REAL AND VIRTUAL PROCESSES IN QUANTUM ELECTRODYNAMICS

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1. *Introduction.* The S-matrix formulation of quantum electrodynamics, as developed by Feynman (3) and Dyson (1), expresses the matrix element for any process as a power series in the coupling constant, the coefficients of the series being, in general, rather complicated multiple integrals. These integrals contain singularities in their integrands, and in certain circumstances the coincidence of such singularities gives the S-matrix a non-analytic behaviour as a function of, for example, the total energy. The threshold behaviour of the S-matrix in the neighbourhood of energy values at which this phenomenon occurs has been investigated by Eden (2), who shows that the non-analytic behaviour is connected with the possible commencement of new physical processes, such as the creation of a particle.

The present paper attempts to show how the coincident singularities, however complicated, can be classified, and how they can be interpreted physically. It is, first of all, necessary to investigate the effect of the coincident singularities on the integral containing them; and after studying a few examples it becomes clear how the contribution arising from any combination of singularities can be picked out from the rest of the integral. A graphical notation is developed to facilitate the classification of the coincidences of singularities. Any graph of the type introduced by Feynman and Dyson can be expressed as the sum of *graph parts*; a graph part corresponding to a particular coincidence, or lack of coincidence, of singularities. It appears that the only coincidences that are of importance correspond to graph parts which describe two (or more) physical processes taking place successively, for example, the emission and re-absorption of a photon. Such graph parts are to be called *real*, while that graph part which cannot be broken down into two physical processes represents a purely virtual process and is called a *virtual* part. The classification of coincidences is then straightforward, even in the case of the finite parts of self-energies and vertex terms. It is shown how any real graph part can be expressed as the sum of its constituent virtual parts.

A particularly important graph part is the *external part* which is a diagonal element of the S-matrix, and leaves the state function unaltered. In the S-matrix theory such external parts do not, in general, give the identity operator; and this is so, simply because the theory is unitary. It is obvious that the formal development of Dyson (1) gives a unitary S-matrix, and it is easy to see that the mass and charge renormalizations are unitary operations, so the renormalized theory is unitary. The unitary nature of the final form of the theory is demonstrated for a simple case with the aid of the various concepts outlined above. A brief section is included on the relation between the unitary condition and the convergence of the S-matrix expansion.

Finally, it is shown how any matrix element can be expressed in terms of the

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virtual graph parts by means of an integral equation, whose form makes the unitary condition obvious.

2. *Separation of the integrals.* Propagation factors $D_F(p)$ and $S_F(p)$ are ascribed to each internal photon line and electron line of a given graph G, p being the four-vector energy-momentum associated with the line. These functions are

$$D_F(p) = \frac{1}{2\pi i}\frac{1}{p^2}, \quad S_F(p) = \frac{1}{2\pi i}\frac{i\gamma p - \kappa}{p^2+\kappa^2},$$

where κ is the rest mass of the electron. The conservation of energy and momentum at each vertex results in a reduction in the number of variables of integration occurring in the integral which gives the matrix element corresponding to G. This integral is of the form

$$\int_{-\infty}^{\infty} d^4p_1 \dots \int_{-\infty}^{\infty} d^4p_f R(p_1, \dots, p_f; t_1, \dots, t_r), \tag{2·1}$$

where $p_1, \dots, p_f$ are the 'free variables' and $t_1, \dots, t_r$ are the external line variables. R is a rational function of p, t and the Dirac matrices γ, the denominator being a product of the denominators of D_F and S_F.

The integration around the poles of $D_F(p)$ and $S_F(p)$ is correctly performed on replacing these denominators by $p^2 - i\epsilon$ and $p^2+\kappa^2 - i\eta$ respectively, where ϵ, η are positive numbers which are to tend to zero after the integration has been completed. It is easy to see that if $f(\mathbf{p}, p_0)$ is an integral function of the energy p_0 and the momentum $\mathbf{p}$ then

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dp_0 \frac{f(\mathbf{p},p_0)}{p^2-i\epsilon} = \frac{\pi i}{2|\mathbf{p}|}\int_{-\infty}^{\infty} dp_0 f(\mathbf{p},p_0)\{\delta(p_0+|\mathbf{p}|)+\delta(p_0-|\mathbf{p}|)\} + \mathscr{P}\int_{-\infty}^{\infty} dp_0 \frac{f(\mathbf{p},p_0)}{\mathbf{p}^2-p_0^2},$$

where

$$\mathscr{P}\int_{-\infty}^{\infty} dp_0 R(p_0) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dp_0 \tfrac{1}{2}\{R(p_0+i\epsilon)+R(p_0-i\epsilon)\}. \tag{2·2}$$

$\mathscr{P}$ denotes a generalized principal value integral. It is important to notice that the integrals of type (2·1) which occur in electrodynamics converge absolutely when integrated over p_0, so that, always

$$\mathscr{P}\int_{-\infty}^{\infty} dp_0 \frac{f(\mathbf{p},p_0)}{p^2} = 0.$$

When there are two or more quadratic factors in the denominator of the function R occurring in (2·1) it is possible for the singularities to coincide when $\epsilon\to 0, \eta\to 0, \dots$. The contributions arising from these *coincident poles* will be separated off from the rest of the integral. In the simplest types of integrals the separation can easily be carried out. For example,

