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Frequently hypercyclic random vectors for C_0-semigroups

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Abstract. We show that, under certain conditions, a strongly continuous semigroup admits an almost surely frequently hypercyclic random vector defined as a stochastic integral in Fréchet spaces with respect to the Brownian motion. Two criteria are given. We will apply the second criterion to three examples: translation semigroups on spaces of integrable functions, the exponential of weighted shifts, and the translation operators on the space of entire functions. This last example, with a stochastic approach, seems to be new in the literature. Some other examples are given.

Key words: frequent hypercyclicity, semigroup of operators, ergodic theory, stochastic integral
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1. *Introduction*

Let E be a separable Fréchet space. A continuous and linear operator T is said to be *hypercyclic* if there exists a vector $x \in E$ such that the set $\{T^n(x) \mid n \in \mathbb{N}\}$, called the *orbit* of x, is dense in E. Such a vector x is said to be *hypercyclic* for T. The first known operators to be hypercyclic were the translation operators [6], the differentiation operator [26], and the multiples of the backward shift [33].

An operator T is *chaotic* if it is hypercyclic and the set of periodic points is dense.

Bayart and Grivaux [2] introduced the notion of frequent hypercyclicity. Recall that the *lower density* of a set $A \subseteq \mathbb{N}$ is the quantity $\underline{\lim}_{N\to\infty} |A \cap \{0, \dots, N\}|/(N+1)$.

Definition 1.1. Let E be a Fréchet space. An operator $T : E \longrightarrow E$ is *frequently hypercyclic* if there exists $x \in E$ such that, for every non-empty open set U of E, the set $\{n \in \mathbb{N} \mid T^n(x) \in U\}$ has positive lower density. Such a vector is called a *frequently hypercyclic vector* for T.

Of course, frequent hypercyclicity implies hypercyclicity. All the operators cited above are frequently hypercyclic.

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The continuous counterpart of a single operator is the concept of semigroups. A family of operators $(T_t)_{t\geq 0}$ on a Fréchet space E is called a C_0-*semigroup* if $T_0 = I$, $T_{t+s} = T_t T_s$ for all $s, t \geq 0$, and $\lim_{s\to t} T_s(x) = T_t(x)$ for all $x \in E$ and $t \geq 0$.

Naturally, we say that a C_0-semigroup $(T_t)_{t\geq 0}$ is *hypercyclic* if there exists $x \in E$ such that the *orbit* $\{T_t(x) \mid t \geq 0\}$ is dense in E. Such a vector is called a *hypercyclic vector* for $(T_t)_{t\geq 0}$. A C_0-semigroup $(T_t)_{t\geq 0} T$ is *chaotic* if it is hypercyclic and the set of periodic points is dense.

For frequent hypercyclicity, we must redefine the lower density: the *lower density* of a measurable set $A \subseteq [0, \infty[$ is the quantity $\underline{\lim}_{N\to\infty} \lambda(A \cap [0, N])/N$, where λ is the Lebesgue measure on $[0, \infty[$.

Definition 1.2. Let E be a Fréchet space. A C_0-semigroup $(T_t)_{t\geq 0}$ on E is *frequently hypercyclic* if there exists $x \in E$ such that, for every non-empty open set U of E, the set $\{t \geq 0 \mid T_t(x) \in U\}$ has positive lower density. Such a vector is called a *frequently hypercyclic vector* for $(T_t)_{t\geq 0}$.

Rolewicz [33] was the first to prove the existence of a hypercyclic C_0-semigroup. Desch, Schappacher, and Webb [16] initiated the study of dynamical properties of semigroups.

We refer to the books [4, 20] for much more details about linear dynamics. For further reading on semigroups on Banach spaces, see for example the book by Engel and Nagel [18]. For the case of Fréchet spaces, see the book by Yosida [42, Ch. IX], and the papers by Kômura [24] and Ôuchi [32].

In a previous paper, the author [1, Theorems 1.3 and 3.3] proved that for an operator T on a Fréchet space E, under some hypothesis, it is possible to construct a random vector that is almost surely frequently hypercyclic for T. It turns out that the measure induced by this random vector is strongly mixing: a measure μ on the Borel sets $\mathscr{B}(E)$ of E is *strongly mixing* for an operator T if it is T-*invariant*, that is, $\mu(T^{-1}(A)) = \mu(A)$ for every $A \in \mathscr{B}(E)$, and $\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B)$ for every $A, B \in \mathscr{B}(E)$. An intermediate result states that a random vector with a particular form is necessarily frequently hypercyclic for the operator and its induced measure is strongly mixing.

PROPOSITION 1.1. [1, Proposition 2.5] *Let $T : E \longrightarrow E$ be an operator and let $(u_n)_{n\in\mathbb{Z}}$ be a sequence in E such that $T(u_n) = u_{n-1}$ for every $n \in \mathbb{Z}$. Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of independent and identically distributed (i.i.d.) random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that the random vector*

$$v := \sum_{n=-\infty}^{\infty} X_n u_n$$

is almost surely well defined and v is of full support (that is, $\mathbb{P}(v \in O) > 0$ for every non-empty open subset O of E). Then, v is almost surely frequently hypercyclic for the operator T and induces a strongly mixing measure with full support for T.

We investigate in this paper the case of C_0-semigroups. Two approaches to finding a frequently hypercyclic vector are presented. A measure μ on $\mathscr{B}(E)$ is *strongly mixing* for

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the C_0-semigroup if it is $(T_t)_{t\geq 0}$-*invariant*, that is, $\mu(T_t^{-1}(A)) = \mu(A)$ for every $t \geq 0$ and $A \in \mathscr{B}(E)$, and $\lim_{t\to\infty} \mu(T^{-t}(A) \cap B) = \mu(A)\mu(B)$ for every $A, B \in \mathscr{B}(E)$.

The idea of the first approach is to try to mimic what has been done in [1]. However, only a result analogous to Proposition 1.1 for semigroups will be proved, namely Proposition 3.2. The sequence of random variables $(X_n)_{n\in\mathbb{Z}}$ in [1, Proposition 2.5] is replaced by the Brownian motion.

Definition 1.3. Let $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ be two independent real Brownian motions. The stochastic process $(B_t)_{t\geq 0} := (B_t^1 + i\, B_t^2)_{t\geq 0}$ is called a *complex Brownian motion*.

This means that for semigroups, we are *a priori* quite restrictive since only the normal distribution is considered. For a C_0-semigroup $(T_t)_{t\geq 0}$ on a separable Fréchet space E, we will assume that there exists a family $(u_t)_{t\in\mathbb{R}}$ of vectors of E such that $T_s(u_t) = u_{t-s}$ for every $s \geq 0$ and $t \in \mathbb{R}$. Instead of a series as in Proposition 1.1, we will integrate this family of vectors with respect to the Brownian motion, see Proposition 3.2. It is thus a stochastic integral for functions with values in a Fréchet space.

The second approach is inspired by the results of Chakir and El Mourchid [11] who used stochastic integrals for scalar-valued functions and considered the particular case of semigroups on the spaces $L^p(T, \Sigma, \sigma)$, where (T, Σ, σ) is a σ-finite measure space. Once again, we will construct a frequently hypercyclic random vector via a stochastic integral, see Theorem 4.1. The family $(u_t)_t$ will consist of eigenvectors of the generator of the semigroup with respect to purely imaginary eigenvalues. Consequently, only complex Fréchet spaces will be considered. We have three examples of this approach: the translation semigroups, the exponential of weighted shifts, and the translation semigroup on the space of entire functions. As far as we know, our approach for this last example is new in the literature.

The stochastic integral for scalar-valued functions is the well-known Itô integral, see [23] or [31], for example, for more details. The case of Fréchet space-valued integrands is more delicate and seems to be quite recent in the literature. We will use the definition of van Neerven and Weis [41]. Their stochastic integral is defined in a Pettis manner for Banach space-valued functions defined on a bounded interval. Nevertheless, it can be extended for Fréchet space-valued functions defined on an arbitrary interval, and their results still hold with the same proofs.

Remark 1.2. By the proof of [28, Proposition 2.1] and by [13, Theorem 3.2], one can prove that a vector $x \in E$ is frequently hypercyclic for $(T_t)_{t\geq 0}$ if and only if it is frequently hypercyclic for each T_t, $t > 0$. Therefore, we could apply [1, Theorem 3.3] to T_1, but this would not guarantee that the resulting measure is strongly mixing for the semigroup.

Section 2 defines and states some properties of the stochastic integral for Fréchet space-valued functions. Some proofs that differ from [41] are postponed to an appendix. Sections 3 and 4 present the first and second approaches explained above, respectively. Finally, we will consider an example of a semigroup for which we cannot apply Theorem 4.1 for 'natural' choices of $(u_t)_t$ in §5, indicating that a more general approach could still be found.

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1.1. *Notation.* In the following, let E be a separable Fréchet space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $(B_t)_{t\in\mathbb{R}}$ be a Brownian motion over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. This stochastic process is real (respectively complex) if E is real (respectively complex).

2. *Stochastic integral in Fréchet spaces*

This section is devoted to the definition and some properties of the stochastic integral in vector spaces developed by van Neerven and Weis [41], generalized to functions on an arbitrary interval and with values in a real or complex Fréchet space. For measurable functions taking values in a separable Banach space, the definition of the stochastic integral from [41] coincides with that of Rosiński and Suchanecki [34], see Theorem 2.3.

For the sake of completeness, proofs of the results from [41] stated below are given in the appendix for the case of Fréchet spaces, and also because the case of Fréchet spaces seems to be new in the literature. In this section, we will only prove some results that are not given in [41].

For the remainder of this section, let $I \subseteq \mathbb{R}$ be an interval, which can be unbounded.

Definition 2.1. Let $\phi : I \longrightarrow E$ be a weakly L^2 function. Then, ϕ is called *stochastically integrable* if for all measurable sets $A \in \mathcal{B}(I)$, there exists a random vector $Y_A : \Omega \longrightarrow E$ such that for all $x^* \in E^*$, one has

$$x^*(Y_A) = \int_I \mathbf{1}_A(t) x^*(\phi(t))\, dB_t$$

almost surely. In that case, we write $Y_A = \int_A \phi(t)\, dB_t$.

The random vectors Y_A are Gaussian since $x^*(Y_A)$ is a Gaussian random variable for any $x^* \in E^*$. Furthermore, they are uniquely determined almost surely. Indeed, assume that Y is a random vector such that $x^*(Y) = 0$ almost surely for any $x^* \in E^*$. Since E is separable and metrizable, the topological space $(E^*, \sigma(E^*, E))$ is also separable by [37, §IV.1.7]. Let $(x_n^*)_{n\in\mathbb{N}} \subseteq E^*$ be a dense sequence in $(E^*, \sigma(E^*, E))$, which thus separates points of E. Then, almost surely, $x_n^*(Y) = 0$ for all $n \in \mathbb{N}$, which is equivalent to $Y = 0$ almost surely.

By the Fernique theorem, see [7, Corollary 2.8.6], we have $Y_A \in L^p(\Omega; E)$ for every $1 \leq p < \infty$. The spaces $L^p(\Omega; E)$ of p-summable functions with values in a Fréchet space E are defined as for Banach spaces, see [21, Ch. 1], and the proof of their completeness follows the same lines as for the scalar case, see [35, Theorem 3.11].

Definition 2.2. Let $1 \leq p < \infty$. We define the space $L^p(\Omega, \mathcal{A}, \mathbb{P}; E)$ as the space of all equivalence classes of measurable functions $f : \Omega \longrightarrow E$ for which

$$\int_\Omega \|f\|^p \, d\mathbb{P} < \infty$$

for all continuous seminorms $\|\cdot\|$ on E. This space is also noted as $L^p(\Omega; E)$.

For a given non-decreasing sequence $(\|\cdot\|_k)_{k\geq 1}$ of seminorms of E generating its topology, we define for each real number $1 \leq p < \infty$ and integer $k \geq 1$, the seminorm $\|\cdot\|_{p,k}$ by

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$$\|f\|_{p,k} := \left(\int_\Omega \|f\|_k^p \, d\mathbb{P}\right)^{1/p}$$

for every $f \in L^p(\Omega; E)$, and the sequence of seminorms $(\|\cdot\|_{p,k})_{k\geq 1}$ defines the topology of $L^p(\Omega; E)$.

Here are two easy consequences of Definition 2.1.

LEMMA 2.1. [41, pp. 138 and 139] *Let $\phi, \psi : I \longrightarrow E$ be two stochastically integrable functions, let $a, b \in \mathbb{K}$ and $T : E \longrightarrow F$ be a continuous linear map, where F is another separable Fréchet space. Then, the functions $a\phi + b\psi$ and $T\phi$ are stochastically integrable, and we have*

$$\int_I (a\phi(t) + b\psi(t)) \, dB_t = a \int_I \phi(t) \, dB_t + b \int_I \psi(t) \, dB_t$$

and

$$\int_I T\phi(t) \, dB_t = T \int_I \phi(t) \, dB_t.$$

In the case of scalar-valued functions, any square-integrable function is stochastically integrable. For a Banach space, this is no longer true: for every $1 \leq p < 2$, there exists a bounded measurable function defined on [0, 1] with values in ℓ^p that is not stochastically integrable, see [34, Example 3.1].

