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Multifractal formalism for some parabolic maps

KENTARO NAKAISHI
Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153, Japan

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Abstract. We study multifractal spectra of a simple class of interval transformations with a neutral fixed point. The analyticity of their spectral curves depends on whether the absolutely continuous invariant measure is infinite or not.

1. *Introduction and results*

Recent progress of computers has made it possible to visualize fractal sets. In producing a computer picture by iteration, one often notices that some points are easy to plot while others are not. This observation suggests that, from the dynamical point of view, an invariant set may have no uniform structure. One of the motivations for the multifractal analysis is to uncover these fine structures. A recent work of Pesin and Weiss [**13**] gave a solid mathematical foundation for conformal hyperbolic systems. We refer to [**14**] for general information on the multifractal analysis.

The aim of this paper is twofold. First we give examples of the multifractal analysis for non-hyperbolic systems. Second, it is observed in physical literature [**8, 16**] that the spectrum of free energy exhibits a phase transition (i.e. the spectrum fails to be real analytic) for a class of maps with a neutral fixed point. Based on the analogy of thermodynamics, one expects that the existence of a phase transition should reflect the geometric and statistical properties of the dynamical system. We present some results along this direction.

In this paper we concern the following piecewise monotone interval maps f on $[0, 1]$:

(A1) there is a partition $0 = a_0 < a_1 < a_2 = 1$ such that each $f|_{(a_{i-1},a_i)}$ is strictly monotone and extends to a C^1 function f_i on $[a_{i-1}, a_i]$ with $f_i[a_{i-1}, a_i] = [0, 1]$;
(A2) $f_1(0) = 0$ and f_1 is C^2 except at $z = 0$;
(A3p) f_2 is C^2 and orientation-preserving, or
(A3r) f_2 is C^2 and orientation-reversing;
(A4) $Df_1(0) = 1$ and $Df_1(z) > 1$ for $z \in [0, a_1]$;
(A5) $|Df_2(z)| > 1$ on $f_2^{-1}([a_1, 1])$ and $|Df_2(z)| \geq 1$ otherwise;
(A6) $D^2 f_1$ has an asymptotic expansion around zero of the form

$$D^2 f(z) = bz^{\tau-1} + o(z^{\tau-1}), \quad b > 0,\ \tau > 0.$$

Example. (Farey map) $f_1(z) = z/(1 - z)$, $f_2(z) = (1 - z)/z$ and $a_1 = 1/2$.

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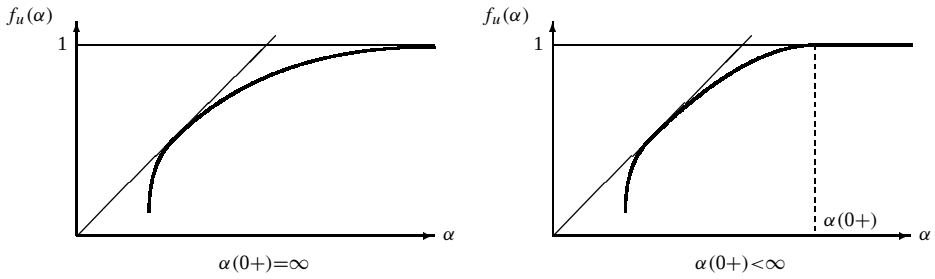


FIGURE 1. Typical spectral shapes.

It is well known [**11**] that the topological entropy of f is $\log 2$ and that there exists a unique invariant probability measure u attaining this entropy. We call *u the measure of maximal entropy for f.* According to general theories [**20, 23**], f possesses another invariant σ-finite measure which is absolutely continuous with respect to Lebesgue measure (*a.c.i.m.* for brevity).

Definition. For $\alpha \in \mathbb{R}^+$ define

$$f_u(\alpha) = \mathrm{HD}\left\{z \in [0,1] : \lim_{r\to 0} \frac{\log u(B(z,r))}{\log r} = \alpha\right\},$$
$$l(\alpha) = \mathrm{HD}\left\{z \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \log|Df^n(z)| = \alpha\right\}$$

where HD stands for Hausdorff dimension. $f_u(\alpha)$ is called *the dimension spectrum of u* and $l(\alpha)$ *the Lyapunov spectrum* for f.

MAIN THEOREM. *Let f be a map satisfying the hypotheses (A1)–(A6) and let u be the measure of maximal entropy for f. Then there exists a continuous function $T(q)$ on $[0,\infty)$ so that:*

(a) *$T(0) = 1, T(1) = 0$ and $T(q)$ is real analytic on $\mathbb{R}^+$ with $D^2T(q) > 0$;*

(b) *the domain of $f_u(\alpha)$ is unbounded and the pair $(f_u(\alpha), T(q))$ forms a Legendre transform pair*

$$f_u(\alpha(q)) = T(q) + q\alpha(q)$$

where $\alpha(q) = -DT(q) > 0$;

(c) *if $\alpha(0+)$ is finite then $f_u(\alpha) = 1$ for every $\alpha \geq \alpha(0+)$;*

(d) *$l(\alpha) = f_u(\log 2/\alpha)$.*

COROLLARY. *The following are equivalent (see Figure 1).*

(a) *$\alpha(0+) = \infty$.*

(b) *The spectra $f_u(\alpha)$ and $l(\alpha)$ are real analytic.*

(c) *The absolutely continuous invariant measure is infinite.*

(d) *The Lyapunov exponent with respect to the a.c.i.m. is zero.*

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2. *Preliminaries*

Put $J = [a_1, 1]$, $I_\infty = \{0\}$ and let $I_k = f_1^{-k}(J)$ for all $k \geq 0$. Define the subintervals $\{J_k\}$ of J by $J_k = f_2^{-1}(I_{k-1})$. Let F be the first return map to J; formally† we can describe F as

$$F|_{J_k}(z) = f^k(z)$$

with $F(1) = F(0) = 1$.

LEMMA 2.1. *For arbitrarily small $\epsilon > 0$ there exist constants C_1, C_2 so that*

$$C_1 k^{(1-\epsilon)(1+\tau)/\tau} \leq Df^k(z) \leq C_2 k^{(1+\epsilon)(1+\tau)/\tau}$$

for all $k \in \mathbb{N}$ and all $z \in I_k$.

