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ON MEASURE OF SYSTEM INTERVAL AVAILABILITY

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1. INTRODUCTION

Let $\mathcal{S}$ be a system that can be in one of two states, up or down. We can interpret the up state as the working state and the down state as the nonworking state for system $\mathcal{S}$. Assume the system $\mathcal{S}$ is up at time $t = 0$. Denote the elapsed time from zero until $\mathcal{S}$ enters down state by U_1, and the elapsed time from then until $\mathcal{S}$ is up again by D_1. The interval $[0, U_1 + D_1)$ is called cycle 1, and, at that point, cycle 2 starts and the sequence is repeated so on and so forth. In general, the up time and down time in the jth cycle are denoted by U_j and D_j, respectively, for $j \geq 1$. We can introduce a binary process $\{X(t), t \geq 0\}$ to describe the state of $\mathcal{S}$ at time t using $X(t) = 1(0)$ to indicate $\mathcal{S}$ is up (down). Many problems require finding the probability that $\mathcal{S}$ is up at time t, that is, $P(X(t) = 1)$. However, an even more interesting question to answer is what the probability is that $\mathcal{S}$ is in the up state in an interval of length w starting at time t, that is, $P(X(s) = 1, t \leq s \leq t + w)$. Due to the complexity of this problem, no explicit expressions are available for most systems even in the case where $(U_j, D_j), j \geq 1$ are i.i.d. Fortunately, in practice engineers are more interested in the long-run properties of the above two probabilities.

By assuming $(U_j, D_j), j \geq 1$ are i.i.d., it has been shown under mild conditions that $\lim_{t\to\infty} P(X(t) = 1) = \mu/(\mu + \nu)$ and $\lim_{t\to\infty} P(X(s) = 1, t \leq s \leq t + w) = \int_w^\infty \bar{F}(x)\, dx/(\mu + \nu)$, where $F(\cdot)$ is the common distribution of U_j's, $\bar{F}(x) \equiv 1 - F(x)$, $\mu \equiv E(U_1)$, and $\nu \equiv E(D_1)$. The two limits are called the stable (point) availability and stable interval availability for $\mathcal{S}$, respectively. The proofs are based on properties of renewal processes. More details can be found in Barlow and Proschan [1] and Baxter [2]. Bergman [3] mentioned that recent buyers require the vendors to guarantee the availability performance. For this purpose Takács [10] and Rényi [8]

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defined a quantity $\bar{A}(t)$, which is the proportion of up time of system in the interval $[0, t]$, and showed that $\bar{A}(t)$ is asymptotically normal with mean $\mu/(\mu + \nu)$ and variance $(\nu^2\sigma_U^2 + \mu^2\sigma_D^2)/(\mu + \nu)^3$t, where μ and σ_U^2, and ν and σ_D^2 are the mean and variance for the up time and for the down time, respectively. Rise [9] used the asymptotic properties of the average availability $\bar{A}(t)$ for reliability test plans. Gut and Janson [5] considered the case when $\{(U_n, D_n), n \geq 1\}$ are i.i.d. but with U_n and D_n possibly dependent, and showed that $\bar{A}(t)$ is also asymptotically normal. Mi [7] generalized some results about point availability to the situation when $(U_j, D_j), j \geq 1$ are independent but not necessarily identically distributed.

The present article again considers the case of independent but not necessarily identically distributed $(U_j, D_j), j \geq 1$, and examines interval availability for such systems. The concept that a sequence of distributions is dominated by a nonnegative bounded function introduced in Mi [7] will be used without redefining it repeatedly in this article. Two quantities that can be used to measure interval availability are defined. One is $\bar{A}_n(w)$ and the other is $\bar{A}(t; w)$ and they are called the average interval availability. Both definitions are given in the next section. For each of these two average availabilities the almost sure convergence property is discussed in Section 2, and the asymptotic normality is established in Section 3. For the exponential family of distributions, a commonly used distribution family, a sufficient condition that guarantees certain assumptions needed for asymptotic normality to hold is given in Section 4. Some examples are also shown in that section.

2. ALMOST SURE CONVERGENCE

Suppose $\mathcal{S}$ is a two-state system: up and down. Let the up time U_j in the jth cycle follow distribution F_j and the down time D_j in the jth cycle follow distribution $G_j (j \geq 1)$. For any fixed $w > 0$ we define

$$X_j \equiv (U_j - w) I_{(U_j > w)} \qquad j \geq 1,$$

where I_A is the indicator function of event A.

As a measure of the system interval availability we can use average interval availability in the first n cycles $\bar{A}_n(w)$ defined by

$$\bar{A}_n(w) = \frac{\sum_{j=1}^{n} X_j}{\sum_{j=1}^{n} U_j + \sum_{j=1}^{n} D_j}. \tag{1}$$

THEOREM 1. *Suppose*

i. $\{F_j, j \geq 1\}$ *converges to a distribution function* F *on* $[0,\infty)$

ii. *both* $\{F_j, j \geq 1\}$ *and* $\{G_j, j \geq 1\}$ *are dominated by* H *that has finite first moment*

iii. $\nu \equiv \lim_{n\to\infty} \sum_{j=1}^{n} \nu_j / n$ *exists, where* $\nu_j = E(D_j)$

iv. $\mu + \nu > 0$, *where* $\mu \equiv \int_0^\infty \bar{F}(t)\, dt$ *and* $\bar{F}(t) \equiv 1 - F(t)$.

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Then with probability one

$$\lim_{n\to\infty} \bar{A}_n(w) = \frac{\mu(w)}{\mu+\nu}, \tag{2}$$

where

$$\mu(w) \equiv \int_w^\infty \bar{F}(t)\,dt. \tag{3}$$

PROOF: First of all, note that $E(U_j) < \infty$ and $E(D_j) < \infty$ because of assumption (ii). Further, from the definition of dominance it follows that $\sum_{j=1}^n F_j/n$ is also dominated by H.

