Theorem 2: $T_{\{\alpha,\beta\}}$ is primitive for all possible κ-pairs $\{\alpha, \beta\}$ if, and only if, $k = \dfrac{1-n}{n}$ for some positive integer n.

Proof: As a simple corollary to Lemma 1, we see that if $m + n = 1$ then d is also equal to 1. On the other hand, Lemma 2 tells us that if $m + n > 1$, there always exists a κ-pair $\{\alpha, \beta\}$ for which $d = n + m$. The fact that $k = \dfrac{m}{n}$ now gives us our result.

Finally, we note that the conditions $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$ are essential for ensuring the validity of the above theorem. In order to show that this is the case, we let $\alpha = \dfrac{1}{1}$, $\beta = \dfrac{2(n-m)}{n}$. Then $\alpha\beta = \kappa$ and the corresponding integer triangle $T_{\{\alpha,\beta\}}$ has sides $4n - 2m$, $3n$ and $5n - 4m$.

This is in itself an interesting triangle since its sides form an arithmetic progression with first term $3n$ and common difference $n - 2m$, thus providing us with a straightforward method for calculating the sides.

However, although $\gcd(\alpha_1, \alpha_2) = 1$, we note that $\gcd(\beta_1, \beta_2) = 1$ when n is odd but 2 when n is even and thus, even if $m + n = 1$, it does not necessarily follow that $T_{\{\alpha,\beta\}}$ is primitive. If we let $n = 2$, $m = -1$, it follows from above that $T_{\{\alpha,\beta\}}$ has sides 6, 10, 14. This is an integer triangle containing an angle of 120° but is not primitive.

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99.13 Solving the quintic in radicals

Introduction

It is well known that the general quintic equation cannot be solved in a finite sequence of radicals. This was proved by Abel and Galois in the 19th century and brought to a close that particular line of research. The important word above is 'finite'. If we remove that word then the above statement is false, as we shall show in this Note.

The general quintic equation can be transformed into $x^5 + ax + b = 0$ by means of a transformation that only uses square and cube roots. This was proved by the Swedish professor of history E. S. Bring in the 18th century and independently by the English mathematician G. B. Jerrard in the 19th. There are two cases to consider which depend on the sign of a.

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(1) The first case is $x^5 - ax - b = 0$ where a and b are both positive reals. By Descartes' rule of signs this equation has just one positive root α (say). Clearly $\alpha > a^{1/4}$.

We can write

$$x = (ax + b)^{1/5} = F(x)$$

so that

$$F'(x) = \frac{a}{5(ax + b)^{4/5}}$$

and

$$F'\left(a^{1/4}\right) = \frac{a}{5\left(a^{5/4} + b\right)^{4/5}} < \frac{a}{5a} = \frac{1}{5}.$$

Clearly $F'(x)$ is monotonically decreasing for $x \geqslant 0$ and so

$$F'(x) < \frac{1}{5} \text{ for } x \geqslant a^{1/4}.$$

From which it follows that the iteration scheme $x_{n+1} = (ax_n + b)^{1/5}$ with $x_0 = a^{1/4}$ converges, because $x_{n+1} > x_n$. We have

$$x_0 = a^{1/4},$$

$$x_1 = \left(a^{5/4} + b\right)^{1/5},$$

$$x_2 = \left(a\left(a^{5/4} + b\right)^{1/5} + b\right)^{1/5},$$

$$x_3 = \left(a\left(a\left(a^{5/4} + b\right)^{1/5} + b\right)^{1/5} + b\right)^{1/5}, \quad \text{and so on}$$

So we have the radical formula

$$\alpha = \ldots \left(a\left(a\left(a\left(a^{5/4} + b\right)^{1/5} + b\right)^{1/5} + b\right)^{1/5} + b\right)^{1/5} \ldots$$

We can see what is happening by looking at the following diagram.