Proof. If z is close to zero then the hypothesis (A6) gives

$$Df_1(z) \geq 1 + b'\sqrt{1-\epsilon}(1+\tau)z^\tau$$

where $b' = b\tau^{-1}(1+\tau)^{-1}$. Let $c_n = f_1^{-n}(a_1)$. As in [**21**] or [**2**] one can show that $\lim_{n\to\infty}(b'\tau n)^{1/\tau}c_n = 1$ and we have

$$\begin{aligned} Df_1^k(z) &\geq (\text{const.}) \prod_{l=0}^{k-1} (1 + b'\sqrt{1-\epsilon}(1+\tau)(f^l z)^\tau) \\ &\geq (\text{const.}) \prod_{l=0}^{k-1} (1 + b'\sqrt{1-\epsilon}(1+\tau)c_{k-l}^\tau) \\ &\geq (\text{const.}) \prod_{n=1}^{k} (1 + (1-\epsilon)(1+\tau)/(\tau n)). \end{aligned}$$

Then the left-hand side inequality follows from an asymptotic estimate;

$$\prod_{n=1}^{k} \left(1 + \frac{a}{n}\right) \asymp k^a, \quad a > 0 \quad (k \to \infty).$$

The right-hand side is similar. □

Let $\mathcal{P}^n$ be the partition of depth n, which consists of *cylinders*

$$J_{z_0 \ldots z_{n-1}} = J_{z_0} \cap F^{-1} J_{z_1} \cap \cdots \cap F^{-(n-1)} J_{z_{n-1}}.$$

LEMMA 2.2. (Bounded distortion) *There is a constant $C_3 > 0$ such that for each $k \in \mathbb{N}$, each $J_* \in \mathcal{P}^k$ and $z, w \in J_*$*

$$\left| \log \frac{|DF^k(z)|}{|DF^k(w)|} \right| \leq C_3 |F^k z - F^k w|.$$

Proof. We verify Adler's condition,

$$M = \sup_k \sup_{z \in J_k} \frac{|D^2 f^k(z)|}{|Df^k(z)|^2} < \infty,$$

† Throughout this paper, a boundary point accompanies a unique subinterval whenever it matters.

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from which the lemma follows by standard arguments (see, for example, [**5**]). Set

$$M_0 = \sup_{k\geq 2} \sup_{w\in I_{k-1}} \frac{|D^2 f_1^{k-1}(w)|}{|Df_1^{k-1}(w)|^2}, \quad M_1 = \sup_{z\in J} \frac{|D^2 f_2(z)|}{|Df_2(z)|^2}.$$

Since M is bounded by $M_0 + M_1$, and since $M_1 < \infty$ by definition, we only have to see that $M_0 < \infty$. For $w \in I_{k-1}(k \geq 2)$, using the logarithmic derivative as usual, we have

$$\frac{|D^2 f_1^{k-1}(w)|}{|Df_1^{k-1}(w)|^2} \leq \sum_{l=0}^{k-2} \left| \frac{D^2 f_1(f^l w)}{Df_1^{k-l-1}(f^l w)} \right|.$$

Now assume $\tau < 1$ (see (A6)). The other cases are easily supplied. By (A6) we have

$$|D^2 f_1(f^l w)| \leq \text{const.}(f^l w)^{\tau-1} \leq \text{const.}(k-l-1)^{-1+1/\tau}$$

and Lemma 2.1 shows that $|Df_1^{k-l-1}(f^l w)| \geq A_1(k-l-1)^{(1-\epsilon)(1+1/\tau)}$. Hence M_0 is bounded by const. $\sum_{l=1}^{\infty} l^{-2+\epsilon(1-\tau+2/\tau)}$. □

3. *Induced systems and equilibrium states*

Let $D = \{s \in \mathbb{R} : s > s_0\}$ where $s_0 = \inf\left\{s > 0 : \sum_{k=1}^{\infty} |Df^k(F|_{J_k}^{-1}z)|^{-s} < \infty,\ \forall z \in J\right\}$ and let $J_0 = \bigcup_{k=1}^{\infty} \operatorname{int} J_k$ ($\operatorname{int} J_k$ denotes the interior of J_k). Denote by $C(J_0)$ the space of continuous functions on J_0. If we put

$$\psi(z) = 2^{-k}, \quad z \in \operatorname{int} J_k$$

then $\psi \in C(J_0)$. For $(s, q) \in \mathbb{R} \times \mathbb{R}^+ \cup D \times \{0\}$ one can define *the transfer operator* $\mathsf{M}_{s,q} : C(J) \to C(J)$ as follows.

(1) Define $\phi_{s,q} \in C(J_0)$ by

$$\phi_{s,q}(z) = q \log \psi - s \log|DF(z)|.$$

(2) Restrict F on J_0. Then

$$\mathsf{M}_{s,q} g(z) = \sum_{w\in F^{-1}z} (\exp \phi_{s,q}(w)) g(w)$$

defines an operator $\mathsf{M}_{s,q}$ acting on uniformly continuous functions on J_0 and can be extended to $C(J)$. If

$$\phi_{s,q}^{(n)}(z) = \phi_{s,q}(z) + \cdots + \phi_{s,q}(F^{n-1}z)$$

then Lemma 2.2 implies that for each $x' = F^n w$, $w \in J_* \in \mathcal{P}^n$,

$$C(x, x') := \sup_{n\geq 1} \sup_{F^n z = x, z\in J_*} |\phi_{s,q}^{(n)}(z) - \phi_{s,q}^{(n)}(w)| \leq |s|C_3|x - x'|. \tag{3.1}$$

Then it is well known [**22**] that there exists a unique triple $(\lambda(s, q), h, \mu)$ where $\lambda(s, q) \in \mathbb{R}^+$, $h \in C(J)$ and μ is a probability measure satisfying:

(i) $\mu(h) = 1$,

(ii) $\mathsf{M}_{s,q}^* \mu = \lambda(s, q)\mu$,

(iii) $\mathsf{M}_{s,q} h = \lambda(s, q)h$,

(iv) $h \cdot \mu$ is F-invariant,

(v) $\lambda(s, q)^{-n} \mathsf{M}_{s,q}^n g$ converges uniformly to $h \cdot \mu(g)$ ($\forall g \in C(J)$),

(vi) $\log \lambda(s, q) = \lim_{n\to\infty} n^{-1} \log(\mathsf{M}_{s,q}^n 1)$.

The uniqueness is characterized by (i), (ii) and (iii).

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Definition. We call the invariant measure above the equilibrium state for $\phi_{s,q}$ and the number $P(s,q) = \log \lambda(s,q)$ the pressure for $\phi_{s,q}$.

Denote the length of an interval I by $|I|$.

THEOREM 3.1.

