)+A)^r$. The set of all elements of Q analytically dependent on A is an ideal of Q, denoted by $\hat{A}$. An ideal C of Q containing A is contained in $\hat{A}$ if and only if, for some r, $AC^r = C^{r+1}$; hence, for any two ideals A, B of Q, $\hat{A}\hat{B} \subseteq \widehat{AB}$.

Lemma 1. *If B is a basic ideal, then, for any integer r,*

$$\widehat{MB^r} \cap B^r = MB^r.$$

Proof. Let $\{b_1, \ldots, b_m\}$ be a set minimally generating B, and let $y \in B^r \setminus MB^r$. Then there is a polynomial $V(X_1, \ldots, X_m)$, homogeneous of degree r and having at least one coefficient not in M, such that $y = V(b_1, \ldots, b_m)$. If $y \in \widehat{MB^r}$, then, for some s,

$$y^s \in MB^r((y)+MB^r)^{s-1} \subseteq MB^{rs}.$$

Since the set $\{b_1, \ldots, b_m\}$ is analytically independent, it follows that every coefficient of $(V(X_1, \ldots, X_m))^s$ is in M, which is impossible. Hence the result follows.

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A proper ideal of Q generated by k elements $g_1, \ldots, g_k$ such that

$$0 : g_1 = 0 \quad \text{and} \quad (g_1, \ldots, g_{i-1}) : g_i = (g_1, \ldots, g_{i-1}) \quad \text{for} \quad 2 \leqslant i \leqslant k$$

is called a *general ideal of rank k*.

LEMMA 2. *Let B be a basic ideal, and suppose given a set $\{g_1, \ldots, g_k\}$ of elements of M, and a sequence $C_1, C_2, \ldots$ of ideals of Q, such that, for each r,*

$$B^{r+1} \subseteq C_{r+1} \subseteq C_r \subseteq \widehat{B^r},$$

and the ideal of Q/C_r generated by the set of residues in Q/C_r of $g_1, \ldots, k_k$, is a general ideal of rank k.

Then, if G is the ideal of Q generated by $\{g_1, \ldots, g_k\}$, B and G are analytically disjoint.

Proof. Consider first a polynomial $K(X_1, \ldots, K_k)$, homogeneous of degree p, such that $K(g_1, \ldots, g_k) \in C_s$, for some s.

Then, since the ideal of Q/C_s generated by the residues of $g_1, \ldots, g_k$ is general of rank k in Q/C_s, it is easily seen (by induction on k and p) that K takes the form

$$\sum_{1 \leqslant i < j \leqslant k} (g_i X_j - g_j X_i) L_{ij}(X_1, \ldots, X_k) + N(X_1, \ldots, X_k)$$

where L_{ij} is homogeneous of degree $(p-1)$, and N homogeneous of degree p with every coefficient in C_s; thus, we see that

(i) $K(g_1, \ldots, g_k) \in G^p C_s$, whence $G^p \cap C_s$ is shown to be $G^p C_s$;

(ii) every coefficient of K is in $G + C_s$.

We now show that, if B is minimally generated by the set $\{b_1, \ldots, b_m\}$, then the set $\{b_1, \ldots, b_m, g_1, \ldots, g_k\}$ is analytically independent. If this were not so, there would be a polynomial $V(Y_1, \ldots, Y_m, X_1, \ldots, X_k)$, homogeneous (of degree r, say) in $Y_1, \ldots, Y_m$, $X_1, \ldots, X_k$, such that $V(b_1 \ldots, b_m, g_1, \ldots, g_k) = 0$, having some coefficient not in M.

Let $V(b_1, \ldots, b_m, X_1, \ldots, X_k) = H_r(X_1, \ldots, X_k) + \ldots + H_0(X_1, \ldots, X_k)$, where, for each p, H_p is a homogeneous polynomial of degree p. Then, for each p, each coefficient of H_p is in B^{r-p}; and, since $\{b_1, \ldots, b_m\}$ is analytically independent and V has some coefficient not in M, some coefficient of some H_p is not in MB^{r-p}.

