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PRICING AND HEDGING FOR A STICKY DIFFUSION

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Abstract

We introduce a financial market model featuring a risky asset whose price follows a sticky geometric Brownian motion and a riskless asset that grows with a constant interest rate $r \in \mathbb{R}$. We prove that this model satisfies no arbitrage and no free lunch with vanishing risk only when $r = 0$. Under this condition, we derive the corresponding arbitrage-free pricing equation, assess the replicability, and give a representation of the replication strategy. We then show that all locally bounded replicable payoffs for the standard Black–Scholes model are also replicable for the sticky model. Last, we evaluate via numerical experiments the impact of hedging in discrete time and of misrepresenting price stickiness.

Keywords: Sticky geometric Brownian motion; no-arbitrage condition; derivatives pricing; arbitrage-free valuation; hedging time-granularity; model mismatch

2010 Mathematics Subject Classification: Primary 91G20
Secondary 91G30; 60J60

1. Introduction

The seminal work of Black and Scholes [11] stands as the founding result in the theory of derivatives valuation and forms the basis of most pricing models in mathematical finance. The model assumes that the price of the risky asset is a geometric Brownian motion and that the interest rate is constant. Under these conditions, it proposes a riskless replication (or hedging) strategy of the payoff of a European contingent claim through a self-financing portfolio composed of the underlying risky asset and the non-risky asset. In accordance with the arbitrage-free pricing principle, the claim's price is equal to the cost of the replication strategy. This allows effective market risk exposure management and fair pricing of financial products.

Most continuous models in mathematical finance extend the Black–Scholes framework to capture additional features of stock and derivatives price dynamics. The assumptions in the Black–Scholes model, such as constant volatility, drift, and interest rate, are unrealistic when applied to both historical and implied price behavior. Several models have been developed to address these limitations. Local volatility models [26, 27] and stochastic volatility models [33–35, 46, 48] account for the stochastic nature of volatility and provide a better fit for the implied volatility structure. Models incorporating jumps (e.g. [8]) and stochastic interest rates (e.g. [6]) offer further refinements to market dynamics. In addition, skew-threshold dynamics and reflecting-threshold dynamics have been explored (see [16, 18, 23, 44] and references

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therein). Rough volatility models have been considered to better fit statistical observations [9, 32]. Models addressing liquidity and transaction costs have been proposed [1, 40], lifting additional assumptions of the Black–Scholes model.

An effect that is observed on price series is the presence of thresholds where the dynamic seems to linger or even spend a positive amount of time there. This is observed in the case of corporate takeovers, where the takeover price is considered to be fair by the market. This is mentioned in [7, Introduction] as motivation for his work on the description of stochastic differential equations with a sticky point. An example of this occurrence is given in [20] on Hansen Technologies Limited between June 7, 2021 and September 6, 2021, where a takeover offer was made and then canceled. Softer psychological barriers are also observed in other markets, especially in commodities (see, for example, gold and silver).

In this paper we consider a variant of the Black–Scholes model where the price-dynamic S of the risky asset is a sticky geometric Brownian motion. The process S is a diffusion that behaves like a geometric Brownian motion away from a certain price threshold $\zeta > 0$, where it spends a positive amount time upon contact. We refer to this model as the *sticky Black–Scholes model*. We establish that there is *no arbitrage* (NA) and *no free lunch with vanishing risk* (NFLVR) in the model if and only if the interest rate r is zero. In this case, we derive the arbitrage-free pricing equation, assess the existence of a replication strategy, and provide an explicit representation of it in terms of the solution of the pricing problem. We then prove additional properties of the prices on claims and replicability. In particular, that the prices of claims with convex payoffs are decreasing and the class of locally bounded replicable payoffs is increasing with respect to the stickiness parameter. This yields that, for this model, we can consider all locally bounded payoff functions that can be replicated in the standard Black–Scholes model. Last, we conduct numerical evaluations of discrete-time hedging and analyze the hedging error incurred when ignoring or misrepresenting price stickiness (model mismatch) as realized volatility or local volatility in a smooth model.

The proof of the sufficient and necessary condition for (NFLVR) is done by constructing an arbitrage strategy in the case $r \neq 0$ and by finding an equivalent local martingale measure (ELMM) in the case $r = 0$. The construction of the arbitrage strategy is based on the proof of [25, Theorem 12.3.5] and relies on the presence of a local time term in the discounted price process that cannot be offset by the quadratic variation of the martingale part. This technique has been previously used to demonstrate the existence of arbitrage in [16], for a diffusion with a reflecting price threshold, and in [44], for a diffusion with skew returns. We note that, unlike these models, in a sticky price model the local time term appears only for the discounted price process when $r \neq 0$.

In our analysis, we show that the sticky Black–Scholes model is an example of a model where there is no uniqueness of (ELMM) for the discounted price process and no uniqueness of payoff replication strategy. These peculiar effects arise from the ability to trade in the infinitesimal range of $\mathbf{1}_{S_t=\zeta}\,\mathrm{d}t$, the times when the price process is located at the sticky threshold. Due to the stickiness, this set has positive Lebesgue measure. The arbitrage strategies in the case $r \neq 0$ are necessarily partially or fully supported on the set $\mathbf{1}_{S_t=\zeta}\,\mathrm{d}t$. In the case $r = 0$, the (ELMM) and the payoff replication strategy are equivalence classes of processes that coincide over the range of $\mathbf{1}_{S_t\neq\zeta}\,\mathrm{d}t$.

First discovered by Feller [29], sticky diffusions are Markov processes with continuous sample paths that spend a positive amount of time at some threshold(s). They are characterized by the presence of atom(s) in their speed measures [7, 12] and can be expressed as time delays of non-sticky diffusions (see [2, 28, 45]). The time delay is proportional to the local time of the

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diffusion at the sticky threshold. This leads in some cases to path-space representations similar to classical stochastic differential equations (SDEs) for absolutely continuous diffusions [28, 45] (see also [4, Section 4]).

In finance, sticky diffusions have been studied to model other phenomena like near-zero short-rate dynamics [41, 50] and winning-streak dynamics [30]. For other applications, see [13, 17, 22, 49, 51].

The paper is organized as follows. In Section 2, we define the sticky Black–Scholes model. In Section 3, we show that the sticky Black–Scholes model satisfies (NFLVR) if and only if $r = 0$. Under this condition, we identify an equivalent local martingale measure for the price process. In Section 4 we first derive the arbitrage-free pricing equation for the case $r = 0$ and prove the existence of the associated payoff replication strategy. Section 5 explores monotonicity properties in both prices and the classes of replicable payoffs. In Section 6, we numerically evaluate properties of discrete-time hedging and assess the impact of model mismatch. Finally, Section 7 provides some concluding remarks.

2. The model

The standard Black–Scholes model makes the following assumptions:

(A1) infinite liquidity, divisibility of assets, and borrowing/lending capacity;
(A2) frictionless market (no transaction costs, no borrowing or lending rate);
(A3) possibility of continuous-time trading;
(A4) constant interest rate r;
(A5) geometric Brownian motion price dynamics.

In the sticky Black–Scholes model, Assumption (A5) is modified such that the price process of the risky asset S follows a sticky geometric Brownian motion. In particular, the prices of risky and non-risky assets (S, S^0) solve the system

$$\begin{cases} \mathrm{d}S_t = \mathbf{1}_{S_t \neq \zeta} S_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t), \\ \mathbf{1}_{S_t=\zeta}\,\mathrm{d}t = \rho\,\mathrm{d}L_t^{\zeta}(S), \\ \mathrm{d}S_t^0 = rS_t^0\,\mathrm{d}t, \end{cases} \tag{1}$$

where $\mu, r \in \mathbb{R}$, $\sigma, \zeta > 0$, $\rho > 0$, B is a standard Brownian motion, and $(L_t^y(S); t \geq 0, y > 0)$ is the local time field of S.

The process S defined in this way has the following properties:

(i). It is a semimartingale with the explicit Doob–Meyer decomposition

$$S_t = \mu \int_0^t S_s \mathbf{1}_{S_s \neq \zeta}\,\mathrm{d}s + \sigma \int_0^t S_s \mathbf{1}_{S_s \neq \zeta}\,\mathrm{d}B_s.$$

(ii). It is the regular diffusion on $(0, \infty)$ defined through scale and speed (s,m) [4, Section 4], where s,m are defined for all $x > 0$ by

$$s'(x) = \exp\left\{-\frac{2\mu}{\sigma^2}\int^x \frac{1}{\zeta}\,\mathrm{d}\zeta\right\}, \qquad m(dx) = \frac{1}{s'(x)}\frac{1}{\sigma^2 x^2}\,\mathrm{d}x + \frac{\rho}{s'(\zeta)}\delta_\zeta(dx). \tag{2}$$

We note that for [4, (4.2)] to be correct, the singular part of the expression should be divided by $s'(0)$. The author thanks David Criens and his student, Sebastian Hahn, for identifying and communicating this correction.

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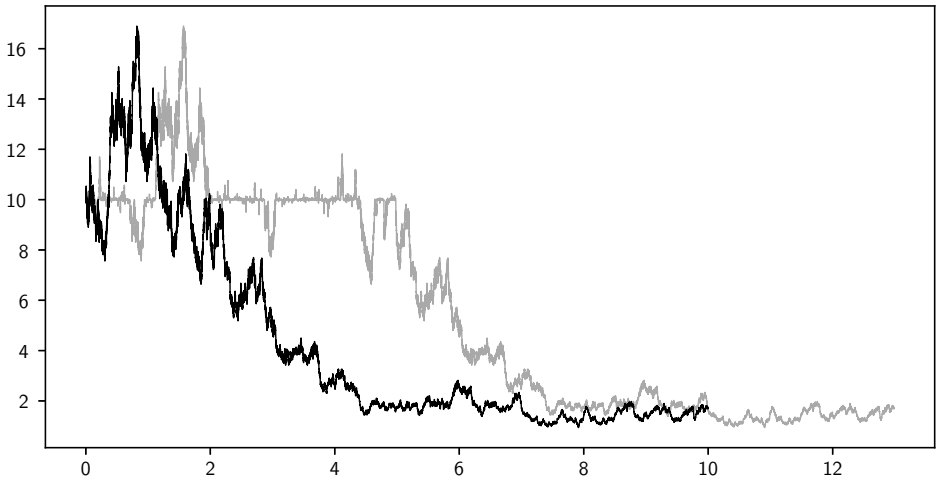


FIGURE 1. Simulated sample paths of the sticky (black) geometric Brownian motion of parameters ($\mu = 0$, $\sigma = 0.25$, $\rho = 2$, $\zeta = 10$) and the non-sticky (gray) geometric Brownian motion of parameters ($\mu = 0$, $\sigma = 0.25$). The approximation method used for the simulation is from [5].