Definition 2.3. Let $\phi : I \longrightarrow E$ be a weakly L^2 function. We define the conjugate-linear map $I_\phi : L^2(I) \longrightarrow (E^*)'$ by

$$I_\phi(f) : E^* \longrightarrow \mathbb{K}, x^* \longmapsto \int_I x^*(\phi(t))\overline{f(t)} \, dt$$

for every $f \in L^2(I)$.

Here, $(E^*)'$ denotes the algebraic dual of E^*.

We will say that I_ϕ is *γ-radonifying* if I_ϕ takes values in E and the linear map $I_\phi S$ is γ-radonifying, where $S : L^2(I)^* \longrightarrow L^2(I)$ is the canonical isometry. This is equivalent to saying that $\sum_{n\geq 0} g_n I_\phi(f_n)$ converges almost surely in E, where $(g_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. standard Gaussian random variables and $(f_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(I)$. Note that $L^2(I)$ is separable by [21, Proposition 1.2.29]. See [22, Ch. 2] for γ-radonifying operators on Banach spaces.

The map I_ϕ associated with a weakly L^2 function ϕ characterizes its stochastic integrability, as the next result says. Its proof is postponed to the appendix.

THEOREM 2.2. [41, Theorem 2.3] *Let $\phi : I \longrightarrow E$ be a weakly L^2 function. The following assertions are equivalent, where $c = 1$ if $\mathbb{K} = \mathbb{R}$ and $c = 2$ if $\mathbb{K} = \mathbb{C}$:*

(i) *ϕ is stochastically integrable;*

(ii) *there exists a random vector $Y : \Omega \longrightarrow E$ such that for every $x^* \in E^*$, we have*

$$x^*(Y) = \int_I x^*(\phi(t)) \, dB_t$$

almost surely;

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(iii) *there exists a Gaussian measure* μ *on* E *with covariance operator* $Q : E^* \longrightarrow E$ *such that for every* $x^* \in E^*$,

$$c \int_I |x^*(\phi(t))|^2 \, dt = x^* Q x^*;$$

(iv) *there exist a separable Hilbert space* H *and a* γ*-radonifying operator* $T : H \longrightarrow E$ *such that for every* $x^* \in E^*$,

$$c \int_I |x^*(\phi(t))|^2 \, dt \leq \|T^*(x^*)\|_H^2;$$

(v) *the map* I_ϕ *takes values in* E*, and is continuous and* γ*-radonifying.*

If one of these assertions holds, then ϕ *is Pettis integrable on every bounded interval included in* I*, and* μ *is the distribution of the random vector* $\int_I \phi(t) \, dB_t$.

This theorem also shows that if a function is stochastically integrable with respect to some Brownian motion, then it is stochastically integrable with respect to any Brownian motion.

Although the stochastic integral is defined in a Pettis manner, any stochastically integrable function can be approximated by step functions.

Definition 2.4. A *step function* $\phi : I \longrightarrow E$ is a function of the form $\phi = \sum_{i=1}^n a_i \mathbf{1}_{]t_{i-1}, t_i[}$, where $a_i \in E$ for all $1 \leq i \leq n$, $n \in \mathbb{N}_0$, and $t_0 \leq \cdots \leq t_n \in I$.

THEOREM 2.3. [41, Theorem 2.5] *Let* $\phi : I \longrightarrow E$ *be a weakly* L^2 *function. Then,* ϕ *is stochastically integrable if and only if there exists a sequence* $(\phi_n)_{n \in \mathbb{N}}$ *of step functions such that:*

(i) *for all* $x^* \in E^*$, $\lim_{n \to \infty} x^* \phi_n = x^* \phi$ *in measure;*

(ii) *there exists a random vector* $Y : \Omega \longrightarrow E$ *such that*

$$Y = \lim_{n \to \infty} \int_I \phi_n(t) \, dB_t$$

in probability.

In that case, we have $Y = \int_I \phi(t) \, dB_t$*, the convergence in part (i) is in* $L^2(I)$*, and the convergence in part (ii) is in* $L^p(\Omega; E)$ *for every* $1 \leq p < \infty$.

See the appendix for the proof of Theorem 2.3. This characterization of the stochastic integral has been taken as its definition in [34] for Banach space-valued measurable functions.

The proof of the next result follows the same lines as [41, Corollary 2.8].

PROPOSITION 2.4. *Let* $\phi : [0, \infty[\longrightarrow E$ *be stochastically integrable. Then,*

$$\lim_{t \to \infty} \int_0^t \phi(s) \, dB_s = \int_0^\infty \phi(s) \, dB_s$$

in $L^p(\Omega; E)$ *for every* $1 \leq p < \infty$.

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Proof. Let $1 \leq p < \infty$ and let $\|\cdot\|$ be a continuous seminorm. Let $(t_n)_{n\geq 0}$ be a sequence of positive numbers converging to ∞. Let $n \geq 0$. Denote by R_n the covariance operator of the distribution ν_n of $\int_{t_n}^{\infty} \phi(s)\, dB_s$, and let Q be the covariance operator of the distribution μ of $\int_0^{\infty} \phi(s)\, dB_s$. By Theorem 2.2(iii), we get

$$x^* R_n x^* = c \int_{t_n}^{\infty} |x^*\phi|^2\, ds \leq c \int_0^{\infty} |x^*\phi|^2\, ds = x^* Q x^*,$$

where $c = 1$ if $\mathbb{K} = \mathbb{R}$ and $c = 2$ if $\mathbb{K} = \mathbb{C}$. By [40, Theorem 8.8] (stated for Banach spaces, but the proof given there carries over verbatim to Fréchet spaces), this implies that the sequence $(\nu_n)_{n\geq 0}$ is uniformly tight. However, we also have $\lim_{n\to\infty} x^* R_n x^* = 0$. By [4, Theorem 5.9(b)] (whose proof also extends to Fréchet spaces), we deduce that $\lim_{n\to\infty} \widehat{\nu_n}(x^*) = 1$ for all $x^* \in E^*$. Therefore, by [39, Theorems I.3.6 and IV.3.1], we get that $\lim_{n\to\infty} \nu_n = \delta_0$ weakly, where δ_0 is the Dirac measure at 0 (that is, $\delta_0(A) = 1$ if $0 \in A$ and $\delta_0(A) = 0$ if $0 \notin A$ for all $A \in \mathscr{B}(E)$).

We conclude the proof by using [7, Corollary 3.8.8]. We just need to check that

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{\{x\in E, \|x\|^p > R\}} \|x\|^p\, d\nu_n(x) = 0.$$

Let $\varepsilon > 0$. Since $(\nu_n)_{n\geq 0}$ is uniformly tight, there exists a compact set $K \subseteq E$, thus bounded, such that $\nu_n(K) > 1 - \varepsilon$ for all $n \geq 0$. Let $R > 0$ be so large that $K \subseteq \{x \in E, \|x\|^p \leq R\}$. For all $n \geq 0$, by the Cauchy–Schwarz inequality and [40, Theorem 8.8], we get that

$$\begin{aligned}\int_{\{x\in E, \|x\|^p > R\}} \|x\|^p\, d\nu_n(x) &\leq \left(\int_E \|x\|^{2p}\, d\nu_n(x)\right)^{1/2} \left(\int_E \mathbf{1}^2_{\{x\in E, \|x\|^p > R\}}\, d\nu_n(x)\right)^{1/2} \\ &\leq \left(\int_E \|x\|^{2p}\, d\mu(x)\right)^{1/2} \varepsilon^{1/2}.\end{aligned}$$

We conclude that $\sup_{n\geq 0} \int_{\{x\in E, \|x\|^p > R\}} \|x\|^p\, d\nu_n(x) \leq M\varepsilon^{1/2}$ for some constant $M \geq 0$. Since ε was arbitrary, we are done. □

LEMMA 2.5. *Let $\phi : I \longrightarrow E$ be stochastically integrable. Then, for any $s \in \mathbb{R}$, $\phi(\cdot - s)$ is stochastically integrable on $I + s$ and the random vectors $\int_{I+s} \phi(t - s)\, dB_t$ and $\int_I \phi(t)\, dB_t$ are identically distributed.*

Proof. For all $x^* \in E^*$, we have by a change of variables and Theorem (2.2) that

$$c \int_{I+s} |x^*(\phi(t-s))|^2\, dt = c \int_I |x^*(\phi(t))|^2\, dt = x^* Q x^*,$$

where $c = 1$ if $\mathbb{K} = \mathbb{R}$ and $c = 2$ if $\mathbb{K} = \mathbb{C}$, and Q is the covariance operator of the distribution μ of $\int_I \phi(t)\, dB_t$. By Theorem 2.2, we deduce that $\phi(\cdot - s)$ is stochastically integrable and μ is the distribution of $\int_{I+s} \phi(t - s)\, dB_t$. □

LEMMA 2.6. *Let $A, B \in \mathscr{B}(I)$ and $\phi, \varphi : I \longrightarrow E$ be two stochastically integrable functions. If A and B are disjoint, then the random vectors $\int_A \phi(t)\, dB_t$ and $\int_B \varphi(t)\, dB_t$ are independent.*

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Proof. The random vectors $X := \int_A \phi(t)\, dB_t$ and $Y := \int_B \varphi(t)\, dB_t$ are independent if and only if $x^*(X)$ and $y^*(Y)$ are independent for all $x^*, y^* \in E^*$, see [10, p. 23]. By Lemma 2.1 and Itô isometry, we have

$$\begin{aligned}\langle x^*(X), y^*(Y)\rangle_{L^2(\Omega)} &= \left\langle \int_A x^*(\phi(t))\, dB_t, \int_B y^*(\varphi(t))\, dB_t \right\rangle_{L^2(\Omega)} \\ &= c\langle x^*\phi, y^*\varphi\rangle_{L^2(I)} = 0,\end{aligned}$$

where $c = 1$ if $\mathbb{K} = \mathbb{R}$ and $c = 2$ if $\mathbb{K} = \mathbb{C}$. Since $(x^*(X), y^*(Y))$ is a Gaussian random vector by linearity of the stochastic integral, this implies that $x^*(X)$ and $y^*(Y)$ are independent by [22, Proposition E.2.12]. □

We end this section with a result characterizing the stochastically integrable functions with values in a space of functions. This will be useful for concrete examples.

THEOREM 2.7. *Let $(S, \mathcal{B}, \mu)$ be a σ-finite measure space, let $1 \le p < \infty$, and set $E := L^p(S, \mathcal{B}, \mu)$. Then, a measurable function $\phi : I \longrightarrow E$ is stochastically integrable if and only if*

$$\left\| \left(\int_I |\phi(t, \cdot)|^2\, dt \right)^{1/2} \right\|_E < \infty.$$

Theorem 2.7 is a direct consequence of [41, Corollary 2.10] by noticing that $L^p(S, \mathcal{B}, \mu)$ has finite cotype for any $1 \le p < \infty$, see [22, Proposition 7.1.4].

3. *Random vector: first method*

The final parts of the proofs of the main results of §§3 and 4 are the same, so we put it in a lemma.

LEMMA 3.1. *Let $(T_t)_{t\ge 0}$ be a C_0-semigroup on E. Let $v : \Omega \longrightarrow E$ be a random vector such that its probability distribution is ergodic for $(T_t)_{t\ge 0}$ and is of full support. Then, v is almost surely frequently hypercyclic for $(T_t)_{t\ge 0}$.*

Proof. Let μ be the probability distribution of v. Let O be a non-empty open subset of E. The Birkhoff ergodic theorem for semigroups, see [14, Ch. 1, §2, Theorem 1], can be applied to T and μ, and gives

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N (\mathbf{1}_O \circ T_t)(x)\, dt = \mu(O)\ \mu\text{-almost surely (a.s.)}.$$

Let A be a Borel subset of E such that $\mu(A) = 1$ and the previous equality holds everywhere on A. Then, if we set $B := v^{-1}(A) \subseteq \Omega$, we have $\mathbb{P}(B) = \mathbb{P}(v^{-1}(A)) = \mu(A) = 1$ and

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N (\mathbf{1}_O \circ T_t)(v)\, dt = \mathbb{P}(v \in O) > 0$$

on B. Since E is separable, we can take a countable base of open subsets of E and get that almost surely, the set $\{t \ge 0 \mid T_t(v) \in O\}$ has positive lower density for every non-empty

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open subset O of E. The random vector v is therefore almost surely frequently hypercyclic for $(T_t)_{t\geq 0}$. □

The next proposition is the analogous result to Proposition 1.1. Their proofs are the same, but we need to use the Birkhoff ergodic theorem for semigroups through Lemma 3.1, and replace all series with stochastic integrals.

