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SOME FELLER SEMIGROUPS ON $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ GENERATED BY PSEUDO-DIFFERENTIAL OPERATORS

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To Professor V. G. Maz'ya on the occasion of his 75th birthday

Abstract. In this paper we construct some Feller semigroups, hence Feller processes, with state space $\mathbb{R}^n \times \mathbb{Z}^m$ starting with pseudo-differential operators having symbols defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{T}^m$.

§0. *Introduction.* The investigations in this note have two quite different origins. In discussions with Yuan on Markov processes with Markov switching (compare [11]), the problem emerges whether it is possible to construct by analytical means Feller semigroups with state space $\mathbb{R}^n \times \mathbb{Z}^m$ in such a way that the symbol of the generator can be used to study the corresponding Feller process which in some sense should behave as a combination of a Markov chain and a Feller process with state space $\mathbb{R}^n$. Most of all the interest is in examples which are not merely "superpositions" of two independent objects operating on the product space. The second question is related to geometric attempts to understand transition functions of Feller processes with state space $\mathbb{R}^n$. Following ideas of the second named author, in [10] it was suggested to use two families of metrics to describe the transition function of, say, a Lévy process by

$$p_t(x, y) = p_t(x - y) = \lambda^{(n)}(B^{d_\psi}_{1/\sqrt{t}}(0))\mathrm{e}^{-\delta^2_{\psi,t}(x,y)}$$

where d_ψ and $\delta_{\psi,t}$ are two metrics related to the symbol of the generator of the process. The surprise was that two metrics are needed; in the case of a Gaussian essentially one is used, namely the Euclidean. This turns out to be related to the fact that Gaussians are fixed points of the Fourier transform and the "two" metrics are considered on different domains: the tangent space and the co-tangent space of the state space, which of course in the case of $\mathbb{R}^n$ can be identified. While for general Lévy processes two different metrics are needed, still the identification of the tangent space with the co-tangent space (both as a copy of $\mathbb{R}^n$) makes it sometimes difficult to understand the underlying structure. For this reason in [5] (see also [6]), it was suggested to change the state space and move from $\mathbb{R}^n$ to $\mathbb{Z}^m$. Thus it becomes natural to combine both cases and to study processes with state space $\mathbb{R}^n \times \mathbb{Z}^m$.

Let X be a Hausdorff space countable at infinity and denote by $C_\infty(X)$ the space of all continuous functions vanishing at infinity. A Feller semigroup

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$(T_t)_{t\geqslant 0}$ is a strongly continuous contraction semigroup of linear operators $T_t : C_\infty(X) \to C_\infty(X)$ which are positivity preserving. Note that some authors also require in the definition of a Feller semigroup the conservativity of T_t, i.e. $T_t 1 = 1$. However, by passing to the one-point compactification of X, if necessary, we can extend each Feller semigroup as defined here to a conservative one. The generator $(A, D(A))$ of a Feller semigroup must satisfy the positive maximum principle, i.e. $u(x_0) = \sup u(x) \geqslant 0$ implies $(Au)(x_0) \leqslant 0$, and can be identified for $X = \mathbb{R}^n$, as well as in many cases for $X = \mathbb{Z}^m$ and hence for many cases with $X = \mathbb{R}^n \times \mathbb{Z}^m$, as a certain pseudo-differential operator. When considering the translation invariant case on $\mathbb{R}^n$ we end up with Feller semigroups induced by convolution semigroups of (sub-)probability measures $(\mu_t)_{t\geqslant 0}$, which are characterized by negative definite functions $\psi : \mathbb{R}^n \to \mathbb{C}$ with the help of their Fourier transform, i.e.

$$\hat{\mu}(\xi) = \mathrm{e}^{-t\psi_r(\xi)},$$

and $-\psi_r$ is the symbol of the generator (see [3] or [8]). Similar observations hold for some Markov chains with state space $\mathbb{Z}^m$ (see [5]), where the symbol $-\psi_z$ must be associated with a continuous negative definite function $\psi_z : \mathbb{T}^m \to \mathbb{C}$, $\mathbb{T}^m$ being the m-dimensional torus, i.e. $\mathbb{T}^m \cong (\mathbb{Z}^m)^*$, the dual group of $\mathbb{Z}^m$. Clearly a pseudo-differential operator with symbol $-q_r - q_z : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{C}$, $(-q_r - q_z)(\xi, \omega) = -q_r(\xi) - q_z(\omega)$ will give a Feller semigroup on the product space; this we refer to as the trivial case. In fact any continuous negative definite function $\psi : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{C}$ gives via $-\psi(D_x, D_k)$ a generator of a translation invariant (Feller) semigroup on $C_\infty(\mathbb{R}^n \times \mathbb{T}^m)$. A proper "coupling" of the Lévy process (state space $\mathbb{R}^n$) and the Markov chain (state space $\mathbb{Z}^m$) shall come by introducing coefficients, i.e. by trying to construct a Feller semigroup on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ with generator $-q(x, k; D_x, D_k)$ having a symbol $-q : \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{C}$, $(x, k; \xi, \omega) \mapsto -q(x, k; \xi, \omega)$, such that for $(x, k) \in \mathbb{R}^n \times \mathbb{Z}^m$ the function $(\xi, \omega) \mapsto q(x, k; \xi, \omega)$ is negative definite. The basic idea is to apply the Hille–Yosida–Ray theorem to $-q(x, k; D_x, D_k)$,