$$\begin{aligned}\lim_{\epsilon\to 0,\,\eta\to 0}\int_{-\infty}^{\infty} dt_0 \int_{-\infty}^{\infty} dp_0 \frac{f(\mathbf{p},p_0;\,\mathbf{t},t_0)}{\{p^2-i\epsilon\}\{(p+t)^2-i\eta\}} &= \frac{\pi i}{2|\mathbf{p}|}\frac{2\pi i}{2|\mathbf{p}+\mathbf{t}|}\int_{-\infty}^{\infty} dt_0 \int_{-\infty}^{\infty} dp_0 f(\mathbf{p},p_0;\,\mathbf{t},t_0)\\ &\quad\times\{\delta(p_0+|\mathbf{p}|)\,\delta(p_0+t_0-|\mathbf{p}+\mathbf{t}|)+\delta(p_0-|\mathbf{p}|)\,\delta(p_0+t_0+|\mathbf{p}+\mathbf{t}|)\}\\ &\quad+\frac{\pi i}{2|\mathbf{p}|}\mathscr{P}\int_{-\infty}^{\infty} dt_0 \int_{-\infty}^{\infty} dp_0 f(\mathbf{p},p_0;\,\mathbf{t},t_0)\{\delta(p_0+|\mathbf{p}|)+\delta(p_0-|\mathbf{p}|)\}\frac{1}{(p+t)^2}.\end{aligned} \tag{2·3}$$

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In the second term the principal part sign $\mathscr{P}$ applies to the t_0 integration, and this second term, as it stands, will vanish. It is more usual to encounter integrals of the type (2·3) in which t_0 has a fixed value (as can be secured by putting a δ-function containing t_0 in $f(\mathbf{p}, p_0; \mathbf{t}, t_0)$), and the second integration will then be over one of the components of $\mathbf{p}$ rather than over t_0. The second term in (2·3) then does not, in general, vanish. The first term in (2·3) which contains the sum of two pairs of δ-functions in the integrand is just the contribution arising from the coincident poles. Another simple example of the separation is the integral

$$\lim_{\epsilon\to 0,\,\eta\to 0,\,\zeta\to 0}\int_{-\infty}^{\infty}dq_0\int_{-\infty}^{\infty}dt_0\int_{-\infty}^{\infty}dp_0\,\frac{f(\mathbf{p},p_0;\,\mathbf{t},t_0;\,\mathbf{q},q_0)}{\{p^2-i\epsilon\}\{(p+t)^2-i\eta\}\{(t+q)^2-i\zeta\}}$$
$$=\frac{\pi i}{2\,|\,\mathbf{p}\,|}\,\frac{2\pi i}{2\,|\,\mathbf{p}+\mathbf{t}\,|}\,\frac{2\pi i}{2\,|\,\mathbf{t}+\mathbf{q}\,|}\int_{-\infty}^{\infty}dq_0\int_{-\infty}^{\infty}dt_0\int_{-\infty}^{\infty}dp_0 f(\mathbf{p},p_0;\,\mathbf{t},t_0,\mathbf{q},q_0)$$
$$\times\{\delta(p_0+|\,\mathbf{p}\,|)\,\delta(p_0+t_0-|\,\mathbf{p}+\mathbf{t}\,|)\,\delta(t_0+q_0+|\,\mathbf{t}+\mathbf{q}\,|)$$
$$+\delta(p_0-|\,\mathbf{p}\,|)\,\delta(p_0+t_0+|\,\mathbf{p}+\mathbf{t}\,|)\,\delta(t_0+q_0-|\,\mathbf{t}+\mathbf{q}\,|)\}+\ldots, \qquad (2\cdot4)$$

where the exact form of the terms containing single δ-functions or the product of two δ-functions is not of importance. The first term arises from three coincident poles and consists of the sum of two products of three δ-functions. The fact that the first term in (2·4) contains only two products of three δ-functions, and that the signs + and − alternate in the arguments of these functions is to be expected from the form of the limiting process. On the last integration terms in the denominator which contain the combinations $\pm i(\epsilon+\eta-\zeta)$, $\pm i(\epsilon-\eta-\zeta)$, $\pm i(\epsilon-\eta+\zeta)$ must vanish or cancel. Only the two factors containing $\pm i(\epsilon+\eta+\zeta)$ give a non-zero contribution, and this must be so since the order in which the limiting processes $\epsilon\to 0$, $\eta\to 0$, $\zeta\to 0$ is made is arbitrary.

When the factor in the denominator is $p^2+\kappa^2$ instead of p^2, the formulae on the right-hand side of (2·3) and (2·4) hold on replacing $|\,\mathbf{p}\,|$ by $\{\mathbf{p}^2+\kappa^2\}^{\frac{1}{2}}$, etc., everywhere. The general form which occurs when any number of poles coincide can readily be seen. The first pole gives a factor of the type $\pi i/2\,|\,\mathbf{p}\,|$, and each subsequent coincident pole gives a factor of the type $2\pi i/2\,|\,\mathbf{p}+\mathbf{t}\,|$. There will also be the sum of two products of δ-functions, and in each product arguments of the δ-functions of the type $(p_0\pm|\,\mathbf{p}\,|)$ will alternate.

3. *The graphical notation.* Consider a typical graph describing the scattering of two electrons as in G_5 (Fig. 1). This graph can be broken up into a number of constituent parts, to be called *graph parts*, according as there are, or are not, coinciding poles in the corresponding integral. In G_5 there are two free variables p_1, p_2, corresponding to the two cycles 1254, 2365 respectively. Poles corresponding to the lines 12 and 45 can coincide giving an integral in the form of the first term on the right-hand side of equation (2·3). The two δ-functions ensure that the energy and momentum associated with the lines 12 and 45 satisfy the relation

$$p^2+\kappa^2=0;$$

and so these lines, in this case, correspond to real particles. This part of the integral will be denoted by the graph part G_6, in which a *cut line AA* divides the graph into two portions by intersecting the lines 12 and 45.

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In general a cut line will be defined as a line dividing a graph into two portions by intersecting internal lines. It will be understood that this graph part represents those terms in the integral which arise from coincident poles belonging to the intersected internal lines. These terms in the integral contain products of δ-functions relating to the lines.