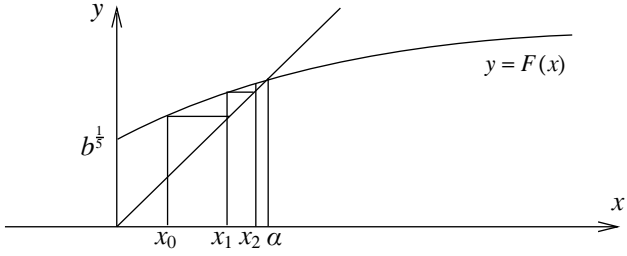


FIGURE 1

We have $F''(x) < 0$ so that $y = F(x)$ is concave and $F'(x) \to 0$ as $x \to \infty$. We also have $x_{n+1} > x_n$ and so the iteration converges to α.

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Alternatively, we can write

$$\alpha - x_{n+1} = \alpha - F(x_n) = F(\alpha) - F(x_n)$$
$$= (\alpha - x_n)F'(\theta) \quad \text{where } x_n < \theta < \alpha.$$

But $F'(\theta) < \frac{1}{5}$ so that $\alpha - x_{n+1} < \frac{1}{5}(\alpha - x_n)$ for all n. Therefore x_n tends to α. We can estimate the rate of convergence as follows. We have

$$\alpha - x_n < \frac{\alpha - x_{n-1}}{5} < \frac{\alpha - x_{n-2}}{5^2} < \ldots < \frac{\alpha - x_0}{5^n} = \frac{\alpha - a^{1/4}}{5^n}$$

so that $\alpha - x_n < k/5^n$ where k is some positive constant. We can take this a little further by finding a value for k by determining an upper bound for α. If we use the Newton-Raphson method, which is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ where } f(x) = x^5 - ax - b \text{ and } x_0 = a^{1/4}$$

then
$$x_1 = a^{1/4} - \left(\frac{-b}{5a - a}\right) = a^{1/4} + \frac{b}{4a}.$$

The function $x^5 - ax - b$ is convex, with $f(a^{1/4}) < 0$, so that the tangent at $x_0 = a^{1/4}$ intersects the x-axis to the right of α. So we have that $a^{1/4} < \alpha < a^{1/4} + \frac{b}{4a}$. If we now substitute this bound in $\frac{\alpha - a^{1/4}}{5^n}$, we have

$$\alpha - x_n < \frac{b}{4a\,5^n} \text{ for } n \geqslant 0.$$

Convergence will therefore be relatively rapid.

We have the solution α given as an infinitely nested collection of radicals. From this result we can immediately deduce the solution of $x^5 - ax + b = 0$ where again $a > 0$ and $b > 0$. We have α is the positive root of $x^5 - ax - b = 0$ so that $-\alpha$ is the root of $(-x)^5 - a(-x) - b = 0$ or $-x^5 + ax - b = 0$. Therefore $x^5 - ax + b = 0$ has the one negative root $-\alpha$.

(2) We now consider the case $x^5 + ax - b = 0$, with $\alpha > 0$ and $b > 0$, but we cannot immediately apply the previous method because $(b - ax)^{1/5}$ does not have the required properties. Instead we put $x = \frac{1}{z}$ to give $z^5 - \frac{a}{b}z^4 - \frac{1}{b} = 0$. Now put $\frac{a}{b} = A$ and $\frac{1}{b} = B$ to give $z^5 - Az^4 - B = 0$ where A and B are both positive. We see again that this equation has just one positive root β, where clearly $\beta > A$. Write, as before,

$$z = (Az^4 + B)^{1/5} = F(z)$$

where $F(z)$ is monotonically increasing for $z \geqslant 0$, but this time it is initially convex before becoming concave since $F''(z)$ changes sign where $Az^4 = 15B$. Now

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$$F'(z) = \frac{4Az^3}{5(Az^4 + B)^{4/5}} < \frac{4Az^3}{5A^{4/5}z^{16/5}} = \frac{4}{5}\left(\frac{A}{z}\right)^{1/5}.$$

We have $F'(z) < \frac{4}{5}$ for $z \geqslant A$ and $F'(z) \to 0$ as $z \to \infty$.