(iii). It spends a positive amount of time at ζ upon contact. Since S is a regular diffusion on $(0, \infty)$, this follows from the change of scale [42, Proposition VII.3.5, Exercise VI.1.14] and the Markov property. This can be observed in the simulated path of the sticky geometric Brownian motion shown in Figure 1.

(iv). It has an alternative path-wise representation akin to that of the geometric Brownian motion. Indeed, from the Itô formula of [4, Lemma 4.5], the process $(X, X^0) = (\log S, \log S^0)$ solves

$$\begin{cases} \mathrm{d}X_t = \left(\mu - \frac{\sigma^2}{2}\right)\mathbf{1}_{X_t \neq \log\zeta}\,\mathrm{d}t + \sigma \mathbf{1}_{X_t \neq \log\zeta}\,\mathrm{d}B_t, \\ \mathbf{1}_{X_t = \log\zeta}\,\mathrm{d}t = \frac{\rho}{\zeta}\,\mathrm{d}L_t^{\log\zeta}(X), \\ \mathrm{d}X_t^0 = r\,\mathrm{d}t. \end{cases}$$

Consequently, we derive the following expressions for the price dynamics:

$$\begin{cases} S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\int_0^t \mathbf{1}_{S_t \neq \zeta}\,\mathrm{d}t + \sigma \int_0^t \mathbf{1}_{S_t \neq \zeta}\,\mathrm{d}B_t\right), \\ \mathbf{1}_{S_t = \zeta}\,\mathrm{d}t = \rho\,\mathrm{d}L_t^{\zeta}(S), \\ S_t^0 = S_0^0 \exp(rt). \end{cases} \tag{3}$$

Example 1. (*Standard Black–Scholes model.*) If $\rho = 0$, then S is a geometric Brownian motion and the model (S, S^0) is the standard Black–Scholes model. Indeed, in this case, (s, m) defined in (2) are the scale function and speed measure of the geometric Brownian motion [12, p. 132]. This can also be seen from (1), where if $\rho = 0$, the process S has a geometric Brownian dynamic away from ζ and the time it spends at ζ is of Lebesgue measure 0.

Remark 1. The process (S, S^0) is a two-dimensional sticky diffusion that, when $r \neq 0$, cannot be reduced to a time change. The stickiness affecting the dynamic of the risky asset's price S slows down its motion on ζ, while the log-value of the zero bond $\log(S^0)$ steadily increases with rate r. This characteristic results in the violation the no-arbitrage hypothesis for the model when $r \neq 0$.

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3. (No) arbitrage

In this section we study the model in terms of arbitrage. The notion of arbitrage of interest is the free lunch with vanishing risk and the absence of it is called no free lunch with vanishing risk or (NFLVR). The reason we focus on (NFLVR) is that it is the key notion related to the existence of an equivalent local martingale measure [25, Theorem 12.1.2] and the induced arbitrage-free pricing framework.

First, we recall all the necessary notions related to arbitrage along with some useful results. Then, we adapt these for the sticky Black–Scholes model. In particular, we prove that the model satisfies (NFLVR) if and only if $r = 0$. In this case, we find an equivalent local martingale measure for the discounted price process $\widetilde{S} = S/S^0$.

3.1. Preliminary notions

We start by introducing several concepts from the arbitrage theory for semimartingales. Let X be a semimartingale defined on the probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is assumed both right-continuous and complete. This means that

- for all $t \geq 0$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$;
- $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible events.

Standing Assumption 1. *These assumptions on $(\mathcal{F}_t)_{t\geq 0}$ will apply to all filtrations considered throughout this paper.*

We may now define the following objects.

- A *trading strategy* is a predictable process θ with respect to $(\mathcal{F}_t)_{t\geq 0}$, i.e. for all stopping times τ, θ_τ is $\mathcal{F}_{\tau-}$-measurable.
- An *elementary trading strategy* is a trading strategy of the form $\theta_t = \sum_{i=0}^{n} \theta^{(i)} \mathbf{1}_{t\in(\tau_i,\tau_{i+1}]}$ with $n \in \mathbb{N}$, $(\tau_i)_{i\leq n}$ a family of stopping times, and $(\theta^{(i)})_i$ a family of random variables such that, for all i, $\theta^{(i)}$ is $\mathcal{F}_{\tau_i}$-measurable. Such strategies are also called simple trading strategies.
- An *M-admissible trading strategy*, for some $M \geq 0$, is an adapted predictable process θ with finite variation such that, for all $t \geq 0$, $\int_0^t \theta_s \, \mathrm{d}X_s > -M$ almost surely (a.s.). An *admissible strategy* is a strategy that is *M-admissible* for some $M \geq 0$. We denote the set of all M-admissible strategies by $\mathcal{A}_M$.
- A probability measure $\mathbb{Q}$ is called an *equivalent local martingale measure* (ELMM) for X if $\mathbb{Q} \sim \mathbb{P}$ and X is a local martingale on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$.

We now introduce three notions of arbitrage. For more on these, see [24, 39]. We note that arbitrage notions in a diffusion market with non-trivial interest rates should be expressed in terms of the discounted price process. Otherwise, for the case $r \in \mathbb{R}$, depending on the sign of r, longing or shorting the riskless asset results in an arbitrage strategy.

Definition 1. (*(No) arbitrage*.) A trading strategy θ is an *arbitrage opportunity* for X on $[0, T]$ if $\theta \in \mathcal{A}_0$ and $\mathbb{P}\big(\int_0^T \theta_s \, \mathrm{d}X_s > 0\big) > 0$. A market is *arbitrage-free*, or satisfies (NA), if, for all $\theta \in \mathcal{A}_0$, the probability $\mathbb{P}\big(\int_0^T \theta_s \, \mathrm{d}X_s > 0\big) = 0$.

Definition 2. (*(No) free lunch with vanishing risk*.) A sequence $(\theta^n)_n$ of trading strategies is said to realize a free lunch with vanishing risk if, for each n, θ_n is ϵ_n-admissible and there exists

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some $M > 0$ such that $\lim_{n\to\infty} \epsilon_n = 0$ and, for all $n \in \mathbb{N}$, $\mathbb{P}\big(\int_0^t \theta_s^n \, \mathrm{d}X_s > M\big) > 0$. A market satisfies no *free lunch with vanishing risk* (NFLVR) if there is no sequence of strategies that is free lunch with vanishing risk.

Definition 3. (*(No) unbounded profit with bounded risk*.) A sequence $(\theta^n)_n$ is said to realize an *unbounded profit with bounded risk* if the payoff is uniformly bounded in (n, t) from below but unbounded in probability from above, i.e. there exists some $M > 0$ such that $(\theta^n)_n \subset \mathcal{A}_M$ and the payoff sequence is unbounded in probability, i.e. for all $M' > 0$, there exists $n \in \mathbb{N}$ and $t > 0$ such that $\mathbb{P}\big(\int_0^t \theta_s^n \, \mathrm{d}X_s > M'\big) > 0$. A market satisfies no unbounded profit with bounded risk (NUPBR) if there is no sequence of strategies that is an unbounded profit with bounded risk.

The reason we are interested in (NFLVR) is the following result, which qualifies it as a sufficient and necessary condition for the existence of an (ELMM). The latter implies an arbitrage-free pricing framework.

Theorem 1. ([25, Theorem 12.1.2].) *Let X be a locally bounded semimartingale defined on the probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The following are equivalent:*

(i). *there exists an* (ELMM) *for X;*

(ii). *X satisfies* (NFLVR)*;*

(iii). *X satisfies* (NA) *and* (NUPBR).

We observe that an arbitrage strategy also realizes a free lunch with vanishing risk. This confirms that (NFLVR) implies (NA). Hence, the following result provides a necessary condition for the existence of an (ELMM).

Theorem 2. (One-dimensional version of [25, Theorem 12.3.5].) *Let X be a locally bounded semimartingale that satisfies* (NA)*. Then, if $X = M + A$ is the Doob–Meyer decomposition of X, $\mathrm{d}A = h \, \mathrm{d}\langle M\rangle$, where h is a predictable process.*

3.2. The case of the sticky Black–Scholes model

We now study the sticky Black–Scholes model in terms of (NA) and (NFLVR). We first prove that (NFLVR) holds when $r = 0$, by finding an (ELMM) for the discounted price $\widetilde{S} = S/S^0$. We then prove that (NA), and hence (NFLVR), do not hold when $r \neq 0$, by establishing an arbitrage strategy.

We first elaborate on the dynamic of the discounted risky asset. From (1) and the Itô formula [42, Theorem IV.3.3],

$$\mathrm{d}\widetilde{S}_t = \mathrm{d}(\mathrm{e}^{-rt} S_t) = \mathrm{e}^{-rt} \, \mathrm{d}S_t - r\mathrm{e}^{-rt} S_t \, \mathrm{d}t = \big(\mu \mathbf{1}_{S_t \neq \zeta} - r\big)\widetilde{S}_t \, \mathrm{d}t + \sigma \widetilde{S}_t \mathbf{1}_{S_t \neq \zeta} \, \mathrm{d}B_t. \tag{4}$$

This qualifies $\widetilde{S}$ as a semimartingale.

Proposition 1. *We consider the sticky Black–Scholes model (S, S^0), as defined in Section 2, defined on the probability space $\mathcal{P}_x = (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ such that, $\mathbb{P}_x$-a.s., $S_0 = x$. If $r = 0$, then $S = \widetilde{S}$ and the following hold:*

(i) *The measure $\mathbb{Q}_x$ defined by $d\mathbb{Q}_x = \mathcal{E}(\theta)_T \, \mathrm{d}\mathbb{P}_x$, where*

$$\mathcal{E}(\theta)_T = \exp\left(\int_0^T \theta_s \, \mathrm{d}B_s - \frac{1}{2}\int_0^T \theta_s^2 \, \mathrm{d}s\right) \text{ and } \theta_t = \frac{\mu}{\sigma}, \quad t \geq 0, \tag{5}$$

is an (ELMM) *for S.*

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(ii) *The process S solves*

$$\begin{cases} \mathrm{d}S_t = \sigma S_t \mathbf{1}_{S_t \neq \zeta}\, \mathrm{d}\widetilde{B}_t, \\ \mathbf{1}_{S_t=\zeta}\, \mathrm{d}t = \rho\, \mathrm{d}L_t^{\zeta}(S), \end{cases} \tag{6}$$

where $\widetilde{B} = (B_t - (\mu/\sigma)t)_{t\geq 0}$ is a $\mathbb{Q}_x$-standard Brownian motion.