(a) $\mathcal{R} = \{[0, a_1), [a_1, 1]\}$ *generates the Borel field, and the induced map* F *on* J *is a Markov map with a countable partition* J_k *such that* F *has uniform bounded distortion on* J *(Lemma 2.2).*

(b) *For each* $(s,q) \in \mathbb{R} \times \mathbb{R}^+ \cup D \times \{0\}$ *there exists an* F*-invariant ergodic probability measure* $\nu_{s,q}$ *on* J *so that for each* $n > 0$ *there is a constant* $\Delta(n) \geq 1$ *satisfying* $\lim_{n\to\infty} \log \Delta(n)/\log \sup_{K\in\mathcal{P}^n} |K| \to 0$ *and*†

$$\nu_{s,q}(F^n A) \overset{\Delta(n)}{\sim} \int_A \exp\left(nP(s,q) - \phi_{s,q}^{(n)}\right) d\nu_{s,q}$$

whenever $F^n : A \to F^n A$ *is invertible.*

Proof. Define $\mathcal{R}^n = \mathcal{R} \vee f^{-1}\mathcal{R} \vee \cdots \vee f^{-(n-1)}\mathcal{R}$. Then it is readily seen that $\lim_{n\to\infty} \sup_{B\in\mathcal{R}^n} |B| = 0$, proving (a).

It is enough to prove (b) when $A = J_{a_0 \ldots a_{n-1}}$. Let $\nu_{s,q} = h \cdot \mu$ be the equilibrium state for $\phi_{s,q}$. If F^n is restricted on A then it follows from (ii) that $d(\mu F^n)|_A / d\mu|_A$ is equal to $\lambda(s,q)^n \exp(-\phi_{s,q}^{(n)})$. By (3.1), $\mathsf{M}_{s,q}^n 1(z) \leq \exp(C_3|s|)\mathsf{M}_{s,q}^n 1(w)$. Hence $h(z) \leq \Delta h(w)$ with $\Delta = \exp(C_3|s|)$. This proves (b). □

Remark. The rate of convergence of (vi) can be estimated from above,

$$|\log \lambda(s,q) - m^{-1} \log(\mathsf{M}_{s,q}^m 1)| \leq m^{-1}(C_3|s|). \tag{3.2}$$

Since a weak limit of atomic measures $\delta_n = 2^{-n} \sum_{b\in f^{-n}(a)} \delta_b$ ($a \in [0,1]$) gives the measure of maximal entropy u, the Jacobians can be described explicitly,

$$u(f^n B) = 2^n u(B) \quad (B \subset A \in \mathcal{R}^n). \tag{3.3}$$

In particular, $u(J_{z_0 \ldots z_{n-1}}) = 2^{-(z_0+\cdots+z_{n-1})} u(J)$. Observe that u gives an F-invariant measure on J and satisfies $\mathsf{M}_{0,1}^* u = u$. A simple computation shows that $P(0,1) = 0$ and $\mathsf{M}_{0,1} 1 = 1$. Consequently, $u(J)^{-1}u$ is the equilibrium state for $\phi_{0,1} = \log \psi$.

4. *Ball lemma*

Let $J_{\text{inv}} = \bigcup_{n\geq 0} F^{-n}\{1\}$. Since J_{inv} is countable, it has no influence on the spectra. For $z \in J \setminus J_{\text{inv}}$ one can encode the orbit of z by $\{z_n\}$ with $F^n z \in J_{z_n}$.

Let $B_{n,L}(z) = B(z, L|DF^n(z)|^{-1})$. It is easily seen that $J_{z_0 \ldots z_{n-1}} \subset B_{n,K}(z)$ for some uniform constant $K > 0$.

LEMMA 4.1. *For each* $z \in J \setminus J_{\text{inv}}$ *there exists a constant* $L > 0$ *so that for all* n*:*

(a) *if* $z_{n-1} \geq 2$ *then* $B_{n,L}(z) \subset \bigcup_{k=-1}^{\infty} J_{z_0 \ldots z_{n-2}(z_{n-1}+k)}$*;*

† $f \overset{\Delta}{\sim} g$ means $\Delta^{-1} g \leq f \leq \Delta g$. We simply write $\sim$ when the constant is inessential.

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(b) *if* $z_{n-1} = 1$ *then*
(A3r) *case:* $B_{n,L}(z) \subset J_{z_0 \dots z_{n-2}2} \cup J_{z_0 \dots z_{n-2}1} \cup J_{z_0 \dots z_{n-3}(z_{n-2}+1)}$,
(A3p) *case:* $B_{n,L}(z) \subset J_{z_0 \dots z_{n-2}2} \cup J_{z_0 \dots z_{n-2}1} \cup \mathcal{J}_N$,
where $\mathcal{J}_N = \bigcup_{k \geq N} J_{z_0 \dots z_{n-l-1}(z_{n-l}-1)k}$, *and* l, N *are chosen so that* $z_{n-1} = \cdots = z_{n-l+1} = 1$, $z_{n-l} > 1$ *and* $|\mathcal{J}_{N+1}| < L|DF^n z|^{-1} \leq |\mathcal{J}_N|$.

Proof. First note that $f^{k-1} I_{k-1} = J$. Lemma 2.1 shows that

$$|I_{k-1}| \leq |I_k| \leq \text{const.} k^{-1+2\epsilon(1+1/\tau)} \sum_{l>k} |I_l| \quad (k \geq 2).$$

Then, Lemma 2.2, we find constants Δ, Δ' such that

$$\Delta |J_{z_0 \dots z_m (k-1)}| \leq |J_{z_0 \dots z_m k}| \leq \Delta' \sum_{l>k} |J_{z_0 \dots z_m l}| \quad (m \geq 1, k \geq 2)$$

and (a) follows immediately.

For the (A2r) case, notice that the partition $\mathcal{P}^{n+1}$ divides $\mathcal{P}^n$ in reverse order and use the same argument as above. To prove the (A2p) case we can find $z_{n-l} > 1$ for sufficiently large n since we exclude boundary points. In this case the partitions preserve the order. So use the neighbour of $J_{z_0 \dots z_{n-l}}$ to cover the ball. □

5. *Multifractal formalism I (proof of the Main theorem (b), (d))*

If $\beta = \inf_k \inf_{z \in J_k} |DF(z)|$ then we have $\beta > 1$ by construction.

LEMMA 5.1. *For each* $q \in \mathbb{R}^+ \cup \{0\}$, *the function* $s \mapsto P(s, q)$ *is convex, strictly decreasing* (*hence continuous*) *and has a unique zero* $T(q)$.

