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Frequently hypercyclic operators and vectors

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Abstract. We study frequently hypercyclic operators, a natural new concept in hypercyclicity that was recently introduced by F. Bayart and S. Grivaux. We derive a strengthened version of their Frequent Hypercyclicity Criterion, which allows us to obtain examples of frequently hypercyclic operators in a straightforward way. Moreover, Bayart and Grivaux have noted that the frequent hypercyclicity setting differs from general hypercyclicity in that the set of frequently hypercyclic vectors need not be residual. We show here that, under weak assumptions, this set is only of first category. Motivated by this we study the question of whether one may write every vector in the underlying space as the sum of two frequently hypercyclic vectors. This investigation leads us to the introduction of a new notion, that of Runge transitivity.

1. *Introduction*

Bayart and Grivaux [**1, 3**] have recently introduced an interesting new notion in the theory of hypercyclic operators. We recall that a (continuous and linear) operator T on a topological vector space X is said to be *hypercyclic* if there exists a vector $x \in X$, also called *hypercyclic*, whose orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X. Thus, x is a hypercyclic vector if its orbit meets every non-empty open subset U of X. Now, Bayart and Grivaux call x frequently hypercyclic if it meets every such set U 'often' in the sense of positive lower density.

Definition 1.1. [**1, 3**] Let X be a topological vector space and $T : X \to X$ an operator. Then a vector $x \in X$ is called *frequently hypercyclic* for T if, for every non-empty open subset U of X, the set

$$\{n \in \mathbb{N} : T^n x \in U\}$$

has positive lower density. The operator T is called *frequently hypercyclic* if it possesses a frequently hypercyclic vector. The set of all frequently hypercyclic vectors for T is denoted by $FHC(T)$.

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We recall that the *lower density* of a subset A of $\mathbb{N}$ is defined as

$$\underline{\text{dens}}\,(A) = \liminf_{N\to\infty} \frac{\#\{n \in A : n \le N\}}{N},$$

where # denotes the cardinality of a set. The *lower density* of a strictly increasing sequence (n_k) of positive integers is defined as the lower density of the corresponding subset of $\mathbb{N}$, that is,

$$\underline{\text{dens}}\,(n_k) = \liminf_{N\to\infty} \frac{\#\{k \in \mathbb{N} : n_k \le N\}}{N}.$$

Since, for any N with $n_k \le N < n_{k+1}$, we have $\#\{j \in \mathbb{N} : n_j \le N\}/N = k/N$, we obtain the following simple fact.

LEMMA 1.2. *Let (n_k) be a strictly increasing sequence of positive integers. Then we have*

$$\underline{\text{dens}}\,(n_k) = \liminf_{k\to\infty} \frac{k}{n_k}.$$

As a consequence we have the following well-known result.

LEMMA 1.3. *A strictly increasing sequence (n_k) of positive integers is of positive lower density if and only if*

$$\sup_{k\ge 1} \frac{n_k}{k} < \infty.$$

This, in turn, leads to an alternative, and useful, formulation of frequent hypercyclicity.

Observation 1.4. A vector $x \in X$ is frequently hypercyclic for an operator T on X if and only if, for every non-empty open subset U of X, there is a strictly increasing sequence (n_k) of positive integers and some $C > 0$ such that

$$n_k \le Ck \quad \text{and} \quad T^{n_k}x \in U \quad \text{for all } k \in \mathbb{N}.$$

In this paper we shall study the new concept of frequent hypercyclicity by taking up the investigations started by Bayart and Grivaux. We first present a strengthened form of their Frequent Hypercyclicity Criterion (§2). In §3 we introduce the notion of Runge transitivity and show that every Runge transitive operator is frequently hypercyclic. Next, we consider various examples of frequently hypercyclic operators (§4). As noted by Bayart and Grivaux, the Baire category theorem cannot be used in the same way as in hypercyclicity because, under suitable assumptions, the set of frequently hypercyclic vectors is non-residual. We show in §5 that, under even weaker assumptions, it is only of first category. In addition, we study the 'bigness' of the set $FHC(T)$ of frequently hypercyclic vectors in the sense of asking if $X = FHC(T) + FHC(T)$.

2. *The Frequent Hypercyclicity Criterion*

Bayart and Grivaux have derived a sufficient condition for an operator to be frequently hypercyclic (see [**3**, Theorem 2.1]). Their criterion was motivated by the well-known Hypercyclicity Criterion (see, for instance, [**13**]). We derive here a strengthened form of their criterion.

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We first need to clarify the following. We recall that an *F-space* is a topological vector space whose topology is induced by a complete translation-invariant metric. One may assume that this metric is defined by an *F-norm*, that is, a positive mapping $\|\cdot\| : X \to \mathbb{R}_+$ so that, for all $x, y \in X$ and all scalars c,

$$\begin{aligned} \|x\| &> 0 \quad \text{if } x \neq 0, \\ \|cx\| &\leq \|x\| \quad \text{if } |c| \leq 1, \\ \|cx\| &\to 0 \quad \text{for } c \to 0, \\ \|x + y\| &\leq \|x\| + \|y\|, \end{aligned}$$

see [**16**].

In an F-space X, a series $\sum_{k=1}^{\infty} x_k$ is said to be *unconditionally convergent* if for every $\varepsilon > 0$ there is some $N \geq 1$ such that for every finite set $F \subset \mathbb{N}$ with $F \cap \{1, 2, \ldots, N\} = \emptyset$ we have

$$\left\| \sum_{k \in F} x_k \right\| < \varepsilon.$$

THEOREM 2.1. (Frequent Hypercyclicity Criterion) *Let T be an operator on a separable F-space X. Suppose that there are a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that:*

(i) $\sum_{n=1}^{\infty} T^n x$ *converges unconditionally, for all* $x \in X_0$*;*

(ii) $\sum_{n=1}^{\infty} S^n x$ *converges unconditionally, for all* $x \in X_0$*;*

(iii) $TSx = x$ *for all* $x \in X_0$.

Then T is frequently hypercyclic.

We first want to comment on various aspects of this result.

Remark 2.2. (a) Since $\sum_{k=1}^{\infty} \|x_k\| < \infty$ implies the unconditional convergence of $\sum_{k=1}^{\infty} x_k$, Theorem 2.1 implies [**3**, Theorem 2.1].

If X is a Fréchet space with a sequence (p_n) of seminorms and an F-norm $\|\cdot\|$, each defining the topology, then we have that

$$\begin{aligned} \sum_{k=1}^{\infty} \|x_k\| < \infty &\Longrightarrow \sum_{k=1}^{\infty} p_n(x_k) < \infty \quad \text{for all } n \in \mathbb{N}, \\ &\Longrightarrow \sum_{k=1}^{\infty} x_k \quad \text{converges unconditionally.} \end{aligned}$$

The intermediate convergence is known as the *absolute convergence* of $\sum_{k=1}^{\infty} x_k$. The first implication follows from the fact that

$$x \mapsto \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x)}{1 + p_n(x)}$$

defines an equivalent F-norm on X. We note that even the first implication is, in general, strict. To see this we consider the Fréchet space $X = \mathbb{C}^{\mathbb{N}}$ with seminorms $p_n(x) = \sup_{j \leq n} |x_j|$, $n \in \mathbb{N}$ for $x = (x_j)$. When we define

$$x^{(k)} = (0, 0, \ldots, 0, 1, 1, 1, \ldots)$$

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for $2^n \le k < 2^{n+1}$, $n \ge 0$, with n leading zeros, then we have $\sum_{k=1}^{\infty} p_n(x^{(k)}) = 2^n - 1 < \infty$ for all $n \in \mathbb{N}$, while

$$\sum_{k=1}^{\infty} \|x^{(k)}\| = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x^{(k)})}{1 + p_n(x^{(k)})} = \infty.$$

This shows that the conditions in Theorem 2.1 of [**3**] are in fact very strong, so that one may wonder if they even imply a stronger form of frequent hypercyclicity.