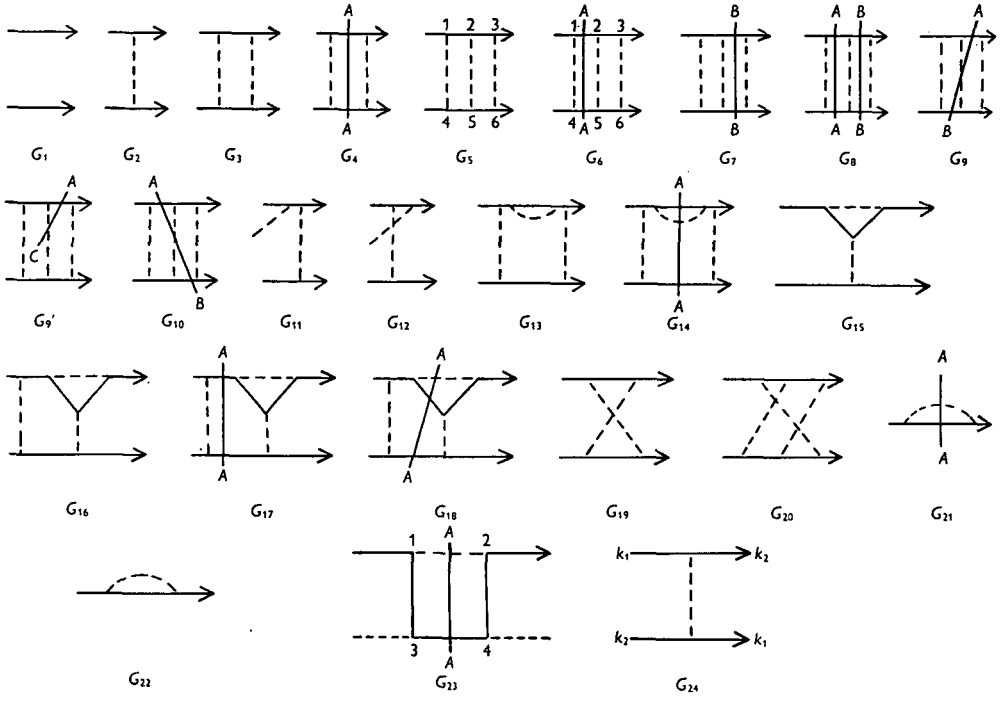


Fig. 1.

Possible cut lines associated with the graph G_5 are shown in G_6 to G_{10}. It is easy to see that no other cut lines are permissible, since the external lines of G_5 represent real particles. In G_6 the two δ-functions arising from the cut line reduce the integral over p_1 from a four-fold to a two-fold integral. The integral over p_2 remains a four-fold integral, and that part of the latter integral which corresponds to coincident poles in the lines 23, 56 is denoted by G_8. G_8 corresponds to two independent sets of coincident poles.

The complete analysis of the integral corresponding to the graph G_5 can be denoted by the graph parts G_5 to G_{10}; G_5 itself denoting that part of the integral which contains no coincident poles. A graph part containing no cut line will be called a *virtual graph part*, and corresponds to a purely virtual process. Graph parts containing cut lines will be called *real graph parts*, and they correspond to a process in which real scattering, creation or annihilation occurs at some intermediate stage. A graph without a cut line can contain coincident poles; for example, the line AC in G_9' indicates coincident poles in 23 and 25. As was noted in § 2 each cycle corresponding to a free variable gives rise, at least, to a δ-function relating to one line. G_9' is merely one example of the distribution of the two δ-functions arising from the two cycles occurring in G_5.

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Suppose a cut line intersects r internal lines of a graph. It is clear that these r lines must contain $(r-1)$ free variables, and there will be (at least) two coincident poles for each free variable. Some of these poles will be common to several variables.

4. *The decomposition of real processes.* It will be shown that a real graph part, for example, G_4, can be decomposed into two graph parts. The external and internal energy and momenta of the graph part G_4 are shown in Fig. 2. The integral corresponding to this graph is of the form

$$\int d^4p \frac{f(\mathbf{p}, p_0; k_1; k_2)}{\{(k_1+p)^2+\kappa^2\}\{(k_2-p)^2+\kappa^2\}},$$

and the coincidence of poles belonging to lines 12 and 34 gives the term

$$\int d^3\mathbf{p} \frac{\pi i}{2\sqrt{[(\mathbf{k}_1+\mathbf{p})^2+\kappa^2]}} \frac{2\pi i}{2\sqrt{[(\mathbf{k}_2-\mathbf{p})^2+\kappa^2]}} \int_{-\infty}^{\infty} dp_0 f(\mathbf{p}, p_0; k_1; k_2)$$
$$\times \{\delta(p_0+k_{10}-\sqrt{[(\mathbf{k}_1+\mathbf{p})^2+\kappa^2]})\,\delta(k_{20}-p_0+\sqrt{[(\mathbf{k}_2-\mathbf{p})^2+\kappa^2]})$$
$$+\delta(p_0+k_{10}+\sqrt{[(\mathbf{k}_1+\mathbf{p})^2+\kappa^2]})\,\delta(k_{20}-p_0-\sqrt{[(\mathbf{k}_2-\mathbf{p})^2+\kappa^2]})\}. \quad (4\cdot1)$$

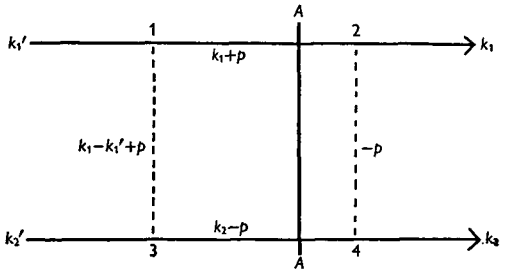


Fig. 2.