$$\begin{aligned} &-q(x, k; D_x, D_k)u(x, k) \\ &\quad = -(2\pi)^{-n/2-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} \mathrm{e}^{\mathrm{i}x\xi} \mathrm{e}^{\mathrm{i}k\omega} q(x, k; \xi, \omega)\tilde{u}(\xi, \omega)\, d\omega\, d\xi \end{aligned} \tag{0.1}$$

where $\tilde{u}$ denotes the Fourier transform of $u : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{C}$. The nature of the symbols makes such an application not straightforward, not even in the case of $\mathbb{R}^n$ alone (see [2, 8] or [9]). Thus we aim to get a first class of non-trivial examples. We suggest a type of perturbation theory [7], i.e. to look at symbols

$$q(x, k; \xi, \omega) = q_1(\xi, \omega) + q_2(x, k; \xi, \omega)$$

with certain smallness conditions on q_2; see conditions A.2.1. To apply a variant of the Hille–Yosida–Ray theorem (see [8]), we need to assume that $-q(x, k; D_x, D_k)$ satisfies the positive maximum principle; sufficient conditions for this are discussed in §2. The harder problem is to solve the equation

$$q(x, k; D_x, D_k)u + \lambda u = f$$

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for some $\lambda > 0$ on a suitable domain in $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ and a dense set of elements $f \in C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$. We employ the approach suggested in [7] for the case $\mathbb{R}^n$ and use some observations made in [6] for the case $q(x; D_x, D_k)$. The basic estimates needed are given in §3. Since some of the estimates, i.e. those for the part $q_1(D_x, D_k)$, are essentially an application of the Plancherel theorem, we concentrate in proofs on the non-trivial cases only. In §4 we combine these estimates and add a regularity consideration to eventually establish our main results, Theorem 4.3 and Corollary 4.4, which state that under suitable, non-trivial (and non-empty) conditions, (0.1) indeed extends to a generator of a Feller semigroup on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$.

When the second named author started with his investigations on pseudo-differential operators generating Markov processes around 30 years ago, on the one hand it soon became apparent that many classical results, for example on Sobolev spaces, must be modified or cannot work. In many cases the books of Maz'ya on these spaces which culminated in [12], were inspirational, as were discussions with him at various conferences, beginning with one organized by our Czech colleagues in the early to mid 1990s. For these reasons it gives the second named author great pleasure to express his appreciation to Professor Maz'ya, the mathematician and the man, by dedicating this paper to him on the occasion of his 75th birthday.

§1. *Notation and some auxiliary results.* The dual group of the locally compact Abelian group $\mathbb{R}^n \times \mathbb{Z}^m$ is identified with $\mathbb{R}^n \times \mathbb{T}^m$, $\mathbb{T}^m$ being the m-dimensional torus. By $S(\mathbb{R}^n \times \mathbb{Z}^m)$ we denote all functions $u : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{R}$ (or $\mathbb{C}$) which are continuous and for every $\alpha \in \mathbb{N}_0^n$ and $j, l \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, there exists $c = c_{n,\alpha,j,l}$ such that

$$|\partial_x^\alpha u(x,k)| \leqslant c_{n,\alpha,j,l}(1+|x|^2)^{-j/2}(1+|k|^2)^{-l/2}.$$

The Fourier transform on $\mathbb{R}^n$ is defined by

$$\hat{u}(\xi) = (F_{x\mapsto\xi}u)(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\xi}u(x)\,dx$$

with inverse

$$(F^{-1}_{\xi\mapsto x}v)(x) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi}v(\xi)\,d\xi.$$

On $\mathbb{Z}^m$ the Fourier transform is given by

$$(F_{k\mapsto\omega}u)(\omega) := \sum_{k\in\mathbb{Z}^m} u(k)\mathrm{e}^{-\mathrm{i}k\omega}$$

with inverse (defined on $\mathbb{T}^m$) given by

$$(F^{-1}_{\omega\mapsto k}v)(k) = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} \mathrm{e}^{\mathrm{i}k\omega}v(\omega)\,d\omega.$$