(b) The conditions in Theorem 2.1 moreover imply that T is topologically mixing and chaotic.

An operator T on an F-space X is called *topologically mixing* if for any pair of non-empty open subsets U, V of X there exists some $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \ge N$. It is well known that T is hypercyclic if and only if it is *topologically transitive*, that is, if for any pair of non-empty open subsets U, V of X there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$, see [**13**]. Thus, every topologically mixing operator is hypercyclic, but the converse is not true (see [**8**]). Now if T is an operator that satisfies the Frequent Hypercyclicity Criterion then, by conditions (i) and (ii), $T^n x \to 0$ and $S^n x \to 0$ for all $x \in X_0$. This implies that T is topologically mixing (see [**8**, Theorem 1.1] and [**12**]).

An operator T on an F-space X is called *chaotic* if it is hypercyclic and it has a dense set of periodic points, that is, vectors $x \in X$ such that $T^n x = x$ for some $n \in \mathbb{N}$. Suppose that T satisfies the Frequent Hypercyclicity Criterion. Then it is, in particular, hypercyclic, and we need only construct a dense set of periodic points. For any $x \in X_0$ we define, depending on x,

$$y_k = \sum_{n=1}^{\infty} S^{nk} x + x + \sum_{n=1}^{\infty} T^{nk} x, \quad k \in \mathbb{N}.$$

By conditions (i) and (ii) both series converge, and by (iii) we have that $T^k y_k = y_k$, so that each y_k is a periodic point for T. Moreover, it follows from (i) and (ii) that $y_k \to x$ as $k \to \infty$. Since X_0 is dense we see that T has a dense set of periodic points and hence is chaotic.

(c) The argument for chaoticity in the previous remark is essentially as described by Taniguchi [**22**, Proposition 1]. He shows there that under conditions that are stronger than those in Theorem 2.1 an operator on a separable Banach space is chaotic. He applies his criterion to deduce that for any hyperbolic automorphism ϕ of the unit disk $\mathbb{D}$ the composition operator C_ϕ is chaotic on the Hardy space $H^p(\mathbb{D})$ for $1 \le p < \infty$, while for any parabolic automorphism ϕ the operator C_ϕ is chaotic on the Hardy space $H^p(\mathbb{D})$ for $1 \le p < 2$. It follows then from Theorem 2.1 that these operators are also frequently hypercyclic. We add that, based on their eigenvalue criterion (cf. Theorem 5.4) Bayart and Grivaux [**3**, Corollary 3.7] have shown that any hyperbolic and any parabolic composition operator is frequently hypercyclic on $H^2(\mathbb{D})$.

We shall give the proof of Theorem 2.1 in a more general context that will be useful for future applications [**7**]. As usual, a concept of hypercyclicity can be generalized to a corresponding concept of universality when one replaces the sequence (T^n) of iterates of an operator T by an arbitrary sequence (T_n) of mappings. In this case one may allow different domain and range spaces.

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Definition 2.3. Let X and Y be topological spaces and $T_n : X \to Y, n \in \mathbb{N}$, mappings. Then an element $x \in X$ is called *frequently universal* for the sequence (T_n) if, for every non-empty open subset U of Y, the set

$$\{n \in \mathbb{N} : T^n x \in U\}$$

has positive lower density. The sequence (T_n) is called *frequently universal* if it possesses a frequently universal element. The set of all frequently universal elements for (T_n) is denoted by $FU(T_n)$.

It may be of interest to note that examples of frequently universal elements were already obtained in the literature some 20 years ago in the context of the universality of zeta-functions. Reich [**20, 21**] (see also [**13**, §6]) showed that the Riemann zeta-function ζ has the following frequent universality property: for any zero-free holomorphic function g in $|z| < 1/4$, any $r < 1/4$ and any $\varepsilon > 0$ the set

$$\left\{n \in \mathbb{N} : \max_{|z| \le r} |\zeta(\tfrac{3}{4} + in + z) - g(z)| < \varepsilon\right\}$$

has positive lower density.

We shall need the following concept. In an F-space X with F-norm $\|\cdot\|$, a collection of series $\sum_{k=1}^{\infty} x_{k,j}$, $j \in I$, is said to be *unconditionally convergent, uniformly in* $j \in I$, if for every $\varepsilon > 0$ there is some $N \ge 1$ such that for every finite set $F \subset \mathbb{N}$ with $F \cap \{1, 2, \ldots, N\} = \emptyset$ and every $j \in I$ we have

$$\left\| \sum_{k \in F} x_{k,j} \right\| < \varepsilon.$$

THEOREM 2.4. (Frequent Universality Criterion) *Let X be an F-space, Y a separable F-space and $T_n : X \to Y, n \in \mathbb{N}$, continuous mappings. Suppose that there are a dense subset Y_0 of Y and mappings $S_n : Y_0 \to X, n \in \mathbb{N}$, such that:*

(i) $\sum_{n=1}^{\infty} T_{k+n} S_k y$ *converges unconditionally in Y, uniformly in $k \in \mathbb{N}$, for all $y \in Y_0$;*

(ii) $\sum_{n=1}^{\infty} T_k S_{k+n} y$ *converges unconditionally in Y, uniformly in $k \in \mathbb{N}$, for all $y \in Y_0$;*

(iii) $\sum_{n=1}^{\infty} S_n y$ *converges unconditionally in X, for all $y \in Y_0$;*

(iv) $T_n S_n y \to y$ *for all $y \in Y_0$.*

Then the sequence (T_n) is frequently universal.

Theorem 2.1 is a special case of Theorem 2.4. The proof of Theorem 2.4 will be an adaptation of that of Bayart and Grivaux [**3**, Theorem 2.1]. In particular, we shall need their crucial Lemma 2.2 on the existence of a certain sequence (n_k). In fact, we give here a strengthened version of this lemma that may be useful for future applications. In addition, its proof, which will be given at the end of this section, is simpler than that in [**3**].

LEMMA 2.5. *Let $\rho(l, \nu), l, \nu \ge 1$, be non-negative numbers such that*

$$\sum_{l,\nu=1}^{\infty} \frac{\rho(l, \nu)}{2^{l+\nu}} < \infty.$$

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Then there are pairwise disjoint sets $A(l,\nu) \subset \mathbb{N}, l, \nu \geq 1$, of positive lower density such that, for all $n \in A(l,\nu), m \in A(k,\mu)$,

$$n \geq \rho(l,\nu),$$
$$|n-m| \geq \rho(l,\nu) + \rho(k,\mu) \quad \text{if } n \neq m.$$

We note that this result implies [**3**, Lemma 2.2] if we take $\rho(l,\nu) = \nu$ and we define (n_k) as the increasing sequence of the elements of $\bigcup_{l,\nu\geq 1} A(l,\nu)$, $R_k = \nu$ and $m_k = l$ if $n_k \in A(l,\nu)$.