PROPOSITION 3.2. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on E and let $(u_t)_{t\in\mathbb{R}}$ be a family of vectors in E such that $T_s(u_t) = u_{t-s}$ for every $s \geq 0$ and $t \in \mathbb{R}$. Let $(B_t)_{t\in\mathbb{R}}$ be a Brownian motion. Assume that $t \longmapsto u_t$ is stochastically integrable and set*

$$v := \int_{\mathbb{R}} u_t \, dB_t.$$

If v has full support, then v is almost surely frequently hypercyclic for the C_0-semigroup $(T_t)_{t\geq 0}$ and it induces a strongly mixing measure with full support for $(T_t)_{t\geq 0}$.

Proof. Define the probability measure

$$\mu : \mathscr{B}(E) \longrightarrow [0, 1], A \longmapsto \mathbb{P}(v \in A).$$

Let $s \geq 0$. We first show that μ is T_s-invariant. Let $A \in \mathscr{B}(E)$. By the definitions of μ and v and by Lemmas 2.1 and 2.5, we have

$$\mu(T_s^{-1}(A)) = \mathbb{P}(T_s(v) \in A) = \mathbb{P}\left(\int_{\mathbb{R}} u_{t-s} \, dB_t \in A\right) = \mathbb{P}\left(\int_{\mathbb{R}} u_t \, dB_t \in A\right).$$

We conclude that $\mu(T_s^{-1}(A)) = \mathbb{P}(v \in A) = \mu(A)$. The measure μ is thus T_s-invariant.

Now, we claim that μ is $(T_t)_{t\geq 0}$-strongly mixing. Let f and g be two bounded and continuous real-valued functions defined on E. We aim to show that

$$\lim_{t\to\infty} \int_E (f \circ T_t) g \, d\mu = \int_E f \, d\mu \int_E g \, d\mu.$$

Since the set of bounded continuous functions on E is dense in $L^2(E, \mu)$ by [**15**, Theorem 18.1], this will imply the claim by [**15**, p. 26]. First, by definition of μ, this is equivalent to showing that

$$\lim_{t\to\infty} \int_\Omega f(T_t(v)) g(v) \, d\mathbb{P} = \int_\Omega f(v) \, d\mathbb{P} \int_\Omega g(v) \, d\mathbb{P}. \tag{3.1}$$

Let $\varepsilon > 0$. Since f and g are continuous and bounded, by Proposition 2.4 and the dominated convergence theorem, there exists $N \geq 0$ such that

$$\left\| g\left(\int_{-\infty}^{N} u_t \, dB_t\right) - g(v) \right\|_{L^1(\Omega,\mathbb{P})} < \varepsilon \tag{3.2}$$

and

$$\left\| f\left(\int_{-N}^{\infty} u_t \, dB_t\right) - f(v) \right\|_{L^1(\Omega,\mathbb{P})} < \varepsilon. \tag{3.3}$$

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Let $s > 2N$ and write

$$\begin{aligned} f(T_s(v))g(v) &= f(T_s(v))g(v) - f(T_s(v))g\left(\int_{-\infty}^{N} u_t\, dB_t\right) \\ &\quad + f(T_s(v))g\left(\int_{-\infty}^{N} u_t\, dB_t\right) - f\left(\int_{-N+s}^{\infty} u_{t-s}\, dB_t\right)g\left(\int_{-\infty}^{N} u_t\, dB_t\right) \\ &\quad + f\left(\int_{-N+s}^{\infty} u_{t-s}\, dB_t\right)g\left(\int_{-\infty}^{N} u_t\, dB_t\right). \end{aligned} \tag{3.4}$$

For the first two terms, using the assumption that f is bounded and the inequality (3.2) yields

$$\begin{aligned} &\left|\int_\Omega f(T_s(v))g(v)\, d\mathbb{P} - \int_\Omega f(T_s(v))g\left(\int_{-\infty}^{N} u_t\, dB_t\right) d\mathbb{P}\right| \\ &\qquad \le \|f\|_\infty \left\| g\left(\int_{-\infty}^{N} u_t\, dB_t\right) - g(v)\right\|_{L^1(\Omega,\mathbb{P})} \le \|f\|_\infty \varepsilon. \end{aligned}$$

Now, for the third and fourth terms, define the random vectors

$$\begin{aligned} X &:= \int_{-\infty}^{-N+s} u_{t-s}\, dB_t, \quad & Y &:= \int_{-N+s}^{\infty} u_{t-s}\, dB_t, \\ U &:= \int_{-\infty}^{-N} u_t\, dB_t, & V &:= \int_{-N}^{\infty} u_t\, dB_t. \end{aligned}$$

Then, Lemma 2.5 implies that X and U are identically distributed, and so are Y and V. By Lemma 2.6, X and Y are independent, and so are U and V. Therefore, the random variables $f(X+Y) - f(Y)$ and $f(U+V) - f(V)$ are identically distributed. Now, by using Lemma 2.1, we get that

$$\begin{aligned} &\left|\int_\Omega \left(f(T_s(v))g\left(\int_{-\infty}^{N} u_t\, dB_t\right) - f\left(\int_{-N+s}^{\infty} u_{t-s}\, dB_t\right)g\left(\int_{-\infty}^{N} u_t\, dB_t\right)\right) d\mathbb{P}\right| \\ &\qquad \le \|g\|_\infty \left\| f\left(\int_{-\infty}^{\infty} u_{t-s}\, dB_t\right) - f\left(\int_{-N+s}^{\infty} u_{t-s}\, dB_t\right)\right\|_{L^1(\Omega,\mathbb{P})} \\ &\qquad = \|g\|_\infty \left\| f\left(\int_{-\infty}^{\infty} u_t\, dB_t\right) - f\left(\int_{-N}^{\infty} u_t\, dB_t\right)\right\|_{L^1(\Omega,\mathbb{P})} \\ &\qquad \le \|g\|_\infty \varepsilon, \end{aligned}$$

where we have used (3.3) for the last inequality. For the last term of (3.4), since $s > 2N$, and by using Lemma 2.6 and then Lemma 2.5, one has

$$\begin{aligned} &\int_\Omega f\left(\int_{-N+s}^{\infty} u_{t-s}\, dB_t\right)g\left(\int_{-\infty}^{N} u_t\, dB_t\right) d\mathbb{P} \\ &\qquad = \int_\Omega f\left(\int_{-N+s}^{\infty} u_{t-s}\, dB_t\right) d\mathbb{P} \int_\Omega g\left(\int_{-\infty}^{N} u_t\, dB_t\right) d\mathbb{P} \\ &\qquad = \int_\Omega f\left(\int_{-N}^{\infty} u_t\, dB_t\right) d\mathbb{P} \int_\Omega g\left(\int_{-\infty}^{N} u_t\, dB_t\right) d\mathbb{P}. \end{aligned}$$

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Therefore, using again (3.2) and (3.3) gives

$$\begin{aligned}&\left|\int_\Omega f\left(\int_{-N+s}^{\infty} u_{t-s}\,dB_t\right)g\left(\int_{-\infty}^{N} u_t\,dB_t\right)d\mathbb{P}-\int_\Omega f(v)\,d\mathbb{P}\int_\Omega g(v)\,d\mathbb{P}\right|\\&\quad\le\|f\|_\infty\left\|g\left(\int_{-\infty}^{N} u_t\,dB_t\right)-g(v)\right\|_{L^1(\Omega,\mathbb{P})}+\|g\|_\infty\left\|f\left(\int_{-N}^{\infty} u_t\,dB_t\right)-f(v)\right\|_{L^1(\Omega,\mathbb{P})}\\&\quad\le\|f\|_\infty\varepsilon+\|g\|_\infty\varepsilon.\end{aligned}$$

We can finally conclude that

$$\left|\int_\Omega f(T_s(v))g(v)\,d\mathbb{P}-\int_\Omega f(v)\,d\mathbb{P}\int_\Omega g(v)\,d\mathbb{P}\right|\le 2\|f\|_\infty\varepsilon+2\|g\|_\infty\varepsilon,$$

and since $\varepsilon>0$ was arbitrary, we eventually get (3.1). The measure μ is thus $(T_t)_{t\ge0}$-strongly mixing.

The result now follows by Lemma 3.1. □

We have the following corollary.

THEOREM 3.3. *Let $(T_t)_{t\ge0}$ be a C_0-semigroup on a separable Banach space E of type 2. Let $(u_t)_{t\in\mathbb{R}}$ be a family of vectors in E such that $T_s(u_t)=u_{t-s}$ for every $s\ge0$ and $t\in\mathbb{R}$, and $\operatorname{span}\{u_t\mid t\notin A\}$ is dense in E for every $A\in\mathscr{B}(\mathbb{R})$ with zero Lebesgue measure. Let $(B_t)_{t\in\mathbb{R}}$ be a Brownian motion. Assume that $t\longmapsto u_t\in L^2(\mathbb{R};E)$. Then, the random vector*

$$v:=\int_{\mathbb{R}} u_t\,dB_t$$

is almost surely well defined and frequently hypercyclic for $(T_t)_{t\ge0}$, and it induces a strongly mixing measure with full support for $(T_t)_{t\ge0}$.

Proof. We first prove that v is almost surely well defined. Define the linear map $J:\phi\longmapsto\int_{\mathbb{R}}\phi(t)\,dB_t$ defined on the space of E-valued step functions. Let $\phi=\sum_{i=1}^n a_i\mathbf{1}_{A_i}$ be a step function. Since E has type 2, we have

$$\|J(\phi)\|^2_{L^2(\Omega;E)}\le C\sum_{i=1}^n\lambda(A_i)\|a_i\|_E^2=C\|\phi\|^2_{L^2(\mathbb{R};E)},$$

where $C>0$ is some constant depending only on E and λ is the Lebesgue measure on $\mathbb{R}$. Therefore, J is a continuous embedding, and the density of the space of step functions in $L^2(\mathbb{R};E)$, see [**21**, Remark 1.2.20], allows us to extend J on $L^2(\mathbb{R};E)$. Theorem 2.3 then implies that $J(\phi)=\int_{\mathbb{R}}\phi(t)\,dB_t$ for every $\phi\in L^2(\mathbb{R};E)$. This shows that v is almost surely well defined since the map $t\longmapsto u_t$ belongs to $L^2(\mathbb{R};E)$.

Let Q be the covariance operator of the probability distribution $\mathbb{P}_v$ of v. By [**4**, Proposition 5.18], it suffices to show that Q is one-to-one to prove that $\mathbb{P}_v$ has full support. Let $x^*\in\operatorname{Ker}(Q)$. By definition of Q, we have

$$0=x^*(Q(x^*))=\int_E|x^*|^2\,d\mathbb{P}_v=\int_\Omega|x^*(v)|^2\,d\mathbb{P},$$

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implying that $x^*(v) = 0$ almost surely. By Lemma 2.1, we get that

$$0 = x^*(v) = \int_{\mathbb{R}} x^*(u_t)\, dB_t$$

almost surely. Therefore, Itô's isometry yields that $x^*(u_t) = 0$ for every $t \in \mathbb{R}$ outside some set of zero Lebesgue measure. By assumption, we get $x^* = 0$.

The result now follows by Proposition 3.2. □

Remark 3.4. In the proof of Theorem 3.3, to show that v is almost surely well defined, we actually proved that $L^2(\mathbb{R}; E)$ is continuously embedded in the space of stochastically integrable functions. This was already known in [34, Proposition 5.2].

As mentioned above, we have not been able to prove analogous results to [1, Theorem 1.3] or [1, Theorem 3.3] for C_0-semigroups. In the proof of Proposition 3.2, the independent and stationary increments of the Brownian motion play a crucial role. The idea was then to prove the existence of a stochastic process with respect to which $t \longmapsto u_t$ is stochastically integrable, under some deterministic conditions on $(u_t)_{t\in\mathbb{R}}$. A sought condition was the Pettis integrability of $t \longmapsto u_t$. If this plan could work, one would then hope to find another proof of the frequent hypercyclicity criterion for semigroups [30, Theorem 3], as we did for operators in [1].

4. *Random vector: second method*

In our second approach, we will make assumptions close to those of the main result of Chakir and El Mourchid [11, Theorem 3.2]. The semigroup will be defined on a complex Fréchet space since we will consider purely imaginary eigenvalues of the generator.