Proof. Hölder's inequality with the counting measure on $\mathbb{N}^m$ implies that

$$\mathsf{M}^m_{ts_1+(1-t)s_2,q} 1 \leq \mathsf{M}^m_{ts_1,tq} 1 \mathsf{M}^m_{(1-t)s_2,(1-t)q} 1 \quad (0 \leq t \leq 1).$$

Then (vi) yields the convexity. If $s_1 < s_2$ then $\mathsf{M}^m_{s_2,q} 1 \leq (\beta^{s_2-s_1})^m \mathsf{M}^m_{s_1,q} 1$ and the strict decrease follows also by (vi). Notice that $P(0, q) = -\log(2^q - 1)$ and

$$\lambda(s, q) = \int \sum_{k=1}^{\infty} (1/2)^{kq} |Df^k (F|_{J_k}^{-1} z)|^{-s} \, d\mu(z).$$

The existence of a unique zero is ready to be seen. □

Consider the one-parameter family of mappings $\phi_{T(q),q}$ for $q \in \mathbb{R}^+ \cup \{0\}$. Let ν_q be the equilibrium state for $\phi_{T(q),q}$. Theorem 3.1(b) shows that

$$\begin{aligned} \int \log|DF| \, d\nu_q &\sim \sum_{k=1}^{\infty} \log|Df^k(w_k)| \nu_q(J_k) \\ &\sim \sum_{k=1}^{\infty} \log|Df^k(w_k)| (1/2)^{kq} |Df^k(w_k)|^{-T(q)}. \end{aligned}$$

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In view of Lemma 2.1 this implies that $\log|DF(z)|$ is ν_q-integrable and that its integral value is *positive* for $q \geq 0$. Similarly $\log\psi \in L^1(\nu_q)$ for $q > 0$. Since ν_q is ergodic, for each $q > 0$ there exists a positive constant $\alpha = \alpha(q)$ such that for almost every $z \in J$

$$\sum_{l=0}^{k-1} -\log\psi(F^l z) \Big/ \sum_{l=0}^{k-1} \log|DF(F^l z)| \to \alpha(q)$$
$$= -\int \log\psi \, d\nu_q \Big/ \int \log|DF| \, d\nu_q (k \to \infty).$$

Define the set

$$K(\delta) = \left\{ z \in J \backslash J_{\text{inv}} : \lim_{k\to\infty} \frac{\sum_{l=0}^{k-1} -\log\psi(F^l z)}{\sum_{l=0}^{k-1} \log|DF(F^l z)|} = \delta \right\}$$

and denote the Hausdorff dimension of X(set or measure) by $\mathrm{HD}(X)$.

Definition. Let ν be a measure. We say that the balls and cylinders are *ν-compatible at z* if there are a uniform constant L and constants $\Delta(n) > 0$ for each $n > 0$ such that

$$\nu(B(z, L|E(z)|)) \overset{\Delta(n)}{\sim} \nu(E(z))$$

and $\lim_{n\to\infty} \log\Delta(n)/\log|E(z)| = 0$ where $E(z)$ is the element of $\mathcal{P}^n$ containing z.

PROPOSITION 5.2. *For $q \in \mathbb{R}^+$:*

(a) *the balls and cylinders are ν_q-compatible at $z \in K(\alpha(q))$;*

(b) $\mathrm{HD}(\nu_q) = \mathrm{HD}(K(\alpha(q))) = T(q) + q\alpha(q)$.

Proof. For $z \in K(\delta)$ with $\delta > 0$, Lemma 2.1 implies that

$$\lim_{n\to\infty} \frac{\log|DF^n(z)|}{\log|DF^{n+1}(z)|} = 1 - \delta^{-1} \lim_{n\to\infty} \frac{\log|Df^{z_n}(F^n(z))|}{z_0 + \cdots + z_n} = 1. \tag{5.1}$$

Hence $\lim_{n\to\infty} z_{n-1}/\log|DF^n z|^{-1} = 0$. Theorem 3.1 implies that

$$\nu_q(J_{z_0\ldots z_{n-2}})/\nu_q(J_{z_0\ldots z_{n-1}}) \sim 2^{qz_{n-1}} |Df^{z_{n-1}}(F^n z)|^{T(q)}$$

and by Lemma 4.1 we have $J_{z_0\ldots z_{n-1}} \subset B_{n,L}(z) \subset J_{z_0\ldots z_{n-3}}$ except for the (A3p) case. Then (a) follows from Lemma 2.1. For the (A3p) case, another estimate is needed. Lemma 2.2 implies that $|DF^l(F^{n-l}z)|^{-1} \sim \sum_{k\geq N} |DF^2(F^{n-l}w_k)|^{-1}$ and

$$\nu_q(\mathcal{J}_N) \sim \nu_q(J_{z_0\ldots z_{n-1}}) \sum_{k\geq N} 2^{q(l-k)} \frac{|DF^l(F^{n-l}z)|^{T(q)}}{|DF^2(F^{n-l}w_k)|^{T(q)}} \sim \nu_q(J_{z_0\ldots z_{n-1}}) 2^{q(l-N)}.$$

Since $|DF^{l-1}(F^{n-l+1}z)| \geq \beta^{l-1}$ it follows that N must grow exponentially in l. Thus (a) holds even for the (A3p) case.

To prove (b), notice that $\nu_q(J_{z_0\ldots z_{n-1}}) \sim \exp(-\phi^{(n)}_{T(q),q})$ by Theorem 3.1. This implies $\lim_{n\to\infty} \log\nu_q(J_{z_0\ldots z_{n-1}})/\log|DF^n z|^{-1} = T(q) + q\alpha(q)$ for every $z \in K(\alpha)$. For small $r > 0$ there is a unique integer n such that $L|DF^{n+1}(z)|^{-1} < r \leq L|DF^n(z)|^{-1}$. By (5.1) it follows that

$$\lim_{r\to 0} \frac{\log\nu_q(B(z,r))}{\log r} = \lim_{r\to 0} \frac{\log\nu_q(B_{n,L}(z))}{\log|DF^n z|^{-1}} \quad (z \in K(\alpha)).$$

Hence (b) follows immediately from (a). □

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Formula (3.3) shows that $u(B) = 2^{-k}u(f^k B)$ for every Borel set B in I_k. So the local dimension of u at $w \in I_k$ equals that at $z = f^k w \in J$; the same is true of the Lyapunov spectrum. Since bi-Lipschitz maps preserve the Hausdorff dimension, the spectra on the whole interval are the same as on J. The next lemma will relate F with f in the multifractal formalism.