(iii) *The process S is a martingale under $\mathbb{Q}_x$.*

Proof. Since $r = 0$, $\widetilde{S} = S$. From the Girsanov theorem of [4, Lemma 4.4], S solves (6) under $\mathbb{Q}_x$. This proves (i) and (ii).

The process S is a local martingale under $\mathbb{Q}_x$ as the stochastic integral of a predictable process with respect to a standard Brownian motion. From [42, Chapter IV, Corollary 1.25], to prove that S is a martingale it suffices to show that $\mathbb{E}^{\mathbb{Q}_x}(\langle S\rangle_t) < \infty$. Indeed, from (3), we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_x}(\langle S\rangle_t) &= \mathbb{E}^{\mathbb{Q}_x}\left(\int_0^t \sigma^2 S_s^2 \mathbf{1}_{S_s\neq\zeta}\, \mathrm{d}s\right) \\ &\leq \sigma^2 \int_0^t \mathbb{E}^{\mathbb{Q}_x}\left(S_s^2\right) \mathrm{d}s \\ &= \sigma^2 S_0^2 \int_0^t \mathbb{E}^{\mathbb{Q}_x}\left(\exp\left(-\sigma^2 \int_0^s \mathbf{1}_{S_u\neq\zeta'}\, \mathrm{d}u + 2\sigma \int_0^s \mathbf{1}_{S_u\neq\zeta'}\, \mathrm{d}B_u\right)\right) \mathrm{d}s, \end{aligned}$$

where, from [42, Corollary VIII.1.16], the quantity under the expectancy is a martingale. Hence, for all $t \geq 0$, $\mathbb{E}^{\mathbb{Q}_x}(\langle S\rangle_t) \leq \sigma^2 S_0^2 t$, which is finite. This proves (iii), which completes the proof. □

Theorem 3. *The sticky Black–Scholes model satisfies* (NA) *and* (NFLVR) *if and only if $r = 0$.*

Proof. We recall that we work under Standing Assumption 1. From Theorem 1 and Proposition 1, if $r = 0$ both (NA) and (NFLVR) are satisfied. From Theorem 1, it thus suffices to prove the existence of an arbitrage strategy when $r \neq 0$. Inspired by the proof of [25, Theorem 12.3.5], we construct an arbitrage strategy.

We consider the trading strategy H defined by

$$H_t = \operatorname{sgn}(-r)\mathrm{e}^{rt}\mathbf{1}_{S_t=\zeta}, \quad t > 0. \tag{7}$$

We now show that, for all $t \geq 0$, $\int_0^t H_s\, \mathrm{d}\widetilde{S}_s \geq 0$, and that there exists some $t > 0$ such that $\mathbb{P}\big(\int_0^t H_s\, \mathrm{d}\widetilde{S}_s > 0\big) > 0$. From (1), (4), and (7),

$$\int_0^t H_s\, \mathrm{d}\widetilde{S}_s = \int_0^t H_s \mathbf{1}_{S_s=\zeta}\, \mathrm{d}\widetilde{S}_s = -r\zeta \int_0^t H_s \mathrm{e}^{-rs} \mathbf{1}_{S_s=\zeta}\, \mathrm{d}s = |r|\zeta \int_0^t \mathbf{1}_{S_s=\zeta}\, \mathrm{d}s = |r|\rho\zeta L_t^0(S).$$

We observe that, for all $t \geq 0$, $\int_0^t H_s\, \mathrm{d}\widetilde{S}_s \geq 0$, i.e. $H \in \mathcal{A}_0$. From [38, Corollary 29.18] and [15, Theorem 1.1], $\mathbb{P}_x\big(\int_0^t H_s\, \mathrm{d}\widetilde{S}_s > 0\big) = \mathbb{P}_x(L_t^0(S) > 0) = \mathbb{P}_x(\inf\{s > 0 \colon S_s = \zeta\} < t) > 0$. These qualify H as an arbitrage strategy. This completes the proof. □

Example 2. (*Standard Black–Scholes ($\rho = 0$); continuation of Example 1.*) In the standard Black–Scholes model the discounted price process has dynamic $\mathrm{d}\widetilde{S}_t = (\mu - r)\widetilde{S}_t\, \mathrm{d}t + \sigma \widetilde{S}_t\, \mathrm{d}B_t$. An (ELMM) for $\widetilde{S}$ is the measure $\mathbb{Q}_x$ defined by $\mathrm{d}\mathbb{Q}_x = \mathcal{E}(\theta)_T\, \mathrm{d}\mathbb{P}_x$, where

$$\mathcal{E}(\theta)_T = \exp\left(\int_0^T \frac{\mu - r}{\sigma}\, \mathrm{d}B_s - \frac{1}{2}\int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 \mathrm{d}s\right).$$

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Hence, (NFLVR) is satisfied.

We note also that in the standard Black–Scholes model, if $A + M$ is the Doob–Meyer decomposition of the discounted price process $\widetilde{S}$, then $\mathrm{d}A_t \ll \mathrm{d}\langle M\rangle_t$. Indeed, for all $t \geq 0$,

$$A_t = \int_0^t (\mu - r)\widetilde{S}_s \,\mathrm{d}s, \qquad M_t = \int_0^t \sigma \widetilde{S}_s \,\mathrm{d}B_s.$$

Remark 2. From Proposition 1, we note that the price process under the (ELMM) $\mathbb{Q}_x$ is a driftless sticky geometric Brownian motion, sticky at $\zeta > 0$, of stickiness parameter ρ. Therefore, under $\mathbb{Q}_x$ the process S is also a diffusion on $(0, \infty)$. This also implies that stickiness persists under the (ELMM) $\mathbb{Q}_x$.

Remark 3. In the sticky Black–Scholes model ($\rho > 0$), there is no uniqueness of (ELMM) for S. Indeed, we observe that taking any θ in (5) such that, for all $t \geq 0$,

$$\int_0^t \theta_s \mathbf{1}_{S_s \neq \zeta} \,\mathrm{d}s = \int_0^t \frac{\mu}{\sigma} \mathbf{1}_{S_s \neq \zeta} \,\mathrm{d}s,$$

also results in an equivalent local martingale measure $\mathbb{Q}^\theta$ for S. In this case, the process S solves

$$\begin{cases} \mathrm{d}S_t = \sigma S_t \mathbf{1}_{S_t \neq \zeta} \,\mathrm{d}B_t^\theta, \\ \mathbf{1}_{S_t = \zeta} \,\mathrm{d}t = \rho \,\mathrm{d}L_t^\zeta(S), \end{cases}$$

where $\widetilde{B}^\theta = (B_t - \int_0^t \theta_s \,\mathrm{d}s)_{t \geq 0}$ is a $\mathbb{Q}^\theta$-Brownian motion.

Remark 4. The arbitrage strategy constructed in the proof of Theorem 3 is highly erratic: it is supported on $\{t \in [0, T] \colon S_t = \zeta\}$. It is therefore not a simple arbitrage (an arbitrage that is an elementary strategy). In an upcoming paper with David Criens and Mikhail Urusov we prove that there is no simple arbitrage strategy in this model for any $r \in \mathbb{R}$. This leads to two key implications. First, arbitrage strategies are practically unattainable as they require an infinite number of transactions. Second, by Theorem 1, the presence of such an unattainable arbitrage precludes the existence of an (ELMM) for $\widetilde{S}$ and of an arbitrage-free pricing framework.

4. Pricing and hedging

This section is dedicated to the pricing and hedging problem in the sticky Black–Scholes model. We first prove the pricing equation and deduce pricing and hedging expressions. These require the existence of a unique classical solution to the pricing problem. Next, we apply the martingale representation theorem to prove, under an integrability condition, the existence of a replication strategy and the representation of the claim's price as the expectancy of the payoff under the (ELMM). In our analysis, we observe another distinctive feature of the sticky model: the non-uniqueness of the replication strategy.

These derivations are only valid for the case $r = 0$. When $r \neq 0$, the (NFLVR) condition is not met and, as per Theorem 1, there is no (ELMM) for the discounted price process. In this case, this pricing framework does not apply.

We recall the concept of the arbitrage pricing principle.

Definition 4 (*[37, Section 2.1.9].*) The arbitrage pricing principle, or (APP), refers to the concept that the price of a contingent claim is determined in such a way that no party involved can achieve an arbitrage.

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For the remainder of this section, we consider the sticky Black–Scholes model defined in Section 2 with $r = 0$, defined on the probability space $\mathcal{P}_x = (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ so that $S_0 = x$, $\mathbb{P}_x$-almost surely. Let $\mathbb{Q}_x$ be the (ELMM) for S derived in Proposition 1.

We also consider the notation of Markovian families used in [31] for the diffusion S under the equivalent local martingale measure $\mathbb{Q}_x$. Let $(s_{\mathbb{Q}}, m_{\mathbb{Q}})$ be the scale and speed of S under $\mathbb{Q}_x$. We denote by $\left(Q_x^{S,\mathbb{Q}},\ x \in J\right)$ the canonical diffusion, of scale and speed $(s_{\mathbb{Q}}, m_{\mathbb{Q}})$, on the path-space $C([0, \infty), \mathbb{R})$; Y is the coordinate process. Hence, for all almost surely finite stopping times τ, we have $Q_{X_\tau}^{S,\mathbb{Q}}(h(Y_t)) = \mathbb{E}^{\mathbb{Q}_x}(h(X_{\tau+t}) \mid \mathcal{F}_\tau)$.

4.1. Pricing equation

We consider the generic pricing problem

$$\begin{cases} v_t = -\frac{\sigma^2 x^2}{2} v_{xx} & \text{for all } (t, x) \in [0, T) \times (0, \infty), \\ \frac{\sigma^2 \zeta^2}{2} v_{xx}(t, \zeta-) = \frac{\sigma^2 \zeta^2}{2} v_{xx}(t, \zeta+) = \frac{1}{2\rho}(v_x(t, \zeta+) - v_x(t, \zeta-)) & \text{for all } t \in [0, T), \\ v(T, x) = h(x) & \text{for all } x \in (0, \infty). \end{cases} \tag{8}$$

We now define the space of classical solutions of (8). This is justified by Remark 6.