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For $u \in S(\mathbb{R}^n \times \mathbb{Z}^n)$ we denote its Fourier transform by $\tilde{u}$ and it holds

$$\tilde{u}(\xi, \omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^m} u(x, k) \mathrm{e}^{-\mathrm{i}k\omega} \mathrm{e}^{-\mathrm{i}x\xi} \, d\xi$$
$$= (2\pi)^{-n/2} \sum_{k \in \mathbb{Z}^m} \int_{\mathbb{R}^n} u(x, k) \mathrm{e}^{-\mathrm{i}x\xi} \, d\xi \mathrm{e}^{-\mathrm{i}k\omega}$$

with inverse

$$(F^{-1}_{\xi \mapsto x} F^{-1}_{\omega \mapsto k} v)(x, k) = (2\pi)^{-(n/2)-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} v(\xi, \omega) \mathrm{e}^{\mathrm{i}k\omega} \mathrm{e}^{\mathrm{i}\xi x} \, d\omega \, d\xi.$$

Moreover, we write

$$\hat{u}(\xi, k) := (F_{x \mapsto \xi} u(\cdot, k))(\xi, k).$$

For $u \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ the Fourier transform is defined with the help of Plancherel's theorem, i.e. the identity

$$\|u\|_{L^2(\mathbb{R}^n \times \mathbb{Z}^m)} = \|\tilde{u}\|_{L^2(\mathbb{R}^n \times \mathbb{T}^m)}$$

or

$$\int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^m} |u(x, k)|^2 \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} |\tilde{u}(\xi, \omega)|^2 \, d\xi \, d\omega,$$

which holds for all $u \in S(\mathbb{R}^n \times \mathbb{Z}^m)$ or $u \in L^1(\mathbb{R}^n \times \mathbb{Z}^m) \cap L^2(\mathbb{R}^n \times \mathbb{Z}^m)$. By $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ we denote the space obtained by completing $S(\mathbb{R}^n \times \mathbb{Z}^m)$ with respect to the sup-norm; clearly, these are the continuous functions vanishing at infinity.

For a locally compact Abelian group G we call $\psi : G \to \mathbb{C}$ a continuous negative definite function if ψ is continuous, $\psi(0) \geqslant 0$ and for all $t > 0$ the function $\xi \mapsto \mathrm{e}^{-t\psi(\xi)}$ is positive definite in the sense of Bochner (compare [1] or [8]). For a continuous negative definite function on $\mathbb{R}^n \times \mathbb{T}^m$ we find

$$|\psi(\xi, \omega)| \leqslant c_\psi (1 + |\xi|^2)$$

and

$$\frac{1 + |\psi(\xi, \omega)|}{1 + |\psi(\eta, \tau)|} \leqslant 2(1 + |\psi(\xi - \eta, \omega - \tau)|).$$

Let $\psi : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{R}$ be a fixed continuous negative definite function and $s \geqslant 0$. We define the norm $\|u\|_{\psi,s}$ by

$$\|u\|^2_{\psi,s} := (2\pi)^{-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} (1 + \psi(\xi, \omega))^s |\tilde{u}(\xi, \omega)|^2 \, d\omega \, d\xi$$
$$= (2\pi)^{-m} \int_{\mathbb{T}^m} \int_{\mathbb{R}^n} (1 + \psi(\xi, \omega))^s |\tilde{u}(\xi, \omega)|^2 \, d\xi \, d\omega.$$

It follows that

$$H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m) := \{u \in L^2(\mathbb{R}^n \times \mathbb{Z}^m) \mid \|u\|_{\psi,s} < \infty\}$$

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is a Banach space and for $t > s \geqslant 0$ we have

$$H^{\psi,t}(\mathbb{R}^n \times \mathbb{Z}^m) \subset H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m) \subset L^2(\mathbb{R}^n \times \mathbb{Z}^m)$$

and $S(\mathbb{R}^n \times \mathbb{Z}^m) \subset H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ is a dense subspace. Since for $u \in S(\mathbb{R}^n \times \mathbb{Z}^m)$ we have

$$u(x,k) = (F^{-1}_{\xi \mapsto x} F^{-1}_{x \mapsto \xi} u)(x,k),$$

it follows that for $s > n/2\rho$ the condition

$$(1 + |\xi|^2)^{\rho/2} \leqslant \lambda_0 (1 + \psi(\xi,\omega))^{1/2}$$

implies

$$\|u\|_\infty \leqslant c \|u\|_{\psi,s},$$

i.e. in this case $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ is continuously embedded into $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$. For more about these spaces we refer to [6, 13, 14].

§2. *The positive maximum principle.* Generators $(A, D(A))$ of Feller semigroups must satisfy the *positive maximum principle*, i.e. for $u \in D(A)$ it follows from $u(x_0) = \sup u(x) \geqslant 0$ that $(Au)(x_0) \leqslant 0$, compare [4] or [8]. Furthermore, the positive maximum principle implies dissipativity in the space of continuous functions vanishing at infinity. Hence for generators (and pre-generators) of Feller semigroups we can replace the dissipativity condition in the Hille–Yosida theorem by the positive maximum principle. This result is often referred to as the Hille–Yosida–Ray theorem. Moreover it is known that if a generator of a strongly continuous contraction semigroup in the space of continuous functions vanishing at infinity satisfies the positive maximum principle, then the semigroup must be positivity preserving, i.e. a Feller semigroup (compare [4] or [8]).