The complete integrand of the matrix element for G_4 can be written

$$\left(\frac{e}{\hbar c}\right)^4 \overline{\psi}_\alpha(k_1)\,\overline{\psi}_\beta(k_2)\,\gamma^\mu_{\alpha\xi} \frac{1}{(2\pi)^4 i}\left(\frac{i\gamma(k_1+p)-\kappa}{(k_1+p)^2+\kappa^2}\right)_{\xi\eta} \gamma^\mu_{\beta\sigma} \frac{1}{(2\pi)^4 i}\left(\frac{i\gamma(k_2-p)-\kappa}{(k_2-p)^2+\kappa^2}\right)_{\sigma\tau}$$
$$\times \gamma^\nu_{\eta\zeta}\gamma^\nu_{\tau\rho}\psi_\rho(k_2')\,\psi_\zeta(k_1')\frac{\hbar c}{2}D_F(p)\frac{\hbar c}{2}D_F(k_1'-k_1-p), \quad (4\cdot2)$$

where the notation of Dyson (1) has been used. The first coincident pole contribution converts the two S_F functions into the term

$$\frac{1}{2}\frac{2\pi i}{2\sqrt{[(\mathbf{p}+\mathbf{k}_1)^2+\kappa^2]}}\frac{1}{(2\pi)^4 i}[i\gamma(k_1+p)-\kappa]_{\xi\eta}\,\delta(k_{10}+p_0-\sqrt{[(\mathbf{k}_1+\mathbf{p})^2+\kappa^2]})$$
$$\times \frac{2\pi i}{2\sqrt{[(\mathbf{p}-\mathbf{k}_2)^2+\kappa^2]}}\frac{1}{(2\pi)^4 i}[i\gamma(k_2-p)-\kappa]_{\sigma\tau}\,\delta(p_0-k_{20}-\sqrt{[(\mathbf{p}-\mathbf{k}_2)^2+\kappa^2]}).$$

Further, on integration over p_0,

$$\frac{i\gamma(k_1+p)-\kappa}{2\sqrt{[(\mathbf{k}_1+\mathbf{p})^2+\kappa^2]}}\,\delta(p_0+k_{10}-\sqrt{[(\mathbf{p}+\mathbf{k}_1)^2+\kappa^2]})$$

becomes

$$-\frac{1}{2}\left\{1+\frac{(\boldsymbol{\alpha}.\mathbf{k}_1+\mathbf{p})+\beta\kappa}{\sqrt{[(\mathbf{k}_1+\mathbf{p})^2+\kappa^2]}}\right\}\beta,$$

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where $\boldsymbol{\alpha}, \beta$ are the usual Dirac matrices. This last expression is just the projection operator $P_+(\mathbf{k}_1+\mathbf{p})$ for positive energy electron states of momentum $\mathbf{k}_1+\mathbf{p}$, multiplied to the right by $(-\beta)$. Similarly, the second S_F function reduces to the operator $-P_+(\mathbf{k}_2-\mathbf{p})\beta$. The other pair of δ-functions in (4·1) reduce the S_F functions to the negative-energy projection operators P_- (multiplied by β).

Assume that initially there are two electrons of momenta k_1', k_2' present. The positive frequency components of the two ψ-functions to the right of (4·2) annihilate these electrons. The two projection operators can be written in the form

$$(P_+(\mathbf{k}_1+\mathbf{p})\beta)_{\xi\eta}(a_u\tilde{a}_u+\tilde{a}_u a_u)(P_+(\mathbf{k}_2-\mathbf{p})\beta_{\sigma\tau}(a_v\tilde{a}_v+\tilde{a}_v a_v), \tag{4·3}$$

where a, $\tilde{a}$ are the usual creation and annihilation operators for the electron-positron field. u and v refer to the positive frequencies corresponding to the energies $\{(\mathbf{p}+\mathbf{k}_1)^2+\kappa^2\}^{\frac{1}{2}}$ and $\{(\mathbf{k}_2-\mathbf{p})^2+\kappa^2\}^{\frac{1}{2}}$ respectively. The operators $\tilde{a}_u$ and a_v which act on the vacuum state function to left and right respectively, disappear, and only the operator $a_u\tilde{a}_u a_v\tilde{a}_v$ remains. Thus (4·3) becomes

$$(P_+(\mathbf{p}+\mathbf{k}_1)\beta)_{\xi\eta}a_v a_u\tilde{a}_u\tilde{a}_v(P_+(\mathbf{k}_2-\mathbf{p})\beta)_{\sigma\tau}. \tag{4·4}$$

It follows that (4·2) can be written as the product of two matrix elements of type G_2, viz.

$$\left(\frac{e}{\hbar c}\right)^4\overline{\psi}_\alpha(k_1)\,\overline{\psi}_\beta(k_2)\,\gamma^\mu_{\alpha\xi}\frac{\hbar c}{2}D_F(k_1''-k_1)\,\gamma^\mu_{\beta\sigma}(P_+(\mathbf{k}_1''))_{\xi\eta}\,(P_+(k_2''))_{\sigma\tau}\,\psi_\tau(k_2'')\,\psi_\eta(k_1'')$$
$$\times\,\overline{\psi}_\eta(k_1'')\,\overline{\psi}_\tau(k_2'')\,\gamma^\nu_{\eta\zeta}\frac{\hbar c}{2}D_F(k_1'-k_1'')\,\gamma^\nu_{\tau\rho}\psi_\rho(k_2')\,\psi_\zeta(k_1')\ldots. \tag{4·5}$$

The β factors in (4·4) give rise to the two new $\overline{\psi}$ operators in (4·5). The remaining integration over $\mathbf{p}$ in (4·1) has the effect of summing (4·5) over all *real* particle states k_1'', k_2'', which are consistent with the conservation of energy and momentum required by G_2. Allowing for the density of these states the result of decomposing the real graph part G_4 can be written

$$\langle k_1k_2\,|\,G_4\,|\,k_2'k_1'\rangle=\tfrac{1}{2}\sum_{k_1'',k_2''}\langle k_1k_2\,|\,G_2\,|\,k_2''k_1''\rangle\langle k_1''k_2''\,|\,G_2\,|\,k_2'k_1'\rangle. \tag{4·6}$$

The factor $\frac{1}{2}$ in front of the summation arises from the presence of a term $(2\pi i)^{r-1}(\pi i)$ rather than $(2\pi i)^r$ in front of the δ-function product corresponding to r coincident poles (cf. § 2).