Proof of Theorem 2.4. Since Y is separable we may assume that Y_0 is countable,

$$Y_0 = \{y_1, y_2, y_3, \dots\}.$$

The conditions (i)–(iv) imply that there are $N_l \in \mathbb{N}, l \geq 1$, such that, for all $\lambda \leq l$, all $k \in \mathbb{N}$ and all finite subsets F of $\mathbb{N}$ with $F \cap \{1, 2, \dots, N_l - 1\} = \emptyset$ we have

$$\left\| \sum_{n\in F} T_{k+n} S_k y_\lambda \right\| \leq \frac{1}{l2^l}, \tag{1}$$

$$\left\| \sum_{n\in F} T_k S_{k+n} y_\lambda \right\| \leq \frac{1}{l2^l}, \tag{2}$$

$$\left\| \sum_{n\in F} S_n y_l \right\| \leq \frac{1}{2^l}, \tag{3}$$

$$\|T_n S_n y_l - y_l\| \leq \frac{1}{2^l} \quad \text{for } n \geq N_l. \tag{4}$$

We let $\rho(l,\nu) = \nu$ for $l, \nu \geq 1$; then by Lemma 2.5 there are pairwise disjoint sets $A(l,\nu), l, \nu \geq 1$, of positive lower density with the properties stated there. We define

$$z_n = \begin{cases} y_l & \text{if } n \in A(l, N_l), \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

and set

$$x = \sum_{n=1}^{\infty} S_n z_n. \tag{6}$$

In order to see that (6) converges in X we note that, for all $l \geq 1$,

$$\sum_{n\in A(l,N_l)} S_n z_n = \sum_{n\in A(l,N_l)} S_n y_l \tag{7}$$

converges unconditionally by condition (iii). Since, by Lemma 2.5,

$$n \geq N_l \quad \text{for } n \in A(l, N_l),$$

we have by (3) that, for any finite subset F of $A(l, N_l)$,

$$\left\| \sum_{n\in F} S_n y_l \right\| \leq \frac{1}{2^l}.$$

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It is easily seen that this, together with the unconditional convergence of (7), implies the (unconditional) convergence of (6).

Now let $n \in A(l, N_l)$ for some $l \in \mathbb{N}$. Then

$$T_n x - y_l = \sum_{j<n} T_n S_j z_j + \sum_{j>n} T_n S_j z_j + (T_n S_n z_n - y_l). \tag{8}$$

We consider the three summands in turn.

We have with (5)

$$\sum_{j<n} T_n S_j z_j = \sum_{\lambda=1}^{l} \sum_{\substack{j\in A(\lambda,N_\lambda)\\ j<n}} T_n S_j y_\lambda + \sum_{\lambda=l+1}^{\infty} \sum_{\substack{j\in A(\lambda,N_\lambda)\\ j<n}} T_n S_j y_\lambda. \tag{9}$$

Since, by Lemma 2.5,

$$|n - j| \geq N_l,$$

we have by (1) for $\lambda \leq l$

$$\left\| \sum_{\substack{j\in A(\lambda,N_\lambda)\\ j<n}} T_n S_j y_\lambda \right\| = \left\| \sum_{\substack{j\in A(\lambda,N_\lambda)\\ j<n}} T_{j+(n-j)} S_j y_\lambda \right\| \leq \frac{1}{l2^l}; \tag{10}$$

since also

$$|n - j| \geq N_\lambda \quad \text{for } j \in A(\lambda, N_\lambda),$$

we have for all λ

$$\left\| \sum_{\substack{j\in A(\lambda,N_\lambda)\\ j<n}} T_n S_j y_\lambda \right\| = \left\| \sum_{\substack{j\in A(\lambda,N_\lambda)\\ j<n}} T_{j+(n-j)} S_j y_\lambda \right\| \leq \frac{1}{\lambda 2^\lambda}. \tag{11}$$

Hence, (9)–(11) imply that

$$\left\| \sum_{j<n} T_n S_j z_j \right\| \leq \sum_{\lambda=1}^{l} \frac{1}{l2^l} + \sum_{\lambda=l+1}^{\infty} \frac{1}{\lambda 2^\lambda} \leq \frac{2}{2^l}. \tag{12}$$

In essentially the same way, using (2) instead of (1), we see that, for all $M > k$,

$$\left\| \sum_{n<j\leq M} T_n S_j z_j \right\| \leq \frac{2}{2^l},$$

hence

$$\left\| \sum_{j>n} T_n S_j z_j \right\| \leq \frac{2}{2^l}. \tag{13}$$

Finally, by (4) and (5) we have

$$\|T_n S_n z_n - y_l\| = \|T_n S_n y_l - y_l\| \leq \frac{1}{2^l} \tag{14}$$

since $n \geq N_l$.

Altogether, (8) and (12)–(14) tell us that

$$\|T_n x - y_l\| \leq \frac{5}{2^l} \quad \text{if } n \in A(l, N_l).$$

Since $A(l, N_l)$ has positive lower density, Y_0 is dense and $5/2^l \to 0$ as $l \to \infty$ we conclude that (T_n) is frequently universal. □

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We note that the proof only uses Lemma 2.5 in the weaker form of Lemma 2.2 of [**3**].

We turn to the proof of Lemma 2.5, where we first need a simple consequence of Lemma 1.3. Since $n_{k_l}/l = (n_{k_l}/k_l)(k_l/l)$, we have the following.

LEMMA 2.6. *If (n_k) and (k_l) are strictly increasing sequences of positive lower density then so is (n_{k_l}).*

Proof of Lemma 2.5. Since $\sum_{l,\nu=1}^{\infty} 1/2^{l+\nu} < \infty$ we may increase each $\rho(l,\nu)$ to its nearest greater positive integer so that we may assume that $\rho(l,\nu) \in \mathbb{N}$ for all l, ν.

In order to define the sets $A(l,\nu)$ we use repeatedly the easiest and most natural splitting of an infinite sequence into two sequences: we put its elements alternately into the one or the other subsequence. This way we first split $\mathbb{N}$ into its odd numbers and its even numbers. Keeping the odd numbers, we split the even numbers into the numbers 2 (mod 4) and 0 (mod 4). Repeating this process we arrive at a partition of $\mathbb{N}$ into the sets

$$I(l) = \{n \in \mathbb{N} : n = 2^{l-1} \pmod{2^l}\} = \{2^{l-1}(2j-1) : j \geq 1\}, \quad l \geq 1.$$

We finally split each set $I(l)$ in the same way into countably many parts. We thus obtain a partition

$$\mathbb{N} = \bigcup_{l,\nu=1}^{\infty} I(l,\nu)$$

given by the sets

$$I(l,\nu) = \{2^{l-1}(2^{\nu}(2j-1)-1) : j \geq 1\}, \quad l, \nu \geq 1.$$

This partition of $\mathbb{N}$, of course, is well known.