THEOREM 4.1. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable complex Fréchet space E. Let A be the generator of $(T_t)_{t\geq 0}$ and let $(u_t)_{t\in I}$ be a family of vectors in* $\mathrm{Dom}(A)$ *such that $A(u_t) = itu_t$ for every $t \in I$, where I is a non-empty interval of $\mathbb{R}$, and* $\mathrm{span}\{u_t \mid t \notin B\}$ *is dense in E for every $B \in \mathscr{B}(I)$ with zero Lebesgue measure. Let $(B_t)_{t\in I}$ be a complex Brownian motion. Assume that $t \longmapsto u_t$ is stochastically integrable and set*

$$v := \int_I u_t\, dB_t.$$

Then, v is almost surely frequently hypercyclic for the C_0-semigroup $(T_t)_{t\geq 0}$ and it induces a strongly mixing measure with full support for $(T_t)_{t\geq 0}$.

Proof. Let Q be the covariance operator of the probability distribution $\mathbb{P}_v$ of v.

Let $x^*, y^* \in E^*$. By using the definition of Q and Lemma 2.1, we have

$$\begin{aligned} y^*Qx^* &= \int_E y^*(x)\overline{x^*(x)}\, d\mathbb{P}_v(x) = \int_\Omega y^*\left(\int_I u_t\, dB_t\right)\overline{x^*\left(\int_I u_t\, dB_t\right)}\, d\mathbb{P} \\ &= \mathbb{E}\left(\int_I y^*(u_t)\, dB_t \overline{\int_I x^*(u_t)\, dB_t}\right) = 2\int_I y^*(u_t)\overline{x^*(u_t)}\, dt; \end{aligned} \tag{4.1}$$

note that we have used the linearization of the Itô isometry.

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We first prove that $\mathbb{P}_v$ is $(T_t)_{t\geq 0}$-invariant. Let $x^*, y^* \in E^*$ and $t \geq 0$. By using (4.1) twice and the assumption on $(u_s)_{s\in I}$, we have

$$\begin{aligned} y^* T_t Q T_t^* x^* &= T_t^* y^* Q T_t^* x^* = 2\int_I T_t^* y^*(u_s)\overline{T_t^* x^*(u_s)}\, ds \\ &= 2\int_I y^* T_t(u_s)\overline{x^* T_t(u_s)}\, dt = 2\int_I e^{its} y^*(u_s) e^{-its}\overline{x^*(u_s)}\, ds \\ &= y^* Q x^*. \end{aligned}$$

By [4, Proposition 5.22], this shows that $\mathbb{P}_v$ is $(T_t)_{t\geq 0}$-invariant.

Now, let us show that $\mathbb{P}_v$ is strongly mixing for $(T_t)_{t\geq 0}$. Let $x^*, y^* \in E^*$ and $t \geq 0$. By using (4.1) and the assumption on $(u_t)_{t\in I}$, we have

$$\begin{aligned} y^* Q T_t^*(x^*) &= 2\int_I y^*(u_s)\overline{T_t^* x^*(u_s)}\, ds = 2\int_I y^*(u_s)\overline{x^*(e^{its}u_s)}\, ds \\ &= 2\int_I e^{-its} y^*(u_s)\overline{x^*(u_s)}\, ds. \end{aligned}$$

Since $s \longmapsto y^*(u_s)\overline{x^*(u_s)}$ is integrable, by the Riemann–Lebesgue lemma, see [21, Lemma 2.4.3], we get that $\lim_{t\to\infty} y^* Q T_t^*(x^*) = 0$. We conclude that $\mathbb{P}_v$ is strongly mixing for $(T_t)_{t\geq 0}$ by [4, Theorem 5.24].

Finally, P_v has full support by the same arguments as in the proof of Theorem 3.3.

We conclude the proof by using Lemma 3.1. □

As a first application, let us apply Theorem 4.1 to the chaotic translation semigroups. Let $I \in \{[0,\infty[, \mathbb{R}\}$. For a given real $1 \leq p < \infty$ and a measurable function $\rho : I \longrightarrow]0,\infty[$, we define the Banach space

$$L^p_\rho(I) := \left\{ f : I \longrightarrow \mathbb{K} \mid \int_I |f(x)|^p \rho(x)\, dx < \infty \right\}$$

endowed with the norm $\|f\|_{L^p_\rho(I)} := (\int_I |f(x)|^p \rho(x)\, dx)^{1/p}$, $f \in L^p_\rho(I)$. An *admissible weight function* ρ is a measurable function $\rho : I \longrightarrow]0,\infty[$ such that there exist $M \geq 1$ and $w \in \mathbb{R}$ such that $\rho(s) \leq M e^{wt}\rho(t+s)$ for every $s \in I$ and $t \geq 0$.

For an admissible weight function ρ, the *translation semigroup* on $L^p_\rho(I)$ is defined by

$$T_t(f)(x) := f(x+t), \quad x \in I, t \geq 0,$$

for every $f \in L^p_\rho(I)$. Its generator is the differentiation operator defined on the space of absolutely continuous functions in $L^p_\rho(I)$ with derivative in $L^p_\rho(I)$, see [18, p. 51, Proposition II.1].

It is shown in [11, Example 4.1] that these semigroups admit a strongly mixing Gaussian measure, by constructing a complex Gaussian process. We reuse their arguments to apply Theorem 4.1.

THEOREM 4.2. *Let $1 \leq p < \infty$, $I \in \{[0,\infty[, \mathbb{R}\}$, and an admissible weight $\rho : I \longrightarrow]0,\infty[$. If the integral $\int_I \rho(x)\, dx$ is finite, then for any $f \in L^2(\mathbb{R};\mathbb{C})$ with $f \neq 0$ almost everywhere, the random vector*

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$$\int_{\mathbb{R}} f(t)e_t \, dB_t,$$

where $e_t(x) = e^{itx}$, $x \in I$, $t \in \mathbb{R}$, *is almost surely frequently hypercyclic for the translation semigroup* $(T_t)_{t\geq 0}$ *on* $L^p_\rho(I)$, *and it induces a strongly mixing measure with full support for* $(T_t)_{t\geq 0}$.

Proof. For every $s \in \mathbb{R}$, define the function $u_s : I \longrightarrow \mathbb{C}$ by

$$u_s(x) := f(s)e^{isx}$$

for all $x \in I$. Since $\int_I \rho(x)\, dx < \infty$, we get that $u_s \in L^p_\rho(I)$ for all $s \in \mathbb{R}$.

It is clear that $T_t(u_s) = e^{ist}u_s$ for every $s \in \mathbb{R}$ and $t \geq 0$. This implies that $u_s \in \mathrm{Dom}(A)$ and $A(u_s) = isu_s$ for every $s \in \mathbb{R}$, where A is the generator of the semigroup.

Now, let $\phi \in (L^p_\rho(I))^* = L^q_{\rho^{-q/p}}(I)$, where $1/p + 1/q = 1$, be such that $\phi(u_s) = 0$ for all $s \notin B$, where $B \in \mathscr{B}(\mathbb{R})$ has zero Lebesgue measure. Then,

$$0 = \phi(u_s) = \int_I f(s)e^{isx}\phi(x)\, dx,$$

and hence, $\int_I e^{isx}\phi(x)\, dx = 0$ for all $s \notin B$. Since $\phi \in L^1(I)$ by the fact that $\int_I \rho(x)\, dx < \infty$ and Hölder's inequality, we get that $\phi = 0$ almost everywhere by [35, Theorem 9.11], and thus, $\phi = 0$. This shows that $\mathrm{span}\{u_s \mid s \notin B\}$ is dense in $L^p_\rho(I)$ for any $B \in \mathscr{B}(\mathbb{R})$ with zero Lebesgue measure.

We now prove that

$$\left\| \left(\int_{\mathbb{R}} |u_s(\cdot)|^2 \, ds \right)^{1/2} \right\|_{L^p_\rho(I)} < \infty.$$

For any $x \in I$, we have

$$\int_{\mathbb{R}} |u_s(x)|^2 \, ds = \int_{\mathbb{R}} f(s)^2 \, ds,$$

and thus,

$$\left\| \left(\int_{\mathbb{R}} |u_s(\cdot)|^2 \, ds \right)^{1/2} \right\|^p_{L^p_\rho(I)} = \int_I \left(\int_{\mathbb{R}} f(s)^2 \, ds \right)^{p/2} \rho(x)\, dx,$$

which is finite since $\int_I \rho(x)\, dx < \infty$ and $f \in L^2(\mathbb{R}; \mathbb{C})$.

By Theorem 2.7, the map $s \longmapsto u_s$ is stochastically integrable. We conclude the proof by applying Theorem 4.1. □

Remark 4.3. By [27, Theorems 3.9 and 3.10], the translation semigroup on $L^p_\rho(I)$ is chaotic if and only if $\int_I \rho(x)\, dx$ is finite. Therefore, by Theorem 4.2, we have found a frequently hypercyclic random vector for any chaotic translation semigroup on $L^p_\rho(I)$.

For these chaotic translation semigroups, in the case where the underlying space is of type 2, one can apply Theorem 3.3 as follows. Define $F : \mathbb{R} \longrightarrow L^p_\rho(I)$ as

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$F(\theta)(x) = \varphi(\theta)e^{i\theta x}$ for all $\theta \in \mathbb{R}$ and $x \in I$, where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a positive, smooth, and integrable function on $\mathbb{R}$. This implies that F is Bochner-integrable, and thus,

$$u_t := \int_{\mathbb{R}} e^{-it\theta} F(\theta)\, d\theta$$

is well defined for all $t \in \mathbb{R}$. We then have $T_s F(\theta) = e^{is\theta} F(\theta)$ for all $\theta \in \mathbb{R}$ and $s \geq 0$, which implies that $T_s u_t = u_{t-s}$ for all $s \geq 0$ and $t \in \mathbb{R}$. The spanning property is proved in a similar way as in the proof of Theorem 4.2. Furthermore, we have $\|u_t\| \leq C/(1+t^2)$ for all $t \in \mathbb{R}$ for some constant $C > 0$ if F is twice-differentiable by [**21**, Lemma 2.4.3 and its proof], and this is guaranteed provided that $\int_I |x|^{2p}\rho(x)\, dx$ is finite. Hence, one can apply Theorem 3.3. This idea, suggested by the referee, might also be used to apply Proposition 3.2 or Theorem 3.3 to other examples in subsequent research.

The second example was studied by El Mourchid and Latrach in [**17**, Proposition 3.4], who proved the existence of a strongly mixing Gaussian measure for the semigroup under consideration. It turns out that their arguments are the same to check the assumptions of Theorem 4.1.

THEOREM 4.4. *Let $1 \leq p < \infty$ and A be the weighted shift on ℓ^p with sequence of weights $(w_n)_{n\geq 1}$. Set $\beta_n := w_1 \cdots w_n$ for all $n \geq 1$ and $\beta_0 := 1$. If the quantity $\limsup_{n\to\infty}(1/|\beta_n|)^{1/n}$ is finite, then there exists $\eta > 0$ such that the random vector*

$$\int_{-\eta}^{\eta} \left(\frac{i^n t^n}{\beta_n}\right)_{n\geq 0} dB_t$$

is almost surely frequently hypercyclic for the C_0-semigroup $(T_t)_{t\geq 0} := (e^{tA})_{t\geq 0}$, and it induces a strongly mixing measure with full support for $(T_t)_{t\geq 0}$.

Proof. For all $\lambda \in \mathbb{C}$, define the sequence $v_\lambda := (\lambda^n/\beta_n)_{n\geq 0}$. Then, $v_\lambda \in \ell^p$ if and only if $\sum_{n\geq 0} |\lambda|^{np}/|\beta_n|^p < \infty$, and in that case, we have $A(v_\lambda) = (\lambda^{n+1}/\beta_n)_{n\geq 0} = \lambda v_\lambda$. The power series $\sum_{n\geq 0} \lambda^n/\beta_n$ has a radius of convergence $R := 1/\rho$, where $\rho := \limsup_{n\to\infty}(1/|\beta_n|)^{1/n} < \infty$.

Let $\eta \in]0, R[$ and define $u_s := v_{is}$ for every $-\eta < s < \eta$. By our previous calculations, we have $u_s \in \ell^p$ and $A(u_s) = isu_s$ for all $-\eta < s < \eta$.

Let $\phi = (\phi_n)_{n\geq 0} \in (\ell^p)^* = \ell^q$, where $1/p + 1/q = 1$, be such that $\phi(u_s) = 0$ for almost all $-\eta < s < \eta$. Define the analytic function $F : D(0, R) \longrightarrow \mathbb{C}$ by

$$F(\lambda) := \sum_{n=0}^{\infty} \frac{\lambda^n \phi_n}{\beta_n} = \phi(v_\lambda)$$

for all $\lambda \in D(0, R)$. Since $F = 0$ almost everywhere on $i]-\eta, \eta[$, we conclude that $F = 0$ on $D(0, R)$ and $\phi = 0$. This means that $\operatorname{span}\{u_t \mid t \notin B\}$ is dense in ℓ^p for any $B \in \mathscr{B}(]-\eta, \eta[)$ with zero Lebesgue measure.