Hence,

$$\frac{1}{n}\sum_{j=1}^n E(U_j) = \int_0^\infty \left(\frac{1}{n}\sum_{j=1}^n \bar{F}_j(t)\right) dt \to \int_0^\infty \bar{F}(t)\,dt \equiv \mu$$

as $n \to \infty$ by the Dominance Convergence Theorem. Applying Lemma 1 of Mi [7], we obtain

$$\frac{1}{n}\sum_{j=1}^n U_j \to \mu \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^n D_j \to \nu \tag{4}$$

with probability one as $n \to \infty$.

Clearly the sequence $\{X_j, j \geq 1\}$ is also dominated by H. In order to see this, note that for any $t > 0$

$$\begin{aligned} P(X_j > t) &= P(U_j > w + t) \\ &\leq \bar{H}(w+t) \leq \bar{H}(t). \end{aligned}$$

The mean of X_j is given as

$$E(X_j) = \int_0^\infty (x-w) I_{(x>w)}\,dF_j(x) = \int_w^\infty \bar{F}_j(x)\,dx \equiv \mu_j(w), \qquad j \geq 1.$$

Thus,

$$\frac{1}{n}\sum_{j=1}^n E(X_j) = \int_w^\infty \left(\frac{1}{n}\sum_{j=1}^n \bar{F}_j(x)\right) dx \to \int_w^\infty \bar{F}(x)\,dx$$

as $n \to \infty$. Applying Lemma 1 of Mi [7] again gives

$$\frac{1}{n}\sum_{j=1}^n X_j \to \int_w^\infty \bar{F}(x)\,dx \equiv \mu(w) \tag{5}$$

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with probability one as $n \to \infty$. Combining the above results of Eqs. (4) and (5), we conclude that

$$\lim_{n\to\infty} \bar{A}_n(w) = \frac{\int_w^\infty \bar{F}(t)\,dt}{\mu + \nu}$$

with probability one. ■

Remark: In the i.i.d. case certainly $\bar{A}_n(w)$ converges to the system stable interval availability.

COROLLARY: *$\bar{A}_n(w)$ converges to $\int_w^\infty \bar{F}(t)\,dt/(\mu+\nu)$ in L_p norm.*

PROOF: The result is true because $0 \le \bar{A}_n(w) \le 1$. ■

In addition to $\bar{A}_n(w)$ we can consider another measure $\bar{A}(w;t)$ of system interval availability, the average interval availability in interval $[0,t]$. As in [7], we define $W_j = U_j + D_j, S_0 = 0, S_n = \sum_{j=1}^n W_j$,

$$N(t) = \sup\{n \ge 0 : S_n \le t\}$$

$$\tau(t) = N(t) + 1$$

and $\lambda(t)$ to be the indicator function of the event

$$\{S_{N(t)} + U_{\tau(t)} \le t < S_{\tau(t)}\}.$$

We further define

$$\alpha(t) \equiv \left(\sum_{j=1}^{\tau(t)} X_j\right)\lambda(t) + \left(\sum_{j=1}^{N(t)} X_j + (t - S_{N(t)} - w)I_{(t-S_{N(t)}>w)}\right)(1-\lambda(t)).$$

Then $\bar{A}(w,t)$ is defined as

$$\bar{A}(w;t) \equiv \frac{\alpha(t)}{t}. \tag{6}$$

THEOREM 2: *Under the conditions* (i)–(iii) *of Theorem* 1, *it holds that*

$$\bar{A}(w;t) \to \frac{\int_w^\infty \bar{F}(x)\,dx}{\mu+\nu} \text{ a.s.} \quad \text{and in } L_p \tag{7}$$

as $t \to \infty$.

PROOF: From the definitions of $N(t), \tau(t)$ and the assumption $\mu + \nu > 0$ it follows that $N(t) \to \infty, \tau(t) \to \infty$ a.s. as $t \to \infty$.

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As in the proof of Theorem 2 of Mi [7] the following holds:

$$\frac{\tau(t)}{t} \to \frac{1}{\mu+\nu} \text{ a.s.}, \quad \frac{N(t)}{t} \to \frac{1}{\mu+\nu} \text{ a.s.},$$
$$\frac{W_{\tau(t)}}{t} \to 0 \text{ a.s.}, \quad \frac{t - S_{N(t)}}{t} \to 0 \text{ a.s.}$$

as $t \to \infty$.

Now, by [7, Lemma 1], it can be seen that

$$\frac{\sum_{j=1}^{N(t)} X_j}{t} = \frac{N(t)}{t} \cdot \frac{\sum_{j=1}^{N(t)} X_j}{N(t)} \to \frac{\int_w^\infty \bar{F}(x)\,dx}{\mu+\nu}, \text{ a.s.}$$

and

$$\frac{\sum_{j=1}^{\tau(t)} X_j}{t} = \frac{\tau(t)}{t} \cdot \frac{\sum_{j=1}^{\tau(t)} X_j}{\tau(t)} \to \frac{\int_w^\infty \bar{F}(x)\,dx}{\mu+\nu}, \text{ a.s.}$$

Therefore, from the definition of $\alpha(t)$, we obtain

$$\frac{\alpha(t)}{t} \to \frac{\int_w^\infty \bar{F}(x)\,dx}{\mu+\nu}, \text{ a.s.}$$

as $t \to \infty$. ■

3. ASYMPTOTIC NORMALITY

In the present section we consider the asymptotic normality property of $\bar{A}_n(w)$ and $\bar{A}(t;w)$. For the sake of simplicity, in addition to the independence of $U_1, U_2, \ldots$, and that of $D_1, D_2, \ldots$, we further assume U_j and D_j are independent of each other for all $j \geq 1$. The case of dependent U_j and D_j can be studied by modifying the discussion for the case of independent U_j and D_j. The notation is the same as introduced in the previous sections.

First we shall establish two auxiliary results needed later.