In the same way as in case (1) we have

$$z_0 = A,$$

$$z_1 = \left(A^5 + B\right)^{1/5},$$

$$z_2 = \left(A\left(A^5 + B\right)^{4/5} + B\right)^{1/5},$$

$$z_3 = \left(A\left(A\left(A^5 + B\right)^{4/5} + B\right)^{4/5} + B\right)^{1/5}, \text{ and so on}$$

So we have

$$\beta^4 = \ldots\left(A\left(A\left(A\left(A^5 + B\right)^{4/5} + B\right)^{4/5} + B\right)^{4/5} + B\right)^{4/5}\ldots$$

which gives the explicit algebraic solution in radicals,

$$\alpha^4 = \frac{1}{\beta^4} = \frac{1}{\ldots\left(A\left(A\left(A\left(A^5 + B\right)^{4/5} + B\right)^{4/5} + B\right)^{4/5} + B\right)^{4/5}\ldots}.$$

In exactly the same way as the previous case we have

$$\beta - z_n < (\beta - A)\left(\frac{4}{5}\right)^n$$

and, using the Newton-Raphson method with $z_0 = A$, we get an upper bound for β given by $z_1 = A + \frac{B}{A^4}$. This gives

$$\beta - z_n < \frac{B}{A^4}\left(\frac{4}{5}\right)^n \text{ for } n \geqslant 0.$$

Clearly the convergence is slower in this case.

For the equation $x^5 + ax + b = 0$, with a and b both positive the one negative root is again given by $-\alpha$.

The algebraic solution is therefore expressed in terms of two infinitely nested sequences of radicals for all values of the parameters a and b. These formulae are of course evaluated by means of the recurrence relationships until there is no further change which will depend upon the number of decimal places used. As the general quintic can be reduced to one of these four equations then it's solution can be written in terms of such infinitely nested radical formulae.

Then [1] gives the first formula in this article in the form

$$x = \left(-b - a\left(-b - a\left(-b - a(-b - a)^{1/5}\right)^{1/5}\right)^{1/5}\right)^{1/5}\ldots$$

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This is equivalent to starting the iteration $x_{n+1} = (-ax_n - b)^{1/5}$ at the value $x_0 = 1$, but the article [1] does not give conditions for the convergence of this process.

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99.14 A recurrence relation for the limit point of nested triangles

There are several unsolved problems in the literature related to the limit point of various nested triangles, see Croft, Falconer, and Guy [1, 2, 3]. In this note, a recursive sequence is found for the limit point of nested triangles obtained by angle bisectors. We consider only the special case of an isosceles triangle.

Let T_1 be the given isosceles triangle with vertices $A_1B_1C_1$ and let C_1 be the vertex angle. Orient T_1 as shown in Figure 1 with C_1 at the origin. Let z_1, $\bar{z}_1$ represent the points A_1, B_1, respectively. See Figure 1.

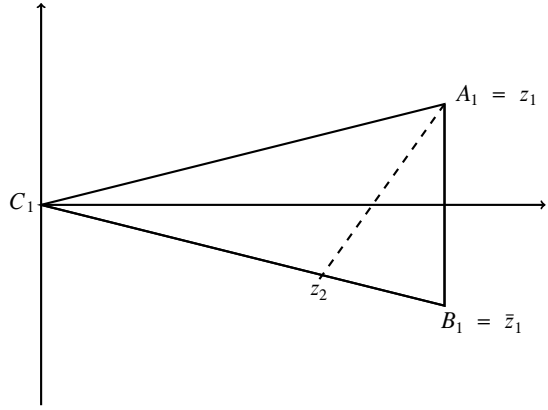


FIGURE 1: The initial triangle T_1

If z_2 is the point where the angle bisector of A_1 intersects side B_1C_1, then by the ratio theorem (or section formula) combined with the angle bisector theorem, z_2 may be expressed as

$$z_2 = \frac{|z_1|\,\bar{z}_1}{|z_1| + |2\ \mathrm{Im}\,(z_1)|}. \tag{1}$$

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