For operators acting on functions defined on $\mathbb{R}^n$ and satisfying the positive maximum principle we know the structure due to results of Courrège [3]. If they map test functions onto bounded continuous functions, they must be pseudo-differential operators $-q(x, D)$ where the symbol $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is continuous and with respect to the co-variable negative definite, i.e. for all $x \in \mathbb{R}^n$ the function $\xi \mapsto q(x,\xi)$ is a continuous negative definite function (compare also [8] or [2]). Behind this result is essentially the Lévy–Khinchin formula.

Markov chains have only very recently been considered as "Feller semigroups" generated by a pseudo-differential operator. Their state space is $\mathbb{Z}^m$ and hence the symbol's co-variable lives on the dual group $\mathbb{T}^m$. We refer to [5, 6]. Again, at least in cases of bounded operators, the positive maximum principle must hold on, say, $l_p(\mathbb{Z}^m) \subset C_\infty(\mathbb{Z}^m)$, and in many cases this refers to negative definite symbols in the co-variable of the symbol (see [5]). In the case where $-q(\xi,\omega)$ is the symbol of a translation invariant generator of a Feller semigroup on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ the techniques applied in [8, Theorem 4.5.6], to prove the positive maximum principle, transfer to symbols $-q(x,k;\xi,\omega)$. However, examples for symbols $-q(\xi,\omega)$ or $-q(x,k;\xi,\omega)$ are not

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easy to write down. One class of examples is given by symbols of the type $-q(\xi,\omega) = -q_r(\xi) - q_z(\omega)$, and then we may consider in the next step symbols

$$q(x,k;\xi,\omega) = a(x,k)q_r(\xi) + b(x,k)q_z(\omega). \tag{2.1}$$

A further class is obtained by looking at

$$-q^f(\xi,\omega) = -f(q_r(\xi) + q_2(\omega)) \tag{2.2}$$

where f is a Bernstein function. This is the symbol corresponding to the subordinate semigroup $(T_t^f)_{t\geqslant 0}$ if $(T_t)_{t\geqslant 0}$ is generated by $-q(D_x, D_k)$. Clearly we can now construct a lot of examples by combining (2.1) and (2.2). Unfortunately we do not (yet) have a result ensuring that if for every $(x,k) \in \mathbb{R}^n \times \mathbb{Z}^m$ the mapping $(\xi,\omega) \mapsto q(x,k;\xi,\omega)$ is positive definite on $\mathbb{R}^n \times \mathbb{T}^m$ then $-q(x,k;D_x,D_k)$ with a suitable domain in $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ satisfies the positive maximum principle. For this reason, in Corollary 4.4 we add the positive maximum principle as an extra condition instead of posing a condition on the symbol along the lines of our example.

§3. *Symbols and basic estimates.* We want to investigate operators $q(x,k;D_x,D_k)$ with symbols splitting according to

$$q(x,k;\xi,\omega) = q_1(\xi,\omega) + q_2(x,k;\xi,\omega) \tag{3.1}$$

where $q_1 : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{R}$ is a continuous negative definite function and $q_2 : \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{R}$ is a continuous function such that for $(x,k) \in \mathbb{R}^n \times \mathbb{Z}^m$ the function $(\xi,\omega) \mapsto q(x,k;\xi,\omega)$ is negative definite. The idea is that such a decomposition derives from freezing coefficients:

$$q(x,k;\xi,\omega) = q(x_0,k_0;\xi,\omega) + (q(x,k;\xi,\omega) - q(x_0,k_0;\xi,\omega)).$$

For a fixed continuous negative definite function $\psi : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{R}$ satisfying

$$1 + \psi(\xi,\omega) \geqslant \lambda_0(1 + |\xi|^2)^{\rho/2} \tag{3.2}$$

for all $(\xi,\omega) \in \mathbb{R}^n \times \mathbb{T}^m$ with λ_0 independent of ξ and ω, we make the following assumptions.