LEMMA 5.3. *For $z \in J \setminus J_{\text{inv}}$ each of the following three implies the other two:*

(a) $\lim_{r\to 0} \dfrac{\log u(B(z,r))}{\log r} = \alpha;$

(b) $\lim_{n\to\infty} \dfrac{-\sum_{l=0}^{n-1} \log \psi(F^l z)}{\sum_{l=0}^{n-1} \log|DF(F^l z)|} = \alpha;$

(c) $\lim_{n\to\infty} \dfrac{1}{n} \log |Df^n(z)| = \dfrac{\log 2}{\alpha}.$

Proof. (b) $\Leftrightarrow$ (a) is contained in Proposition 5.2 and (c) $\Rightarrow$ (b) is obvious. We will show (b) $\Rightarrow$ (c). Assume (b). Decompose $n = z_0 + \cdots + z_{k-1} + l$, $0 \le l < z_k$. Then

$$\lim_{n\to\infty} \frac{l}{z_0 + \cdots + z_{k-1}} \le \lim_{n\to\infty} \frac{z_k}{z_0 + \cdots + z_{k-1}} = 1 - \lim_{n\to\infty} \frac{\log|DF^{k+1}(z)|}{\log|DF^k(z)|} = 0.$$

So $l/n \to 0$. Observe

$$\frac{1}{n} \log|Df^n(z)| = \frac{1}{n} \log |Df^l(f^{z_0+\cdots+z_{k-1}}(z))| + \frac{1}{n} \log|Df^{z_0+\cdots+z_{k-1}}(z)|.$$

The first term of the right-hand side is bounded by $l \log|Df|_\infty / n$ where $|\cdot|_\infty$ denotes the supremum norm. This yields (c). □

6. *Analyticity of spectra (proof of Corollary)*

Put $U = \{(s,q) \in \mathbb{C}^2 : \operatorname{Re} q > 0\}$ and let $BV(J)$ be the Banach space of $\mathbb{C}$-valued functions on J of bounded variation with the norm

$$\|g\|_{BV} := \operatorname*{var}_J(g) + |g|_\infty$$

where

$$\operatorname*{var}_J(g) := \sup \left\{ \sum_{i=1}^{n} |g(a_i) - g(a_{i-1})| : n \ge 0, c \le a_0 < \cdots < a_n \le 1, a_i \in J \right\}.$$

Let $CBV(J)$ denote the Banach space of $\mathbb{C}$-valued continuous functions of bounded variation with the same norm. (To clarify, we denote $\|g\|_{CBV}$.)

For $(s,q) \in U$ define the operators $\mathsf{M}_{s,q}$, $\mathsf{M}'_{s,q}$ on $BV(J)$ by

$$\mathsf{M}_{s,q} g(z) = \sum_{k=1}^{\infty} (1/2)^{kq} |Df^k(F|_{J_k}^{-1} z)|^{-s} g(F|_{J_k}^{-1} z)$$

$$\mathsf{M}'_{s,q} g(z) = \sum_{k=1}^{\infty} (1/2)^{kq} |Df^k(z)|^{-s} g(F|_{J_k} z)$$

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where $g(F|_{J_k}z) = 0$ if $z \notin J_k$. Set

$$R_{s,q} = \lim_{m\to\infty} |\mathsf{M}_{s,q}^m 1|_\infty^{1/m}, \quad R'_{s,q} = \lim_{m\to\infty} |\mathsf{M}'^m_{s,q} 1|_\infty^{1/m}.$$

Ruelle [**19**] proves that if

$$\sum_{k=1}^{\infty} \operatorname*{var}_{J_k}(2^{-kq}|Df^k|^{-s}) + \sup_{J_k}(|2^{-kq}|Df^k|^{-s}|) < \infty, \tag{6.1}$$

then the essential spectral radius of $\mathsf{M}_{s,q}$ is less than or equal to $R'_{s,q}$. Furthermore, if $R'_{s,q} < R_{s,q}$, then $R_{s,q}$ is the spectral radius of $\mathsf{M}_{s,q}$ and also an eigenvalue of $\mathsf{M}_{s,q}$.

Since $\mathsf{M}_{s,q}$ preserves continuous functions, we can let $\mathsf{M}_{s,q}$ act on $CBV(J)$. From the very definition, the spectral radius and essential spectral radius of $\mathsf{M}_{s,q}$ on $CBV(J)$ are smaller than those on $BV(J)$ [**9**, IV.Theorem 5.28]. If $R'_{s,q} < R_{s,q}$ and (s, q) is real then

$$\lim_{m\to\infty} \|\mathsf{M}_{s,q}^m\|_{CBV}^{1/m} \geq \lim_{m\to\infty} \|\mathsf{M}_{s,q}^m 1\|_{CBV}^{1/m} = \lim_{m\to\infty} |\mathsf{M}_{s,q}^m 1|_\infty^{1/m} = R_{s,q}.$$

Hence $R_{s,q}$ is the spectral radius of $\mathsf{M}_{s,q}$ even on $CBV(J)$.

Note that $\mathsf{M}'_{s,q}$ does not preserve continuous functions at boundary points.

LEMMA 6.1. *(6.1) holds.*

Proof. From Lemma 2.1 the second term decays exponentially in k. As for the first term, we have

$$\begin{aligned}
|2^{-kq}| \operatorname*{var}_{J_k}(|Df^k|^{-s}) &\leq |2^{-kq}| \sup\left\{ \sum_{i=1}^{n} ||Df^k(a_i)|^{-s} - |Df^k(a_{i-1})|^{-s}| : a_i \in J_k \right\} \\
&\leq |2^{-kq}| \sup_{J_k}(||Df^k|^{-s}|) \sup\left\{ \sum \left| 1 - \exp\left(-s \log \frac{|Df^k(a_i)|}{|Df^k(a_{i-1})|} \right) \right| \right\} \\
&\leq |2^{-kq}| \sup_{J_k}(||Df^k|^{-s}|) C_3 |s| |J|.
\end{aligned}$$

For the last line, we used Lemma 2.2. □

LEMMA 6.2. *If $(s, q) \in U$ is real, then $\lambda(s, q)$ is a simple isolated eigenvalue of $\mathsf{M}_{s,q}$.*

Proof. For each $n > 0$, set

$$\theta_n(z; x_0, \ldots, x_{n-1}) = \prod_{k=0}^{n-1} 2^{-qx_k} |Df^{x_k}(F^k z)|^{-s}. \tag{6.2}$$

Notice that $\theta_n(z; x_0 \ldots x_{n-1})$ is continuous on each $J_* \in \mathcal{P}^n$ and that each $z \in J$ is contained in at most two intervals of $\mathcal{P}^n$. Let J_* denote an arbitrary cylinder $J_{x_0 \ldots x_{n-1}}$ and J_*^+ be the one next to J_* (replace x_{n-1} by $x_{n-1} + 1$). Then we have

$$\mathsf{M}'^n_{s,q} 1(z) = \begin{cases} \theta_n(z; x_0, \ldots, x_{n-1}) + \theta_n(z; x_0, \ldots, (x_{n-1} + 1)) & \text{if } z \in J_* \cap J_*^+, \\ \theta_n(z; x_0, \ldots, x_{n-1}) & \text{if } z \in J_*, \end{cases}$$

where the latter case makes sense provided that z is not of the first type.