We now let

$$R_k = \rho(l,\nu) \quad \text{if } k \in I(l,\nu), \tag{15}$$

$$n_k = 2\sum_{p=1}^{k-1} R_p + R_k \quad \text{for } k \geq 1 \tag{16}$$

and

$$A(l,\nu) = \{n_k : k \in I(l,\nu)\}, \quad l, \nu \geq 1.$$

We claim that the sets $A(l,\nu)$ have the required properties.

First, we have for every $k \geq 1$,

$$n_k \geq R_k$$

and, for $k > j \geq 1$,

$$\begin{aligned} n_k - n_j &= R_k + 2\sum_{p=1}^{k-1} R_p - 2\sum_{p=1}^{j-1} R_p - R_j \\ &= R_k + 2\sum_{p=j}^{k-1} R_p - R_j \geq R_k + R_j, \end{aligned}$$

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which, in particular, implies that (n_k) is strictly increasing. We deduce that, for all $n \in A(l,\nu), m \in A(k,\mu)$,

$$n \geq \rho(l,\nu) \quad \text{and} \quad |n-m| \geq \rho(l,\nu) + \rho(k,\mu) \quad \text{if } n \neq m.$$

It remains to prove that each set $A(l,\nu)$ has positive lower density.

We shall first show that the sequence (n_k) has positive lower density, which by Lemma 1.3 is equivalent to saying that

$$\sup_{k\geq 1} \frac{n_k}{k} < \infty.$$

The elements of $I(l,\nu)$ are exactly those positive integers whose binary expansion ends $01\ldots10\ldots0$ with ν consecutive 1's and $l-1$ consecutive 0's. It is then easy to see that

$$\#\{p \in I(l,\nu) : p < 2^N\} \leq 2^{N-l-\nu+1}, \quad N \geq 1.$$

Thus we have for $2^{N-1} \leq k < 2^N$, $N \geq 1$, by (15) and (16),

$$\begin{aligned}\frac{n_k}{k} &\leq \frac{2}{2^{N-1}} \sum_{p=1}^{2^N-1} R_p = \frac{1}{2^{N-2}} \sum_{l,\nu=1}^{\infty} \sum_{\substack{p\in I(l,\nu)\\ p<2^N}} R_p \\ &= \frac{1}{2^{N-2}} \sum_{l,\nu=1}^{\infty} \rho(l,\nu)\,\#\{p \in I(l,\nu) : p < 2^N\} \\ &\leq \frac{2^{N+1}}{2^{N-2}} \sum_{l,\nu=1}^{\infty} \frac{\rho(l,\nu)}{2^{l+\nu}} = 8 \sum_{l,\nu=1}^{\infty} \frac{\rho(l,\nu)}{2^{l+\nu}} < \infty,\end{aligned}$$

so that (n_k) has positive lower density. However, since each set $I(l,\nu)$ also has positive lower density (it has density $1/2^{l+\nu}$), we have by Lemma 2.6 that

$$A(l,\nu) = \{n_k : k \in I(l,\nu)\}$$

has positive lower density for each $l, \nu \geq 1$. □

3. *Further criteria for frequent hypercyclicity*

Our next criterion is very different from the Frequent Hypercyclicity Criterion. It is of a very special nature and only seems to apply to a few operators. However, we shall see in §5 that these special operators enjoy a remarkable property, cf. Theorem 5.9.

The criterion was inspired by the proof of the topological transitivity and hence hypercyclicity of the translation operator $Tf(z) = f(z+1)$ on the space $H(\mathbb{C})$ of entire functions which uses the Runge approximation theorem, see [**5**, Theorem 3.1]. We require here that the underlying space is a Fréchet space.

Definition 3.1. (Runge transitivity) An operator T on a Fréchet space X is called *Runge transitive* if there is an increasing sequence (p_n) of seminorms defining the topology of X and numbers $N_m \in \mathbb{N}$, $C_{m,n} > 0$ for $m, n \in \mathbb{N}$ such that:

(i) for all $m, n \in \mathbb{N}$ and $x \in X$,

$$p_m(T^n x) \leq C_{m,n}\, p_{n+N_m}(x);$$

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(ii) for all $m, n \in \mathbb{N}$, $x, y \in X$ and $\varepsilon > 0$ there is some $z \in X$ such that

$$p_n(z - x) < \varepsilon \quad \text{and} \quad p_m(T^{n+N_m} z - y) < \varepsilon.$$

If, in these conditions, $n + N_m$ were allowed to be arbitrary integers we would just have statements of the continuity of all T^n and the topological transitivity of T. Requiring these numbers to be of the form $n + N_m$ imposes a severe restriction.

In particular, no operator on a Banach space is ever Runge transitive. Indeed, in that case the norm must be dominated by some p_n, that is, $\|x\| \le C p_n(x)$ for some $C > 0$ and all $x \in X$. Hence condition (ii) requires that for all $x, y \in X$ there are $x_k \in X$, $k \in \mathbb{N}$, such that

$$\|x_k - x\| < \frac{1}{k} \quad \text{and} \quad \|T^{n+N_n} x_k - y\| < \frac{1}{k},$$

which is impossible if $y \neq T^{n+N_n} x$.

On certain Fréchet spaces, however, there are Runge transitive operators.

Example 3.2. (a) The translation operator $Tf(z) = f(z + 1)$ on the space $H(\mathbb{C})$ of entire functions is Runge transitive. To see this we fix δ with $0 < \delta < 1$ and set

$$p_n(f) = \sup_{|z| \le n - \delta} |f(z)|.$$

Then we have for all $f \in H(\mathbb{C})$

$$p_m(T^n f) = \sup_{|z-n| \le m-\delta} |f(z)| \le \sup_{|z| \le n+m-\delta} |f(z)| = p_{n+m}(f).$$

Moreover, since the disks $|z| \le n - \delta$ and $|z - (n + m)| \le m - \delta$ are disjoint, the Runge approximation theorem implies that for all $g, h \in H(\mathbb{C})$ and $\varepsilon > 0$ there is some $f \in H(\mathbb{C})$ with

$$\sup_{|z| \le n-\delta} |f(z) - g(z)| < \varepsilon \quad \text{and} \quad \sup_{|z-(n+m)| \le m-\delta} |f(z) - h(z - n - m)| < \varepsilon,$$

hence

$$p_n(f - g) < \varepsilon \quad \text{and} \quad p_m(T^{n+m} f - h) < \varepsilon.$$

This shows that conditions (i) and (ii) hold with $N_m = m$ and $C_{m,n} = 1$.

More generally, each translation operator $T_a f(z) = f(z+a)$, $a \neq 0$, on $H(\mathbb{C})$ is Runge transitive, as can be seen by taking $p_n(f) = \sup_{|z| \le |a| n - \delta} |f(z)|$.

(b) The backward shift operator $B(x_k) = (x_{k+1})$ on the space $\omega = \mathbb{R}^{\mathbb{N}}$ or $\mathbb{C}^{\mathbb{N}}$ of all real or complex sequences is Runge transitive. Here we put

$$p_n(x) = \sup_{k \le n} |x_k|, \; N_m = m \quad \text{and} \quad C_{m,n} = 1.$$

The simple derivations of conditions (i) and (ii) are omitted.

We proceed to the announced criterion.