A.1. For all $(\xi,\omega) \in \mathbb{R}^n \times \mathbb{T}^m$,

$$1 + q_1(\xi,\omega) \geqslant \gamma_0(1 + \psi(\xi,\omega))$$

and

$$|q_1(\xi,\omega)| \leqslant \gamma_1(1 + \psi(\xi,\omega)).$$

A.2.l. For all $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leqslant l$, the function $x \mapsto q_2(x,k;\xi,\omega)$ is l-times continuously differentiable and with $\varphi_\alpha \in L^1(\mathbb{R}^n \times \mathbb{Z}^m)$ we have

$$|\partial_x^\alpha q_2(x,k;\xi,\omega)| \leqslant \varphi_\alpha(x,k)(1 + \psi(\xi,\omega)). \tag{3.3}$$

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Obviously, assuming (3.1), we can replace in (3.3) the symbol q_2 by q. From (3.1) we deduce that $q(x, k; D_x, D_k)$ splits according to

$$q(x, k; D_x, D_k) = q_1(D_x, D_k) + q_2(x, k; D_x, D_k).$$

Standard arguments (compare, for example, [**6**]) yield the following set of estimates for $q_1(D_x, D_k)$.

THEOREM 3.1. *Under A.1, for $s \geqslant 0$ and $u \in H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$,*

$$\|q_1(D_x, D_k)u\|_{\psi,s} \leqslant \gamma_1 \|u\|_{\psi,s+2}.$$

Further there exist $\kappa_0 > 0$ and $\kappa_1 \geqslant 0$ such that

$$\|q_1(D_x, D_k)u\|_{\psi,s} \geqslant \kappa_0 \|u\|_{\psi,s+2} - \kappa_1 \|u\|_0 \tag{3.4}$$

where $\kappa_j = \kappa_j(n, m, s)$. Moreover, for the sesquilinear form

$$B_1(u, v) = (q_1(D_x, D_k)u, v)_0,$$

we have

$$|B_1(u, v)| \leqslant \gamma_1 \|u\|_{\psi,1} \|v\|_{\psi,1}$$

and

$$B_1(u, v) \geqslant \gamma_0 \|u\|^2_{\psi,1} - \|u\|^2_0.$$

Clearly the proof of this theorem uses the Plancherel theorem and the definition of the spaces $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ but not much more.

Before we can handle $q_2(x, k; D_x, D_k)$ we need an observation about commutators. We can apply Theorem 3.1 to $\psi(D_x, D_k)$ and we of course find

$$\|u\|_{\psi,s} = \|(1 + \psi(D_x, D_k))^{s/2}u\|_0 = \|(1 + \psi(\cdot, \cdot))^{s/2}\tilde{u}\|_0. \tag{3.5}$$

The commutator we are interested in is

$$[(1 + \psi(D_x, D_k))^s, q(x, k; D_x, D_k)] = [(1 + \psi(D_x, D_k))^s, q_2(x, k; D_x, D_k)].$$

First we note that under A.2.1 we find for all $p \in \mathbb{N}_0$, $p \leqslant l$, the estimate

$$|\tilde{q}(\eta, \tau; \xi, \omega)| \leqslant \tilde{\gamma}_{\rho,n} \sum_{|\alpha| \leqslant p} \|\varphi_\alpha\|_{L^1} (1 + |\eta|^2)^{-p/2} (1 + \psi(\xi, \eta))$$

where $\tilde{q}(\eta, \tau; \xi, \omega)$ denotes the Fourier transform of $q(x, k; \xi, \omega)$ with respect to $(x, k) \in \mathbb{R}^n \times \mathbb{Z}^m$. Details of the calculations, which are completely analogous to those in [**9**], are given in [**14**].

LEMMA 3.2. *Under A.2.l we have, with $s \geqslant \frac{1}{2}$ and $l > n + 2s$,*

$$\|[(1 + \psi(D_x, D_k))^s, q_2(x, k; D_x, D_k)]u\|_0 \leqslant \kappa_{m,n,l,s,\psi} \sum_{|\alpha| \leqslant l} \|\varphi_\alpha\|_{L^1} \|u\|_{\psi,2s+1}. \tag{3.6}$$

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Proof. Plancherel's theorem yields for $u, v \in S(\mathbb{R}^n \times \mathbb{Z}^m)$,