For I an interval of $[0, \infty)$ and L an interval of $(0, \infty)$, we consider the spaces $C^L(I)$ and $C^{1,L}(I \times L)$ defined by

$$C^L(L) := \left\{ f \in C(L) \cap C^2(L) \colon \frac{\sigma^2 \zeta^2}{2} f''(\zeta-) = \frac{\sigma^2 \zeta^2}{2} f''(\zeta-) = \frac{1}{2\rho}(f'(\zeta+) - f'(\zeta-)) \right\},$$
$$C^{1,L}(I \times L) := \left\{ u \in C(I \times L) \colon \text{for all } x \in L,\ u(\cdot, x) \in C^1(I); \text{for all } t \in I,\ u(t, \cdot) \in C^L(L) \right\},$$

where all the derivatives considered are right derivatives. We can now define the notion of a classical solution of (8).

Definition 5. A function v is said to be a classical solution of (8) if $v \in C^{1,L}([0, T) \times (0, \infty))$ and it solves (8).

We note that the space $C^L((0, \infty))$ is comprised of functions that are differences of convex functions. Indeed, if $f \in C^L((0, \infty))$, the function ϕ defined by

$$\phi(x) = f(x) - \frac{\rho(\zeta\sigma)^2}{2} f''(\zeta)(x - \zeta)_+, \quad x \in (0, \infty),$$

is $C^2((0, \infty))$. It suffices to observe that the regularity of ϕ ensures it is difference of convex functions, and that $[x \to (x - \zeta)_+]$ is convex. This justifies the use of the Itô–Tanaka formula [42, Theorem VI.1.5] on any $f \in C^L((0, \infty))$.

Theorem 4. *Let h be a measurable function such that (8) has a unique classical solution v. Then the following hold:*

(i) *The* (APP) *price at $t \in [0, T]$ of the contingent claim with payoff $h(S_T)$ is $v(t, S_t)$.*

(ii) *A replication strategy for $h(S_T)$ is $\delta = (v_x(t, S_t); t \in [0, T])$.*

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Proof. Let v be the unique classical solution of (8). Since, for all $t \in [0, T)$, $v(t, \cdot) \in C^L((0, \infty))$, from the Itô–Tanaka formula,

$$\begin{aligned} \mathrm{d}v(t, S_t) &= v_t(t, S_t)\,\mathrm{d}t + v_x(t, S_t)\,\mathrm{d}S_t \\ &\quad + \frac{1}{2} v_{xx}(t, S_t)\,\mathrm{d}\langle S\rangle_t + \frac{1}{2}(v_x(t, \zeta+) - v_x(t, \zeta-))\,\mathrm{d}L_t^\zeta(S). \end{aligned}$$

From (6), $\mathrm{d}\langle S\rangle_t = \sigma^2 S_t^2 \mathbf{1}_{S_t \neq \zeta}\,\mathrm{d}t$ and $\mathbf{1}_{S_t=\zeta}\,\mathrm{d}t = \rho\,\mathrm{d}L_t^\zeta(S)$. This, combined with (8), yields

$$\frac{1}{2}(v_x(t, \zeta+) - v_x(t, \zeta-))\,\mathrm{d}L_t^\zeta(S) = \frac{\rho\sigma^2\zeta^2}{4} v_{xx}(t, \zeta)\,\mathrm{d}L_t^\zeta(S) = -v_t(t, \zeta)\mathbf{1}_{S_t=\zeta}\,\mathrm{d}t.$$

Consequently,

$$\begin{aligned} \mathrm{d}v(t, S_t) &= v_t(t, S_t)\mathbf{1}_{S_t\neq\zeta}\,\mathrm{d}t + v_x(t, S_t)\,\mathrm{d}S_t + \frac{\sigma^2 S_t^2}{2} v_{xx}(t, S_t)\mathbf{1}_{S_t\neq\zeta}\,\mathrm{d}t \\ &\quad + v_t(t, S_t)\mathbf{1}_{S_t=\zeta}\,\mathrm{d}t + \frac{1}{2}(v_x(t, \zeta+) - v_x(t, \zeta-))\,\mathrm{d}L_t^\zeta(S) \\ &= v_x(t, S_t)\,\mathrm{d}S_t + \left(v_t(t, S_t) + \frac{\sigma^2 S_t^2}{2} v_{xx}(t, S_t)\right)\mathbf{1}_{S_t\neq\zeta}\,\mathrm{d}t. \end{aligned}$$

Integrating in time yields $h(S_T) = v(t, S_t) + \int_t^T v_x(t, S_s)\,\mathrm{d}S_s$. This proves the two assertions. □

Remark 5. The replication strategy $\delta = (v_x(t, S_t); t \in [0, T])$ in the sticky Black–Scholes model is typically discontinuous at the sticky threshold ζ. Indeed, from (8), if $v_{xx}(t, \zeta) \neq 0$, then $v_x(t, \zeta+) - v_x(t, \zeta-) \neq 0$.

For our next remark, we recall some notation. The semi-group of a diffusion is the family of operators $(P_t)_{t\geq 0}$ defined by $P_t f(x) := \mathbb{E}(f(S_t) \mid S_0 = x)$ for all $f \in C_{\mathrm{b}}((0, \infty))$, $t \geq 0$, $x \in (0, \infty)$. We note that S is a Feller process [19, Theorem 1], i.e. it is strong Markov and its semi-group $(P_t)_{t\geq 0}$ is internal on $C_0((0, \infty))$ and continuous at 0, i.e.

- for all $f \in C_0((0, \infty))$, $P_t f \in C_0((0, \infty))$;
- for all $f \in C_0((0, \infty))$ and $x \in (0, \infty)$, $\lim_{t\to 0} P_t f(x) = f(x)$.

Its semi-group is therefore generated by the operator

$$Lf = \frac{(x\sigma)^2}{2} f'', \quad \mathrm{dom}_{C_0}(L) = C_0((0, \infty)) \cap C^L((0, \infty)).$$

Remark 6. We note that if $h \in \mathrm{dom}_{C_0}(L)$, from the Kolmogorov equations [42, Proposition VII.1.2], the problem (8) has a unique classical solution $v \in C^{1,L}([0, T] \times (0, \infty))$ that admits the representation, for all $(t, x) \in [0, T] \times (0, \infty)$, $v(t, x) = Q_x^{S,\mathbb{Q}}(h(Y_{T-t}))$.

4.2. Representation

In Remark 6, we observed that if $h \in \mathrm{dom}_{C_0}(L)$, then the pricing equation has a unique classical solution, the price function has a representation in terms of an expectancy, and a replication strategy exists. In this section we prove that the two latter assertions are part of a more general principle, which is a consequence of the martingale representation property [21, Theorem 2.7].

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Our motivation is that the condition $h \in \mathrm{dom}_{C_0}(L)$ is quite restrictive. Indeed, classical claims like calls and puts have payoff functions that are neither in $C_0((0,\infty))$ nor in $C^L((0,\infty))$. This holds true even for the standard Black–Scholes model, where $\mathrm{dom}_{C_0}(L) = C_0((0,\infty)) \cap C^2((0,\infty))$. Nevertheless, as depicted in Section 6.1, we obtain satisfactory numerical results for a call payoff in the sticky and the non-sticky model.

Theorem 5. *For all* $h \in \mathsf{H}(S_T) := \{g \text{ measurable}\colon g(S_T) \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q}_x)\}$,

(i) *there exists a trading strategy* δ *in* $L^2_{\mathrm{loc}}(S)$ *(for a definition of* $L^2_{\mathrm{loc}}(S)$*, see [36, I.4.39]), such that, for all* $t \in [0,T]$,

$$\begin{cases} \mathbb{E}^{\mathbb{Q}_x}(h(S_T) \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}_x}(h(S_T)) + \int_0^t \delta_s \,\mathrm{d}S_s, \\ h(S_T) = \mathbb{E}^{\mathbb{Q}_x}(h(S_T) \mid \mathcal{F}_t) + \int_t^T \delta_s \,\mathrm{d}S_s; \end{cases} \tag{9}$$

(ii) *for all* $t \in [0,T]$*, the* (APP) *price of the claim of terminal payoff* $h(S_T)$ *at* t *is* $V_t := \mathbb{E}^{\mathbb{Q}_x}(h(S_T) \mid \mathcal{F}_t) = Q_{S_t}^{S,\mathbb{Q}}(h(Y_{T-t}))$.

Proof. Regarding (i), since $h(S_T) \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q}_x)$, the process M defined as $M_t := \mathbb{E}^{\mathbb{Q}_x}(h(S_T) \mid \mathcal{F}_t)$, $t \in [0,T]$, is a uniformly integrable martingale. From the Markov property of S under $\mathbb{Q}_x$ (see Remark 2), we have that $\mathbb{E}^{\mathbb{Q}}(h(S_T) \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(h(S_T) \mid \sigma(S_t))$. Hence, M is adapted to the canonical filtration of S. Since S is a local martingale, from [21, Theorem 2.7], there exists a predictable process $\delta \in L^2_{\mathrm{loc}}(S)$ such that $M_t = M_0 + \int_0^t \delta_s \,\mathrm{d}S_s$. This proves the first equation of (i). Regarding the second equation, we consider (9) for different values of t. Substracting them results in $h(S_T) = \mathbb{E}^{\mathbb{Q}_x}(h(S_T) \mid \mathcal{F}_{t_0}) - \int_{t_0}^T \delta_s \,\mathrm{d}S_s$.