LEMMA 1: *The inequality below holds*

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} < 2, \qquad \forall n \geq 1.$$

PROOF: It is easy to see that

$$\frac{1}{\sqrt{n}} \sum_{j=2}^{n} \frac{1}{\sqrt{j}} \leq \frac{1}{\sqrt{n}} \int_1^n \frac{1}{\sqrt{x}}\,dx = 2 - \frac{2}{\sqrt{n}}$$

and thus the desired inequality. ■

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LEMMA 2: *Suppose $F, F_n, n \geq 1$ are distribution functions on $[0,\infty)$ and F_n converges to F as $n \to \infty$. Let $0 \leq \xi(t)$ be a nondecreasing bounded function on $[0,\infty)$, $\xi(0) = 0$, $\bar{\xi}(t) \equiv \xi(\infty) - \xi(t)$, and $\int_0^\infty \bar{\xi}(t)\, dt < \infty$. If*

$$|\bar{F}_n(t) - \bar{F}(t)| \leq \bar{\xi}(t) o^*(n^{-1/2}), \qquad \forall t \geq 0, \quad \forall n \geq 1,$$

where the quantity $o^*(n^{-1/2})$ does not depend on t and satisfies $\sqrt{n} \cdot o^*(n^{-1/2}) \to 0$ as $n \to \infty$ (this interpretation of $o^*(n^{-1/2})$ remains the same through out the paper), then

$$\frac{1}{n} \sum_{j=1}^{n} \int_c^\infty |\bar{F}_j(t) - \bar{F}(t)|\, dt = o(n^{-1/2})$$

for any $c \geq 0$.

PROOF: From the assumption for any given $\varepsilon > 0$ there exists $n_0 \geq 1$ such that

$$|\bar{F}_j(t) - \bar{F}(t)| \leq \bar{\xi}(t) \cdot \frac{\varepsilon}{\sqrt{j}}, \qquad \forall t \geq 0, \quad \forall j \geq n_0.$$

Hence,

$$\begin{aligned}
&\sqrt{n} \cdot \frac{1}{n} \sum_{j=1}^{n} \int_c^\infty |\bar{F}_j(t) - \bar{F}(t)|\, dt \\
&\quad = \sqrt{n} \cdot \frac{1}{n} \sum_{j=1}^{n_0} \int_c^\infty |\bar{F}_j(t) - \bar{F}(t)|\, dt + \sqrt{n} \cdot \frac{1}{n} \sum_{j=n_0+1}^{n} \int_c^\infty |\bar{F}_j(t) - \bar{F}(t)|\, dt \\
&\quad \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n_0} \int_c^\infty |\bar{F}_j(t) - \bar{F}(t)|\, dt + \sqrt{n} \cdot \frac{1}{n} \sum_{j=n_0+1}^{n} \frac{\varepsilon}{\sqrt{j}} \int_c^\infty \bar{\xi}(t)\, dt \\
&\quad \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n_0} \int_c^\infty |\bar{F}_j(t) - \bar{F}(t)|\, dt + \frac{\varepsilon}{\sqrt{n}} \int_c^\infty \bar{\xi}(t)\, dt \left(\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \right) \\
&\quad \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n_0} \int_c^\infty |\bar{F}_j(t) - \bar{F}(t)|\, dt + 2\varepsilon \int_c^\infty \bar{\xi}(t)\, dt,
\end{aligned}$$

here the last inequality comes from Lemma 1.

Hence,

$$\limsup_{n \to \infty} \left(\sqrt{n} \cdot \frac{1}{n} \sum_{j=1}^{n} \int_c^\infty |\bar{F}_j(t) - \bar{F}(t)|\, dt \right) \leq 2\varepsilon \int_c^\infty \bar{\xi}(t)\, dt.$$

The arbitrariness of $\varepsilon > 0$ gives

$$\lim_{n \to \infty} \sqrt{n} \cdot \frac{1}{n} \sum_{j=1}^{n} \int_c^\infty |\bar{F}_j(t) - \bar{F}(t)|\, dt = 0.$$

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That is,

$$\frac{1}{n}\sum_{j=1}^{n}\int_{c}^{\infty}|\bar{F}_j(t)-\bar{F}(t)|\,dt=o(n^{-1/2}). \qquad \blacksquare$$

DEFINITION 1: *Suppose random variable X has distribution F. A point $x_0 \in (-\infty,\infty)$ is said to be a support point of F (or X) if for any $\varepsilon > 0$, $P(|X - x_0| < \varepsilon) > 0$. The set consisting of all support points of F is called the support set of F.*

THEOREM 3: *Assume the following conditions hold*:

i. *there exists a distribution F on $[0,\infty)$ and a nondecreasing bounded function $\xi(t)$ such that $\xi(0) = 0$, $\int_0^\infty \bar{\xi}(t)\,dt < \infty$, where $\bar{\xi}(t) \equiv \xi(\infty) - \xi(t)$, and $|\bar{F}_n(t) - \bar{F}(t)| \le \bar{\xi}(t)o^*(n^{-1/2})$, $\forall t \ge 0$*
ii. *the support set of F has at least three points and at least one is in (w,∞)*
iii. *both $\{F_j, j \ge 1\}$ and $\{G_j, j \ge 1\}$ are dominated by H satisfying $\int_0^\infty t^{2+\delta}\,dH(t) < \infty$ for a positive constant δ*
iv. $\sum_{j=1}^{n}\nu_j/n - \nu = o(n^{-1/2})$
v. *the limit $\sigma_D^2 \equiv \lim_{n\to\infty}\sum_{j=1}^{n}\mathrm{Var}(D_j)/n$ exists.*