For all $n \geq 0$, we have

$$\int_{-\eta}^{\eta} |u_s(n)|^2\, ds = \int_{-\eta}^{\eta} \frac{|is|^{2n}}{|\beta_n|^2}\, ds \leq \frac{2\eta\eta^{2n}}{|\beta_n|^2},$$

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implying that

$$\left\| \left(\int_{-\eta}^{\eta} |u_s(\cdot)|^2 \, ds \right)^{1/2} \right\|_{\ell^p}^p \leq \sum_{n=0}^{\infty} \frac{2^{p/2} \eta^{p/2} \eta^{np}}{|\beta_n|^p},$$

which is finite since $\eta < R$. Therefore, by Theorem 2.7, the map $s \longmapsto u_s$ is stochastically integrable.

We conclude the proof by applying Theorem 4.1. □

We now study the translation semigroups on $H(\mathbb{C})$. We will need the notion of *Hilbert seminorm*: it is a seminorm $p : H(\mathbb{C}) \longrightarrow [0, \infty[$ for which there exists a semi-scalar product $\langle \cdot, \cdot \rangle$ on $H(\mathbb{C})$ such that $p(x) = \langle x, x \rangle$ for all $x \in H(\mathbb{C})$. If $f \in H(\mathbb{C})$ and $r > 0$, we define $\|f\|_r := \sup_{|z| \leq r} |f(z)|$.

THEOREM 4.5. *For each $t \geq 0$, define the translation operator*

$$T_t : H(\mathbb{C}) \longrightarrow H(\mathbb{C}),\ f \longmapsto f(\cdot + t).$$

Then, the random vector $\int_0^{2\pi} e^{itz} \, dB_t$ is almost surely frequently hypercyclic for the C_0-semigroup $(T_t)_{t \geq 0}$, and it induces a strongly mixing measure with full support for $(T_t)_{t \geq 0}$.

Proof. It is easy to check that $(T_t)_{t \geq 0}$ is indeed a C_0-semigroup. Furthermore, its generator is the differentiation operator $D : H(\mathbb{C}) \longrightarrow H(\mathbb{C})$.

For every $s \in [0, 2\pi]$, set $u_s(z) := e^{isz}$, $z \in \mathbb{C}$. Clearly, $D(u_s) = isu_s$ for any $s \in [0, 2\pi]$. The space $\operatorname{span}\{u_s \mid s \notin B\}$ is dense in $H(\mathbb{C})$ by [20, Lemma 2.34] for any $B \in \mathscr{B}([0, 2\pi])$ with zero Lebesgue measure. Furthermore, the map $s \longmapsto u_s$ is weakly L^2 since $\{u_s \mid s \in [0, 2\pi]\}$ is bounded in $H(\mathbb{C})$. We now show that the map I_ϕ of Definition 2.3 takes values in $H(\mathbb{C})$ and is γ-radonifying, where $\phi(s) := u_s$, $s \in [0, 2\pi]$.

Let $f \in L^2([0, 2\pi])$. Define the function $F : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$F(z) := \int_0^{2\pi} u_s(z) \overline{f(s)} \, ds$$

for every $z \in \mathbb{C}$. It is well defined since the maps $s \longmapsto u_s(z)$ and f are in $L^2([0, 2\pi])$ for any $z \in \mathbb{C}$. Let $z, h \in \mathbb{C}$ with $h \neq 0$, we have

$$\frac{F(z+h) - F(z)}{h} = \int_0^{2\pi} \frac{e^{is(z+h)} - e^{isz}}{h} \overline{f(s)} \, ds.$$

An application of the dominated convergence theorem then shows that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \int_0^{2\pi} ise^{isz} \overline{f(s)} \, ds,$$

and F is holomorphic on $\mathbb{C}$.

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Let $x^* \in H(\mathbb{C})^*$. We show that $x^*(F) = \int_0^{2\pi} x^*(u_s)\overline{f(s)}\,ds$. Let $N \geq 1$. By linearity of x^* and the integral, one has

$$\begin{aligned} x^*\Big(\sum_{k=0}^{N} \frac{i^k z^k}{k!} \int_0^{2\pi} s^k \overline{f(s)}\,ds\Big) &= \sum_{k=0}^{N} x^*\Big(\frac{i^k z^k}{k!}\Big) \int_0^{2\pi} s^k \overline{f(s)}\,ds \\ &= \int_0^{2\pi} \sum_{k=0}^{N} x^*\Big(\frac{i^k z^k}{k!}\Big) s^k \overline{f(s)}\,ds \\ &= \int_0^{2\pi} x^*\Big(\sum_{k=0}^{N} \frac{i^k s^k z^k}{k!}\Big)\overline{f(s)}\,ds. \end{aligned} \tag{4.2}$$

The right-hand side converges to $\int_0^{2\pi} x^*(u_s)\overline{f(s)}\mathrm{d}s$ by the dominated convergence theorem. As for the left-hand side, let us prove that it converges to $x^*(F)$ as N goes to ∞. Let $r > 0$ be fixed. Then,

$$\begin{aligned} \Big\|\sum_{k=0}^{N} \frac{i^k z^k}{k!} \int_0^{2\pi} s^k \overline{f(s)}\,ds - F(z)\Big\|_r &= \Big\|\int_0^{2\pi} \sum_{k=N+1}^{\infty} \frac{i^k s^k z^k}{k!} \overline{f(s)}\,ds\Big\|_r \\ \leq \int_0^{2\pi} \sum_{k=N+1}^{\infty} \frac{s^k r^k}{k!} |f(s)|\,ds &\leq \sum_{k=N+1}^{\infty} \frac{r^k}{k!} \|s^k\|_{L^2([0,2\pi])} \|f\|_{L^2([0,2\pi])}, \end{aligned}$$

where we have used the Cauchy–Schwarz inequality for the last inequality. The right-hand side converges to 0 when N goes to ∞. By continuity of x^*, we conclude that the left-hand side of (4.2) converges to $x^*(F)$. Therefore, $x^*(F) = I_\phi(f)(x^*)$, so that $I_\phi(f) = F$, and hence, I_ϕ takes values in $H(\mathbb{C})$. The map I_ϕ is also continuous: for every $r \geq 0$, we have $\|I_\phi(f)\|_r \leq \sqrt{2\pi} e^{2\pi r} \|f\|_{L^2([0,2\pi])}$.

By Theorem 2.2, to prove that $t \longmapsto u_t$ is stochastically integrable, it remains to show that I_ϕ is γ-radonifying. Let $(f_n)_{n\in\mathbb{Z}} = (e^{ins}/\sqrt{2\pi})_{n\in\mathbb{Z}}$ be the canonical orthonormal basis of $L^2([0, 2\pi])$. By [**29**, Example 28.9(4) and Lemma 28.1], the space $H(\mathbb{C})$ has a system of Hilbert seminorms generating its topology. Let $\|\cdot\|$ be such a seminorm. Since it is continuous, there exist some $r > 0$ and $C > 0$ such that $\|\cdot\| \leq C\|\cdot\|_r$. Let $(g_n)_{n\in\mathbb{Z}}$ be an i.i.d. Gaussian sequence. Let $0 \leq N \leq M$ be two positive integers; we have, by using the facts that $\|\cdot\|$ is a Hilbert seminorm for the first equality and g_n, $n \geq 0$, are independent for the second one,

$$\begin{aligned} \mathbb{E}\Big(\Big\|\sum_{n=N}^{M} g_n I_\phi(f_n)\Big\|^2\Big) &= \sum_{n,m=N}^{M} \mathbb{E}(g_n \overline{g_m}) \langle I_\phi(f_n), I_\phi(f_m)\rangle \\ &= \sum_{n=N}^{M} \|I_\phi(f_n)\|^2 \leq C \sum_{n=N}^{M} \|I_\phi(f_n)\|_r^2 \\ &= C \sum_{n=N}^{M} \Big\|\int_0^{2\pi} e^{isz} \overline{f_n(s)}\,ds\Big\|_r^2, \end{aligned}$$

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where $\langle \cdot, \cdot \rangle$ is the semi-scalar product associated to $\| \cdot \|$. For each $n \in \mathbb{N}$ such that $n > r$ and all $z \in \mathbb{C}$ such that $|z| = r$, we have

$$\int_0^{2\pi} e^{isz}\overline{f_n(s)}\, ds = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{isz} e^{-isn}\, ds = \frac{1}{\sqrt{2\pi}} \frac{e^{i2\pi(z-n)} - 1}{i(z-n)},$$

and then

$$\mathbb{E}\left(\left\| \sum_{n=N}^{M} g_n I_\phi(f_n) \right\|^2 \right) \leq \frac{C}{2\pi} \sum_{n=N}^{M} \left\| \frac{e^{i2\pi(z-n)} - 1}{i(z-n)} \right\|_r^2 \leq \frac{C}{2\pi} \sum_{n=N}^{M} \frac{(e^{2\pi r} + 1)^2}{(n-r)^2}$$

provided that $N > r$. We can conclude that the series $\sum_{n=0}^{\infty} g_n I_\phi(f_n)$ converges in $L^2(\Omega; E)$. The same arguments show that $\sum_{n \leq 0} g_n I_\phi(f_n)$ also converges in $L^2(\Omega; E)$, and hence, I_ϕ is γ-radonifying by Itô–Nisio's theorem, see for example [22, Corollary 6.4.4] in the case of Banach spaces. By Theorem 2.2, the map $s \longmapsto u_s$ is thus stochastically integrable.

The result now follows by Theorem 4.1. □

Remark 4.6. By [13, Theorem 3.2], any frequently hypercyclic vector for the semigroup $(T_t)_{t \geq 0}$ is also a frequently hypercyclic for each T_t, $t > 0$. Therefore, the random vector $\int_0^{2\pi} e^{itz}\, dB_t$ is almost surely frequently hypercyclic for the translation operators $T_t : H(\mathbb{C}) \longrightarrow H(\mathbb{C})$, $f \longmapsto f(\cdot + t)$, $t \geq 0$. Bonilla and Grosse-Erdmann [9, Theorem 4.2] used the frequent hypercyclicity criterion to prove the frequent hypercyclicity of the translation operators. Therefore, one could also use [1, Theorem 4.9] to prove that each operator T_a, $a \in \mathbb{C} \setminus \{0\}$, has an almost surely frequently hypercyclic random vector.

We repeat our remark made at the end of §3: in Theorem 4.1, we still assume that the map $t \longmapsto u_t$ is stochastically integrable. Then, one may ask if we could find a deterministic assumption to get the stochastic integrability of $t \longmapsto u_t$, possibly replacing the Brownian motion with another stochastic process.

4.1. *Other examples.* Some chaotic semigroups associated to differential equations are presented in [20, §7.5]. It turns out that with essentially the same arguments for the proofs of their chaoticity given in [20], one can apply Theorem 4.1 to those semigroups. We list the results obtained below.

THEOREM 4.7. *Let $1 \leq p < \infty$, $f \in L^2([0, \infty[)$ and let $h : [0, \infty[\longrightarrow \mathbb{R}$ be a bounded, continuous function such that $x \longmapsto \exp(-\int_0^x h(s)\, ds) \in L^p([0, \infty[)$. Then, the random vector*

$$\int_{\mathbb{R}} e^{itx} \exp\left(-\int_0^x h(s)\, ds \right) f(t)\, dB_t,$$

where $f \in L^2(\mathbb{R})$ with $f(t) \neq 0$ for all $t \in \mathbb{R}$ is almost frequently hypercyclic for the semigroup $(T_t)_{t \geq 0}$ on $L^p([0, \infty[)$ given for all $\varphi \in L^p([0, \infty[)$ by

$$T_t\varphi(x) = e^{\int_x^{x+t} h(s)\, ds} \varphi(x + t), \quad x, t \in [0, \infty[.$$

The previous example is taken from [20, Proposition 7.34 and Exercise 7.5.2].

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THEOREM 4.8. *Let $\alpha, \tau, \rho > 0$ be such that $\alpha\tau\rho^2 > 2$. Define for $1 \leq p < \infty$ the Banach space*

$$X_{\rho,p} = \left\{ f : \mathbb{R} \longmapsto \mathbb{C},\ f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n,\ (a_n)_{n\geq 0} \in \ell^p \right\},$$

endowed with the norm $\|f\| := \|(a_n)_{n\geq 0}\|_p$. Let A be the operator on $X_{\rho,p} \oplus X_{\rho,p}$ given by

$$A = \begin{pmatrix} 0 & I \\ (\alpha/\tau)\partial^2/\partial x^2 & -(1/\tau)I \end{pmatrix}.$$

For all $\lambda \in \mathbb{C}$, define $R_\lambda = (\tau\lambda^2 + \lambda)/\alpha$ and $f_1(\lambda) = \sum_{n=0}^{\infty}(R_\lambda^n x^{2n})/(2n)!$, $f_2(\lambda) = \sum_{n=0}^{\infty}(R_\lambda^n x^{2n+1})/(2n+1)!$. Then, there exists $\eta > 0$ such that the random vector

$$\int_{-\eta}^{\eta} (f_1(it), itf_1(it))\, dB_t^{(1)} + \int_{-\eta}^{\eta} (f_2(it), itf_2(it))\, dB_t^{(2)}$$

is almost surely frequently hypercyclic for the semigroup generated by A, where $(B_t^{(1)})_{t\geq 0}$ and $(B_t^{(2)})_{t\geq 0}$ are two independent Brownian motions.