For such p, we have

$$H_r(g_1, \ldots, g_k) + \ldots + H_p(g_1, \ldots, g_k) \in B^{r-p+1} \cap G^p \leqslant C_{r-p+1} \cap G^p = C_{r-p+1} G^p.$$

Hence $H_p(g_1, \ldots, g_k) \in C_{r-p+1} G^p + G^{p+1}$.

Hence every coefficient of H_p is contained in

$$\begin{aligned} B^{r-p} \cap (C_{r-p+1} + G) &= B^{r-p} \cap (C_{r-p+1} + (B^{r-p} + C_{r-p+1}) \cap G) \\ &\leqslant B^{r-p} \cap (C_{r-p+1} + C_{r-p} \cap G) = B^{r-p} \cap (C_{r-p+1} + C_{r-p} G) \\ &\leqslant B^{r-p} \cap (\widehat{B^{r-p+1}} + C_{r-p} G) \leqslant B^{r-p} \cap \widehat{MB^{r-p}} \\ &= MB^{r-p}, \text{ by Lemma 1.} \end{aligned}$$

Hence we have a contradiction, and the result follows, since the analytic spread of an ideal cannot exceed the number of elements in a set minimally generating it.

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I now extend this result to deal with the situation where the set $\{g_1, \ldots, g_k\}$ is not specified, but where a sequence $\{C_1, C_2, \ldots\}$ of ideals of Q is given satisfying

$$B^{r+1} \leqslant C_{r+1} \leqslant C_r \leqslant \widehat{B^r}$$

for each r, and such that codim $Q/C_r \geqslant k$ for each r. The existence of elements $g_1, \ldots, g_k$ of Q whose residues in Q/C_r generate a general ideal of rank k in Q/C_r is not then obvious, even in the case $k = 1$; for the relations $B^{r+1} \leqslant C_{r+1} \leqslant C_r \leqslant \widehat{B^r}$ do not appear to imply that an embedded prime ideal of C_r is necessarily associated with C_{r+s} for all $s \geqslant 0$, so that there is no obvious reason why the set of all prime ideals of Q associated with some C_r should be finite; hence, if Q is countable, we cannot assume the existence of an element g of Q such that the residue of g in Q/C_r is not a zero-divisor in Q/C_r for any r.

I shall therefore require to establish the following Lemma:

LEMMA 3. *Suppose that Q is a complete local ring, and that $\{P_i\}_{i=1,2,\ldots}$ is a countable collection of prime ideals of Q, none of which is equal to M.*

Then there is an element x of M which is not contained in any P_i.

Proof. Since Q is Noetherian, any ascending chain

$$P_{i_1} \subseteq P_{i_2} \subseteq \ldots \subseteq P_{i_r} \subseteq \ldots$$

of ideals of the collection has a maximal element; hence

$$\bigcup_{i=1}^{\infty} P_i = U\{P_i\colon P_i \subseteq P_j \Rightarrow P_i = P_j\}.$$

We may thus suppose without loss of generality that if $P_i \subseteq P_j$, then $P_i = P_j$.

We construct inductively a sequence $\{x_r\}$ of elements of M as follows:

Choose $x_1 \in M \setminus P_1$. Suppose we have chosen $x_1, \ldots, x_{r-1}$ in M such that, if $i \leqslant s \leqslant r-1$, then $x_s \notin P_i$ and $x_i - x_s \in M^i \cap P_i$.

If $x_{r-1} \notin P_r$, set $x_r = x_{r-1}$; if $x_{r-1} \in P_r$, choose y_r not in P_r but in $P_1 \cap \ldots \cap P_{r-1}$ (which is possible since P_r contains none of $P_1, \ldots, P_{r-1}$), and set $x_r = x_{r-1} + y_r^r$.