Then as $n \to \infty$

$$\frac{\bar{A}_n(w)-\mu(w)/(\mu+\nu)}{\sigma/((\mu+\nu)^2\sqrt{n})} \to N(0,1) \qquad \textbf{(8)}$$

in law as $n \to \infty$, where σ^2 is defined by

$$\sigma^2 = \left\{(\mu+\nu)^2\left[2\int_w^\infty t\bar{F}(t)\,dt - 2w\mu(w) - \mu^2(w)\right] + \mu^2(w)\sigma_U^2 - 2(\mu+\nu)\mu(w)\left[2\int_w^\infty t\bar{F}(t)\,dt - (w+\mu)\mu(w)\right]\right\} + \mu^2(w)\sigma_D^2 \qquad \textbf{(9)}$$

and

$$\sigma_U^2 \equiv \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathrm{Var}(U_j).$$

PROOF: We have

$$\frac{\bar{A}_n(w)-\mu(w)/(\mu+\nu)}{\sigma/((\mu+\nu)^2\sqrt{n})} = \frac{n(\mu+\nu)}{\sum_{j=1}^{n}(U_j+D_j)}\cdot\frac{\sum_{j=1}^{n}(Y_j-\mu(w)D_j)}{\sigma\sqrt{n}},$$

where

$$Y_j \equiv (\mu+\nu)X_j - \mu(w)U_j.$$

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It is obvious from assumption (iii) and [7, Lemma 1] that

$$\frac{n(\mu+\nu)}{\sum_{j=1}^{n}(U_j+D_j)} \to 1 \text{ a.s.}$$

Hence, by Slutsky's theorem, in order to show the desired asymptotic normality (8) it suffices to show that

$$\frac{\sum_{j=1}^{n}(Y_j-\mu(w)D_j)}{\sigma\sqrt{n}} \to N(0,1) \quad \text{in law} \tag{10}$$

as $n \to \infty$.

By simple computation, the mean of Y_j is given as

$$E(Y_j-\mu(w)D_j) = \mu\mu_j(w) - \mu(w)\mu_j + \nu\mu_j(w) - \mu(w)\nu_j,$$

where $\mu_j(w) = \int_w^\infty \bar{F}_j(x)\,dx$. Hence,

$$\frac{\sum_{j=1}^{n}(Y_j-\mu(w)D_j)}{\sigma\sqrt{n}} = \frac{\sum_{j=1}^{n}[(Y_j-\mu(w)D_j)-E(Y_j-\mu(w)D_j)]}{\sigma\sqrt{n}} + \frac{\sum_{j=1}^{n}E(Y_j-\mu(w)D_j)}{\sigma\sqrt{n}}. \tag{11}$$

Let us consider $\sum_{j=1}^{n}E(Y_j-\mu(w)D_j)/\sigma\sqrt{n}$. We have

$$\begin{aligned}\frac{\sum_{j=1}^{n}E(Y_j-\mu(w)D_j)}{\sigma\sqrt{n}} &= \frac{\mu\sum_{j=1}^{n}\mu_j(w) - \mu(w)\sum_{j=1}^{n}\mu_j + \nu\sum_{j=1}^{n}\mu_j(w) - \mu(w)\sum_{j=1}^{n}\nu_j}{\sigma\sqrt{n}} \\ &= \frac{n\mu\left(\sum_{j=1}^{n}\mu_j(w)/n - \mu(w)\right) + n\mu(w)\left(\mu - \sum_{j=1}^{n}\mu_j/n\right)}{\sigma\sqrt{n}} \\ &\quad + \frac{n\nu\left(\sum_{j=1}^{n}\mu_j(w)/n - \mu(w)\right) + n\mu(w)\left(\nu - \sum_{j=1}^{n}\nu_j/n\right)}{\sigma\sqrt{n}}.\end{aligned}$$

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Applying Lemma 2, we obtain $\sum_{j=1}^{n}\mu_j/n - \mu = o(n^{-1/2})$ and $\sum_{j=1}^{n}\mu_j(w)/n - \mu(w) = o(n^{-1/2})$. Combining these results with assumption (iv), we then have

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n}E(Y_j-\mu(w)D_j)}{\sigma\sqrt{n}}=0. \tag{12}$$

Thus, from (11) and (12), it is sufficient to show that

$$\frac{\sum_{j=1}^{n}[(Y_j-\mu(w)D_j)-E(Y_j-\mu(w)D_j)]}{\sigma\sqrt{n}}\to N(0,1) \tag{13}$$

in law as $n\to\infty$ in order to show the result of the theorem.

Let $Z_j \equiv Y_j - \mu(w)D_j$ and $B_n^2 = \sum_{j=1}^{n}\mathrm{Var}(Z_j)$. It is not difficult to obtain

$$\begin{aligned}\mathrm{Var}(Y_j) &= \mathrm{Var}((\mu+\nu)X_j-\mu(w)U_j)\\ &= (\mu+\nu)^2\left[2\int_w^\infty t\bar{F}_j(t)\,dt - 2w\mu_j(w)-\mu_j^2(w)\right]+\mu^2(w)\,\mathrm{Var}(U_j)\\ &\quad -2(\mu+\nu)\mu(w)\left[2\int_w^\infty t\bar{F}_j(t)\,dt-(w+\mu_j)\mu_j(w)\right].\end{aligned} \tag{14}$$

From (14) it follows that

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\mathrm{Var}(Y_j) &= (\mu+\nu)^2\left[\frac{2}{n}\sum_{j=1}^{n}\int_w^\infty t\bar{F}_j(t)\,dt-\frac{2w}{n}\sum_{j=1}^{n}\mu_j(w)-\frac{1}{n}\sum_{j=1}^{n}\mu_j^2(w)\right]\\ &\quad+\mu^2(w)\cdot\frac{1}{n}\sum_{j=1}^{n}\mathrm{Var}(U_j)-2(\mu+\nu)\mu(w)\\ &\quad\times\left[\frac{2}{n}\sum_{j=1}^{n}\int_w^\infty t\bar{F}_j(t)\,dt-\frac{w}{n}\sum_{j=1}^{n}\mu_j(w)-\frac{1}{n}\sum_{j=1}^{n}\mu_j\,\mu_j(w)\right].\end{aligned} \tag{15}$$