THEOREM 4.9. *Let $1 \leq p < \infty$ and let $(\alpha_n)_{n\geq 0}, (\beta_n)_{n\geq 0} \subseteq]0, \infty[$ be two bounded sequences such that*

$$\sup_{n\geq 0} \alpha_n < \liminf_{n\to\infty} \beta_n.$$

Let $A : \ell^p \longrightarrow \ell^p$ be the map given by

$$A(f) = (-\alpha_n f_n + \beta_n f_{n+1})_{n\geq 0}$$

for all $f \in \ell^p$. Then, there exists $\eta > 0$ such that the random vector

$$\int_{-\eta}^{\eta} \left(\prod_{k=0}^{n-1} \frac{it + \alpha_k}{\beta_k} \right)_{n\geq 0} dB_t$$

is almost surely frequently hypercyclic for the semigroup generated by A.

Using similar arguments given in [**20**, §7.5], which also allow us to prove the above results, one can apply Theorem 4.1 to the examples given in [**20**, Exercise 7.5.3] (wave equation) and [**20**, Exercise 7.5.4] (classical heat equation), yielding results in the same way as above.

Finally, let us mention that, once again, by using the arguments given in [**12**], one can apply Theorem 4.1 to the one-dimensional Ornstein–Uhlenbeck operator.

5. *A counterexample*

We find in this section a counterexample of semigroups that do not admit a 'canonical' frequently hypercyclic random vector.

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Let $\rho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a locally bounded positive function on $\mathbb{R}_+ := [0, \infty[$. Define the Banach space

$$\mathcal{C}_0(\mathbb{R}_+, \rho) = \left\{ f \in \mathcal{C}_0(\mathbb{R}_+) \mid \lim_{x\to\infty} f(x)\rho(x) = 0 \right\},$$

endowed with the norm $\|f\|_{\mathcal{C}_0(\mathbb{R}_+,\rho)} := \|f\rho\|_\infty$, $f \in \mathcal{C}_0(\mathbb{R}_+, \rho)$.

We say that ρ is an *admissible weight* if

$$C(t) := \sup_{x\geq 0} \frac{\rho(x)}{\rho(x+t)} < \infty$$

for all $t \geq 0$ and $C : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is locally bounded on $\mathbb{R}_+$.

Define the translation semigroup

$$T_t : \mathcal{C}_0(\mathbb{R}_+, \rho) \longrightarrow \mathcal{C}_0(\mathbb{R}_+, \rho), t \longmapsto f(\cdot + t), \quad t \geq 0.$$

A 'natural choice' of vectors $(u_t)_t$ for our example would be $u_t := a(t)e^{itx}$ $(a(t) \neq 0)$, which belongs to $\mathcal{C}_0(\mathbb{R}_+, \rho)$ if $\lim_{x\to\infty} \rho(x) = 0$ for all $t \in \mathbb{R}$. We will assume that $\lim_{x\to\infty} \rho(x) = 0$, which is equivalent to saying that $(T_t)_{t\geq 0}$ is strongly mixing in the Gaussian sense by [5, Proposition 8.4]. We then have $T_s(u_t) = e^{it(x+s)} = e^{its}u_t$ for all $s, t \in \mathbb{R}$.

LEMMA 5.1. *The space* $\operatorname{span}\{u_t \mid t \in \mathbb{R}\}$ *is dense in* $\mathcal{C}_0(\mathbb{R}_+, \rho)$.

Proof. Let $\varphi : \mathcal{C}_0(\mathbb{R}_+, \rho) \longrightarrow \mathbb{C}$ be linear and continuous such that $\varphi(u_t) = 0$ for all $t \in \mathbb{R}$. By the Riesz representation theorem, see [35, Theorem 6.19], there exists a (unique) regular complex Borel measure μ such that

$$\varphi(f) = \int_0^\infty f(x)\rho(x)\, d\mu(x)$$

for all $f \in \mathcal{C}_0(\mathbb{R}_+, \rho)$. Define the measure $d\nu = \rho\, d\mu$. Then, ν is finite and we have

$$0 = \varphi(u_t) = \int_0^\infty a(t)e^{itx}\rho(x)\, d\mu(x) = a(t)\widehat{\nu}(t)$$

for all $t \in \mathbb{R}$. Therefore, $\nu = 0$ by [8, Proposition 3.8.6], and since ρ is positive, we deduce that $\mu = 0$, implying that $\varphi = 0$. □

Our aim is to show that $(u_t)_{t\in\mathbb{R}}$ is not stochastically integrable; hence, one cannot apply Theorem 4.1. Our arguments are an adaptation of [3, Example 3.13].

Set $\phi : \mathbb{R} \longrightarrow \mathcal{C}_0(\mathbb{R}_+, \rho), t \longmapsto u_t$. We have

$$I_\phi(f)[x^*] = \int_{-\infty}^\infty x^*(u_t)\overline{f(t)}\, dt = \int_{-\infty}^\infty a(t)x^*(e^{itx})\overline{f(t)}\, dt.$$

Let $(f_n)_n \subseteq L^2(\mathbb{R}; \mathbb{C})$ be an orthonormal basis. Assume that $\operatorname{Im}(I_\phi) \subseteq \mathcal{C}_0(\mathbb{R}_+, \rho)$ and that $\sum_n g_n I_\phi(f_n)$ converges almost surely, that is, that ϕ is stochastically integrable. Then, we can define the Gaussian process

$$Z_{x^*} := \sum_n g_n x^*(I_\phi(f_n)) \in L^2(\Omega; \mathbb{C}), \quad x^* \in (\mathcal{C}_0(\mathbb{R}_+, \rho))^*.$$

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We have $\mathbb{E}(\sup_{\|x^*\|\leq 1} \|Z_{x^*}\|) < \infty$ and thus by [25, Theorem 3.18], we deduce that

$$\sup_{\varepsilon>0} \varepsilon\sqrt{\log(N(\varepsilon, B_{(\mathcal{C}_0(\mathbb{R}_+,\rho))^*}, d))}$$

is finite, where d is the L^2-metric of the process,

$$d(x^*, y^*) = \|Z_{x^*} - Z_{y^*}\|_{L^2(\Omega;\mathbb{C})}, \quad x^*, y^* \in (\mathcal{C}_0(\mathbb{R}_+, \rho))^*,$$

and $N(\varepsilon, B_{(\mathcal{C}_0(\mathbb{R}_+,\rho))^*}, d)$ is the entropy number.

We have $I_\phi^* = K$, where $K : (\mathcal{C}_0(\mathbb{R}_+, \rho))^* \longrightarrow L^2(\mathbb{R}; \mathbb{C})$, $x^* \longmapsto x^* \circ \phi$.

For any $x^*, y^* \in (\mathcal{C}_0(\mathbb{R}_+, \rho))^*$, we have

$$\|Z_{x^*} - Z_{y^*}\|^2_{L^2(\Omega;\mathbb{C})} = \sum_n |\langle K(x^* - y^*), f_n\rangle|^2 = \|K(x^* - y^*)\|^2_{L^2([0,\infty[;\mathbb{C})}.$$

Define

$$x^* : \mathcal{C}_0(\mathbb{R}_+, \rho) \longrightarrow \mathbb{C},\ f \longmapsto \int_0^\infty f(x)h(x)\,dx.$$

Then, $(x^* \in (\mathcal{C}_0(\mathbb{R}_+, \rho))^*$ and $\|x^*\| \leq 1)$ if $\|h/\rho\|_{L^1([0,\infty[;\mathbb{C})} \leq 1$. Similarly for

$$y^* : \mathcal{C}_0(\mathbb{R}_+, \rho) \longrightarrow \mathbb{C},\ f \longmapsto \int_0^\infty f(x)g(x)\,dx.$$

Let us consider $\rho = \sum_{j\geq 1} \rho_j \mathbf{1}_{[t_j,t_{j+1}[}$, $t_j = 2\pi j$, $j \geq 1$, where $\rho_j := 1/n$ if $j \in [2^{n^3}, 2^{(n+1)^3}[$. We will show that ρ is an admissible weight. Note also that $\lim_{x\to\infty} \rho(x) = 0$.

Let $t \geq 0$. Let $x \geq 0$ be large, with $x \in [2\pi j, 2\pi(j+1)[$, $j \in [2^{n^3}, 2^{(n+1)^3}[$, and $n \geq 1$. We have $t = 2\pi k + r$ for some $k \geq 0$ and $r \in [0, 2\pi[$; thus, $x + t \in [2\pi(j+k), 2\pi(j+k+2)[$. Now, if j is large, then $j + k + 2 \in [2^{n^3}, 2^{(n+2)^3}[$, implying that

$$\frac{\rho(x)}{\rho(x+t)} \leq \frac{n+1}{n} \leq 2.$$

This shows that $C(t) = \sup_{x\geq 0}(\rho(x)/\rho(x+t))$ is finite.

Now, we prove that C is locally bounded. Remark that ρ is non-increasing. Let $t_0 > 0$. If $0 \leq t \leq t_0$, then

$$\frac{\rho(x)}{\rho(x+t)} \leq \frac{\rho(x)}{\rho(x+t_0)}$$

for all $x \geq 0$, and $C(t) \leq C(t_0)$. This shows that C is locally bounded.

We will try to give a lower bound for

$$\begin{aligned}\|K(x^* - y^*)\|^2_{L^2(\mathbb{R};\mathbb{C})} &= \int_{-\infty}^\infty |K(x^* - y^*)(t)|^2\,dt \\ &= \int_{-\infty}^\infty \left|\int_0^\infty a(t)e^{itx}h(x)\,dx - \int_0^\infty a(t)e^{itx}g(x)\,dx\right|^2 dt.\end{aligned}$$

Let $\widetilde{n} \geq 0$. We take $h = \rho_j \mathbf{1}_{[t_j,t_{j+1}[}e^{inx}$ and $g = \rho_k \mathbf{1}_{[t_k,t_{k+1}[}e^{imx}$ with $j, k \in [2^{\widetilde{n}^3}, 2^{(\widetilde{n}+1)^3}[$ and $j \neq k$. Then, $\|h/\rho\|_{L^1} \leq 1$, $\|g/\rho\|_{L^1} \leq 1$, and $x^*, y^* \in B_{(\mathcal{C}_0(\mathbb{R}_+,\rho))^*}$. Therefore, we now have

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$$\|K(x^*-y^*)\|^2_{L^2(\mathbb{R};\mathbb{C})} \geq \int_0^\infty |a(t)|^2 \left| \rho_j \left[\frac{e^{inx+itx}}{i(n+t)} \right]_{t_j}^{t_{j+1}} - \rho_k \left[\frac{e^{imx+itx}}{i(m+t)} \right]_{t_k}^{t_{k+1}} \right|^2 dt$$
$$= \int_0^\infty |a(t)|^2 \left| \rho_j \left(\frac{e^{i(n+t)2\pi(j+1)} - e^{i(n+t)2\pi j}}{i(n+t)} \right) - \rho_k \left(\frac{e^{i(m+t)2\pi(k+1)} - e^{i(m+t)2\pi k}}{i(m+t)} \right) \right|^2 dt$$
$$= \int_0^\infty |a(t)|^2 \left| \frac{\rho_j e^{it2\pi j}}{n+t}(e^{2\pi ti}-1) - \frac{\rho_k e^{it2\pi k}}{m+t}(e^{2\pi ti}-1) \right|^2 dt.$$

Since $j, k \in [2^{\widetilde{n}^3}, 2^{(\widetilde{n}+1)^3}[$, we then obtain

$$\|K(x^*-y^*)\|^2_{L^2(\mathbb{R};\mathbb{C})} \geq \widetilde{n}^{-2} \int_0^\infty |a(t)|^2 \left| \frac{e^{it2\pi j}}{n+t} - \frac{e^{it2\pi k}}{m+t} \right|^2 |e^{2\pi ti}-1|^2 \, dt.$$

Now, we take $n=1$ and $m=2$, and we obtain

$$\|K(x^*-y^*)\|^2_{L^2(\mathbb{R};\mathbb{C})} \geq \widetilde{n}^{-2} \int_0^\infty |a(t)|^2 \left(\frac{1}{n+t} - \frac{1}{m+t} \right)^2 |e^{2\pi ti}-1|^2 \, dt$$
$$= 2\widetilde{n}^{-2} \int_0^\infty \frac{|a(t)|^2}{(t+1)^2(2+t)^2}(1-\cos(2\pi t)) \, dt.$$

The last integral is positive and we conclude that $\|K(x^*-y^*)\|_{L^2(\mathbb{R};\mathbb{C})} \geq A/\widetilde{n}$, where $A>0$ is some constant.