Regarding (ii), from the (APP) (see Definition 4) and the payoff replication relations (9), the price of the claim at t is necessarily $\mathbb{E}^{\mathbb{Q}_x}(h(S_T) \mid \mathcal{F}_t)$. Any other value would result in an arbitrage opportunity for one of the two parties. This completes the proof. □

Remark 7. We note that in the sticky Black–Scholes model ($\rho > 0$) we have no uniqueness of the payoff replication strategy. Indeed, let δ be a payoff replication strategy and δ' a strategy such that $\delta \neq \delta'$ and, for all $t \geq 0$, $\delta_t \mathbf{1}_{S_t \neq \zeta} = \delta'_t \mathbf{1}_{S_t \neq \zeta}$. Then, we have

$$\mathbb{E}^{\mathbb{Q}_x}(h(S_T) \mid \mathcal{F}_t) - \mathbb{E}^{\mathbb{Q}_x}(h(S_T)) = \int_0^t \delta_s \,\mathrm{d}S_s = \int_0^t \delta'_s \,\mathrm{d}S_s.$$

Corollary 1. *Let* $h \in \mathsf{H}(S_T)$ *such that (8) has a unique classical solution* v*. Then, from the* (APP)*, for all* $(t,x) \in [0,T] \times (0,\infty)$*,* $v(t,x) = Q_x^{S,\mathbb{Q}}(h(Y_{T-t}))$*, and if* δ *is a replication strategy for* $h(S_T)$*, then, for all* $t \in [0,T]$*,* $v_x(t,S_t)\mathbf{1}_{S_t \neq \delta} = \delta_t \mathbf{1}_{S_t \neq \delta}$ *and* $(v_x(t,S_t); t \in [0,T]) \in L^2_{\mathrm{loc}}(S)$.

Example 3. (*Standard Black–Scholes* ($\rho = 0$)*; continuation of Examples 1 and 2.*) For the standard Black–Scholes model ($\rho = 0$), the risky asset and discounted risky asset price dynamics $S, \widetilde{S}$ under $\mathbb{Q}_x$ are $\mathrm{d}S_t = S_t(r\,\mathrm{d}t + \sigma\,\mathrm{d}B_t)$ and $\mathrm{d}\widetilde{S}_t = \sigma \widetilde{S}_t \,\mathrm{d}B_t$. The pricing equation [47, Theorems 6.4.3 and 7.3.1] is

$$\begin{cases} v_t(t,x) + rxv_x(t,x) + \frac{\sigma^2 x^2}{2} v_{xx}(t,x) = rv(t,x) \quad \text{for all } (t,x) \in (0,T) \times (0,\infty), \\ v(T,x) = h(x). \end{cases} \tag{10}$$

If $h \in \mathsf{H}(S_T)$, there exists a replication strategy in L^2 and the price at $t \in [0,T]$ of the contingent claim of payoff at maturity $h(S_T)$ is given by $V_t := \mathbb{E}^{\mathbb{Q}_x}(\mathrm{e}^{-r(T-t)} h(S_T) \mid \mathcal{F}_t)$. If (10) has a unique classical solution v, the (APP) price of the contingent claim at $t \in [0,T]$ is $V_t = v(t,S_t)$, and the replication strategy is the process δ defined [47, Remark 7.3.3] by $\delta_t := v_x(t,S_t)$, $t \in [0,T]$.

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5. Price monotonicity and payoff inclusion

In this section we prove price monotonicity and inclusion properties for a class of replicable payoffs with respect to the stickiness parameter.

We first prove that higher stickiness leads to lower prices for European derivatives with convex payoff functions. We deduce the respective inclusion on the subset of convex positive replicable payoffs. Second, we generalize the latter and prove that within the class of locally bounded ($L^{\infty}_{\mathrm{loc}}$) payoffs, the same inclusion holds. The proof relies on fine properties of the transition kernel. These inclusions justify the use of classical payoffs in the sticky Black–Scholes model.

We first define the following objects. For each $\rho \geq 0$, let $S^{(\rho)}$ be the solution of (1) and $\mathsf{H}(S_T^{(\rho)})$ be the class of replicable payoff functions on $S_T^{(\rho)}$ given by

$$\mathsf{H}\big(S_T^{(\rho)}\big) := \big\{h \text{ measurable}\colon h\big(S_T^{(\rho)}\big) \in L^1(\Omega, \mathcal{F}_T, \mathbb{Q}_x)\big\}.$$

Proposition 2. *(Price monotonicity.) Let* $\mathrm{Opt}_h(S^{(\rho)}, x, T)$ *be the* (APP) *price at* $t=0$ *of the contingent claim on the underlying asset* $S^{(\rho)}$ *with payoff function h, maturity* $T>0$*, and spot price* $S_0^{(\rho)} = x$*, i.e.* $\mathrm{Opt}_h(S^{(\rho)}, x, T) = \mathbb{E}^{\mathbb{Q}_x}\big(h(S_T^{(\rho)})\big)$*. If h is convex, then*

(i) *for all* $x, T>0$*, the application* $[\rho \to \mathrm{Opt}_h(S^{(\rho)}, x, T)]$ *is decreasing;*

(ii) *for all* $x>0$ *and* $\rho \geq 0$*, the application* $[T \to \mathrm{Opt}_h(S^{(\rho)}, x, T)]$ *is increasing.*

The proof relies on the following result, which allows a sticky diffusion to be expressed as a time change of a non-sticky diffusion. Although this result can be inferred from the proof of [3, Proposition 2.1.17], we provide a more detailed proof in Appendix A.1.

Lemma 1. *Let X be a regular diffusion on J, an open interval of* $\mathbb{R}$*, on a natural scale of speed measure m so that* $\rho := m(\{\zeta\}) \in (0, \infty)$*. We suppose X to be defined on the probability space* $\mathcal{P}_x = (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$*. There exists a regular diffusion Z on J, on a natural scale, with speed measure* $m_Z = m - \rho\delta_\zeta$*, defined on an extension of* $\mathcal{P}_x$*, such that, for all* $t \geq 0$*,* $X_t = Z_{\gamma_\rho(t)}$ *and* $A_\rho(t) = t + \rho L_t^\zeta(Z)$*, where* γ_ρ *is the right inverse of* A_ρ*.*

Proof of Proposition 2. Let $S^{(0)}$ be defined on the probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ such that $S^{(0)} = x$, $\mathbb{P}_x$-almost surely. Let $\mathbb{Q}_x$ be the (ELMM) for $S^{(0)}$ defined in Proposition 1. From Theorem 5(ii), $\mathrm{Opt}_h(S^{(0)}, x, T) = \mathbb{E}^{\mathbb{Q}_x}\big(h\big(S_T^{(0)}\big)\big)$. For all $\rho \geq 0$, let A_ρ be the random time change defined by $A_\rho(t) = t + \rho L_t^\zeta(S^{(0)})$, and let γ_ρ be its right inverse. We observe that

- if $\rho' < \rho$, then $0 \leq \gamma_\rho(T) \leq \gamma_{\rho'}(T) \leq T$;
- from Lemma 1, $\big(S^{(0)}_{\gamma_\rho(t)}\big)_{t\geq 0} = \big(S_t^{(\rho)}\big)_{t\geq 0}$ in law;
- γ_ρ^{-1} and γ_ρ are stopping times;
- since $S^{(0)}$ is a $\mathbb{Q}_x$-martingale (Proposition 1(iii), $\big(h\big(S_t^{(0)}\big)\big)_{t\geq 0}$ is a $\mathbb{Q}_x$-submartingale.

Thus, from the stopping theorem, if $\rho' < \rho$,

$$\begin{aligned}\mathrm{Opt}_h(S^{(\rho')}, x, T) = \mathbb{E}^{\mathbb{Q}_x}\big(h\big(S_T^{(\rho')}\big)\big) = \mathbb{E}^{\mathbb{Q}_x}\big(h\big(S^{(0)}_{\gamma_{\rho'}(T)}\big)\big) &\leq \mathbb{E}^{\mathbb{Q}_x}\big(h\big(S^{(0)}_{\gamma_\rho(T)}\big)\big)\\ &= \mathbb{E}^{\mathbb{Q}}_x\big(h\big(S_T^{(\rho)}\big)\big) = \mathrm{Opt}_h(S^{(\rho)}, x, T).\end{aligned}$$

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This proves (i). With similar arguments, if $0 \le T \le T'$,

$$\mathrm{Opt}_h(S^{(\rho)}, x, T) = \mathbb{E}^{\mathbb{Q}_x}\big(h\big(S_T^{(\rho)}\big)\big) \le \mathbb{E}^{\mathbb{Q}_x}\big(h\big(S_{T'}^{(\rho)}\big)\big) = \mathrm{Opt}_h(S^{(\rho)}, x, T').$$

This proves (ii), which completes the proof. □

Remark 8. Lemma 1 yields that if C^+ is the cone of positive convex functions, then

$$\mathsf{H}\big(S_T^{(\rho)}\big) \cap C^+ \subset \mathsf{H}\big(S_T^{(\rho')}\big) \cap C^+ \quad \text{for all } 0 \le \rho < \rho'.$$

In particular, this holds for call and put payoff functions.

The next proposition goes beyond this observation.

Proposition 3. *Let $L^\infty_{\mathrm{loc}}((0,\infty))$ be the family of locally bounded functions on $(0,\infty)$, i.e. the functions that are bounded on all compacts of $(0,\infty)$. For all $0 \le \rho < \rho'$,*

$$\mathsf{H}\big(S_T^{(\rho)}\big) \cap L^\infty_{\mathrm{loc}}((0,\infty)) \subset \mathsf{H}\big(S_T^{(\rho')}\big) \cap L^\infty_{\mathrm{loc}}((0,\infty)).$$

For this, we use the following lemma. For a proof, we refer the reader to Appendix A.2.

Lemma 2. *Let $[(t,x,y) \to p_0(t,x,y)]$ and $m_0(dy) = \dot m_0(y)\,\mathrm{d}y$ be the probability transition kernel and speed measure of a geometric Brownian motion. For all $x > 0$ and $T > 0$, there exists a compact $K_{x,T}$ such that, on $(t,y) \in [0,T] \times K^{\mathrm{c}}_{x,T}$, $[t \to p_0(t,x,y)\dot m_0(y)]$ is increasing.*

Proof of Proposition 3. For all $\rho \ge 0$, let m_ρ be the speed measure and $[(t,x,y) \to p_\rho(t,x,y)]$ the probability transition kernel of the process $S^{(\rho)}$. Let γ_ρ be the right inverse of the time change $A_\rho(t) = t + \rho L_t^{(0)}(S)$, $t \ge 0$. This yields in particular that, for all $t \ge 0$, $\gamma_\rho(t) \le t$.