In either case, x_r is in none of $P_1, \ldots, P_r$; and $x_r - x_{r-1}$ is in $M^r \cap P_1 \cap \ldots \cap P_{r-1}$, so that, if $i < r$, then $x_i - x_r$ is in $M^i \cap P_i$.

Suppose x_r chosen in this fashion for all r. The sequence $\{x_r\}$ is then a Cauchy sequence in the complete local ring Q, and hence has a limit x in Q. Since each P_i is closed in Q, and $x_i - x_s$ is in P_i for all $s \geqslant i$, we have $x_i - x$ in P_i for each i; but $x_i \notin P_i$, so $x \notin P_i$ for any i.

COROLLARY. *If $\{C_r\}_{r=1,2,\ldots}$ is a countable collection of ideals of the complete local ring Q, and if* codim $Q/C_r \geqslant k$ *for each r, then there is a set $\{g_1, \ldots, g_k\}$ of elements of M such that, for each r, the ideal of Q/C_r generated by the residues of $g_1, \ldots, g_k$ is a general ideal of rank k in Q/C_r.*

Proof. If $k = 0$, there is nothing to prove. Suppose therefore that, for some $k > 0$, the result is known to be true when k is replaced by $k-1$.

Then, if codim $Q/C_r \geqslant k$ for each r, we can choose $g_1, \ldots, g_{k-1}$ in M such that, for each r, the ideal of Q/C_r generated by their residues is a general ideal of rank $k-1$; hence, for each r, codim $Q/(C_r + (g_1, \ldots, g_{k-1})) \geqslant 1$. The set of all prime ideals of Q associated

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with $C_r + (g_1, \ldots, g_{k-1})$ for some r, which is clearly countable, cannot therefore include M. Thus, since Q is complete, it follows from the Lemma that there exists an element g_k of M not in any associated prime of $C_r + (g_1, \ldots, g_{k+1})$ for any r; whence, by the choice of $g_1, \ldots, g_{k-1}$, the ideal of Q/C_r generated by the residues of $g_1, \ldots, g_k$ is a general ideal for each r, as required.

I shall require also the following Lemma, for which I am indebted to D. Rees.

LEMMA 4. *No proper ideal of Q can have analytic spread greater than* dim Q.

Proof. If $\dim Q = 0$, then every proper ideal of Q is nilpotent and thus of analytic spread zero. Suppose therefore that $\dim Q > 0$.

For any ideal A of Q, we can find a basic ideal B of analytic spread $l(A)$ such that $B \leqslant A$ and $\hat{A} = \hat{B}$ (a *minimal reduction* of A; see (1), § 2, Theorem 1, and § 7, Theorem 4). Hence it is sufficient to show that no basic ideal can have analytic spread greater than $\dim Q$. Let B be a basic ideal minimally generated by a set $\{b_1, \ldots, b_m\}$.

Suppose first that Q is not a domain, and let $N_1, \ldots, N_s$ be the minimal prime ideals of Q. Write m_i for the analytic spread of $B + N_i$ in Q/N_i. Then it is clear that $m \geqslant m_i$ for each i. If $m > m_i$ for each i, then for each i there is a polynomial $V_i(X_1, \ldots, X_m)$, homogeneous in $X_1, \ldots, X_m$, with some coefficient not in M, such that $V_i(b_1, \ldots, b_m) \in N_i$; whence, for some r,

$$(V_1(b_1, \ldots, b_m) \ldots V_s(b_1, \ldots, b_m))^r = 0.$$

But this is impossible, since the set $\{b_1, \ldots, b_m\}$ is analytically independent and $(V_1 \ldots V_s)^r$ has some coefficient not in M.

Thus $m = \max m_i$, so we may suppose without loss of generality that Q is a domain.