Set $\varepsilon_{\widetilde{n}} = A/(2\widetilde{n})$ for each $\widetilde{n} \geq 1$. We conclude that $N(\varepsilon_{\widetilde{n}}, B_{(\mathcal{C}_0(\mathbb{R}_+,\rho))^*}, d) \geq 2^{\widetilde{n}^3}$ for all $\widetilde{n} \geq 1$. This implies that

$$\sup_{\widetilde{n}\geq 1} \varepsilon_{\widetilde{n}} \sqrt{\log(N(\varepsilon_{\widetilde{n}}, B_{(\mathcal{C}_0(\mathbb{R}_+,\rho))^*}, d))} \gtrsim \sup_{\widetilde{n}\geq 1} \widetilde{n}^{1/2} = \infty,$$

which is a contradiction.

This (counter-)example shows that we might expect to integrate the family $(u_t)_t$ with respect to another stochastic process than the Brownian motion. Therefore, our example might indicate that a more general criterion might exist, as in the case for single operators studied in [1].

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A. *Appendix. Stochastic calculus in Fréchet spaces*

We will give a proof of Theorems 2.2 and 2.3 in this appendix. Their proofs follow the same lines as in [41], but we give them for the sake of completeness. For the definition of weak topologies on Fréchet spaces and their duals, see [29, 37].

Recall that a map $R : E^* \longrightarrow E$ is *positive* if $x^*Rx^* \geq 0$ for all $x^* \in E^*$, and is *symmetric* if $x^*Ry^* = \overline{y^*Rx^*}$ for all $x^*, y^* \in E^*$.

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LEMMA A.1. [38, Lemma 4 and its proof] *Let $R : E^* \longrightarrow E$ be a positive symmetric map. Then, there exists a Hilbert space H_R and a continuous linear map $i_R : H_R^* \longrightarrow E$ such that $R = i_R I i_R^*$, where $I : H_R \longrightarrow H_R^*$ is the canonical conjugate-linear operator. More precisely, H_R is the completion of E^*/M under the norm induced by R, given by $\langle x^* + M,\ y^* + M\rangle_{H_R} = x^* R y^*$, $x^*, y^* \in E^*$, where $M = \{x^* \in E^* \mid x^* R x^* = 0\}$, and $i_R^* : E^* \longrightarrow H_R$, $x^* \longmapsto x^* + M$ is the inclusion map.*

A *positive symmetric* map $Q : E^* \longrightarrow (E^*)'$ is defined in the same way as a positive symmetric map from E^* to E: for all $x^*, y^* \in E^*$, one has $(Qx^*)(x^*) \geq 0$ and $(Qy^*)(x^*) = \overline{(Qx^*)(y^*)}$.

Proposition A.2 below is proved in the given reference for Banach spaces, but we give the proof for the sake of completeness.

PROPOSITION A.2. [19, Proposition 2.2] *Let $Q : E^* \longrightarrow (E^*)'$ be a positive symmetric map. Suppose that there exists a positive symmetric map $R : E^* \longrightarrow E$ such that $(Qx^*)(x^*) \leq x^* R x^*$ for every $x^* \in E^*$. Then, $Q(E^*) \subseteq E$.*

Proof. Let $i_R : H_R^* \longrightarrow E$ be the map given by Lemma A.1. Fix $x^* \in E^*$ and define the linear map

$$\phi_{x^*} : i_R^*(E^*) \longrightarrow \mathbb{K},\ i_R^*(y^*) \longmapsto (Qx^*)(y^*).$$

Let $y^* \in E^*$. By the Cauchy–Schwarz inequality applied to the sesquilinear form $(x^*, y^*) \longmapsto (Qx^*)(y^*)$, the assumption, and Lemma A.1, we get that

$$|\phi_{x^*}(i_R^*(y^*))| \leq ((Qx^*)(x^*))^{1/2}((Qx^*)(y^*))^{1/2} \leq \|i_R^*(x^*)\|_{H_R}\|i_R^*(y^*)\|_{H_R}.$$

This shows that ϕ_{x^*} is well defined and continuous on $(i_R^*(E^*), \|\cdot\|_{H_R})$. Therefore, we can continuously extend ϕ_{x^*} on H_R, and the extension is still noted as ϕ_{x^*}. Now, for every $y^* \in E^*$, we have

$$y^*(i_R(\phi_{x^*})) = (y^* \circ i_R)(\phi_{x^*}) = \phi_{x^*}(y^* \circ i_R) = (Qx^*)(y^*).$$

We deduce that $Qx^* = i_R(\phi_{x^*}) \in E$. □

By convention, the inner product on a vector space is linear in the first argument and conjugate-linear in the second argument.

Proof of Theorem 2.2. The implication (i) $\implies$ (ii) is trivial.

Assume that item (ii) holds. Then, for all $x^* \in E^*$, the random variable $x^*(Y)$ is Gaussian and, hence, Y is Gaussian. Let Q be the covariance operator of its distribution μ. For every $x^* \in E^*$, we have by the Itô isometry,

$$x^* Q x^* = \int_E |x^*|^2\, d\mu = \int_\Omega |x^*(Y)|^2\, d\mathbb{P} = c \int_I |x^*(\phi(t))|^2\, dt,$$

where $c = 1$ in the real case and $c = 2$ in the complex case. This shows item (iii).

Now, assume that item (iii) holds. Set $H := (\overline{E^*})^*$, where the closure is taken in the space $L^2(\mu)$. It is a separable space by [21, Proposition 1.2.29]. Take the γ-radonifying

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operator $T := K : H \longrightarrow E$ given by [4, Theorem 5.13] (the proof carries over verbatim to Fréchet spaces). From $Q = KJK^*$, where $J : H^* \longrightarrow H$ is the canonical conjugate-linear identification operator, and by hypothesis, it follows that

$$\begin{aligned} c\int_I |x^*(\phi(t))|^2\,dt = x^*Qx^* &= (x^*\circ K)JK^*(x^*) \\ &= \|K^*(x^*)\|_H^2 = \|T^*(x^*)\|_H^2 \end{aligned}$$

for all $x^* \in E^*$, and item (iv) holds.

We now show (iv) $\implies$ (v). Define the conjugate-linear map $Q : E^* \longrightarrow (E^*)'$ by

$$Qx^* : E^* \longrightarrow \mathbb{K},\ y^* \longmapsto \int_I y^*(\phi(t))\overline{x^*(\phi(t))}\,dt$$

for all $x^* \in E^*$. It is well defined since ϕ is weakly L^2 and is clearly a positive symmetric map. Let $J : H^* \longrightarrow H$ be the canonical conjugate-linear identification operator. We finish the proof of the implication in five steps.

(a) By hypothesis, we have for all $x^* \in E^*$ that

$$c\int_I |x^*(\phi(t))|^2\,dt \le \|T^*(x^*)\|_H^2 = (T^*(x^*))(JT^*(x^*)) = x^*(TJT^*(x^*)).$$

Hence, setting $R := TJT^* : E^* \longrightarrow E$, we have

$$c(Qx^*)(x^*) \le x^*Rx^* \tag{A.1}$$

for all $x^* \in E^*$. By Proposition A.2, we conclude that $Q(E^*) \subseteq E$.

(b) By hypothesis and [4, Theorem 5.13], R is a Gaussian covariance operator. Then, by [40, Theorem 8.8], inequality (A.1), and since $Q(E^*) \subseteq E$, Q is also a Gaussian covariance operator.

(c) Let us prove that I_ϕ takes values in E. Define $G := \{x^* \circ \phi \mid x^* \in E^*\} \subseteq L^2(I)$. Let $f \in L^2(I)$. Then,

$$\begin{aligned} f \in \mathrm{Ker}(I_\phi) &\iff I_\phi(f) = 0, \\ &\iff \text{for all } x^* \in E^*,\ I_\phi(f)(x^*) = 0, \\ &\iff \text{for all } x^* \in E^*,\ \int_I x^*(\phi(t))\overline{f(t)}\,dt = 0, \\ &\iff f \in G^\perp. \end{aligned}$$

Therefore, $\mathrm{Ker}(I_\phi) = G^\perp$ and $L^2(I) = \overline{G} \oplus \mathrm{Ker}(I_\phi)$. Since $\mathrm{Im}(Q) \subseteq E$ by step (A), we have that $I_\phi(G \oplus \mathrm{Ker}(I_\phi)) \subseteq E$.

It remains to prove that $I_\phi(\overline{G}) \subseteq E$. Let $g \in \overline{G}$. There exists a sequence $(g_n)_{n\ge1} \subseteq G$ converging to g in $L^2(I)$. Since $I_\phi(G) \subseteq E$, for each $n \ge 1$, there exists $x_n \in E$ such that $I_\phi(g_n) = x_n$. We show that $(x_n)_{n\ge1}$ is Cauchy in E. Let $\|\cdot\|$ be a continuous seminorm on E. For all $x^* \in E^*$ and every integers $n, m \ge 1$, we have

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$$\begin{aligned}|x^*(x_n - x_m)| &= |(I_\phi(g_n) - I_\phi(g_m))(x^*)| \\ &= \left| \int_I x^*(\phi(t))\overline{g_n(t)}\, dt - \int_I x^*(\phi(t))\overline{g_m(t)}\, dt \right| \\ &\leq \|x^*\phi\|_{L^2(I)} \|g_n - g_m\|_{L^2(I)},\end{aligned}$$

where the inequality is justified by the Cauchy–Schwarz inequality. Define the set $V := \{x \in E, \|x\| < 1\}$. By [36, Theorems 3.15 and 3.16], the polar V° of V is $\sigma(E^*, E)$-compact and metrizable in the $\sigma(E^*, E)$ topology; hence, it is sequentially complete and bounded for $\sigma(E^*, E)$. However, the linear map $(E^*, \sigma(E^*, E)) \longrightarrow L^2(I), x^* \longmapsto x^*\phi$ has a sequentially closed graph. By [38, Lemma 3], we then get $\sup_{x^* \in V^\circ} \|x^*\phi\|_{L^2(I)} < \infty$. Therefore,

$$\|x_n - x_m\| = \sup_{x^* \in V^\circ} |x^*(x_n - x_m)| \leq \sup_{x^* \in V^\circ} \|x^*\phi\|_{L^2(I)} \|g_n - g_m\|_{L^2(I)},$$

the equality holds by [36, Theorem 1.34] and [29, Proposition 22.14]. Since $(g_n)_{n\geq 1}$ is convergent in $L^2(I)$, we conclude that $(x_n)_{n\geq 1}$ is Cauchy for $\|\cdot\|$. Since $\|\cdot\|$ was arbitrary, the sequence $(x_n)_{n\geq 1}$ is thus Cauchy in E. By completeness, there exists $x \in E$ such that $(x_n)_{n\geq 1}$ converges to x in E. Now, let $x^* \in E^*$. By continuity of x^*, we have that $\lim_{n\to\infty} I_\phi(g_n)(x^*) = \lim_{n\to\infty} x^*(x_n) = x^*(x)$. However, by the Cauchy–Schwarz inequality, we have

$$|I_\phi(g_n)(x^*) - I_\phi(g)(x^*)| \leq \|x^*\phi\|_{L^2(I)} \|g_n - g\|_{L^2(I)},$$

implying that $\lim_{n\to\infty} I_\phi(g_n)(x^*) = I_\phi(g)(x^*)$. We conclude that $I_\phi(g)(x^*) = x^*(x)$ for all $x^* \in E^*$, and $I_\phi(g) = x \in E$.

(d) We now prove that I_ϕ is continuous. Define the linear map

$$K : E^* \longrightarrow L^2(I), x^* \longmapsto x^* \circ \phi.$$

Let $S : L^2(I)^* \longrightarrow L^2(I)$ be the canonical conjugate-linear identification operator. We first show that $K^* = I_\phi \circ S$. Let $f^* \in L^2(I)^*$ and $x^* \in E^*$. By definition of K^*, we have

$$K^*(f^*)(x^*) = f^*(K(x^*)) = \langle x^* \circ \phi, S(f^*)\rangle_{L^2(I)} = I_\phi(S(f^*))(x^*).$$

This implies that $K^*(f^*) = I_\phi(S(f^*))$ and, in turn, $K^* = I_\phi \circ S$. By [29, Lemma 23.28], the map $K : (E^*, \sigma(E^*, E)) \longrightarrow (L^2(I), \sigma(L^2(I), L^2(I)^*))$ is continuous if and only if $K^*(L^2(I)^*) \subseteq E$. Since $K^* = I_\phi \circ S$ and I_ϕ takes values in E by (C), K is indeed continuous with respect to the weak topologies. By [29, Lemma 23.29], K^* is then continuous with respect to the Mackey topologies. Since $L^2(I)^*$ and E are Mackey spaces by [37, §IV.3.4], that is, their respective original topologies are the Mackey topology, we conclude that $K^* : L^2(I)^* \longrightarrow E$ is continuous. Finally, I_ϕ is also continuous thanks to the equality $K^* = I_\phi \circ S$.