$$
\begin{aligned}
&|([(1+\psi(D_x, D_k))^s, q_2(x,k; D_x, D_k)]u, v)_0| \\
&\quad = \left| \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} \Big(F_{\substack{x\mapsto\xi \\ k\mapsto\omega}}((1+\psi(\xi,\omega)))^s (q(x,k; D_x, D_k)u)(\xi,\omega) \right. \\
&\qquad \left. - F_{\substack{x\mapsto\xi \\ k\mapsto\omega}}(q(x,k; D_x, D_k))((1+\psi(D_x, D_k))^s u)(\xi,\omega) \Big) \bar{\tilde{v}}(\xi,\omega)\, d\omega\, d\xi \right| \\
&\quad = (2\pi)^{-n/2-m} \left| \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} \tilde{q}(\xi-\eta, \omega-\tau; \eta, \tau) \right. \\
&\qquad \left. \times ((1+\psi(\xi,\omega))^s - (1+\psi(\eta,\tau))^s) \tilde{u}(\eta,\tau) \bar{\tilde{v}}(\xi,\omega)\, d\tau\, d\eta\, d\omega\, d\xi \right| \\
&\quad \leqslant (2\pi)^{-n/2-m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} \tilde{\gamma}_{l,n} \sum_{|\alpha|\leqslant l} \|\varphi_\alpha\|_{L^1} (1+|\xi-\eta|^2)^{-l/2} \\
&\qquad \times (1+\psi(\eta,\tau)) c_{s,\psi} (1+\psi(\eta,\tau))^{(2s-1)/2} \\
&\qquad \times (1+|\xi-\eta|^2)^s |\tilde{u}(\eta,\tau)|\, d\tau\, d\eta\, |\tilde{v}(\xi,\omega)|\, d\omega\, d\xi \\
&\quad = c_{m,n,l,s,\psi} \sum_{|\alpha|\leqslant l} \|\varphi_\alpha\|_{L^1} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} (1+|\xi-\eta|^2)^{-l/2+s} \\
&\qquad \times (1+\psi(\eta,\tau))^{(2s+1)/2} |\tilde{u}(\eta,\tau)||\tilde{v}(\xi,\omega)|\, d\tau\, d\eta\, d\omega\, d\xi \\
&\quad \leqslant \kappa_{m,n,l,s,\psi} \sum_{|\alpha|\leqslant l} \|\varphi_\alpha\|_{L^1} \|(1+\psi(\cdot,\cdot))^{(2s+1)/2} \tilde{u}\|_0 \|v\|_0,
\end{aligned}
$$

implying (3.6). □

COROLLARY 3.3. *Under A.2.l we have, with $s \geqslant 0$, $t \geqslant \frac{1}{2}$ and $l > n+2s+2t$,*

$$
\|[(1+\psi(D_x, D_k))^s, q_2(x,k; D_x, D_k)]u\|_{\psi,2t} \leqslant \tilde{c} \sum_{|\alpha|\leqslant l} \|\varphi_\alpha\|_{L^1} \|u\|_{\psi,2s+2t+1}.
$$

The proof follows from (3.6) in combination (3.5).

THEOREM 3.4. *Under A.2.l we have, with $s \geqslant 0$ and $l > n+s+1$, for $u \in H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$,*

$$
\|q_2(x,k; D_x, D_k)u\|_{\psi,s} \leqslant \tilde{c}_{m,n,l,s,\psi} \sum_{|\alpha|\leqslant m} \|\varphi_\alpha\|_{L^1} \|u\|_{\psi,s+2}. \tag{3.7}
$$

Proof. For $s = 0$ we have for $u, v \in S(\mathbb{R}^n \times \mathbb{Z}^m)$,

$$
\begin{aligned}
&|(q_2(x,k; D_x, D_k)u, v)_0| \\
&\quad = c \left| \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} \int_{\mathbb{R}^n} \int_{\mathbb{T}^m} \tilde{q}(\xi-\eta, \omega-\tau; \eta, \tau) \tilde{u}(\eta,\tau)\, d\tau\, d\eta\, \bar{\tilde{v}}(\xi,\omega)\, d\omega\, d\xi \right|
\end{aligned}
$$

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$$\leqslant c' \sum_{|\alpha|\leqslant n+1} \|\varphi_\alpha\|_{L^1} \int_{\mathbb{R}^n}\int_{T^m}\int_{\mathbb{R}^n}\int_{T^m} (1+|\eta|^2)^{-(n+1)/2} \times (1+\psi(\xi,\omega))|\tilde{u}(\eta,\tau)||\tilde{v}(\xi,\omega)|\,d\tau\,d\eta\,d\omega\,d\xi$$
$$\leqslant c'' \sum_{|\alpha|\leqslant n+1} \|\varphi_\alpha\|_{L^1}\|u\|_{\psi,2}\|v\|_0,$$

implying

$$\|q_2(x,k;D_x,D_k)u\|_0 \leqslant c'' \sum_{|\alpha|\leqslant n+1} \|\varphi_\alpha\|_{L^1}\|u\|_{\psi,2}.$$

The case $s \geqslant 1$ follows from the observation

$$\|q_2(x,k;D_x,D_k)u\|_{\psi,s} \leqslant \|q_2(x,k;D_x,D_k)(1+\psi(D_x,D_k))^{s/2}u\|_0 + \|[(1+\psi(D_x,D_k))^{s/2}, q(x,k;D_x,D_k)]u\|_0$$

and Corollary 3.3. For $0 < s < 1$ we use interpolation. □

With techniques similar to those used above we can show (see [**14**]) the following theorem.

THEOREM 3.5. *Under A.2.l, $l \geqslant n+2$, for all $u, v \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$, we have*

$$|B_2(u,v)| = |(q_2(x,k;D_x,D_k)u, v)_0| \leqslant c_{m,n,\psi} \sum_{|\alpha|\leqslant n+2} \|\varphi_\alpha\|_{L^1}\|u\|_{\psi,1}\|v\|_{\psi,1}.$$

Combining Theorems 3.1, 3.4 and 3.5, we arrive at the following two results.