From Lemma 1, since $S^{(\rho)} = \big(S^{(0)}_{\gamma_\rho(t)}\big)_{t\ge 0}$ in law, for all positive measurable functions f,

$$\mathbb{E}^{\mathbb{Q}_x}\big(f\big(S_T^{(\rho)}\big)\big) = \mathbb{E}^{\mathbb{Q}_x}\big(f\big(S^{(0)}_{\gamma_\rho(T)}\big)\big) = \mathbb{E}^{\mathbb{Q}_x}\bigg(\int_{(0,\infty)} f(y) p_0(\gamma_\rho(T), x, y)\dot m_0(y)\,\mathrm{d}y\bigg). \tag{11}$$

Let $K_{x,T}$ be some compact set of $(0,\infty)$ inferred from Lemma 2. Hence, from (11),

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_x}\big(\big|h\big(S_T^{(\rho)}\big)\big|\big) &= \mathbb{E}^{\mathbb{Q}_x}\big(\mathbf{1}_{S_T^{(\rho)} \in K_{x,T}}\big|h\big(S_T^{(\rho)}\big)\big|\big) + \mathbb{E}^{\mathbb{Q}_x}\big(\mathbf{1}_{S_T^{(\rho)} \in K^{\mathrm{c}}_{x,T}}\big|h\big(S_T^{(\rho)}\big)\big|\big) \\
&\le \big\|\mathbf{1}_{K_{x,T}} h\big\|_\infty + \mathbb{E}^{\mathbb{Q}_x}\big(\mathbf{1}_{S^{(0)}_{\gamma_\rho(T)} \in K^{\mathrm{c}}_{x,T}}\big|h\big(S^{(0)}_{\gamma_\rho(T)}\big)\big|\big) \\
&\le \big\|\mathbf{1}_{K_{x,T}} h\big\|_\infty + \mathbb{E}^{\mathbb{Q}_x}\bigg(\int_0^\infty \mathbf{1}_{y \in K^{\mathrm{c}}_{x,T}} |h(y)| p_0(\gamma_\rho(T), x, y)\dot m_0(y)\,\mathrm{d}y\bigg).
\end{aligned}$$

Let $0 \le \rho < \rho'$. We recall that, for all $T \ge 0$ and $t \in [0,T]$, $0 \le \gamma_{\rho'}(t) \le \gamma_\rho(t) \le T$. Hence, from the definition of $K_{x,T}$,

$$\mathbb{E}^{\mathbb{Q}_x}\bigg(\int_{K^{\mathrm{c}}_{x,T}} |h(y)| p_0(\gamma_{\rho'}(T), x, y)\dot m_0(y)\,\mathrm{d}y\bigg) \le \mathbb{E}^{\mathbb{Q}_x}\bigg(\int_{K^{\mathrm{c}}_{x,T}} |h(y)| p_0(\gamma_\rho(T), x, y)\dot m_0(y)\,\mathrm{d}y\bigg).$$

Since h is locally bounded, $\|\mathbf{1}_{K_{x,T}} h\|_\infty < \infty$. These prove the inclusion $\mathsf{H}\big(S_T^{(\rho)}\big) \cap L^\infty_{\mathrm{loc}}((0,\infty)) \subset \mathsf{H}\big(S_T^{(\rho')}\big) \cap L^\infty_{\mathrm{loc}}((0,\infty))$. This completes the proof. □

Remark 9. Proposition 3 ensures that the L^1 condition of Theorem 5 is met for all locally bounded payoff functions that are replicable in the standard Black–Scholes model, i.e. for all $h \in \mathsf{H}\big(S_T^{(0)}\big) \cap L^\infty_{\mathrm{loc}}((0,\infty))$.

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6. Numerical experiments

This section is dedicated to numerical experiments aimed at evaluating practical aspects of hedging a contingent claim on an asset with a sticky price dynamic. We investigate two key properties. The first property involves the hedging time granularity of the sticky Black–Scholes model, referred to as 'the impact on replication when the agent hedges in discrete time'. The second property is model mismatch, or 'the impact on replication when an agent ignores or misrepresents price stickiness'.

In the rest of this section we consider the asset dynamics (1) with $r = 0$, $\sigma = 0.25$, and various degrees of stickiness $\rho \in \{0, 1, 2\}$ at the threshold $\zeta = 10$. We recall that a stickiness value of $\rho = 0$ corresponds to the standard Black–Scholes dynamic. The claim of interest will be a call option with strike K that delivers $(S_T - K)_+$ at maturity $T = 10$.

The approximation scheme we use to simulate sample paths of the sticky geometric Brownian motion is the space–time Markov chain approximation (STMCA) of one-dimensional diffusions [5].

6.1. Hedging time granularity

The message of the Black–Scholes model is as follows: it is possible to replicate the payoff of a European option using a self-financing portfolio composed of quantities of the risky and non-risky assets. In particular, the option seller can hedge their risk by holding a portfolio that, at each time t, comprises $\delta_t = v_x(t, S_t)$ parts of the risky asset.

However, a practical challenge arises since achieving this requires continuous rebalancing of the portfolio composition over time. This, as mentioned in [10], proves impossible in real-world scenarios as it necessitates an infinite number of operations. One way to address this is to introduce a hedging window $h > 0$, resulting in hedging times $(t_i)_i$ such that $t_{i+1} - t_i = h$. The agent can then rebalance the portfolio at each time t_i so that, for all i, they hold $v_x(t_i, S_{t_i})$ parts of the risky asset on $[t_i, t_{i+1})$. The resulting discretized version of the continuous-time hedging strategy δ is the strategy $\delta^{(N)}$ defined for all t by $\delta_t^{(N)} = \delta_{[Nt]/N}$, where $[x] = \sup\{k \in \mathbb{N} : k \le x\}$.

The profits and losses (PnLs) at t of the strategies δ and $\delta^{(N)}$ are

$$V_t(\delta) - V_0(\delta) = \int_0^t \delta_s \, \mathrm{d}S_s, \qquad V_t(\delta^{(N)}) - V_0(\delta) = \int_0^t \delta_s^{(N)} \, \mathrm{d}S_s,$$

where, from Theorem 5, the option premium is $V_0(\delta) = \mathbb{E}^{\mathbb{Q}}(h(S_T))$ and $V_t(\delta) = \mathbb{E}^{\mathbb{Q}}(h(S_T) \mid \mathcal{F}_t) = Q_{S_t}^{S,\mathbb{Q}}(h(Y_{T-t}))$.

This discrete replication strategy, outlined in Algorithm 1, results in replication errors known as the *tracking error*, defined by $\varepsilon_t^{(N)} := V_t(\delta^{(N)}) - V_t(\delta)$. In [10], the metric used to assess the replication quality is the mean squared tracking error, defined by

$$\mathsf{R}_T^{(N)} := \sqrt{\mathbb{E}^{\mathbb{Q}}\big[\big(\varepsilon_T^{(N)}\big)^2\big]}.$$

Instead of this, we will quantify the replication properties of each model with the pair $\big(\mathbb{E}^{\mathbb{Q}}\big[\varepsilon_t^{(N)}\big], \big(\mathrm{Var}^{\mathbb{Q}}\big[\varepsilon_t^{(N)}\big]\big)^{1/2}\big)$, where $\mathbb{E}^{\mathbb{Q}}\big[\varepsilon_t^{(N)}\big]$ is the expected replication PnL, with the replication PnL standard deviation being $\big(\mathrm{Var}^{\mathbb{Q}}\big[\varepsilon_t^{(N)}\big]\big)^{1/2}$. The reason is that $\mathsf{R}_t^{(N)}$ is more adapted to the case where $\mathbb{E}^{\mathbb{Q}}\big[\varepsilon_t^{(N)}\big] = 0$. If an agent is unsure about the pricing/replication model they apply, they are likely to miscalculate the option premium and end up with non-zero PnL, making the metric $\mathsf{R}_t^{(N)}$ irrelevant. This is the object of Section 6.2. Another reason for this

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Algorithm 1: Delta hedging of an option
Data: Stock price time-series $(S_t)_{t\geq 0}$, strike price K, time to maturity T, volatility σ, stickiness ρ, hedging window Δt
Result: Deltas and portfolio values over time

Compute or approximate the price function $\texttt{u_price}(x, K, T, \sigma, \rho)$;
Compute or approximate the delta function $\texttt{u_delta}(x, K, T, \sigma, \rho)$;
Initialize $cash$ to $\texttt{u_price}(S_0, K, T, \sigma, \rho)$;
Initialize t to 0;
Initialize old_delta to 0;
while $t < T$ **do**
  Update time: $t \leftarrow t + \Delta t$;
  Retrieve current stock price S_t;
  Calculate current option value: $price \leftarrow \texttt{u_price}(S_t, K, T - t, \sigma, \rho)$;
  Calculate current option delta: $delta \leftarrow \texttt{u_delta}(S_t, K, T - t, \sigma, \rho)$;
  Update cash position: $cash \leftarrow cash - (delta - old_delta) \times S_t$;
  Set $old_delta \leftarrow delta$;
  Update portfolio value: $portfolio_value \leftarrow cash + (delta \times S_t) - price$;
  Append $delta$ to the list of deltas;
  Append $portfolio_value$ to the list of portfolio values;
return *List of deltas, list of portfolio values*;

choice is that the pair $\left(\mathbb{E}^{\mathbb{Q}}\left[\varepsilon_t^{(N)}\right], \left(\mathrm{Var}^{\mathbb{Q}}\left[\varepsilon_t^{(N)}\right]\right)^{1/2}\right)$ contains the information in $\mathsf{R}_t^{(N)}$. Indeed, the three quantities are linked by the relation $\mathrm{Var}^{\mathbb{Q}}\left[\varepsilon_t^{(N)}\right] = \left(\mathsf{R}_t^{(N)}\right)^2 - \left(\mathbb{E}^{\mathbb{Q}}\left[\varepsilon_t^{(N)}\right]\right)^2$.

We assess the discrete-time hedging error of the sticky Black–Scholes model as follows. We sample trajectories of the sticky geometric Brownian motion. Then, we apply Algorithm 1 to hedge a call option for various hedging windows. Last, we compute the empiric mean and standard deviation of the PnL with

$$\widehat{\mu}_{\mathrm{MC}}^{(N)} = \frac{1}{N_{\mathrm{MC}}} \sum_{i=1}^{N_{\mathrm{MC}}} \varepsilon_T^{(N)}(\omega_i), \qquad \widehat{\sigma}_{\mathrm{MC}}^{(N)} = \frac{1}{\sqrt{N_{\mathrm{MC}}}} \sqrt{\sum_{i=1}^{N_{\mathrm{MC}}} \left(\varepsilon_T^{(N)}(\omega_i) - \widehat{\mu}_{\mathrm{MC}}\right)^2}.$$

We have no closed-form expressions for the replication strategy and the (APP) prices of claims in the sticky Black–Scholes model. Therefore, we use a finite-difference approximation to numerically solve (8). The simulation results are summarized in Table 1.