(e) For all $x^*, y^* \in E^*$, one has

$$I_\phi(K(x^*))(y^*) = \int_I y^*(\phi(t))\overline{K(x^*)}\, dt = \int_I y^*(\phi(t))\overline{x^*(\phi(t))}\, dt = y^* Q x^*.$$

Therefore, $(I_\phi \circ S) \circ S^{-1} \circ K = I_\phi \circ K = Q$. We also have $(I_\phi \circ S)^* = K$. Since Q is a covariance operator by (B) and any square-root of Q is γ-radonifying, see [4, Remark

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5.16], we deduce that I_ϕ is γ-radonifying. Note that $L^2(I)$ is separable by [21, Proposition 1.2.29].

We have thus proved item (v).

It remains to show (v) $\implies$ (i). Let $A \in \mathscr{B}(I)$ and define $\phi_A := \mathbf{1}_A \phi$ and

$$M_A : L^2(I) \longrightarrow L^2(I),\ f \longmapsto \mathbf{1}_A f.$$

For all $f \in L^2(I)$ and $x^* \in E^*$, we have

$$I_{\phi_A}(f)(x^*) = \int_I x^*(\phi_A(t))\overline{f(t)}\, dt = \int_A x^*(\phi(t))\overline{f(t)}\, dt = I_\phi(M_A(f))(x^*).$$

This shows that $I_{\phi_A} = I_\phi \circ M_A$. Since M_A is obviously continuous, the map I_{ϕ_A} is γ-radonifying by the ideal property, see [22, Theorem 9.1.10].

Let $(f_n)_{n\in\mathbb{N}} \subseteq L^2(I)$ be an orthonormal basis. Denote by

$$\mathcal{I} : L^2(I) \longrightarrow L^2(\Omega),\ f \longmapsto \int_I f(t)\, dB_t$$

the Itô isometry. Since $(\mathcal{I}(f_n))_{n\in\mathbb{N}}$ is a standard Gaussian sequence and I_{ϕ_A} is γ-radonifying, the random series $Y_A := \sum_{n\geq 0} \mathcal{I}(f_n) I_{\phi_A}(f_n)$ converges almost surely. Let $x^* \in E^*$. We have

$$\begin{aligned} x^*(Y_A) &= \sum_{n=0}^{\infty} \mathcal{I}(f_n) x^*(I_{\phi_A} f_n) = \sum_{n=0}^{\infty} \mathcal{I}(x^*(I_{\phi_A} f_n) f_n) \\ &= \sum_{n=0}^{\infty} \mathcal{I}(\langle x^* \circ \phi_A,\, f_n\rangle_{L^2(I)} f_n) = \mathcal{I}\Big(\sum_{n=0}^{\infty} \langle x^* \circ \phi_A,\, f_n\rangle_{L^2(I)} f_n \Big) \\ &= \int_I x^* \phi_A(t)\, dB_t \end{aligned}$$

almost surely, where we have used the continuity of x^* for the first equality, the definition of I_{ϕ_A} for the third one, the linearity and continuity of $\mathcal{I}$ in the fourth one, and the fact that $(f_n)_{n\in\mathbb{N}}$ is an orthonormal basis and the definition of $\mathcal{I}$ for the last one. We then deduce that $x^*(Y_A) = \int_I x^* \phi_A(t)\, dB_t$ almost surely for every $x^* \in E^*$. In conclusion, ϕ is stochastically integrable.

As for the Pettis integrability of ϕ, let $A \in \mathscr{B}(I)$ be a set of finite measure. Let $x^* \in E^*$. Then, $\mathbf{1}_A \in L^2(I)$, $I_\phi(\mathbf{1}_A) \in E$, and

$$x^*(I_\phi(\mathbf{1}_A)) = I_\phi(\mathbf{1}_A)(x^*) = \int_I x^*(\phi(t))\mathbf{1}_A(t)\, dt,$$

and ϕ is Pettis integrable on A. □

We state two results from [41], adapted to the case of functions defined on an arbitrary interval and taking values in a Fréchet space.

THEOREM A.3. [41, Corollary 2.7] *Let $\phi, \psi : I \longrightarrow E$ be two weakly measurable functions on an interval $I \subseteq \mathbb{R}$. Assume that ϕ is stochastically integrable and that*

$$\int_I |x^*\psi(t)|^2\, dt \leq \int_I |x^*\phi(t)|^2\, dt$$

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for every $x^* \in E^*$. Then, ψ is stochastically integrable, and for all $1 \le p < \infty$ and all continuous seminorms $\|\cdot\|$ on E, we have

$$\mathbb{E}\left(\left\|\int_I \psi(t)\, dB_t\right\|^p\right) \le \mathbb{E}\left(\left\|\int_I \phi(t)\, dB_t\right\|^p\right).$$

Proof. By assumption and item (iv) of Theorem 2.2, we get for all $x^* \in E^*$ that

$$c\int_I |x^*\psi(t)|^2\, dt \le c\int_I |x^*\phi(t)|^2\, dt \le \|T^*x^*\|_H^2.$$

Again by item (iv) of Theorem 2.2, we can conclude that ψ is stochastically integrable.

By item (iii) of Theorem 2.2 applied to both ϕ and ψ, we have $x^*Rx^* \le x^*Qx^*$ for all $x^* \in E^*$, where R and Q are the covariance operators of the distributions μ_R and μ_Q of $\int_I \psi(t)\, dB_t$ and $\int_I \phi(t)\, dB_t$, respectively. By [40, Theorem 8.8], we finally get that

$$\int_E \|x\|^p\, d\mu_R(x) \le \int_E \|x\|^q\, d\mu_Q(x). \qquad \square$$

THEOREM A.4. [41, Theorem 6.2] *Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of E-valued stochastically integrable functions defined on an interval $I \subseteq \mathbb{R}$. Assume that there exists a weakly L^2 function $\phi : I \longrightarrow E$ such that for every $x^* \in E^*$, one has*

$$\lim_{n\to\infty} \int_I |x^*\phi_n(t) - x^*\phi(t)|^2\, dt = 0.$$

If there exists a stochastically integrable function $\psi : I \longrightarrow E$ such that for every $x^* \in E^*$ and all $n \ge 0$, one has

$$\int_I |x^*\phi_n(t)|^2\, dt \le \int_I |x^*\psi(t)|^2\, dt,$$

then ϕ is stochastically integrable and for every $1 \le p < \infty$, we have, in $L^p(\Omega; E)$,

$$\lim_{n\to\infty} \int_I \phi_n(t)\, dB_t = \int_I \phi(t)\, dB_t.$$

Proof. By using the assumptions, we get that

$$\int_I |x^*\phi(t)|^2\, dt \le \int_I |x^*\psi(t)|^2\, dt$$

for all $x^* \in E^*$. By Theorem A.3, ϕ is stochastically integrable.

The rest of the proof follows as in [41, Theorem 6.2]. $\square$

Proof of Theorem 2.3. Assume that such a sequence of step functions exists. Let $x^* \in E^*$. By Lemma 2.1, we have $x^*(Y) = \lim_{n\to\infty} \int_I x^*\phi_n(t)\, dB_t$ in probability. The random vectors on the right-hand side are Gaussian and, thus, the convergence takes place in $L^2(\Omega)$ by [34, Lemma 2.1]. By the Itô isometry, the sequence $(x^*\phi_n)_{n\in\mathbb{N}}$ is Cauchy in $L^2(I)$. Thus, by item (i), we necessarily have $\lim_{n\to\infty} x^*\phi_n = x^*\phi$ in $L^2(I)$. Again by the Itô isometry, we get that $x^*(Y) = \int_I x^*\phi(t)\, dB_t$ almost surely. By item (ii) of Theorem 2.2, we conclude that ϕ is stochastically integrable on I with integral Y.

Assume that ϕ is stochastically integrable. If I is a bounded interval, the result is proved in [41, Theorem 2.5] when ϕ is a Banach space-valued function, but the proof carries over

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verbatim to the case of Fréchet spaces. If I is unbounded, for the sake of simplicity, we assume that $I = [0, \infty[$. By Theorem 2.2, ϕ is Pettis integrable on any bounded intervals. For all integers $k \geq 0$, $n \geq 1$, $1 \leq j \leq 2^n$, and $N \geq 1$, define

$$\phi_{n,k,j} := 2^n \int_{k+(j-1)/2^n}^{k+j/2^n} \phi(t)\, dt,$$

$$\phi_n^N := \sum_{k=0}^{N-1} \sum_{j=1}^{2^n} \mathbf{1}_{]k+(j-1)/2^n, k+j/2^n]} \phi_{n,k,j},$$

and $\varphi_N := \phi \mathbf{1}_{[0,N]}$. For all $n, N \in \mathbb{N}_0$, define also $\mathcal{G}_{n,N}$ as the finite σ-algebra on $[0, N]$ generated by the intervals of the form $]k + (j-1)/2^n, k + j/2^n]$, $0 \leq k \leq N-1$, $1 \leq j \leq 2^n$.

Let $x^* \in E^*$. It is an easy task to check that $\mathbb{E}(x^*\varphi_N \mid \mathcal{G}_{n,N}) = x^*\phi_n^N$ almost everywhere on $[0, N]$ for every n, $N \geq 1$, see [8, Example 10.1.2]. Since $(\mathcal{G}_{n,N})_{n\geq 1}$ is increasing and $\sigma(\bigcup_{n\geq 1} \mathcal{G}_{n,N}) = \mathcal{B}([0, N])$, we get that $\lim_{n\to\infty} x^*\phi_n^N = x^*\varphi_N$ in $L^2([0, N]; \mathbb{R})$ by [21, Theorem 3.3.2].

Now, define the step functions $\phi_N := \phi_N^N \mathbf{1}_{[0,N]}$ for every integer $N \geq 1$. Let us show that $\lim_{N\to\infty} x^*\phi_N = x^*\phi$ in $L^2([0, \infty[)$. Notice that $\phi_N^M = \phi_N^N$ on $[0, M]$ for any $N \geq M \geq 1$. Let $\varepsilon > 0$ and let $N_0 \geq 1$ be such that $\int_N^\infty |x^*\phi(t)|^2\, dt \leq \varepsilon^2$ for every $N \geq N_0$. Now, let $M_0 \geq N_0$ be such that $\|x^*\phi_N^{N_0} - x^*\varphi_{N_0}\|_{L^2([0,N_0])} \leq \varepsilon$ for every integer $N \geq M_0$. We have for any $N \geq M_0$,

$$\begin{aligned}\|x^*\phi_N - x^*\phi\|_{L^2([0,\infty[)} &\leq \|x^*\phi_N^{N_0} - x^*\varphi_{N_0}\|_{L^2([0,N_0])} \\ &\quad + \|x^*\phi_N - x^*\phi\|_{L^2([N_0,N])} + \|x^*\phi\|_{L^2([N,\infty[)}.\end{aligned}$$

The first and third terms are smaller than ε since $N \geq M_0 \geq N_0$. For the second term, first note that

$$\|x^*\phi_N - x^*\phi\|_{L^2([N_0,N])} \leq \|x^*\phi_N\|_{L^2([N_0,N])} + \|x^*\phi\|_{L^2([N_0,\infty[)}.$$

The second term on the right-hand side is smaller than ε by definition of N_0. For the first term, notice that $x^*\phi_N = x^*\phi_N^N = \mathbb{E}(x^*\phi \mid \mathcal{G}_N)$ almost everywhere on $[N_0, N]$, where $\mathcal{G}_N$ is the finite σ-algebra on $[N_0, N]$ generated by the intervals of the form $]k + (j-1)/2^N, k + j/2^N]$, $N_0 \leq k \leq N-1$, $1 \leq j \leq 2^N$; see [8, Example 10.1.2]. Therefore, by a corollary of the conditional Jensen inequality, see [21, Corollary 2.6.30], we have $\|x^*\phi_N\|_{L^2([N_0,N])} \leq \|x^*\phi\|_{L^2([N_0,N])}$. We then conclude that $\|x^*\phi_N - x^*\phi\|_{L^2([0,\infty[)} \leq 4\varepsilon$ and, thus, $\lim_{N\to\infty} x^*\phi_N = x^*\phi$ in $L^2([0, \infty[)$.

Again by [21, Corollary 2.6.30], we have $\|x^*\phi_N\|_{L^2([0,\infty[)} \leq \|x^*\phi\|_{L^2([0,\infty[)}$ for every $N \geq 1$. Since every step function is stochastically integrable, Theorem A.4 yields item (ii) with convergence in every $L^p(\Omega; E)$, $1 \leq p < \infty$. □

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