Consider the graph G_{23} which can be cut by AA as shown. The line factor associated with the internal photon line 12 can be written as

$$\frac{\hbar c}{(2\pi)^4 i}\int\frac{d^4p}{p^2}.$$

When this line gives rise to a coincident pole the integrand will be replaced by

$$\frac{2\pi i}{2\,|\,\mathbf{p}\,|}\frac{\hbar c}{(2\pi)^4 i}\delta(p_0-|\,\mathbf{p}\,|), \tag{4·7}$$

plus a similar term. The amplitude of the matrix element for the emission or absorption of a real photon of momentum $\mathbf{p}$ is $\{\hbar c/2\,|\,\mathbf{p}\,|\}^{\frac{1}{2}}$. Removing the square of this matrix element from (4·7) leaves $(2\pi)^{-3}\delta(p_0-|\,\mathbf{p}\,|)$, which gives the correct density function for the emission and reabsorption of a real photon of momentum $\mathbf{p}$ along the line 12.

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It is clear, without going into further details, that formula (4·6) can be extended to any real graph part, except perhaps when the graph contains self-energy or vertex parts. If G is a real graph part which is divided into graph parts G' and G'' by the cut line, then

$$\langle k_1 k_2 \,|\, G \,|\, k_2' k_1' \rangle = \tfrac{1}{2} \sum_{k_1'', k_2'', k_3'' \ldots} \langle k_1 k_2 \,|\, G' \,|\, \ldots k_3'' k_2'' k_1'' \rangle \langle k_1'' k_2'' k_3'' \ldots \,|\, G'' \,|\, k_2' k_1' \rangle, \quad (4{\cdot}8)$$

where $(k_1'', k_2'', k_3'', \ldots)$ refer to real particles consistent with the conservation of energy and momentum, and where there are as many electrons or photons as there are respectively electron or photon lines intersected by the cut line. The case of the primitive divergent graphs will be discussed below.

5. *Symmetry properties.* The general term of the S-matrix is (1)

$$\frac{1}{n!} \left(\frac{-i}{\hbar c}\right)^n \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n P(H_1(x_1), \ldots, H_1(x_n)), \quad (5{\cdot}1)$$

where H_1 is the interaction Hamiltonian and P is the chronological operator. The factor $1/n!$ is just cancelled by the possible ways of assigning $x_1, x_2, \ldots, x_n$ to the vertices of any graph, and on doing this, only topologically distinct graphs are to be considered.

Assume, for simplicity, that there are two electrons in both the initial and final states. If the electrons are labelled 1, 2, and if the initial and final states are α, β and γ, δ respectively, then

$$\frac{1}{\sqrt{2}} \{u_\alpha(1)\, u_\beta(2) - u_\alpha(2)\, u_\beta(1)\}$$

is the correct anti-symmetrical initial state function. u_α, u_β are the usual spinors. The S-matrix element M deduced from (5·1) is symmetric in the electron variables, so using the correct state functions gives

$$\begin{aligned} \langle \frac{1}{\sqrt{2}} \{\overline{u}_\gamma(1)\, \overline{u}_\delta(2) - \overline{u}_\delta(1)\, \overline{u}_\gamma(2)\} \,|\, M \,|\, \frac{1}{\sqrt{2}} \{u_\alpha(1)\, u_\beta(2) - u_\alpha(2)\, u_\beta(1)\} \rangle \\ = \langle \overline{u}_\gamma(1)\, \overline{u}_\delta(2) \,|\, M \,|\, u_\beta(2)\, u_\alpha(1) \rangle - \langle \overline{u}_\delta(1)\, \overline{u}_\gamma(2) \,|\, M \,|\, u_\beta(2)\, u_\alpha(1) \rangle. \quad (5{\cdot}2) \end{aligned}$$

When the initial electrons have energy and momentum k_1', k_2', the total probability for scattering is

$$\tfrac{1}{2} \sum_{k_1'', k_2''} |\langle k_1'' k_2'' \,|\, M \,|\, k_2' k_1' \rangle - \langle k_2'' k_1'' \,|\, M \,|\, k_2' k_1' \rangle|^2,$$

where k_1'', k_2'' are summed over all real particle states consistent with the conservation laws. The factor $\frac{1}{2}$ appears because the ordering of k_1'', k_2'' is unimportant.

6. *External parts and the unitary condition.* An external part of a graph G is a portion of G which, when it is removed by cutting internal lines, leaves a graph G' having external lines representing the same state as the external lines of G. It is clear that an external part is a graph part which arises from a diagonal element of the S-matrix. The simplest external parts for the two-electron problem are G_1 and G_{24}.

The S-matrix expansion of Dyson (1) is formally unitary (and the renormalization process is a unitary transformation), and it is easy to see, with the aid of the ideas of graph parts and external parts, how the unitary condition is fulfilled. For the moment the anti-symmetry property of the electrons will be neglected for simplicity. The unitary condition for the two electron problem can then be written in the form

$$|\langle k_1' k_2' \,|\, E \,|\, k_2' k_1' \rangle|^2 = 1 - \sum_{k_1'', k_2'', \ldots} |\langle k_1' k_2' \,|\, \Sigma G \,|\, \ldots k_2'' k_1'' \rangle|^2, \quad (6{\cdot}1)$$

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where E is the sum of the external parts, and ΣG is the sum of all graphs which have matrix elements connecting the states k_1', k_2' and $k_1'', k_2'', \ldots$. $k_1'', k_2'', \ldots$ can indicate states with any number of electrons or photons.