From the numerical simulations, we observe the following. The replication errors in the sticky and non-sticky models decreases the higher the hedging frequency N is. Convergence in the sticky model seems to occur at the same rate as for the non-sticky one (standard Black–Scholes model). In [10], it is proved that the rate of convergence for the Black–Scholes model is $\mathcal{O}(N^{-1/2})$.

Also, higher stickiness leads to higher tracking errors for coarse hedging time-windows. We attribute this behavior to the inherent discontinuity of the delta function at the sticky threshold. As depicted in Figure 2, higher stickiness at ζ corresponds to more substantial jumps in the delta, resulting in sudden shifts in portfolio composition whenever the price process S crosses

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TABLE 1. Premiums and Monte Carlo estimation of the expected replication error (1000 simulated trajectories) for a call $(K, T) = (10, 10)$ for various stickiness and hedging windows. $N_{\mathrm{MC}} = 1000$.

Stickiness $\rho = 0$				Stickiness $\rho = 1$				Stickiness $\rho = 2$			
N	Premium	$\widehat{\mu}_{\mathrm{MC}}$	$\widehat{\sigma}^{(N)}_{\mathrm{MC}}$	N	Premium	$\widehat{\mu}_{\mathrm{MC}}$	$\widehat{\sigma}^{(N)}_{\mathrm{MC}}$	N	Premium	$\widehat{\mu}_{\mathrm{MC}}$	$\widehat{\sigma}^{(N)}_{\mathrm{MC}}$
2000	3.039 23	−0.034 38	0.062 51	2000	2.436 07	−0.024 26	0.087 83	2000	1.9976	−0.014 87	0.046 02
1000	3.039 23	−0.034 81	0.086 05	1000	2.436 07	−0.015 73	0.177 48	1000	1.9976	−0.015 02	0.184 74
250	3.039 23	−0.036 87	0.167 81	250	2.436 07	−0.019 59	0.294 77	250	1.9976	−0.034 55	0.381 81
100	3.039 23	−0.045 29	0.268 32	100	2.436 07	−0.020 13	0.401 77	100	1.9976	−0.037 31	0.518 32
10	3.039 23	−0.006 01	0.690 78	10	2.436 07	−0.001 82	0.809 82	10	1.9976	0.016 37	0.968 78

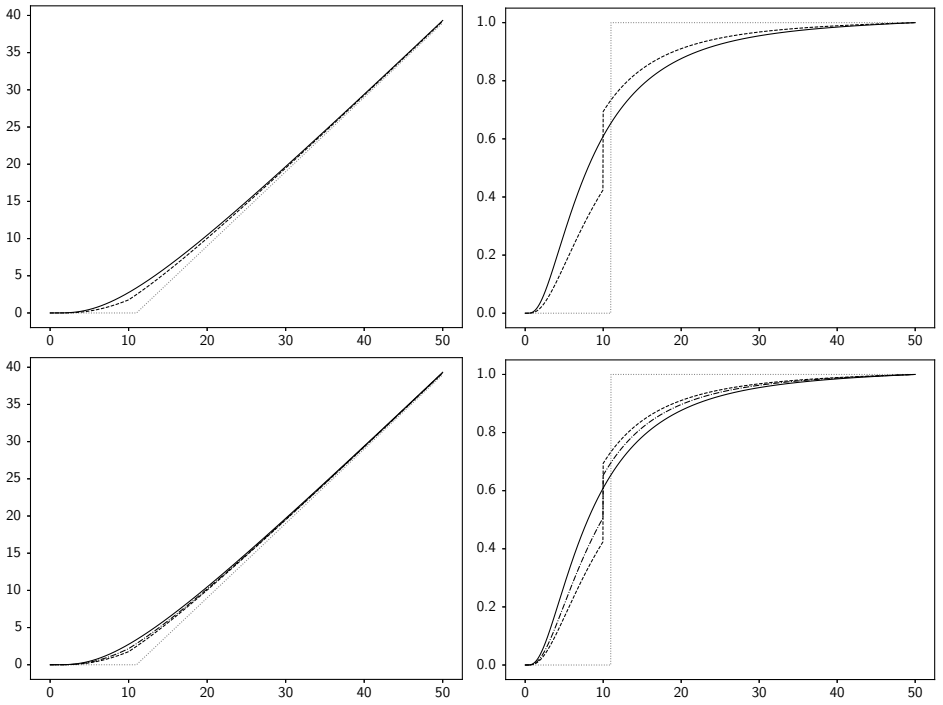


FIGURE 2. Finite difference approximations of the price (left) and delta (right) curves for a call option with strike $K = 11$ and maturity $T = 10$. The graphs compare prices and deltas for three models: the standard Black–Scholes model with $(r, \sigma) = (0, 0.25)$ (solid curves); the sticky Black–Scholes model with $(r, \sigma, \rho, \zeta) = (0, 0.25, 1.0, 10.0)$ (dot-dashed curves); and the sticky Black–Scholes model with $(r, \sigma, \rho, \zeta) = (0, 0.25, 2.0, 10.0)$ (dashed curves).

the sticky threshold. Consequently, the discrete-time hedging portfolio manifests substantial deviations in portfolio composition from the ideal configuration whenever the price crosses the sticky threshold. This is illustrated in Figure 3, where spikes and oscillations in the portfolio composition for fine-grained hedging do not appear in coarser regimes.

6.2. Model mismatch: Constant volatility

Now let's assume that an agent intends to hedge a contingent claim on an underlying asset with a sticky geometric Brownian motion price dynamic with parameters (σ, ρ, ζ). The agent considers the following models:

- Model 1: the standard Black–Scholes model with volatility σ.
- Model 2: the standard Black–Scholes model with volatility $\Sigma_n(X)$, where for some $T' > 0$, $\Sigma_n(X)$ is the estimator of the realized volatility on $[-T', 0)$ of the current sample path. The estimator is defined by

$$\Sigma_n(X) = \frac{1}{\sqrt{T'}} \sqrt{\sum_{i=1}^{[nT']} (\log S_{-i/n} - \log S_{-(i+1)/n})^2}.$$

- Model 3: the sticky Black–Scholes model with parameters (σ, ρ, ζ).

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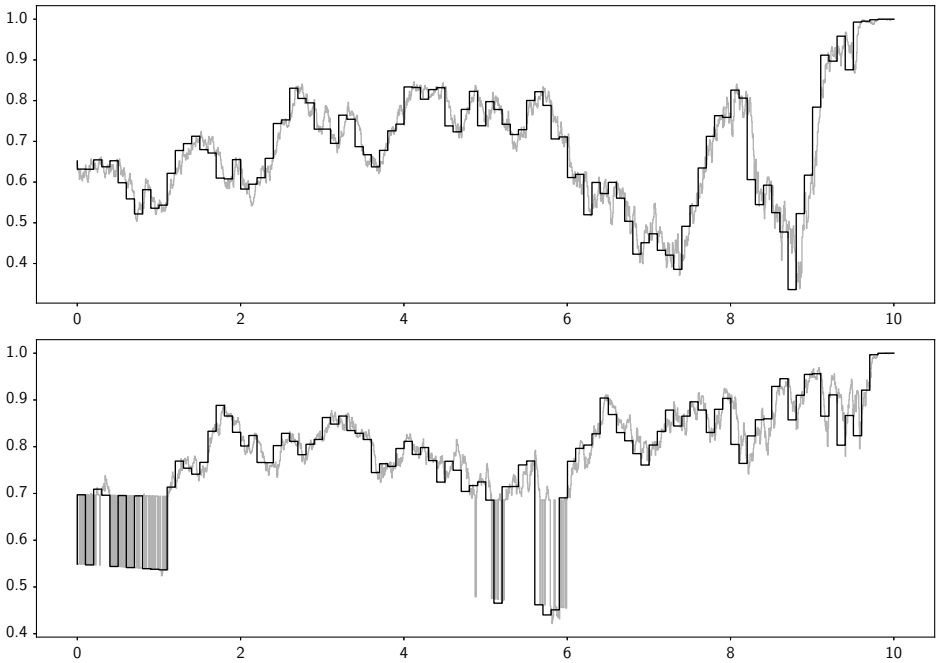


FIGURE 3. (Top): Parts of the risky asset in the payoff replication strategy for one sample path of S with $\rho = 0$, for (black) $N = 100$ and (gray) $N = 2000$. (Bottom): Same, but with $\rho = 1$.

According to Theorem 5, the replication cost for the claim with payoff $h(S_T)$ is $\mathbb{E}^{\mathbb{Q}}[h(S_T)]$. This matches the (APP) price or premium in Model 3, i.e.,

$$\text{(APP) price} = \text{(Premium in Model 3)} = \mathbb{E}^{\mathbb{Q}}[h(S_T)].$$

Therefore, the premium is miscalculated in Models 1 and 2. To address the premium miscalculation in each model, we introduce the mispricing error, defined as

$$\epsilon_{\text{MP}} := \text{(Premium in Model } i\text{)} - \text{(APP) price}.$$

This error compensates for the error made by the agent when using an incorrect model to compute premiums for contingent claims. It is to be compared with the empiric mean PnL of the replication portfolios $\widehat{\mu}_{\text{MC}}$. The discrepency between values of Model 1 and Model 3 when $\rho = 0$ is due to the error incurred from the finite-difference approximations of quantities in Model 3.

We observe in Model 1 that the mispricing error ϵ_{MP} is close to the empiric mean PnL of the replication portfolios $\widehat{\mu}_{\text{MC}}$. This is due to two facts. First, the agent ultimately holds $h(S_T)$ at maturity. Second, given that the price dynamics are martingales on $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, from (9) the self-financing portfolio is also a martingale. Hence, the expected PnL of any self-financing portfolio is 0 and the expected PnL of the option seller is Premium $- \mathbb{E}^{\mathbb{Q}}[h(S_T)]$, which equals 0 only when pricing with the correct model, Model 3. Also, among Models 1, 2, and 3, it is Model 3 that exhibits the lowest hedging error, indicating superior replication properties (see Table 2).

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TABLE 2. Theoretical premiums in each model, (APP) prices, mean PnLs, and tracking errors for a call $(K, T) = (10, 10)$ on assets with price processes $\sigma = 0.25$ sticky at $\zeta = 10$ for various degrees of stickiness ρ in each model. The hedging frequency is $N = 2000$ and the number of simulated sample paths of each process is $N_{\mathrm{MC}} = 1000$.