Evidently, it is true that $\sum_{j=1}^{n}\mu_j(w)/n\to\mu(w)$. On the other hand, for any $c>0$,

$$\int_c^\infty t\bar{F}_j(t)\,dt\to\int_c^\infty t\bar{F}(t)\,dt,\quad \text{as } j\to\infty,$$

since F_j converges to F and $\{F_j, j\ge 1\}$ is dominated by H that satisfies $\int_0^\infty t^{2+\delta}\,dH(t) = (2+\delta)\int_0^\infty t^{1+\delta}\bar{H}(t)\,dt<\infty$ for a $\delta>0$. This implies

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$$\frac{1}{n}\sum_{j=1}^{n}\int_{w}^{\infty} t\bar{F}_j(t)\,dt \to \int_{w}^{\infty} t\bar{F}(t)\,dt \tag{16}$$

and

$$E(U_j^2) = 2\int_{0}^{\infty} t\bar{F}_j(t)\,dt \to 2\int_{0}^{\infty} t\bar{F}(t)\,dt \tag{17}$$

as $j \to \infty$. Since $\mu_n \to \mu$, $\{\mu_n, n \geq 1\}$ must be a bounded sequence, say, $0 < \mu_n \leq L/2 < \infty$, $\forall n \geq 1$. Now the application of Lemma 2 gives

$$\begin{aligned}
\left|\frac{1}{n}\sum_{j=1}^{n}\mu_j^2 - \mu^2\right| &\leq \frac{1}{n}\sum_{j=1}^{n}(\mu_j + \mu)|\mu_j - \mu| \\
&\leq L\cdot\frac{1}{n}\sum_{j=1}^{n}|\mu_j - \mu| \\
&= L\cdot\frac{1}{n}\sum_{j=1}^{n}\left|\int_{0}^{\infty}(\bar{F}_j(t) - \bar{F}(t))\,dt\right| \\
&\leq L\cdot\frac{1}{n}\sum_{j=1}^{n}\int_{0}^{\infty}|\bar{F}_j(t) - \bar{F}(t)|\,dt \\
&= L\cdot o(n^{-1/2})
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n}\mathrm{Var}(U_j) &= 2\cdot\frac{1}{n}\sum_{j=1}^{n}\int_{0}^{\infty} t\bar{F}_j(t)\,dt - \frac{1}{n}\sum_{j=1}^{n}\mu_j^2 \\
&\to 2\int_{0}^{\infty} t\bar{F}(t)\,dt - \left(\int_{0}^{\infty}\bar{F}(t)\,dt\right)^2
\end{aligned} \tag{18}$$

as $n \to \infty$. Similarly, it can be shown that

$$\frac{1}{n}\sum_{j=1}^{n}\mu_j^2(w) \to \left(\int_{w}^{\infty}\bar{F}(t)\,dt\right)^2 \tag{19}$$

and

$$\frac{1}{n}\sum_{j=1}^{n}\mu_j\,\mu_j(w) \to \left(\int_{0}^{\infty}\bar{F}(t)\,dt\right)\left(\int_{w}^{\infty}\bar{F}(t)\,dt\right). \tag{20}$$

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Combining the above results of Eqs. (15)–(20), we see that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathrm{Var}(Y_j) = (\mu+\nu)^2\Bigg[&2\int_w^\infty t\bar{F}(t)\,dt - 2w\int_w^\infty \bar{F}(t)\,dt - \left(\int_w^\infty \bar{F}(t)\,dt\right)^2\Bigg] \\ &+ \left(\int_w^\infty \bar{F}(t)\,dt\right)^2\cdot\left[2\int_0^\infty t\bar{F}(t)\,dt - \left(\int_0^\infty \bar{F}(t)\,dt\right)^2\right] \\ &- 2(\mu+\nu)\int_w^\infty \bar{F}(t)\,dt \\ &\times\left[2\int_w^\infty t\bar{F}(t)\,dt - w\int_w^\infty \bar{F}(t)\,dt\right. \\ &\left. - \left(\int_0^\infty \bar{F}(t)\,dt\right)\left(\int_w^\infty \bar{F}(t)\,dt\right)\right]. \qquad \textbf{(21)}\end{aligned}$$

Let U be a random variable that follows distribution $F(\cdot)$. Comparing (14) and (21), we see that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathrm{Var}(Y_j) = \mathrm{Var}(Y), \qquad \textbf{(22)}$$

where $Y \equiv (\mu+\nu)(U-w)I_{(U>w)} - \mu(w)U$.

By the assumption (v) it follows that

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\mathrm{Var}(Z_j) &= \frac{1}{n}\sum_{j=1}^{n}\mathrm{Var}(Y_j) + \mu^2(w)\,\frac{1}{n}\sum_{j=1}^{n}\mathrm{Var}(D_j) \\ &\to \mathrm{Var}(Y) + \mu^2(w)\sigma_D^2 = \sigma^2. \qquad \textbf{(23)}\end{aligned}$$

We claim that $\sigma^2 = \mathrm{Var}(Y) + \mu^2(w)\sigma_D^2 > 0$. In order to see it, recall that the support set of F has at least three points; hence, this support set has at least two points in $[0,w]$ or at least two points in (w,∞). Suppose $0 \le t_1 < t_2 \le w$ are two support points. Without loss of generality, we can further assume these points actually belong to the range of the above defined random variable U. Then Y has at least two support points $-\mu(w)t_1$ and $-\mu(w)t_2$. These two points are different since $\mu(w) > 0$ under the assumption that at least one support point of F belongs to (w,∞), thus $\mathrm{Var}(Y) > 0$. If $w < t_1 < t_2 < \infty$ are two support points of F, then Y has at least two support points $(\mu+\nu)(t_1-w) - \mu(w)t_1$ and $(\mu+\nu)(t_2-w) - \mu(w)t_2$. These two points are also different since the assumption that the support set of F has at least one point in (w,∞) implies $\mu > \mu(w) > 0$. In any case, it holds that $\mathrm{Var}(Y) > 0$ and, consequently, $\sigma^2 = \mathrm{Var}(Y) + \mu^2(w)\sigma_D^2 > 0$.