In evaluating a probability, for example $|\langle k_1' k_2' \,|\, G \,|\, k_3'' k_2'' k_1''\rangle|^2$, it should be noted that the matrix element contains the product of four δ-functions in the form

$$(2\pi)^4 \delta^{(4)}(k_1'' + k_2'' + k_3'' - k_1' - k_2'), \tag{6·2}$$

which ensures the overall conservation of energy and momentum. On taking the square of the modulus of the matrix element and integrating over the k'' variables, one factor (6·2) picks out the set of final energy-momentum values which satisfy the conservation laws. A constant term $(2\pi)^4 \delta^{(4)}(0)$ is given by the other factor (6·2). This is infinite, but it is easy to see that if a large space V and a large time T replaces the infinite space-time region used in (5·1) then $(2\pi)^4 \delta^{(4)}(0)$ is replaced by VT. The relations

$$2\pi\delta(k) = \int_{-\infty}^{\infty} e^{ikx}\, dx,$$

$$\lim_{k\to 0} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} e^{ikx}\, dx = \lim_{k\to 0} (2/k) \sin(\tfrac{1}{2}kL) = L,$$

make this obvious. To avoid this constantly recurring infinite factor it is only necessary to deal with probabilities per unit volume per unit time. Equation (6·1) will be considered in this sense, and the factor $(2\pi)^4 \delta^{(4)}(0)$ will be equated to unity.

Formula (4·6) shows the relation satisfied by the real graph part G_4. Consider the Hermitian conjugate of $\langle k_1 k_2 \,|\, G \,|\, k_2' k_1'\rangle$, where G is any graph, using $\tilde{A}$ to denote the Hermitian conjugate of A. It is clear that

$$\widetilde{(i\gamma p - \kappa)}\, \gamma^4 = \gamma^4 (i\gamma p - \kappa). \tag{6·3}$$

The Hermitian conjugate of a typical term arising from (5·1),

$$\bar{\psi}(k_1)\, \gamma^4 (i\gamma p_1 - \kappa)\, \gamma^\nu (i\gamma p_2 - \kappa)\, \gamma^\lambda \psi(k_2), \tag{6·4}$$

is

$$\bar{\psi}(k_2)\, \gamma^\lambda \widetilde{(i\gamma p_2 - \kappa)}\, \gamma^\nu \widetilde{(i\gamma p_1 - \kappa)}\, \gamma^4 \psi(k_1). \tag{6·5}$$

Using (6·3) the factor γ^4 occurring in (6·5) can be moved to the extreme left. Then (6·5) becomes

$$(\pm)\, \bar{\psi}(k_2)\, \gamma^4 \gamma^\lambda (i\gamma p_2 - \kappa)\, \gamma^\nu (i\gamma p_1 - \kappa)\, \psi(k_1). \tag{6·6}$$

The factor γ^ν arises from a vertex from which an internal or an external photon line issues. If the photon line is internal there will be another factor γ^ν at the vertex at its other end. This vertex lies on another electron line, so that the factor (-1) which occurs in (6·6) in the case $\lambda = 4$, $\nu \neq 4$ will occur in two electron-line terms, giving $(-1)^2$ in all. If the photon line is external it is clear (remembering the anti-Hermitian property of the fourth component of the electromagnetic field) that a change in sign is introduced by moving γ^4 to the left of γ^ν. It follows that in taking the Hermitian conjugate of a general expression of the type (6·4) as many changes of sign are introduced as there are γ factors corresponding to external photon lines. Therefore, considering only the $\bar{\psi}$, ψ, γ^ν and $(i\gamma p - \kappa)$ factors in the expression corresponding to a graph G,

$$\widetilde{\langle k \,|\, G \,|\, k'\rangle} = (-1)^{E_p} \langle k' \,|\, \bar{G} \,|\, k\rangle, \tag{6·7}$$

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where E_p is the number of external electron lines in G, and where $\bar{G}$ is the graph which is the mirror image of G, except for the fact that the arrows on the electron lines are not to be reversed. A device of this sort has been used by Morette (6).

Now consider the number of factors i which appear in the c-number terms of the matrix element of G. The S_F and D_F functions each contain a factor i, and each δ-function that arises from the poles in the integrand contributes a factor i. Hence the number of changes of sign involved in taking the Hermitian conjugate of a real graph part is the sum of the number of changes of sign introduced in taking the Hermitian conjugates of its constituent virtual graph parts. Thus it is only necessary to study the virtual graph parts. At each of the n vertices of the graph there is a factor i which just removes the term $(-i)^n$ which is outside the integral in (5·1). Suppose that a virtual graph part G contains n vertices and F internal lines. The number of free variables of integration is then $f = F-n+1$. In a virtual part there is one δ-function (and one power of i) for each free variable, so the integrand contains a factor $(i)^{F-f}$. A graph which has E_p external and F_p internal photon lines has n vertices, where $n = E_p + 2F_p$. Hence $$F-f = n-1 = E_p + 2F_p - 1.$$

The complex conjugate of the numerical factors thus gives a factor $(-1)^{E_P-1}$, so for the complete virtual graph part (6·7) is replaced by

$$\widetilde{\langle k \,|\, G \,|\, k' \rangle} = -\langle k' \,|\, \bar{G} \,|\, k \rangle. \tag{6·8}$$

In particular,
$$\langle k_1 k_2 \,|\, G_4 \,|\, k_2 k_1 \rangle = -\tfrac{1}{2} \sum_{k_1'', k_2''} |\langle k_1 k_2 \,|\, G_2 \,|\, k_2'' k_1'' \rangle|^2. \tag{6·9}$$