THEOREM 3.3. *Every Runge transitive operator on a separable Fréchet space is frequently hypercyclic.*

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Proof. We let $\rho(l, \nu) = \nu$ for $l, \nu \geq 1$; then by Lemma 2.5 there are pairwise disjoint sets $A(l, \nu)$ of positive lower density with the properties stated there. Let (n_k) be the increasing sequence of the elements of $\bigcup_{l,\nu\geq 1} A(l, \nu)$, and set $\nu_k = \nu$ and $l_k = l$ if $n_k \in A(l, \nu)$.

Now let T be a Runge transitive operator on a separable Fréchet space. Let (y_k) be an arbitrary sequence in X which will be further specified later, and let p_n, N_m and $C_{m,n}$ be such that the properties (i) and (ii) in the definition of Runge transitivity hold.

We then define recursively a sequence (x_k) in X. We let $x_1 = 0$, and if $x_1, \ldots, x_{k-1}$, $k \geq 2$, have been defined then condition (ii) gives us a vector $x_k \in X$ with

$$p_{n_k-\nu_k}(x_k - x_{k-1}) < \frac{1}{2^k(\max_{j<k}(C_{l_j,n_j}) + 1)} \tag{17}$$

and

$$p_{l_k}(T^{n_k-\nu_k+N_{l_k}}x_k - y_k) < \frac{1}{2^k}. \tag{18}$$

Note that (17) is well defined since by Lemma 2.5 we have that

$$n_k - \nu_k \geq n_{k-1} + \nu_{k-1} \geq 1 \quad \text{for } k \geq 2.$$

This also shows that $n_k - \nu_k \to \infty$. Since the p_n are increasing, (17) therefore implies that for any $n \in \mathbb{N}$ there is some K_n such that

$$p_n(x_k - x_{k-1}) < \frac{1}{2^k} \quad \text{for } k \geq K_n,$$

hence

$$\sum_{k=2}^{\infty} p_n(x_k - x_{k-1}) < \infty \quad \text{for all } n \in \mathbb{N}.$$

The fact that (p_n) defines the topology of X now implies that the series

$$x := \sum_{k=2}^{\infty}(x_k - x_{k-1})$$

converges in X.

Next, we consider $k \in \mathbb{N}$. Since $x_1 = 0$ we also have

$$x = x_k + \sum_{j=k+1}^{\infty}(x_j - x_{j-1}),$$

hence

$$T^{n_k}x = T^{n_k}x_k + \sum_{j=k+1}^{\infty} T^{n_k}(x_j - x_{j-1}). \tag{19}$$

We shall only consider those $k \geq 1$ for which $\nu_k = N_{l_k}$. Then (18) shows that

$$p_{l_k}(T^{n_k}x_k - y_k) < \frac{1}{2^k}. \tag{20}$$

Moreover, we have for $j \geq k + 1$ by condition (i) of Runge transitivity,

$$\begin{aligned} p_{l_k}(T^{n_k}(x_j - x_{j-1})) &\leq C_{l_k,n_k}\, p_{n_k+N_{l_k}}(x_j - x_{j-1}) \\ &= C_{l_k,n_k}\, p_{n_k+\nu_k}(x_j - x_{j-1}). \end{aligned}$$

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By Lemma 2.5 we have $n_k + \nu_k \le n_j - \nu_j$ for $j \ge k+1$. Since the p_n are increasing we conclude with (17) that

$$p_{l_k}(T^{n_k}(x_j - x_{j-1})) \le C_{l_k,n_k}\, p_{n_j-\nu_j}(x_j - x_{j-1}) \le \frac{1}{2^j}. \tag{21}$$

From (19), (20) and (21) we thus obtain that

$$p_{l_k}(T^{n_k}x - y_k) < \frac{2}{2^k}. \tag{22}$$

To finish the proof it suffices to define the sequence (y_k) appropriately. Let (z_k) be a dense sequence in X. When we then define $y_k = z_l$ if $n_k \in A(l, N_l)$ and arbitrarily, otherwise, we have by (22)

$$p_l(T^{n_k}x - z_l) < \frac{2}{2^k} \quad \text{for } n_k \in A(l, N_l).$$

Since each set $A(l, N_l)$ has positive lower density it follows that x is frequently hypercyclic. □

To end this section we note that, as in the case of hypercyclic operators, frequent hypercyclicity transfers from one operator to another if they are linked by a certain commutative diagram. The proof is essentially the same as that in [**18**, Lemma 2.1].

THEOREM 3.4. *Let X and Y be F-spaces and $S : Y \to Y, T : X \to X$ operators. Suppose that there is an operator $\phi : Y \to X$ of dense range such that the following diagram commutes:*

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow{T} & X \end{array}$$

That is, we have $T \circ \phi = \phi \circ S$. If S is frequently hypercyclic then so is T.

4. *Examples of frequently hypercyclic operators*

In [**3**, §2.3], Bayart and Grivaux have obtained the first examples of frequently hypercyclic operators. However, only the first two of their examples uses their Frequent Hypercyclic Criterion. We shall see here that the strengthened version of this criterion, Theorem 2.1, is easily applicable in all of their examples.

THEOREM 4.1. *The operator D of differentiation, $Df = f'$, is frequently hypercyclic on the space $H(\mathbb{C})$ of entire functions.*

Proof. We consider the set X_0 of polynomials, which is dense in $H(\mathbb{C})$, and we define $S : X_0 \to X_0$ by $(Sf)(z) = \int_0^z f(\zeta)\, d\zeta$. Then conditions (i) and (iii) of the Frequent Hypercyclicity Criterion are satisfied, and it suffices to check condition (ii) for all monomials $f_k(z) = z^k$, $k \ge 0$. For these we have

$$\sum_{n=1}^{\infty} S^n f_k(z) = \sum_{n=1}^{\infty} \frac{k!}{(k+n)!} z^{k+n},$$

which converges absolutely, hence unconditionally in $H(\mathbb{C})$. □

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The translation operators on $H(\mathbb{C})$ are Runge transitive and hence frequently hypercyclic (see §3). Their frequent hypercyclicity was already shown by Bayart and Grivaux, using Arakelyan's theorem. We show here that this also follows from the Frequent Hypercyclicity Criterion.

THEOREM 4.2. *The operators T of translation, $T_a f(z) = f(z+a)$, $a \neq 0$, are frequently hypercyclic on the space $H(\mathbb{C})$ of entire functions.*

Proof. It follows easily from Theorem 3.4 that we may assume $a = 1$. We use the idea of Gethner and Shapiro [**10**] in their proof of the hypercyclicity of $T = T_1$. For $m, k \in \mathbb{N}$ we consider the entire functions

$$f_{m,k}(z) = z^m \left(\frac{\sin(z/k)}{(z/k)} \right)^{m+2}.$$

Since $f_{m,k}(z) \to z^m$ in $H(\mathbb{C})$ as $k \to \infty$, we have that $X_0 = \operatorname{span}\{f_{m,k}(z-l) : m, k \in \mathbb{N}, l \in \mathbb{N}_0\}$ is dense in $H(\mathbb{C})$. Moreover, we define $S : X_0 \to X_0$ by $Sf(z) = f(z-1)$. Then, for all m, k and l,

$$\sum_{n=1}^{\infty} T^n(f_{m,k}(z-l)) = k^{m+2} \sum_{n=1}^{\infty} \frac{\sin^{m+2}((z-l+n)/k)}{(z-l+n)^2}$$

converges absolutely, hence unconditionally in $H(\mathbb{C})$, which implies condition (i) of the Frequent Hypercyclicity Criterion. Condition (ii) follows in the same way, while condition (iii) is obvious. □

In [**6**] we have obtained a joint generalization of the last two results, using a different criterion for frequent hypercyclicity (see Theorem 5.4). In fact, every operator on $H(\mathbb{C}^N)$ that commutes with all translation operators T_a, $a \in \mathbb{C}^N$, and that is not a scalar multiple of the identity is frequently hypercyclic.