Now let us verify Liaponov's condition for the sequence $\{Z_n, n \ge 1\}$. By the c_r-inequality (see Chow and Teicher [4]) there are two positive constants c_1, c_2 such that

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$$\sum_{j=1}^{n} E|Z_j|^{2+\delta} = \sum_{j=1}^{n} E|Y_j - \mu(w)D_j|^{2+\delta}$$

$$\leq \sum_{j=1}^{n} [c_1 E|Y_j|^{2+\delta} + c_2 E|D_j|^{2+\delta}]$$

$$\leq \sum_{j=1}^{n} [c_1(\mu+\nu)^{2+\delta} E|U_j|^{2+\delta} + c_2 E|D_j|^{2+\delta}] \quad \textbf{(24)}$$

$$\leq n(2+\delta)[c_1(\mu+\nu)^{2+\delta} + c_2] \int_0^{\infty} t^{1+\delta} \bar{H}(t)\, dt \equiv nK, \quad \textbf{(25)}$$

where $K \equiv (2+\delta)[c_1(\mu+\nu)^{2+\delta} + c_2] \int_0^{\infty} t^{1+\delta} \bar{H}(t)\, dt$, (24) comes from the fact that $|Y_j| \leq (\mu+\nu)U_j$, and (25) follows from assumption (iii). Therefore,

$$\frac{\sum_{j=1}^{n} E|Z_j|^{2+\delta}}{B_n^{2+\delta}} \leq \frac{nK}{\left(\sum_{j=1}^{n} \mathrm{Var}(Z_j)\right)^{1+\delta/2}}$$

$$= \frac{nK}{n^{1+\delta/2}\left(\sum_{j=1}^{n} \mathrm{Var}(Z_j)/n\right)^{1+\delta/2}}$$

$$\to 0$$

as $n \to \infty$ because

$$\frac{1}{n}\sum_{j=1}^{n} \mathrm{Var}(Z_j) \to \mathrm{Var}(Y) + \mu^2(w)\sigma_D^2 > 0$$

by (23). The Liaponov's condition is satisfied for the sequence of independent variables $\{Z_n, n \geq 1\}$ and so the Central Limit Theorem holds for $\{Z_n, n \geq 1\}$. That is,

$$\frac{\sum_{j=1}^{n} (Z_j - E(Z_j))}{B_n} \to N(0,1) \quad \textbf{(26)}$$

in law as $n \to \infty$. Note that

$$\frac{B_n}{\sigma\sqrt{n}} = \sqrt{\frac{\sum_{j=1}^{n} \mathrm{Var}(Z_j)}{n(\mathrm{Var}(Y) + \mu^2(w)\sigma_D^2)}} \to 1.$$

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Finally, by Slutsky's Theorem we obtain

$$\frac{\sum_{j=1}^{n}(Z_j - E(Z_j))}{\sigma\sqrt{n}} \to N(0,1) \tag{27}$$

in law, that is, (13) is true. This ends the proof of the theorem. ■

By modifying the proof of Theorem 4 in [7], the following result regarding the asymptotic normality of $\bar{A}(t;w)$ can be established. We shall just give the result and omit its proof.

THEOREM 4: *Under the same conditions of Theorem* 3,

$$\frac{\bar{A}(t;w) - \mu(w)/(\mu+\nu)}{\sigma/\left((\mu+\nu)\sqrt{t}\right)} \to N(0,1)$$

in law as $t \to \infty$.

In practice it is important to obtain point and/or interval estimations of system stable interval availability $A(w)$. The point estimation can be obtained by using Theorems 1 and 2, and the interval estimation by Theorems 4 and 5.

4. A SUFFICIENT CONDITION

In order to prove the asymptotic normality in Theorem 3 it is important to have

$$\frac{1}{n}\sum_{j=1}^{n}\mu_j - \mu = o(n^{-1/2})$$

$$\frac{1}{n}\sum_{j=1}^{n}\mu_j(w) - \mu(w) = o(n^{-1/2})$$

and

$$\frac{1}{n}\sum_{j=1}^{n}\nu_j - \nu = o(n^{-1/2}).$$

These results are consequences of conditions like

$$|\bar{F}_j(t) - \bar{F}(t)| \le \bar{\xi}(t)o^*(n^{-1/2}) \tag{28}$$

described in Theorem 3, (i). In this section, we shall give sufficient conditions under which (28) is satisfied. The study will apply to the exponential family of distributions whose definition can be found, for instance, in Lehmann [6]. First, we establish an auxiliary result as below.

LEMMA 3: *Consider an exponential family of distributions that has density function* $f(x;\theta)$ *given by*

$$f(x;\theta) = e^{-B(\theta)}e^{-\eta(\theta)T(x)}h(x), \qquad x \ge 0, \quad \forall\theta \in \Theta, \tag{29}$$

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where $\eta(\theta)$ is a monotone function. If there exists a positive number δ such that

$$\lim_{x\to\infty} \frac{T^{\delta}(x)}{x} > 0, \tag{30}$$

then (a) $E_\theta X^k < \infty$, $\forall k \geq 0$ and (b) $E_\theta X^k T^l(X) < \infty$, $\forall k \geq 0$, $l \geq 0$ for any θ belonging to the interior Θ^0 of Θ, where random variable X has density $f(x;\theta)$.