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As in Lemma 6.1 it can be shown that

$$\operatorname*{var}_{J_*}(\theta_n) \le (|s|C_3|J|)|2^{-q(x_0+\cdots+x_{n-1})}|\sup_{J_*}|DF^n|^{-s}.$$

From this and Lemma 2.2 it follows that given $w \in J_*$, $w^+ \in J_*^+$,

$$\begin{aligned}\sup_{z\in J_*}|\theta_n(z)| + \sup_{z\in J_*^+}|\theta_n(z)| &\le |\theta_n(w)| + \operatorname*{var}_{J_*}(\theta_n) + |\theta_n(w^+)| + \operatorname*{var}_{J_*^+}(\theta_n)\\ &\le (1+|s|C_4)(|\theta_n(w)| + |\theta_n(w^+)|), \qquad (6.3)\end{aligned}$$

for some constant $C_4 > 0$. Now we claim that $\forall N \in \mathbb{N}\, \exists n \ge N\, \forall J_* \in \mathcal{P}^n\, \exists y_* \in J_*\, \exists y_*^+ \in J_*^+$

$$|\theta_n(y_*; x_0, \dots, x_{n-1})| + |\theta_n(y_*^+; x_0, \dots, (x_{n-1}+1))| \le \lambda(s,q)^n(1+2|s|C_4)^{-1}. \quad (6.4)$$

Suppose not. Then we could find a cylinder $J_* = J_{e_0\dots e_{n-1}}$ violating (6.4). Since

$$\theta_n(z; e_0, \dots, e_i, \dots, e_{n-1})/\theta_n(w; e_0, \dots, e_i+k, \dots, e_{n-1}) \le \text{const.}2^{-qk}(|J_{e_i+k}|/|J_{e_i}|)^{-s}$$

by Lemma 2.2, we could find a constant C_5 independent of n such that

$$\sum_{\substack{0\le i\le n-1\\ k\ge 1}} \min_z |\theta_n(z; \dots, e_i+k, \dots)| + \min_{z\in J_*}|\theta_n(z)| + \min_{z\in J_*^+}|\theta_n(z)| \ge C_5 n\lambda(s,q)^n. \quad (6.5)$$

If n is large enough then $C_5 n\lambda(s,q)^n > \lambda(s,q)^n$. On the other hand, there is a measure μ with $\mu(\lambda^{-n}\mathsf{M}_{s,q}^n 1) = 1$ for all n and

$$\mathsf{M}_{s,q}^n 1(y) = \sum_{w\in J_{x_0,\dots,x_{n-1}}, F^n w = y} \theta_n(w; x_0, \dots, x_{n-1}).$$

This contradicts (6.5). By (6.3) and (6.4), we conclude that for some $n > 0$

$$|(\mathsf{M}')^n 1(z)|_\infty \le (1+|s|C_4)(1+2|s|C_4)^{-1}\lambda(s,q)^n,$$

and so $R'_{s,q} < \lambda(s,q)$.

To prove $R = \lambda(s,q)$, we only have to remember (vi) $\lambda(s,q) = \lim_{m\to\infty}|\mathsf{M}_{s,q}^m 1|^{1/m}$. Thus $\lambda(s,q)$ is an isolated eigenvalue. Simpleness follows by the mixing property (v). □

Fix $(T(q), q)$, $q > 0$. Let

$$P_{T(q),q} = (2\pi i)^{-1}\int_\Gamma (\mathsf{M}_{T(q),q} - z)^{-1}\, dz$$

where Γ is a circle separating $\lambda(T(q),q)$ from the other spectra of $\mathsf{M}_{T(q),q}$. The operator $P_{T(q),q}$ is the projection to the generalized eigenspace associated to $\lambda(T(q),q)$. The mixing property (v) requires that $(\mathsf{M}_{T(q),q} - \lambda)P_{T(q),q} = 0$ (trivial nilpotent), and hence the range of $P_{T(q),q}$ is one-dimensional (see [**19**]). We can verify that $(s,q) \in \mathrm{U} \mapsto \mathsf{M}_{s,q}$ is an analytic mapping from U to the Banach space $L(CBV(J))$ of bounded linear operators. By standard perturbation techniques [**9**] one can take a complex neighbourhood W of $(T(q),q)$ such that

$$(s,q) \in W \mapsto \frac{\mathsf{M}_{s,q}P_{s,q}1}{P_{s,q}1} \qquad (6.6)$$

is analytic.

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PROPOSITION 6.3.

(a) *$T(q)$ is real analytic for $q > 0$ and satisfies*

$$DT(q) = \frac{\int \log \psi(z)\, dv_q(z)}{\int \log|DF(z)|\, dv_q(z)} = -\alpha(q).$$

(b) *If $\Psi := \phi_{1,DT(q)} = \log\psi - DT(q)\log|DF|$ and $\Psi^{(n)} := \phi^{(n)}_{1,DT(q)}$ then*

$$D^2T(q) = \left(\int \log|DF(z)|\, dv_q(z)\right)^{-1} \sigma^2(\Psi)$$

where

$$\sigma^2(\Psi) = \frac{d^2}{ds^2} P(T(q) + sDT(q), q+s)\Big|_{s=0} = \lim_{n\to\infty} \int \frac{1}{n}(\Psi^{(n)})^2\, dv_q.$$

Proof. $P(T(q), q) = 0$ gives $T(q) = -\partial_q P(T(q), q)/\partial_s P(T(q), q)$. Differentiating (6.6) by s, q respectively and integrating the results by $\nu_q/(P_{s,q}1)$ we have (a).