Next, we turn to weighted shift operators. Let X be an *F-sequence space*, that is, a subspace of the space ω of all (real or complex) sequences that is an F-space and in which each coordinate functional $(x_n)_{n\in\mathbb{N}} \mapsto x_N$, $N \in \mathbb{N}$, is continuous. A *weighted backward shift* is an operator $T : X \to X$ of the form

$$T(x_n) = (w_{n+1} x_{n+1})_{n\in\mathbb{N}},$$

where (w_n) is some sequence of non-zero scalars.

Let e_n, $n \in \mathbb{N}$, denote the canonical unit sequences.

THEOREM 4.3. *Let X be an F-sequence space in which* $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ *is dense, and let $T : X \to X$ be a weighted backward shift with weights w_n, $n \in \mathbb{N}$. If*

$$\sum_{n=1}^{\infty} \frac{1}{w_1 w_2 \cdots w_n} e_n \tag{23}$$

converges unconditionally in X then T is frequently hypercyclic.

Proof. By assumption the set $X_0 = \operatorname{span}\{e_n : n \in \mathbb{N}\}$ is dense in X, and for any $x \in X_0$, $\sum_{n=1}^{\infty} T^n x$ is a finite series, hence unconditionally convergent. For $S : X_0 \to X_0$ we take

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the corresponding forward shift defined by

$$S(x_n) = \left(\frac{x_{n-1}}{w_n}\right)_{n\in\mathbb{N}},$$

where $x_0 = 0$. Then $TSx = x$ for all $x \in X$. Moreover we have for $k \in \mathbb{N}$

$$\sum_{n=1}^{\infty} S^n e_k = \sum_{n=1}^{\infty} \frac{1}{w_{k+1}w_{k+2}\cdots w_{k+n}} e_{k+n}$$
$$= w_1 w_2 \cdots w_k \sum_{n=1}^{\infty} \frac{1}{w_1 w_2 \cdots w_{k+n}} e_{k+n}.$$

The hypothesis implies that this series converges unconditionally, hence so does $\sum_{n=1}^{\infty} S^n x$ for any $x \in X_0$. It remains to apply the Frequent Hypercyclicity Criterion. □

In the special case of $X = l^p$, $1 \le p < \infty$, considered in [**3**], condition (23), reads

$$\sum_{n=1}^{\infty} \frac{1}{(w_1 w_2 \cdots w_n)^p} < \infty.$$

We note that in contrast to [**3**] we do not require the w_n to be bounded away from zero; on the other hand, their boundedness is needed in order that T is well-defined.

A comparison with [**14**, Theorem 8], which characterizes the chaotic weighted backward shifts, leads to the following result.

COROLLARY 4.4. *Let X be an F-sequence space in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. Then every chaotic weighted backward shift is frequently hypercyclic.*

For another application of Theorem 4.3 we identify the space $H(\mathbb{C})$ of entire functions with the sequence space

$$\left\{(x_k) \in \mathbb{C}^{\mathbb{N}_0} : \sum_{k=0}^{\infty} |x_k| R^k < \infty \text{ for all } R > 0\right\}.$$

Then the differentiation operator $Df = f'$ becomes the weighted backward shift with weight sequence (n).

COROLLARY 4.5. *Let T be a weighted backward shift on $H(\mathbb{C})$ with weights w_n, $n \in \mathbb{N}$. If*

$$\left|\prod_{k=1}^{n} w_k\right|^{1/n} \to \infty$$

then T is frequently hypercyclic. In particular, the differentiation operator is frequently hypercyclic on $H(\mathbb{C})$.

We finally consider the space $\omega = \mathbb{R}^{\mathbb{N}}$ or $\mathbb{C}^{\mathbb{N}}$ of all sequences.

COROLLARY 4.6. *Every weighted backward shift on ω is frequently hypercyclic.*

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It would be of great interest to improve Theorem 4.3 to a characterization of frequently hypercyclic weighted backward shifts. At present we only have the following necessary condition, which follows [**3**, Example 2.8]. Conjecture 2.10 of [**3**], by which this necessary condition would also be sufficient in the case $X = l^p$, was recently disproved by Peris and the second author [**15**].

PROPOSITION 4.7. *Let X be a locally convex or locally bounded F-sequence space in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. If $T : X \to X$ is a frequently hypercyclic weighted backward shift with weights $w_n, n \in \mathbb{N}$, then there exists a sequence (n_k) of positive lower density such that*

$$\sum_{k=1}^{\infty} \frac{1}{w_1 w_2 \cdots w_{n_k}} e_{n_k}$$

converges.

Proof. Let x be a frequently hypercyclic vector for T. Since the coordinate functional $(x_n) \mapsto x_1$ is continuous there is a sequence (n_k) of positive lower density such that

$$|[T^{n_k} x] - 1| < \tfrac{1}{2} \quad \text{for all } k \in \mathbb{N},$$

where $[y]_k$ denotes the kth component of the sequence y. Hence

$$|w_2 w_3 \cdots w_{n_k+1} x_{n_k+1} - 1| < \tfrac{1}{2},$$

so that

$$|x_{n_k+1}| \geq \left| \frac{1}{2 w_2 w_3 \cdots w_{n_k+1}} \right|.$$

Since $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis and $\sum_{n=1}^{\infty} x_n e_n$ converges we have that

$$\sum_{k=1}^{\infty} \frac{1}{w_1 w_2 \cdots w_{n_k+1}} e_{n_k+1}$$

converges, see [**17**, 3.3.8 and 3.3.9]. The result follows on replacing $n_k + 1$ by n_k. □

In spite of Remark 2.2(b) and Corollary 4.4 there is no general relationship between frequent hypercyclicity and chaos or the property of topological mixing, respectively. Example 2.8 of [**3**], that is, the special case $X = l^p$ of Proposition 4.7, was used in [**3**, Example 2.9] to show that the backward shift on the Bergman space A^2 is hypercyclic but not frequently hypercyclic. By [**8**, Theorem 1.2] this operator is even topologically mixing. In [**4**, §5.7] one can find an example of a frequently hypercyclic operator (on c_0) that is neither chaotic nor topologically mixing. Only the question if every chaotic operator is frequently hypercyclic remains open [**4**, Question 6.5].

5. *The set of frequently hypercyclic vectors*

It is a well-known and often used fact in hypercyclicity that the set of hypercyclic vectors of an operator is either empty or a dense G_δ-set, hence a residual set, if the underlying space is an F-space. Bayart and Grivaux [**3**] have noted that this ceases to hold, in general, for the set of frequently hypercyclic vectors. This makes the new concept of frequent

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hypercyclicity more interesting rather than less so because it will force the development of new techniques in situations where, in hypercyclicity, the Baire category theorem is invoked.