Equation (6·1) will now be verified to order e^6 for graphs like G_2, G_3, G_5. By choosing the order e^6 most of the important features can be seen without undue complication. The graph parts (of the type just mentioned) which contribute to $|E|^2$ to order e^6 are $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8$ taken with the same initial and final states. Then using (6·8),

$$\begin{aligned} |\langle k_1' k_2' \,|\, E \,|\, k_2' k_1' \rangle|^2 = 1 &+ |\langle k_1' k_2' \,|\, G_2 \,|\, k_2' k_1' \rangle|^2 + 2\langle k_1' k_2' \,|\, G_4 \,|\, k_2' k_1' \rangle \\ &+ \{\langle k_1' k_2' \,|\, G_2 \,|\, k_2' k_1' \rangle \langle k_1' k_2' \,|\, G_3 \,|\, k_2' k_1' \rangle^* + \text{complex conjugate}\} \\ &+ 2\langle k_1' k_2' \,|\, G_6 \,|\, k_2' k_1' \rangle + 2\langle k_1' k_2' \,|\, G_7 \,|\, k_2' k_1' \rangle + O(e^8). \end{aligned} \tag{6·10}$$

Here * denotes the complex conjugate. Further use of (4·8) and (6·8) gives

$$\begin{aligned} |\langle k_1' k_2' \,|\, E \,|\, k_2' k_1' \rangle|^2 = 1 &- \{\sum_{k_1'', k_2''} |\langle k_1' k_2' \,|\, G_2 \,|\, k_2'' k_1'' \rangle|^2 - |\langle k_1' k_2' \,|\, G_2 \,|\, k_2' k_1' \rangle|^2\} \\ &- \{[\sum_{k_1'', k_2''} \langle k_1' k_2' \,|\, G_2 \,|\, k_2'' k_1'' \rangle \langle k_1' k_2' \,|\, G_3 \,|\, k_2'' k_1'' \rangle^* - \langle k_1' k_2' \,|\, G_2 \,|\, k_2' k_1' \rangle \langle k_1' k_2' \,|\, G_3 \,|\, k_2' k_1' \rangle^*] \\ &+ \text{conjugate}\} + O(e^8). \end{aligned} \tag{6·11}$$

It is a simple matter to show how the formula corresponding to (6·11) is derived when the anti-symmetry property of the electron state function, as discussed in § 5, is taken into account.

To order e^6 other graph parts which occur in $|E|^2$ are G_9, G_{10}, G_{13}, G_{14}, G_{15}, G_{16}, G_{17}, G_{18}, and the mirror images of the last three.

7. *Self-energy and vertex parts.* Of the graphs mentioned at the end of § 6, G_{13} to G_{18} contain self-energy or vertex parts from which the infinite terms have to be subtracted. Consider the graph part G_{14}. The electron self energy part (G_{21}) is intersected by the cut line AA, so two δ-functions are introduced into the self-energy part of the integrand.

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The remaining integration over the free variable associated with the self energy part is thus a solid angle integration, so G_{21} and therefore G_{14} converge. G_{21} is proportional to the probability of an electron of energy-momentum p emitting a free photon and becoming a free electron, so its contribution will vanish whenever $p^2+\kappa^2 = 0$. The self energy graph part G_{22} without a cut line is, of course, infinite, and the infinity is subtracted in the usual way. By (6·8) it follows that

$$\overbrace{\langle p \,|\, G_{22} \,|\, p\rangle} = -\langle p \,|\, G_{22} \,|\, p\rangle,$$

while

$$\overbrace{\langle p \,|\, G_{21} \,|\, p\rangle} = \langle p \,|\, G_{21} \,|\, p\rangle,$$

so that the two finite parts are distinguished by the Hermitian property.

The graphs G_{16}, G_{17}, G_{18} containing vertex parts can be treated in the same way, the finite contribution being divided into two portions, one a real and the other a virtual graph part. On checking the unitary condition the correct anti-symmetric matrix elements (5·2) must be used to deal with the diagonal cut lines AB in G_9 and G_{10}, which decompose into graphs of the type G_{11}, G_{12}. Other graphs like G_{19} and G_{20} will give contributions to (6·1) to order e^6, but the effects of these graphs can readily be evaluated. Further details of the verification of the unitary condition are not of great interest and will not be reproduced here.

8. *The unitary condition and convergence.* The convergence of the renormalized S-matrix expanded as a power series in the coupling constant has been investigated directly by Hurst (4, 5). Without reaching a definite conclusion he has put forward arguments which, in the author's opinion, are sufficiently strong to throw considerable doubt on the convergence of the expansion. It is clear from Hurst's work that, at least in electrodynamics, the partial sums will appear to converge up to a considerable power of the coupling constant, but thereafter rapid divergence may occur. It has, however, not been proved that at the least the expansion is an asymptotic series.

The relation between the unitary condition and the convergence problem is of interest. Denoting the coupling constant by λ, any element of the S-matrix can be written in the form

$$\langle k \,|\, S \,|\, k'\rangle = \sum_{n=0}^{\infty} \lambda^n \langle k \,|\, S_n \,|\, k'\rangle, \tag{8·1}$$

where $S_0 = I$. Therefore, formally,

$$|\langle k \,|\, S \,|\, k'\rangle|^2 = \sum_{n=0}^{\infty} \lambda^n \Big\{ \sum_{m=0}^{n} \langle k \,|\, S_{n-m} \,|\, k'\rangle \langle k \,|\, S_m \,|\, k'\rangle^* \Big\};$$

and the unitary property of the S-matrix implies the well-known relations

$$\int dk' \sum_{n=0}^{m} \langle k \,|\, S_{n-m} \,|\, k'\rangle \langle k \,|\, S_m \,|\, k'\rangle^* = 0 \quad (n = 1, 2, 3, \ldots), \tag{8·2}$$

where the integration over k' includes all real states which satisfy the conservation laws. The validity of (8·2) is independent of the convergence of the expansion (8·1).