Model 1						Model 2						Model 3					
ρ	Prem.	(APP)	ϵ_{MP}	$\widehat{\mu}_{\mathrm{MC}}$	$\widehat{\sigma}^{(N)}_{\mathrm{MC}}$	ρ	Prem.	(APP)	ϵ_{MP}	$\widehat{\mu}_{\mathrm{MC}}$	$\widehat{\sigma}^{(N)}_{\mathrm{MC}}$	ρ	Prem.	(APP)	ϵ_{MP}	$\widehat{\mu}_{\mathrm{MC}}$	$\widehat{\sigma}^{(N)}_{\mathrm{MC}}$
0	3.073	3.039	0.034	−0.0018	0.057	0	3.076	3.039	0.037	0.0016	0.077	0	3.039	3.039	0	−0.0343	0.062
1	3.073	2.436	0.637	0.6224	0.515	1	2.670	2.436	0.234	0.2172	0.508	1	2.436	2.436	0	−0.0242	0.087
2	3.073	1.997	1.076	1.0551	0.727	2	2.359	1.997	0.362	0.3238	0.735	2	1.997	1.997	0	−0.0014	0.046

6.3. Model mismatch: Smooth SDE

We now suppose the agent misrepresents price stickiness as local volatility in a smooth SDE model.

It is possible to approximate in law diffusions with sticky features by smooth diffusions. Let X be the diffusion on $(0, \infty)$, on a natural scale, of speed measure

$$m(dx) = \frac{2}{\sigma^2 x^2}\,\mathrm{d}x + \rho\delta_\zeta(dx), \quad x > 0.$$

It corresponds to the law of S under $\mathbb{Q}$, see (6). We can define an approximating sequence $(X^n)_n$ of S in law as follows. For all n, let X^n solve $\mathrm{d}X_t^n = \sigma_n(X_t^n)\,\mathrm{d}B_t^n$ with B^n a standard Brownian motion and σ_n a smooth function, such that

$$\frac{2}{\sigma_n^2(x)}\,\mathrm{d}x \xrightarrow[n\to\infty]{\text{weakly}} m(\mathrm{d}x).$$

From [14], $X^n \to X$ in law.

We make two key observations. First, the prices in the smooth models converge to the prices in the sticky model. Indeed, let v be the price function and $(Q_x^X)_x$ the Markovian family of the sticky model X, and, for all $n \in \mathbb{N}$, let $v^{(n)}$ be the price function and $(Q_x^{X^n})_x$ the Markovian family of the smooth model X^n. The convergence in law implies that

$$\lim_{n\to\infty} v^{(n)}(t,x) = \lim_{n\to\infty} Q_x^{X^n}(h(Y_{T-t})) = Q_x^X(h(Y_{T-t})) = v(t,x),$$

where Y is the coordinate process. Second, while the deltas in the smooth models are continuous, the delta of the sticky model is discontinuous (see Remark 5). Indeed, due to the smoothness of the coefficient σ_n, the function $x \to v^{(n)}(t,x)$ is also smooth, for all $t \in [0,T]$ and $n \in \mathbb{N}$.

In Section 6.1, we approximated the delta function using an interpolation of the finite-difference scheme that preserved the discontinuity. In this section, we use an approximation that does not preserve the discontinuity. Thus, the deltas for the two approximations differ only locally around the threshold of stickiness. We observe that the second method produces a high tracking error (see Table 3, and compare with Table 1). This indicates that hedging a sticky diffusion using a smooth model results in some significant irreductible tracking error. The reason is that a smooth function of the spot price cannot replicate the discontinuity of the true sticky hedging strategy.

7. Conclusion

In conclusion, we have demonstrated that the sticky Black–Scholes model satisfies no free lunch with vanishing risk (NFLVR) and is consistent with arbitrage-free pricing theory only when the interest rate $r = 0$. Under this condition, we proved the existence of a replication strategy and that in this model there is no uniqueness in either the replication strategy or the equivalent local martingale measure (ELMM). Moreover, we established that any locally bounded payoff that can be replicated in the standard Black–Scholes model can also be replicated in the sticky model.

Our numerical experiments indicate that the mean squared tracking error of discrete-time hedging converges at a rate of $\mathcal{O}(N^{1/2})$ to 0 as N diverges to ∞, matching the rate in the

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TABLE 3. As Table 1 but for the locally linearly interpolated price curve around ζ.

Stickiness $\rho = 0$				Stickiness $\rho = 1$				Stickiness $\rho = 2$			
N	Premium	$\widehat{\mu}_{\mathrm{MC}}$	$\widehat{\sigma}^{(N)}_{\mathrm{MC}}$	N	Premium	$\widehat{\mu}_{\mathrm{MC}}$	$\widehat{\sigma}^{(N)}_{\mathrm{MC}}$	N	Premium	$\widehat{\mu}_{\mathrm{MC}}$	$\widehat{\sigma}^{(N)}_{\mathrm{MC}}$
2000	3.039 23	−0.034 38	0.062 51	2000	2.436 07	−0.023 11	0.171 89	2000	1.9976	−0.001 26	0.244 02
1000	3.039 23	−0.034 81	0.086 05	1000	2.436 07	−0.019 09	0.198 02	1000	1.9976	−0.009 39	0.281 32
250	3.039 23	−0.036 87	0.167 81	250	2.436 07	−0.021 65	0.296 81	250	1.9976	−0.027 60	0.405 03
100	3.039 23	−0.045 29	0.268 32	100	2.436 07	−0.020 50	0.387 60	100	1.9976	−0.020 75	0.529 87
10	3.039 23	−0.006 01	0.690 78	10	2.436 07	−0.016 81	0.780 04	10	1.9976	0.003 03	0.948 37

standard Black–Scholes model. We also observed that ignoring or misrepresenting stickiness results in significant irreducible tracking errors, emphasizing the importance of using hedging strategies that accurately account for sticky price features.

This work leaves several topics for future investigation. First, exploring whether the model with $r \neq 0$ remains arbitrage-free under proportional or fixed transaction costs. Second, identifying the class of payoffs for which the pricing problem stated in Theorem 4 admits a unique classical solution. And last, determining the rate of convergence of the mean squared tracking error that arises from discrete-time hedging.

Appendix A. Proof-assisting results

A.1. Proof of Lemma 1

For the reader's convenience, we recall the statement of Lemma 1 here.

Lemma. *Let X be a regular diffusion on J, an open interval of $\mathbb{R}$, on a natural scale of speed measure m so that $\rho := m(\{\zeta\}) \in (0, \infty)$. We suppose X to be defined on the probability space $\mathcal{P}_x = (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$. There exists a regular diffusion Z on J, on a natural scale, with speed measure $m_Z = m - \rho\delta_\zeta$, defined on an extension of $\mathcal{P}_x$, such that, for all $t \geq 0$, $X_t = Z_{\gamma_\rho(t)}$ and $A_\rho(t) = t + \rho L_t^\zeta(Z)$, where γ_ρ is the right inverse of A_ρ.*

Proof. From [44, Theorem V.47.1], there exists a standard Brownian motion W, defined on an extension of $\mathcal{P}_x$, such that, for all $t \geq 0$, $X_t = W_{\gamma_X(t)}$ and $A_X(t) = \int_J L_t^y(W)\, m(\mathrm{d}y)$, where γ_X is the right inverse of A_X. We consider the process Z defined for all $t \geq 0$ by $Z_t = W_{\gamma_Z(t)}$, $A_Z(t) = \int_J L_t^y(W)\, m_Z(\mathrm{d}y)$, where γ_Z is the right inverse of A_Z. From [44, Section V.47, Remark (ii)], Z is a diffusion on J on a natural scale with speed measure m_Z. Thus, $X = (Z_{A_Z\circ\gamma_X(t)})_{t\geq 0}$. From [43, Proposition IV.1.12] and the proof of [44, Theorem V.47.1], for the case of a regular diffusion with unattainable boundaries we have that $\gamma_Z(\,\cdot\,) = \langle Z\rangle$ and that $\langle Z\rangle$ is almost surely strictly increasing. Thus, the time changes $\gamma_Z, A_Z, \gamma_X, A_X$ are all almost surely strictly increasing and hence invertible, i.e. $A_Z = \gamma_Z^{-1}$, $A_X = \gamma_X^{-1}$. This, with [43, Exercise VI.1.27], yield that

$$\gamma_X^{-1}(t) = \int_J L_t^y(W)\, m(\mathrm{d}y) = \int_J L_t^y(W)\, m_Z(\mathrm{d}y) + L_t^\zeta(W)\, m(\{\zeta\}) = \gamma_Z^{-1}(t) + m(\{\zeta\}) L^\zeta_{\gamma_Z^{-1}(t)}(Z)$$

and that $\left(\gamma_Z^{-1}\circ\gamma_X\right)^{-1}(t) = \gamma_X^{-1}\circ\gamma_Z = t + m(\{\zeta\})L_t^\zeta(Z) = A_\rho(t)$. This completes the proof. □

A.2. Proof of Lemma 2

For the reader's convenience, we recall the statement of Lemma 2 here.

Lemma. *Let $[(t, x, y) \to p_0(t, x, y)]$ and $m_0(dy) = \dot{m}_0(y)\,\mathrm{d}y$ be the probability transition kernel and speed measure of a geometric Brownian motion. For all $x > 0$ and $T > 0$, there exists a compact $K_{x,T}$ such that, on $(t, y) \in [0, T] \times K_{x,T}^{\mathrm{c}}$, $[t \to p_0(t, x, y)\dot{m}_0(y)]$ is increasing.*

Proof. From the expressions for the probability transition kernel and speed measure of the geometric Brownian motion [13, Appendix 1.20],

$$\dot{m}(y)p_0(t, x, y) = \frac{1}{y\sigma\sqrt{2\pi t}} \exp\left(-\frac{\left(\log y - \log x - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)^2}{2\sigma^2 t}\right).$$

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Taking the derivative in time yields

$$\partial_t(\dot{m}(y)p_0(t,x,y)) = \left(\frac{\left(\log y - \log x - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)^2}{2\sigma^2 t^2} + 2\frac{\left(\log y - \log x - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)}{2\sigma^2 t}\left(\mu - \frac{1}{2}\sigma^2\right) t - \frac{1}{2t}\right)\dot{m}(y)p_0(t,x,y).$$

We observe that, for all $t, x > 0$, as $y \to 0$ and $y \to \infty$, the quantity $\partial_t p_0(t,x,y)$ becomes positive. This completes the proof. □

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