Let $V \subset \mathbb{C}^2$ be a neighbourhood of $(0, 0)$ such that if

$$\xi(s,t) = \sum_{k=0}^{\infty} \frac{D^kT(q)}{k!} t^k + sDT(q)$$

then $(\xi(s,t), q+s+t) \in U$. Consider the transfer operators $\bar{\mathsf{M}}_{s,t} := \mathsf{M}_{\xi(s,t),q+s+t}$. Denote $\bar\lambda(s,t) = \lambda(\xi(s,t), q+s+t)$ and $\bar P(s,t) = \log\bar\lambda(s,t)$. Since $\bar{\mathsf{M}}_{0,0}$ has a simple isolated eigenvalue equal to one, perturbed ones also have simple isolated eigenvalues $\bar\lambda(s,t)$:

$$\bar{\mathsf{M}}_{s,t}\omega(s,t) = \bar\lambda(s,t)\omega(s,t)$$

where $\omega(s,t)$ are the images of one by the projections associated with $\bar\lambda(s,t)$. Differentiating both sides by s, we have that for real (s,t)

$$\bar{\mathsf{M}}_{s,t}(\Psi\cdot\omega) + \bar{\mathsf{M}}_{s,t}(\partial_s\omega) = \partial_s\bar P(s,t)\exp(\bar P(s,t))\omega + \exp(\bar P(s,t))\partial_s\omega. \tag{6.7}$$

If $\nu_{s,t}$ is the equilibrium state for $\phi_{\xi_1(s,t),q+s+t}$, equation (6.7) yields

$$\int \Psi\, dv_{s,t} = \partial_s\bar P(s,t).$$

In particular, if $s = t = 0$ then $\partial_s\bar P(0,0) = 0$ (note $\nu_{0,0} = \nu_q$). Using Taylor expansion $DT(q+t) = DT(q) + D^2T(q)t + o(t)$, we have

$$\partial_s\bar P(0,t) - \partial_s\bar P(0,0) = \int \Psi\, dv_{0,t} = (D^2T(q)t + o(t)) \int \log|DF|\, dv_{0,t}.$$

So $\partial_t\partial_s\bar P(0,0) = D^2T(q)\int \log|DF|\, dv_{0,0}$. We shall prove that

$$\partial_t^2\bar P(0,t)|_{t=0} = \partial_t\partial_s\bar P(0,0). \tag{6.8}$$

Differentiating (6.7) by t and letting $(s,t) \to 0$,

$$\begin{aligned}&\bar{\mathsf{M}}_{0,0}([\Psi]^2\omega) + \bar{\mathsf{M}}_{0,0}(\Psi\cdot\partial_t\omega) + \bar{\mathsf{M}}_{0,0}(\Psi\cdot\partial_s\omega) + \bar{\mathsf{M}}_{0,0}(\partial_t\partial_s\omega)\\ &\quad = \partial_t\partial_s\bar P(0,0)e^{\bar P}\omega + \partial_t\bar P\partial_s\bar P e^{\bar P}\omega + \partial_s\bar P e^{\bar P}\partial_t\omega + \partial_t\bar P e^{\bar P}\partial_s\omega + e^{\bar P}\partial_t\partial_s\omega.\end{aligned}$$

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$\partial_t \bar{P}(0,0) = 0$ can be proved similarly to $\partial_s \bar{P}(0,0) = 0$. The computation of a first few terms of the power series

$$\omega(s,t) = \int_\Gamma (\bar{\mathsf{M}}_{s,t} - z)^{-1}\, dz 1 = \left(\sum_{m,m'} a_{m,m'} s^m t^{m'} \right) 1$$

implies that $\partial_s \omega(0,0) = \partial_t \omega(0,0)$. As a result,

$$\partial_t \partial_s \bar{P}(0,0) = \int [\Psi]^2\, d\nu_{0,0} + \text{const.} \int \Psi \cdot \partial_t \omega(0,0)\, d\nu_{0,0}.$$

The computation of $\partial_t^2 \bar{P}(0,t)|_{t=0}$ gives the same result, which concludes (6.8). The same steps for $\bar{\mathsf{M}}_{0,t}^n \omega(0,t) = \bar{\lambda}(0,t)^n \omega(0,t)$ yields

$$\partial_t^2 \bar{P}(0,t)|_{t=0} = n^{-1} \int [\Psi^{(n)}]^2\, d\nu_{0,0} + \text{const.} n^{-1} \int \Psi^{(n)} \cdot \partial_t \omega(0,0)\, d\nu_{0,0}.$$

The second term can be rewritten as

$$n^{-1} \int \Psi^{(n)} \cdot \partial_t \omega(0,0)\, d\nu_{0,0} = \int \Psi n^{-1} (1 + \mathsf{M} + \cdots + \mathsf{M}^{n-1})(\partial_t \omega)\, d\nu_{0,0}$$

where M is the normalization of $\bar{\mathsf{M}}_{0,t}$: $\mathsf{M}1 = 1$. By the dominated convergence theorem, it converges to $\nu_{0,0}(\partial_t \omega) \int \Psi\, d\nu_{0,0} = 0$ as $n \to \infty$. This completes the proof. □

It is not difficult to prove that $T(0) = \mathrm{HD}(J) = 1$, using a slightly modified argument of [**17**] (equivalently we can show that ν_0 is equivalent to Lebesgue measure).

Proof of Corollary. By the uniqueness of the equilibrium state, we may assume that $\nu_q \to \nu_0$ weakly as $q \to 0$. Since $\log \psi, \log|DF| \in C(J_0)$, and since $\int \log|DF|\, d\nu_0 < \infty$ we have that

$$\alpha(0+) \text{ is infinite if and only if } \int \log \psi\, d\nu_0 \text{ is infinite.}$$

Lemma 2.2 and Theorem 3.1 show that this integral is equal to

$$\begin{aligned} \sum_{k=0}^{\infty} k|I_k| = |I_1| + |I_2| + |I_3| + \cdots \\ + |I_2| + |I_3| + \cdots \\ + |I_3| + \cdots \end{aligned}$$

up to a constant. Following Schweiger [**7, 20**], this sum is infinite if and only if the absolute continuous invariant measure is infinite. It is easy to show that f is conservative with respect to the a.c.i.m. even if the a.c.i.m. is infinite (Maharam's recurrence theorem [**1**]). Then, applying the same argument as in [**7**] (cf. [**1**, p. 61]), we obtain that the a.c.i.m. is infinite if and only if the Lyapunov exponent with respect to the a.c.i.m. is zero. □

Remark. By Lemma 2.1, if $\tau > 1$, $\alpha(0+)$ is infinite and if $\tau < 1$, $\alpha(0+)$ is finite. In the case of $\tau = 1$, more regularity of the system f seems to be needed. For instance, see [**6**].

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Let M denote the normalization of $\mathsf{M}_{T(q),q}$ and extend M to $L^1(\nu_q)$.

LEMMA 6.4. *The eigenvalue one is still simple for* $\mathsf{M} : L^1 \to L^1$.