Whenever (8·1) diverges the unitary relation

$$\int dk' \,|\langle k \,|\, S \,|\, k'\rangle|^2 = 1,$$

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which is true up to any arbitrary power of λ, does not imply that any term $|\langle k \mid S \mid k' \rangle|^2$ is bounded, as the relations (8·2) are not sufficient to show that

$$\sum_{n=0}^{N} \lambda^n \Big\{ \sum_{m=0}^{n} \langle k \mid S_{n-m} \mid k' \rangle \langle k \mid S_m \mid k' \rangle^* \Big\} \geqslant 0$$

for any N. In fact when (8·1) diverges it is to be expected that some of these partial sums for the probability will become negative.

9. *An integral equation.* It is possible to express any graph as the sum of the product of a number of virtual graph parts. Denote the general graph by G and the general virtual graph part by G_v. Consider, for example, the complete matrix element $\langle k_1' k_2' \mid M \mid k_2 k_1 \rangle$ for the mutual scattering of two electrons. This matrix element can be written in the form

$$\langle k_1' k_2' \mid M \mid k_2 k_1 \rangle = \delta_{k_1' k_1} \delta_{k_2' k_2} + \langle k_1' k_2' \mid U \mid k_2 k_1 \rangle,$$

where by (4·8) $\langle k_1' k_2' \mid U \mid k_2 k_1 \rangle$ satisfies the equation

$$\langle k_1' k_2' \mid U \mid k_2 k_1 \rangle = \sum_v \langle k_1' k_2' \mid G_v \mid k_2 k_1 \rangle + \tfrac{1}{2} \sum_{k_1'', k_2'', k_3'', \ldots} \sum_v \langle k_1' k_2' \mid G_v \mid \ldots k_3'' k_2'' k_1'' \rangle \langle k_1'' k_2'' k_3'' \ldots \mid U \mid k_2 k_1 \rangle. \quad (9{\cdot}1)$$

$\sum_v$ denotes the sum over all virtual graph parts, and $\langle k_1'' k_2'' k_3'' \ldots \mid U \mid k_2 k_1 \rangle$ is the matrix element connecting the states k_1, k_2, and $k_1'', k_2'', k_3'', \ldots$. Clearly (9·1) is one of a set of simultaneous integral equations whose solution would make it possible to express the matrix element $\langle k_1' k_2' \mid M \mid k_2 k_1 \rangle$ in terms of the virtual graph parts G_v.

It is not obvious that this set of integral equations will give a theory which has convergence properties that are much superior to those of original expansion, for the convergence of the matrix elements of $\sum_v G_v$ is difficult to estimate. However, Hurst (5) has discussed a somewhat similar case and again has put forward good reasons why the sum should not converge.

Consider equation (9·1) as given by taking the partial sum $G_v^{(n)}$ of $\sum_v G_v$ up to the order λ^n. For simplicity neglect the creation and annihilation of particles, so that (9·1) becomes

$$\langle k_1' k_2' \mid U \mid k_2 k_1 \rangle = \langle k_1' k_2' \mid G_v^{(n)} \mid k_2 k_1 \rangle + \tfrac{1}{2} \sum_{k_1'', k_2''} \langle k_1' k_2' \mid G_v^{(n)} \mid k_2'' k_1'' \rangle \langle k_1'' k_2'' \mid U \mid k_2 k_1 \rangle.$$

The integral equation for $\langle k_1' k_2' \mid M \mid k_2 k_1 \rangle$ is thus

$$\langle k_1' k_2' \mid M \mid k_2 k_1 \rangle = \langle k_1' k_2' \mid I + \tfrac{1}{2} G_v^{(n)} \mid k_2 k_1 \rangle + \tfrac{1}{2} \sum_{k_1'', k_2''} \langle k_1' k_2' \mid G_v^{(n)} \mid k_2'' k_1'' \rangle \langle k_1'' k_2'' \mid M \mid k_2 k_1 \rangle, \quad (9{\cdot}2)$$

where I is the identity operator. In matrix notation (9·2) can be written

$$(I - \tfrac{1}{2} G_v^{(n)}) M^{(n)} = I + \tfrac{1}{2} G_v^{(n)}, \quad (9{\cdot}3)$$

the index n being used to distinguish the solution derived from $G_v^{(n)}$.

As $G_v^{(n)}$ is the sum of all virtual graph parts up to order λ^n it will contain the mirror image $\bar{G}$ of each graph which is not its own mirror image. Hence, by (6·8)

$$\widetilde{\langle k_1' k_2' \mid G_v^{(n)} \mid k_2 k_1 \rangle} = -\langle k_1 k_2 G_v^{(n)} \mid k_2' k_1' \rangle,$$

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so it is possible to write $\frac{1}{2}G_v^{(n)} = iK_v^{(n)}$,

where $K_v^{(n)}$ is a Hermitian operator. An exact solution $M^{(n)}$ of (9·3) is therefore unitary and gives positive probabilities. Therefore it follows that $|\langle k_1' k_2' | M^{(n)} | k_2 k_1 \rangle|^2$, determined in this manner, is bounded for all n. However, if $G_v^{(n)}$ does not converge as $n \to \infty$ there is no reason to suppose that this bounded probability will converge to any definite value.

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REFERENCES

(1) Dyson, F. J. *Phys. Rev.* (2), 75 (1949), 486–502, 1736–55.
(2) Eden, R. J. *Proc. roy. Soc.* A, 210 (1952), 388–404.
(3) Feynman, R. P. *Phys. Rev.* (2), 80 (1950), 440–57.
(4) Hurst, C. A. *Proc. roy. Soc.* A (in the Press).
(5) Hurst, C. A. *Proc. Camb. phil. Soc.* 48 (1952), 625–39.
(6) Morette, C. *Phys. Rev.* (2), 76 (1949), 1432–9.

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