Let R_B be the subring $Q[b_1 X, \ldots, b_m X]$ of the ring $Q[X]$ of polynomials over Q. Then the ring R_B/MR_B is isomorphic to the ring of polynomials in m variables over Q/M; so MR_B is a prime ideal of R_B. Since Q is a domain, so is $Q[X]$, and hence so is R_B; so the rank of MR_B in R_B is at least one. Let Q' be the homogeneous ring of fractions $(R_B)_{MR_B}$, and let M' be its maximal ideal. Clearly $\dim Q' \geqslant 1$.

Applying a result of Zariski and Samuel ((2), p. 326, Proposition 2), I deduce that, if F, F' are the fields of fractions of Q, Q' and $\operatorname{tr deg}_F F'$ the transcendence degree of F' over F, then

$$\dim Q' + \operatorname{tr deg}_{Q/M} Q'/M' \leqslant \dim Q + \operatorname{tr deg}_F F'.$$

But it is easily seen that Q' has the same field of fractions as Q; and Q'/M' is the homogeneous field of fractions of R_B/MR_B, whence $\operatorname{tr deg}_{Q/M} Q'/M' = m - 1$. Hence $\dim Q \geqslant m$, as required.

THEOREM. *Let B be a basic ideal of Q, and $\{C_1, \ldots, C_r, \ldots\}$ a sequence of ideals of Q such that, for each r,*

$$B^{r+1} \subseteq C_{r+1} \subseteq C_r \subseteq \widehat{B^r}$$

and $\operatorname{codim} Q/C_r \geqslant k$.

Then $l(B) \leqslant \dim Q - k$.

Proof. Let $\bar{Q}$ be the completion of Q, $\bar{B}$ that of B, etc. Then

$$\operatorname{codim} \overline{Q/C_r} = \operatorname{codim} \bar{Q}/\bar{C}_r = \operatorname{codim} Q/C_r,$$

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and, since B^r/MB^r is a vector space over Q/M,

$$B^r/MB^r \text{ is isomorphic to } \overline{B^r/MB^r} = \bar{B}^r/\bar{M}\bar{B}^r,$$

so that the analytic spread of $\bar{B}$ in $\bar{Q}$ is the same as that of B in Q.

It is therefore sufficient to prove that the Theorem holds if Q is complete; but in that case, by the Corollary to Lemma 3, the hypothesis that codim $Q/C_r \geqslant k$ for each r implies that there exist elements $g_1, \ldots, g_k$ of Q such that, for each r, the ideal of Q/C_r generated by the residues of $g_1, \ldots, g_k$ is a general ideal of rank k in Q/C_r. Hence, by Lemma 2,

$$l(B + (g_1, \ldots, g_k)) = l(B) + k,$$

whence, by Lemma 4, $l(B) \leqslant \dim Q - k$, as required.

COROLLARY. *For any ideal A of Q, the following inequalities hold*:

(i) $l(A) \leqslant \dim Q - \min \operatorname{codim} Q/A^r$

(ii) $l(A) \leqslant \dim Q - \min \operatorname{codim} Q/\widehat{A^r}$.

If Q is a Cohen–Macaulay ring, then

$$l(A) \leqslant \min (\max (\operatorname{hd}_Q Q/A^r, \operatorname{hd}_Q Q/\widehat{A^r})).$$

Proof. Let B be a minimal reduction of A. Then, since

$$B^{r+1} \subseteq A^{r+1} \subseteq A^r \subseteq \widehat{B^r} \quad \text{and} \quad B^{r+1} \subseteq \widehat{A^{r+1}} \subseteq \widehat{A^r} \subseteq \widehat{B^r},$$

(i) follows at once from the Theorem on setting $C_r = A^r$, and (ii) follows on setting $C_r = \widehat{A^r}$.

Since, if the Q-module Q/C_r is of finite homological dimension,

$$\operatorname{hd}_Q Q/C_r = \operatorname{codim} Q - \operatorname{codim} Q/C_r,$$

the last inequality follows at once if Q is Cohen–Macaulay.

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