More precisely, Bayart and Grivaux [**3**, Proposition 4.1] have found conditions under which the set $FHC(T)$ of frequently hypercyclic vectors is not residual. We show here that under even weaker conditions this set is even of first category.

We recall that the upper density of a subset A of $\mathbb{N}$ is defined as

$$\overline{\text{dens}}\,(A) = \limsup_{N\to\infty} \frac{\#\{n \in A : n \le N\}}{N}.$$

We have $\overline{\text{dens}}\,(A) = 1 - \underline{\text{dens}}\,(\mathbb{N} \setminus A)$.

THEOREM 5.1. *Let T be an operator on a separable F-space X. Suppose that there exists an open subset U of X with $\overline{U} \neq X$ such that, for all $C < 1$,*

$$\{x \in X : \overline{\text{dens}}\{n \in \mathbb{N} : T^n x \in U\} \ge C\}$$

is dense. Then the set $FHC(T)$ of frequently hypercyclic vectors for T is of first category in X.

Proof. We first note that, for $C < 1$ fixed,

$$\begin{aligned} S_C :&= \{x \in X : \overline{\text{dens}}\{n \in \mathbb{N} : T^n x \in U\} \ge C\} \\ &= \bigcap_{m\ge 1} \bigcap_{K\ge 1} \bigcup_{N\ge K} \left\{x \in X : \#\{n \le N : T^n x \in U\} \ge \left(1 - \frac{1}{m}\right)CN\right\}. \end{aligned}$$

Since the sets

$$\left\{x \in X : \#\{n \le N : T^n x \in U\} \ge \left(1 - \frac{1}{m}\right)CN\right\}$$

are easily seen to be open, each set S_C is a dense G_δ-set in X, hence so is

$$S = \bigcap_{C<1} S_C.$$

Now let x be a frequently hypercyclic vector. Since $X \setminus \bar{U}$ is open and non-empty we have that

$$\underline{\text{dens}}\{n \in \mathbb{N} : T^n x \in X \setminus \bar{U}\} > 0,$$

hence

$$\overline{\text{dens}}\{n \in \mathbb{N} : T^n x \in U\} \le \overline{\text{dens}}\{n \in \mathbb{N} : T^n x \in \bar{U}\} < 1.$$

This shows that $x \notin S$, hence

$$FHC(T) \subset X \setminus S.$$

Consequently, $FHC(T)$ is of first category. □

COROLLARY 5.2. *Let T be an operator on a separable F-space X. If there is an open subset U of X with $\overline{U} \neq X$ and a dense subset X_0 of X such that, for any $x \in X_0$, $T^n x \in U$ for n sufficiently large, then $FHC(T)$ is of first category in X. This is the case, in particular, for all operators satisfying the Frequent Hypercyclicity Criterion.*

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In [**3**, §3] Bayart and Grivaux have developed an eigenvalue criterion for frequent hypercyclicity. We need to recall the pertinent definitions.

Definition 5.3. [**1, 2**] Let T be an operator on a separable complex Banach space X. Then T is said to have a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there is a continuous probability measure σ on $\mathbb{T} = \{z : |z| = 1\}$ such that for every Borel subset A of $\mathbb{T}$ with $\sigma(A) = 1$ the vectors $x \in X$ with $Tx = \lambda x$ for some $\lambda \in A$ span a dense subspace of X.

This definition was motivated by the well-known observation of Godefroy and Shapiro [**11**] (see also [**2**]) that rich supplies of eigenvectors to eigenvalues λ with $|\lambda| < 1$ and to eigenvalues λ with $|\lambda| > 1$ imply hypercyclicity. In addition, the observation that a rich supply of eigenvectors to unimodular eigenvalues implies hypercyclicity was first made by Flytzanis [**9**] who obtained a special case of the hypercyclicity part of the following result.

THEOREM 5.4. [**1–3**] *Let T be an operator on a separable complex Banach space X. If T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues then T is hypercyclic. If X is a Hilbert space then T is even frequently hypercyclic.*

In [**3**], Bayart and Grivaux implicitly ask if the set $FHC(T)$ is also non-residual under the assumptions of their eigenvalue criterion. As another application of Theorem 5.1 we obtain a positive answer.

THEOREM 5.5. *Let T be an operator on a separable Banach space X. If T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues then the set $FHC(T)$ is of first category in X.*

Proof. By the proof of Lemma 2.6 of [**2**] there exists, under the given assumptions, a Banach space E and an operator V on E such that, for any $w \in E$ and $z \in E'$, the dual of E, there exists a sequence (n_k) of density 1 such that

$$\langle V^{n_k} w, z\rangle \to 0 \quad \text{as } k \to \infty.$$

By [**2**, Proposition 2.8] there is, in addition, a compact operator $K : E \to X$ of dense range such that $TK = KV$.

We now fix $y \in X'$, $y \neq 0$, and we set $z = K'y$ and $X_0 = K(E)$, which is dense in X. Let $x \in X_0$. Then $x = Kw$ for some $w \in E$ so that there is a sequence (n_k) of density 1 with

$$\langle V^{n_k} w, K'y\rangle \to 0.$$

On the other hand,

$$\langle V^{n_k} w, K'y\rangle = \langle KV^{n_k} w, y\rangle = \langle T^{n_k} Kw, y\rangle = \langle T^{n_k} x, y\rangle.$$

Hence the set

$$\{n \in \mathbb{N} : |\langle T^n x, y\rangle| < 1\}$$

has density 1. Considering the open set

$$U = \{u \in X : |\langle u, y\rangle| < 1\}$$

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we conclude that for any $x \in X_0$ the set

$$\{n \in \mathbb{N} : T^n x \in U\}$$

has density 1. Since X_0 is dense and $\overline{U} \neq X$, an application of Theorem 5.1 finishes the proof. □

As a consequence of Corollary 5.2 and Theorem 5.5 we see that any concrete frequently hypercyclic operator considered in the papers of Bayart and Grivaux [**2, 3**] and in the present paper has the property that $FHC(T)$ is of first category. We therefore raise the following.

Problem 5.6. Is there a frequently hypercyclic operator for which the set of frequently hypercyclic vectors is not of first category?

Since, as least in many cases, the set $FHC(T)$ is not residual, the proofs of all those results in hypercyclicity that rely on the residuality of the set $HC(T)$ of hypercyclic vectors necessarily break down for frequent hypercyclicity.

We shall consider here one such situation. It follows almost trivially from residuality that for any hypercyclic operator on an F-space X any vector in X is the sum of two hypercyclic vectors, that is,

$$X = HC(T) + HC(T),$$

see, for example, [**13**, Proposition 8]. So the question arises if the same is true for the set $FHC(T)$. It turns out, perhaps surprisingly, that the answer is positive for some operators and negative for others.

We begin by giving a sufficient condition under which we even have that $X \neq FHC(T) + HC(T)$.