Proof. Assume $g \in L^1$ with $\mathsf{M}g = g, \nu_q(g) = 1$. For $\epsilon > 0$ take $\tilde{g} \in C(J)$ with $\|g - \tilde{g}\|_{L^1} < \epsilon$. Then $\|g - 1\|_{L^1} \le \|M^n\|\epsilon + \|M^n\tilde{g} - 1\|_{L^1}$. Since $\|M^n\| = 1$ the lemma follows by the mixing property (vi). □

Proof of Main theorem (a). It remains to prove $D^2T(q) > 0$ ($\forall q > 0$). By Proposition 6.3 we can reduce this to show that $\sigma^2(\Psi) > 0$. Assume $\sigma^2(\Psi) = 0$. By Lemma 2.2, $\mathsf{M}\Psi$ can be extended to a Lipschitz map. So $\mathsf{M}\Psi \in CBV(J)$. The mixing property (vi) implies that one is the only eigenvalue on the unit circle. As $\int \mathsf{M}\Psi \, d\nu_q = 0$, the spectral decomposition shows that there exist $0 < \rho < 1$ and $K > 0$ such that $|\mathsf{M}^n\Psi|_\infty \le K\rho^n$. This implies that the correlation $\int(\Psi \circ F^n)\Psi \, d\nu_q$ decays exponentially fast. Then it is standard that $\sigma^2(\Psi) = 0$ implies the existence of a weak solution $g \in L^2(\nu_q)$ for $\Psi(z) = g(Fz) - g(z)$. Denote $V : L^1 \to L^1$ by $(Vg)(z) = g(Fz)$. Notice that $\mathsf{M} \cdot V$ is equal to identity. Then we can rewrite the equation as $\Psi = (1 - \mathsf{M})Vg$. On the other hand, if $v = \sum_{k\ge1} \mathsf{M}^k\Psi + \Psi$ then $\Psi = (1 - \mathsf{M})v$. By Lemma 6.4, $Vg = v$ a.e., which implies $g = \mathsf{M}v$ a.e. In turn, $(\mathsf{M}v) \circ F = v$ a.e. By continuous extension, $(\mathsf{M}v) \circ F = v$ except for boundary points ∂J_k. So we obtain a nearly-continuous solution

$$\Psi = (\mathsf{M}v) \circ F - \mathsf{M}v.$$

Integrating the last equation by ν_q for $\forall q > 0$, it follows that $DT(q)$ is constant for $\forall q > 0$. Since $T(0) = 1, T(1) = 0$ and by the convexity of $T(q)$, we conclude that $DT(q) = 1$. Take $F^n z = z$. Then $\Psi^{(n)}(z) = 0$. By Theorem 3.1 it leads to the contradiction that two ergodic measures $u = \nu_1$ and ν_0 are the same. □

7. *Multifractal formalism II (proof of Main theorem (c))*

This section treats the remaining case that $\alpha(0+)$ is finite. For each integer N and $(s, q) \in \mathbb{R}^2$ define $\phi_{s,q}[N] \in C(J_0)$ by

$$\phi_{s,q}[N](z) = \begin{cases} q \log \psi(z) - s \log|DF(z)|, & \text{if } z \in \bigcup_{k=1}^N \operatorname{int} J_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then (3.1) holds and the whole argument above can be adapted similarly; if $P_N(s, q)$ denotes the pressure for $\phi_{s,q}[N]$ then $P_N(s, q)$ has a unique zero $T_N(q)$ depending analytically on $q \in \mathbb{R}$. This is, of course, equivalent to considering the hyperbolic system obtained by restricting F on $\bigcup_{k=1}^N J_k$.

PROPOSITION 7.1. *For each* $\alpha > \alpha(0+)$, $\mathrm{HD}(K(\alpha)) = 1$.

LEMMA 7.2. *There exists* $\{q_m\}$ *so that* $\lim_{m\to\infty} q_m = 0$ *and* $DT_m(q_m) = -\alpha$.

Proof. First, we show that $T_N(q)$ converges to $T(q)$ uniformly on compact sets of $[0, \infty)$. Notice that this includes $\lim_{q\to0+} T(q) = T(0)$. Since $N \to P_N(s, q)$ is increasing, it follows that $0 = P_N(T_N(q), q) \le P_{N+1}(T_N(q), q)$. This implies that $N \mapsto T_N(q)$ is increasing. If $M(q) = \sup_N T_N(q)$ then

$$|P(M(q), q)| \le |P(M(q), q) - P(T_N(q), q)| + |P(T_N(q), q) - P_N(T_N(q), q)|.$$

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The first term goes to zero by the continuity of P. As for the second term, (3.2) shows

$$|P(T_N(q),q) - P_N(T_N(q),q)| \le m^{-1}(C_3|T_N(q)|) + m^{-1}(C_3|T_N(q)|) + |m^{-1}\log \mathsf{M}^m_{T_N(q),q}1 - m^{-1}\log \tilde{\mathsf{M}}^m_{T_N(q),q}1|$$

where $\tilde{\mathsf{M}}_{T_N(q),q}$ denotes the transfer operator for $\phi_{T_N(q),q}[N]$. From this, $P(M(q),q) = 0$. Hence $M(q) = T(q)$.

Secondly, we see that $\lim_{N\to\infty} T_N(q) = \infty$ for $q < 0$ since

$$1 = \exp(P_N(T_N(q),q)) \ge 2^{|q|N}|DF(F^N z))|^{-T_N(q)}$$

for some z (see (ii)). Lastly choose a decreasing neighbourhood of zero and apply the mean value theorem to obtain the result. □

Proof of Proposition 7.1. Let ν_q^N be the equilibrium state for $\phi_{T_N(q),q}[N]$ and let $\alpha_N(q) = -DT_N(q)$ for $q \in \mathbb{R}$. Define

$$K_q^N = \left\{ z \in \bigcap_{n\ge 0} F^{-n}\left(\bigcup_{k=1}^N J_k\right) : \lim_{n\to\infty} \frac{-\sum_{l=0}^{n-1}\log\psi(F^l z)}{\sum_{l=0}^{n-1}\log|DF(F^l z)|} = \alpha_N(q) \right\}.$$

As in Proposition 5.2, it can be shown that

$$\mathrm{HD}(\nu_q^m) = \mathrm{HD}(K_{q_m}^m) = T_m(q_m) + q_m\alpha_m(q_m). \tag{7.1}$$

Observe that $\{K_{q_m}^m\}$ is a sequence of increasing subsets of $K(\alpha)$. Since $T_m(q)$ is analytic and $DT_m(q)$ is increasing in q, we have $|T_m(q_m) - T_m(0)| \le \alpha|q_m|$. On the other hand, we showed in the proof of Lemma 7.2 that $T_m(0)$ converges to $T(0) = 1$. So $\lim_{m\to\infty} T_m(q_m) = 1$. By (7.1) and Lemma 7.2 the proof is completed. □

Addendum. The author found that the paper by Pollicott and Weiss [**15**] is closely related to this paper.

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