THEOREM 3.6. *Under A.1 and A.2.l, $l \geqslant n+2$, we have*

$$|B(u,v)| = |B_1(u,v) + B_2(u,v)| \leqslant \delta_0\|u\|_{\psi,1}\|v\|_{\psi,1} \tag{3.8}$$

with δ_0 depending on n, m, ψ and $\|\varphi_\alpha\|_{L^1}$, $|\alpha| \leqslant n+2$. Furthermore, if

$$\delta_1 := \gamma_0 - c_{m,n,\psi} \sum_{|\alpha|\leqslant n+2} \|\varphi_\alpha\|_{L^1} > 0,$$

then

$$|B(u,u)| \geqslant \delta_1\|u\|^2_{\psi,1} - \|u\|^2_0. \tag{3.9}$$

Proof. While (3.8) is clear, for (3.9) observe that

$$|B(u,v)| \geqslant |B_1(u,u)| - |B_2(u,u)| \geqslant \gamma_0\|u\|^2_{\psi,1} - \|u\|^2_0 - c_{n,m,\psi} \sum_{|\alpha|\leqslant n+2} \|\varphi_\alpha\|_{L^1}\|u\|^2_{\psi,1}.$$ □

THEOREM 3.7. *Under A.1 and A.2.l we have, with $s \geqslant 0$ and $l > n+s+1$, for all $u \in H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$,*

$$\|q(x,k;D_x,D_k)u\|_{\psi,s} \leqslant \hat{c}\|u\|_{\psi,s+2}$$

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where $\hat{c}$ depends on n, m, ψ, s and $\|\varphi_\alpha\|_{L^1}$ for $|\alpha| \leqslant l$. Moreover, for $\sum_{|\alpha|\leqslant l} \|\varphi_\alpha\|_{L^1} \leqslant \epsilon = \epsilon(n, m, \psi, s)$ we have, for some $\delta_2 > 0$ and $\lambda_1 \geqslant 0$,

$$\|q(x, k; D_x, D_k)u\|_{\psi,s} \geqslant \delta_2 \|u\|_{\psi,s+2} - \lambda_1 \|u\|_0.$$

(Note that we can give an explicit bound for ϵ, for example

$$\epsilon \leqslant \frac{\kappa_0}{2\tilde{c}_{n,m,l,s,\psi}},$$

with κ_0 from (3.4) and $\tilde{c}_{m,n,l,s,\psi}$ from (3.7).)

§4. *Solving $q(x, k; D_x, D_k)u + \lambda u = f$.* In order to identify $-q(x, k; D_x, D_k)$ as a generator of a Feller semigroup on $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ we want to apply a variant of the Hille–Yosida–Ray theorem. For this we need to solve the equation

$$q(x, k; D_x, D_k)u + \lambda u = f \tag{4.1}$$

for a dense set of right-hand sides in $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ and some $\lambda > 0$. The estimates for the bilinear form B (see Theorems 3.5 and 3.6) immediately imply by the Lax–Milgram theorem the following result.

THEOREM 4.1. *Under A.1 and A.2.l, $l \geqslant n + 2$, for $\lambda \geqslant 1$, there exists for every $f \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ a unique $u \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ satisfying*

$$B_\lambda(u, v) = B(u, v) + \lambda(u, v)_0 = (f, v)_0 \tag{4.2}$$

for all $v \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$.

Assumption (3.2) implies for $s > n/2\rho$ that

$$\|u\|_\infty \leqslant c\|u\|_{\psi,s},$$

i.e. $H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ is under this condition continuously embedded into $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ and it is clearly a dense subspace (compare [**6**]). Thus a regularity result of type $f \in H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ implies for a solution $u \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ of (4.2) that $u \in H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m)$ will give us the identity of (4.1) in $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$. With respect to the discrete variable we need not require improved regularity as long as we can improve regularity with respect to x uniformly in k. For this purpose mollifying u with respect to x only is sufficient. Let J_ϵ denote the usual Friedrichs mollifier (compare [**8**] or [**6**]) with respect to x, i.e.

$$J_\epsilon(u)(x, k) = \int_{\mathbb{R}^n} j_\epsilon(x - y)u(y, k)\, dy.$$

In [**6**] we collect the typical properties:

$$J_\epsilon(u) \in \bigcap_{t\geqslant 0} H^{\psi,t}(\mathbb{R}^n \times \mathbb{Z}^m),$$
$$\|J_\epsilon(u)\|_{\psi,s} \leqslant \|u\|_{\psi,s}$$

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and

$$\lim_{\epsilon\to 0}\|J_\epsilon(u)-u\|_{\psi,s}=0,$$

and since J_ϵ is translation invariant we have $[J_\epsilon, q_1(D_x, D_k)] = 0$. For $q_2(x, k; D_x, D_k)$ commutator estimates are still regularly improving; in fact we have the following theorem.