PROOF: Denote the limit $\lim_{x\to\infty} T^{\delta}(x)/x$ by c. From the assumption there exists $x_0 > 0$ such that

$$\frac{T^{\delta}(x)}{x} > \frac{c}{2} > 0, \qquad \forall x \geq x_0.$$

This implies

$$x < \frac{2}{c} T^{\delta}(x), \qquad \forall x \geq x_0.$$

Hence,

$$\begin{aligned} 0 \leq EX^k &= E(X^k I_{(X\leq x_0)}) + E(X^k I_{(X>x_0)}) \\ &\leq x_0^k + E\left(\left(\frac{2}{c}\right)^k T^{k\delta}(X) I_{(X>x_0)}\right) \\ &\leq x_0^k + \left(\frac{2}{c}\right)^k E T^{k\delta}(X) < \infty, \end{aligned}$$

where the last inequality comes from a property of the exponential family that $T(X)$ has a finite moment of any order provided $\theta \in \Theta^0$ [6]. ■

THEOREM 5: *Let $\{f(x;\theta), \theta \in \Theta\}$ be an exponential family given by* (29) *and C a subset of Θ. Suppose the following hold*

i. *C is a closed set and $C \subseteq \Theta^0$, the interior of Θ*
ii. *$\{\theta_n\} \subseteq C$, and $\theta_n - \theta_0 = o(n^{-1/2})$*
iii. *$T(x) \geq 0$ and $\lim_{x\to\infty} T^{\delta}(x)/x > 0$ for a $\delta > 0$*
iv. *$\eta(\theta)$ is monotone, both $\eta(\theta)$ and $B(\theta)$ are differentiable. Also $A_1 \equiv \sup_{\theta\in C}|B'(\theta)| < \infty$, $A_2 \equiv \sup_{\theta\in C}|\eta'(\theta)| < \infty$, and $A_3 \equiv \max_{\theta\in C}|B(\theta)|$.*

Then there exists an increasing bounded function $\xi(t) \geq 0$ such that (a) $\xi(0) = 0$; (b) $|\bar{F}(t;\theta_n) - \bar{F}(t;\theta_0)| \leq \bar{\xi}(t) o^*(n^{-1/2})$; (c) $\int_0^\infty t^{1+\beta}\bar{\xi}(t)\,dt < \infty$, $\forall \beta \geq 0$.

PROOF: Denote the distribution function and survival function associated with density function $f(x;\theta)$ by $F(x;\theta)$ and $\bar{F}(x;\theta)$, respectively. We have

$$|\bar{F}(t;\theta_n) - \bar{F}(t;\theta_0)| = |\theta_n - \theta_0| \cdot \left| \frac{\partial \bar{F}(t;\theta)}{\partial \theta} \right|_{\theta=\tilde{\theta}}$$

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where $\tilde{\theta} \in \Theta$ is between θ_n and θ_0. Thus,

$$\begin{aligned}
|\bar{F}(t;\theta_n) - \bar{F}(t;\theta_0)| &= |\theta_n - \theta_0| \cdot \left| \int_t^\infty \frac{\partial}{\partial \theta} f(x;\theta)|_{\theta=\tilde{\theta}}\, dx \right| \\
&= |\theta_n - \theta_0| \cdot \left| \int_t^\infty f(x;\tilde{\theta})[-B'(\tilde{\theta}) - \eta'(\tilde{\theta})T(x)]\, dx \right| \\
&\le |\theta_n - \theta_0| \cdot \int_t^\infty e^{-B(\tilde{\theta})} e^{-\eta(\tilde{\theta})T(x)} h(x)[|B'(\tilde{\theta})| + |\eta'(\tilde{\theta})|T(x)]\, dx \\
&\le |\theta_n - \theta_0| \cdot \int_t^\infty e^{-B(\tilde{\theta})} e^{-\eta(\tilde{\theta})T(x)} h(x)[A_1 + A_2 T(x)]\, dx \\
&\le |\theta_n - \theta_0| \cdot \int_t^\infty e^{A_3} e^{-\eta(\theta^*)T(x)} h(x)[A_1 + A_2 T(x)]\, dx, \qquad \textbf{(31)}
\end{aligned}$$

where $\theta^* \in C$ satisfies $\eta(\theta^*) = \min_{\theta \in C} \eta(\theta)$. Hence, from (31),

$$|\bar{F}(t;\theta_n) - \bar{F}(t;\theta_0)| \le |\theta_n - \theta_0| \cdot e^{A_3 + B(\theta^*)} \int_t^\infty f(x;\theta^*)[A_1 + A_2 T(x)]\, dx. \qquad \textbf{(32)}$$

We define

$$\xi(t) \equiv e^{A_3 + B(\theta^*)} \int_0^t f(x;\theta^*)[A_1 + A_2 T(x)]\, dx. \qquad \textbf{(33)}$$

Obviously, $\xi(t) \ge 0$, $\xi(0) = 0$, and $\xi(t)$ is increasing in $t \ge 0$. Note that

$$0 \le \int_0^\infty f(x;\theta^*) T(x)\, dx = E_{\theta^*}(T(X)) < \infty$$

because $\theta^* \in C \subseteq \Theta^0$. It yields

$$\xi(\infty) = e^{A_3 + B(\theta^*)}[A_1 + A_2 E_{\theta^*}(T(X))] < \infty.$$

Further, for any $\beta \ge 0$ it can be shown that

$$\int_0^\infty t^{1+\beta}\, d\xi(t) = (1+\beta) \int_0^\infty t^\beta \bar{\xi}(t)\, dt < \infty, \qquad \textbf{(34)}$$

where $\bar{\xi}(t) \equiv \xi(\infty) - \xi(t) = \alpha \int_t^\infty f(x;\theta^*)[A_1 + A_2 T(x)]\, dx$, and $\alpha \equiv \exp\{A_3 + B(\theta^*)\}$. This is true because

$$\begin{aligned}
(1+\beta) \int_0^\infty t^\beta \bar{\xi}(t)\, dt &= (1+\beta) \int_0^\infty t^\beta \left(\alpha \int_t^\infty f(x;\theta^*)[A_1 + A_2 T(x)]\, dx \right) dt \\
&= (1+\beta) \int_0^\infty \left(\int_0^x t^\beta f(x;\theta^*)[A_1 + A_2 T(x)]\, dt \right) dx \\
&= \int_0^\infty x^{1+\beta}[A_1 + A_2 T(x)] f(x;\theta^*)\, dx \\
&= A_1 E_{\theta^*}(X^{1+\beta}) + A_2 E_{\theta^*}(X^{1+\beta} T(X)).
\end{aligned}$$