THEOREM 5.7. *Let T be an operator on a separable F-space X. Suppose that there are continuous seminorms p and q on X and numbers $C_n > 0$ with $C_n \to \infty$ such that:*

(i) *for all $x \in X$ and $n \in \mathbb{N}$*

$$p(T^n x) \leq C_n q(x);$$

(ii) *there is some $x \in X$ and some $\delta > 0$ such that for any $\gamma < 1$*

$$\liminf_{\substack{m,n\to\infty \\ m\leq \gamma n}} \frac{p(T^n x)}{C_m} > \delta.$$

Then

$$X \neq FHC(T) + HC(T).$$

Proof. We shall in fact see that the point x in condition (ii) does not belong to $FHC(T) + HC(T)$. Suppose, on the contrary, that there are a frequently hypercyclic vector y and a hypercyclic vector z such that $x = y + z$. Then there is some $\eta > 0$ such that

$$\underline{\text{dens}}\{n \in \mathbb{N} : q(T^n y) < \delta\} = \eta > 0,$$

and there is a sequence (n_k) with $n_k \to \infty$ such that

$$p(T^{n_k} z) < 1$$

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for all k. Hence, for all sufficiently large k there is some m_k with

$$\frac{\eta}{2}n_k \le m_k \le \left(1-\frac{\eta}{4}\right)n_k \quad \text{and} \quad q(T^{m_k}y) < \delta;$$

otherwise we would have $\#\{m \le n_k : q(T^m y) < \delta\} \le \frac{3}{4}\eta n_k$ for infinitely many k, which is impossible.

Consequently we have with condition (i)

$$\begin{aligned} p(T^{n_k}x) &= p(T^{n_k}y + T^{n_k}z) \\ &\le p(T^{n_k-m_k}T^{m_k}y) + p(T^{n_k}z) \\ &< C_{n_k-m_k}q(T^{m_k}y) + 1 \\ &< C_{n_k-m_k}\delta + 1, \end{aligned}$$

hence

$$\frac{p(T^{n_k}x)}{C_{n_k-m_k}} < \delta + \frac{1}{C_{n_k-m_k}}.$$

Since $n_k \to \infty, n_k - m_k \to \infty, n_k - m_k \le (1-(\eta/2))n_k$ and $1/C_{n_k-m_k} \to 0$, we have arrived at a contradiction to condition (ii). □

We consider special cases.

Example 5.8. (a) For the differentiation operator D on $H(\mathbb{C})$ we have

$$H(\mathbb{C}) \ne FHC(D) + HC(D).$$

To see this we check the two conditions in the theorem. For condition (i) we note that by the Cauchy estimates we have for $f \in H(\mathbb{C})$

$$\sup_{|z|\le 1} |D^n f(z)| \le n! \sup_{|z|\le 2} |f(z)|.$$

Thus we choose $C_n = n!$.

For condition (ii) we consider the entire function

$$f(z) = \sum_{n=2}^{\infty} \frac{1}{\log 2 \log 3 \cdots \log n} z^n.$$

Fix $\gamma < 1$, and let $m, n \in \mathbb{N}$ with $m \le \gamma n$. We denote by $[\cdot]$ the integer part of a number. Then

$$\begin{aligned} \frac{\sup_{|z|\le 1}|D^n f(z)|}{C_m} &\ge \frac{|D^n f(0)|}{m!} = \frac{n!}{\log 2 \log 3 \cdots \log n\, m!} \\ &= \frac{(m+1)(m+2)\cdots([\gamma n]+1)\cdots n}{\log 2 \log 3 \cdots \log n} \\ &\ge \frac{([\gamma n]+1)\cdots n}{\log 2 \log 3 \cdots \log n} \ge \frac{(\gamma n)^{n-\gamma n}}{\log 2 \log 3 \cdots \log n} =: a_n \to \infty \end{aligned}$$

as $n \to \infty$, because $a_{n+1}/a_n \to \infty$. This establishes the claim.

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(b) For the multiples $T = \lambda B$, $|\lambda| > 1$, of the backward shift $B(x_k) = (x_{k+1})$ on l^p, $1 \le p < \infty$, we have

$$l^p \neq FHC(T) + HC(T).$$

In this case we take the norm for the seminorms in condition (i) and $C_n = |\lambda|^n$. In condition (ii) we choose the sequence $x = (1/k^2)$. Fix $\gamma < 1$, and let $m, n \in \mathbb{N}$ with $m \le \gamma n$. Then

$$\frac{\|T^n x\|}{C_m} = \frac{|\lambda|^n}{|\lambda|^m}\left(\sum_{k=n+1}^{\infty} \frac{1}{k^{2p}}\right)^{1/p} \ge |\lambda|^{n-\gamma n}\left(\frac{1}{2p-1}\frac{1}{(n+1)^{2p-1}}\right)^{1/p} \to \infty$$

as $n \to \infty$.

Next, we look for operators T that do have the property that $X = FHC(T) + FHC(T)$. It was the search for such operators that has led us to the definition of Runge transitivity in §4.

THEOREM 5.9. *Let T be a Runge transitive operator on a separable Fréchet space X. Then*

$$X = FHC(T) + FHC(T).$$

Proof. Once more we use Lemma 2.5 to construct sets $A(l, \nu)$ for the choice of $\rho(l, \nu) = \nu$. In the proof of Theorem 3.3 we have shown that to every sequence (y_k) in X there exists a vector $x \in X$ that satisfies (22), that is,

$$p_l(T^{n_k}x - y_k) < \frac{2}{2^k} \quad \text{if } n_k \in A(l, N_l), \tag{24}$$

where the p_l and N_l are those appearing in the definition of Runge transitivity.

Let $\xi \in X$ be an arbitrary vector. We define the sequence (y_k) in the following way. Let (x_l) be a dense sequence in X. After splitting each set $A(l, N_l)$ into two disjoint sets A_l^1, A_l^2 of positive lower density, we set

$$y_k = \begin{cases} x_l & \text{if } n_k \in A_l^1, \\ T^{n_k}\xi - x_l & \text{if } n_k \in A_l^2, \end{cases}$$

and y_k arbitrary, otherwise. Let $x \in X$ be a vector that satisfies (24) for this sequence (y_k). We then have

$$p_l(T^{n_k}x - x_l) < \frac{2}{2^k} \quad \text{if } n_k \in A_l^1.$$

Since A_l^1 has positive lower density, since (p_l) is an increasing sequence defining the topology of X and since (x_l) is dense, we have that x is frequently hypercyclic. On the other hand, we have by (24) for $n_k \in A_l^2$

$$p_l(T^{n_k}(\xi - x) - x_l) = p_l(T^{n_k}x - (T^{n_k}\xi - x_l)) < \frac{2}{2^k}.$$

This shows that also $\xi - x$ is frequently hypercyclic. Hence

$$\xi = x + (\xi - x) \in FHC(T) + FHC(T),$$

which had to be shown. □

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In view of Example 3.2 we have the following.

Example 5.10. The translation operators $Tf(z) = f(z + a), a \neq 0$, on the space $X = H(\mathbb{C})$ of entire functions and the backward shift operator $T(x_k) = (x_{k+1})$ on the space $X = \omega$ of all sequences satisfy

$$X = FHC(T) + FHC(T).$$

We have noted in §4 that no operator on a Banach space can be Runge transitive. It is thus natural to ask the following.

Problem 5.11. Is there a frequently hypercyclic operator T on a Banach space X for which $X = FHC(T) + FHC(T)$ holds?

One might be even more demanding. Recall that by Read [**19**] there is an operator on l^1 for which every non-zero vector is hypercyclic.

Problem 5.12. Is there an operator on an F-space for which every non-zero vector is frequently hypercyclic?

It also is not known if Read's operator is frequently hypercyclic.

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