THEOREM 4.2. *Under A.2.l, $l > n + 1 + (s - 1)$, we have for all $u \in H^{\psi,s+1}(\mathbb{R}^n \times \mathbb{Z}^m)$,*

$$\|[J_\epsilon, q_2(x, k; D_x, D_k)]u\|_{\psi,s} \leqslant \tilde{c}\|u\|_{\psi,s+1}$$

with a constant $\tilde{c}$ independent of $\epsilon \in (0, 1]$.

We can mimic the proofs of [**9**, Theorem 2.3.16 and Corollary 2.3.18]. Our conditions on q_2 assure the uniformity with respect to k that is needed; for details we refer to [**14**].

Now let $u \in H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$ be a solution to (4.2) with $f \in H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m)$ and let $(u_l)_{l\in\mathbb{N}}$ be a sequence in $S(\mathbb{R}^n \times \mathbb{Z}^m)$ converging to u in $H^{\psi,1}(\mathbb{R}^n \times \mathbb{Z}^m)$. We find $J_\epsilon(u_l) \in \bigcap_{t\geqslant 0} H^{\psi,t}(\mathbb{R}^n \times \mathbb{Z}^m) \subset H^{\psi,2}(\mathbb{R}^n \times \mathbb{Z}^m)$, and further, for $v \in S(\mathbb{R}^n \times \mathbb{Z}^m)$,

$$\begin{aligned}
B_\lambda(J_\epsilon(u_l), v) &= (q(x, k; D_x, D_k)J_\epsilon(u_l) + \lambda J_\epsilon(u_l), v)_0\\
&= (J_\epsilon((q(x, k; D_x, D_k) + \lambda)u_l), v)_0\\
&\quad - ([J_\epsilon, q(x, k; D_x, D_k)]u_l, v)_0\\
&= B_\lambda(u_l, J_\epsilon(v)) - ([J_\epsilon, q_2(x, k; D_x, D_k)]u_l, v)_0.
\end{aligned}$$

Since

$$\|[J_\epsilon, q_2(x, k; D_x, D_k)]u_l\|_0 \leqslant c\|u_l\|_{\psi,1} \leqslant \tilde{c}(\|u\|_{\psi,1})$$

we conclude that $([J_\epsilon, q_2(x, k; D_x, D_k)]u_l)_{l\in\mathbb{N}}$ converges weakly in $L^2(\mathbb{R}^n \times \mathbb{Z}^m)$ to some $w_\epsilon \in L^2(\mathbb{R}^n \times \mathbb{Z}^m)$, $\|w_\epsilon\|_0 \leqslant \tilde{c}$. Thus for $l \to \infty$ we find

$$\begin{aligned}
B_\lambda(J_\epsilon(u), v) &= B_\lambda(u, J_\epsilon(v)) - (w_\epsilon, v)_0\\
&= (f, J_\epsilon(v))_0 - (v_\epsilon, v)_0 = (J_\epsilon(f), v)_0 - (w_\epsilon, v)_0.
\end{aligned}$$

It now follows that

$$\|(q(x, k; D_x, D_k) + \lambda)J_\epsilon(u)\|_{L^2} \leqslant \|J_\epsilon(f)\|_0 + \|w_\epsilon\|_0,$$

and with Theorem 3.7 we arrive at

$$\delta_2\|J_\epsilon(u)\|_{\psi,2} \leqslant c(\|f\|_0 + \tilde{c}\|u\|_{\psi,1}),$$

which yields $u \in H^{\psi,2}(\mathbb{R}^n \times \mathbb{Z}^m)$. An application of Theorem 4.2 gives the following theorem.

THEOREM 4.3. *Under A.1 and A.2.l, $l > n + 3 + s$, $s > n/2\rho$, with $\lambda \geqslant 0$ sufficiently large, equation (4.1) has for every $f \in H^{\psi,s}(\mathbb{R}^n \times \mathbb{Z}^m) \subset C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$ a unique solution $u \in H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m) \subset C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$.*

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The Hille–Yosida theorem in a variant of Lumer and Phillips yields the following corollary.

COROLLARY 4.4. *If* $(-q(x,k;D_x,D_k), H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m))$ *satisfies the positive maximum principle then* $(-q(x,k;D_x,D_k), H^{\psi,s+2}(\mathbb{R}^n \times \mathbb{Z}^m))$ *extends to a generator of a Feller semigroup on* $C_\infty(\mathbb{R}^n \times \mathbb{Z}^m)$.

We can now apply our discussions in §2 to derive sufficient conditions for the positive maximum principle to hold for $-q(x,k;D_x,D_k)$.

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