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However, from Lemma 3, we have

$$0 \le E_{\theta^*}(X^{1+\beta}) < \infty$$

and

$$0 \le E_{\theta^*}(X^{1+\beta}T(X)) < \infty$$

since $\theta^* \in \Theta^0$. Therefore, (34) holds. Combining (32) and (34), we see that

$$|\bar{F}(t;\theta_n) - \bar{F}(t;\theta_0)| \le |\theta_n - \theta_0|\bar{\xi}(t)$$

and $\xi(t)$ has the desired properties. Finally, considering assumption (ii), we obtain

$$|\bar{F}(t;\theta_n) - \bar{F}(t;\theta_0)| \le \bar{\xi}(t)o^*(n^{-1/2}). \quad \blacksquare$$

COROLLARY: *In addition to the conditions of Theorem* 5, *if* $\int_0^\infty t^{1+\beta}\bar{F}(t;\theta_0)\,dt < \infty$ *for a* $\beta > 0$, *then* $\{F(t;\theta_n)n \ge 1\}$ *is also dominated by a bounded increasing function* H *that satisfies* $\int_0^\infty t^{1+\beta}\bar{H}(t)\,dt < \infty$.

PROOF: Define $H(t) \equiv F(t;\theta_0) + \xi(t)$ where $\xi(t)$ is as described in Theorem 5. It is easy to see that $H(0) = 0, H(t)$ increases in $t \ge 0$, and $H(\infty) = 1 + \xi(\infty) < \infty$. Moreover, denoting $\bar{H}(t) = H(\infty) - H(t)$, we have $\bar{H}(t) = \bar{F}(t;\theta_0) + \bar{\xi}(t)$ and

$$\int_0^\infty t^{1+\beta}\bar{H}(t)\,dt = \int_0^\infty t^{1+\beta}\bar{F}(t;\theta_0)\,dt + \int_0^\infty t^{1+\beta}\bar{\xi}(t)\,dt < \infty. \quad \blacksquare$$

In the following we consider some examples of commonly used distribution families.

Example 1 (Gamma family): The density function of the gamma family of distributions is given as

$$f(x;\theta,\alpha) = \frac{1}{\Gamma(\alpha)\theta^\alpha}\,x^{\alpha-1}e^{-x/\theta}, \qquad t \ge 0, \tag{35}$$

where $\theta > 0$, $\alpha > 0$ are scale and shape parameters, respectively. We assume α is known. Then $\{f(x;\theta,\alpha), \theta \in \Theta\}$ with $\Theta = (0,\infty)$ is obviously an exponential family. Comparing (35) with (29), we see that

$$B(\theta) = \alpha \ln \theta, \quad \eta(\theta) = \frac{1}{\theta}, \quad T(x) = x, \quad h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}.$$

If $\{\theta_n, n \ge 1\} \subseteq C \equiv [a,b] \subset (0,\infty) = \Theta$, then

$$A_1 = \sup_{a\le\theta\le b} |B'(\theta)| = \frac{\alpha}{a} < \infty,$$

$$A_2 = \sup_{a\le\theta\le b} |\eta'(\theta)| = \frac{1}{a^2} < \infty$$

$$A_3 = \max_{a\le\theta\le b} |B(\theta)| = \alpha \max\{|\ln a|, |\ln b|\} < \infty$$

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and

$$\frac{T(x)}{x} = \frac{x}{x} = 1, \qquad \forall x > 0.$$

Therefore, if $\theta_n - \theta_0 = o(n^{-1/2})$ with $\theta_n \in [a,b]$, then according to Theorem 5 we know that there exists $\xi(t)$ satisfying all the requirements described in Theorem 5 and, in particular,

$$|\bar{F}(t;\theta_n,\alpha) - \bar{F}(t;\theta_0,\alpha)| \le \xi(t) o^*(n^{-1/2}).$$

Because $\int_0^\infty t^{1+\beta}\bar{F}(t;\theta,\alpha)\,dt < \infty$ for any $\beta \ge 0$, $\theta \in \Theta$, we further conclude that the sequence $\{F(t;\theta_n,\alpha), n \ge 1\}$ is also dominated by an H satisfying $\int_0^\infty t^{1+\beta}\bar{H}(t)\,dt < \infty$. Hence, the result of Theorem 3 holds for the sequence $\{F(t;\theta_n,\alpha), n \ge 1\}$.

Example 2 (Weibull Family): The density function of this family is defined by

$$f(x;\theta,\gamma) = \frac{\gamma}{\theta} x^{\gamma-1} e^{-x^\gamma/\theta}, \qquad x > 0.$$

where parameters $\theta > 0$, $\gamma > 0$. Suppose γ is known and consider $\{\theta_n, n \ge 1\} \subseteq C = [a,b] \subset (0,\infty) = \Theta$. Then $\{f(x;\theta,\gamma), \theta \in \Theta\}$ is, of course, an exponential family with

$$B(\theta) = \ln\theta, \quad \eta(\theta) = \frac{1}{\theta}, \quad T(x) = x^\gamma, \quad h(x) = \gamma x^{\gamma-1}.$$

It can be easily verified that the sequence $\{F(t;\theta_n,\gamma), n \ge 1\}$ satisfies all the assumptions in Theorem 5 if $\theta_n - \theta_0 = o(n^{-1/2})$ and $\{\theta_n, n \ge 1\} \subseteq [a,b]$.

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