Ergod. Th. & Dynam. Sys. (2017), **37**, 2643–2670 © *Cambridge University Press, 2016*
doi:10.1017/etds.2016.10

Limit drift

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(*Received* 26 *February* 2015 *and accepted in revised form* 12 *January* 2016)

Abstract. We study the problem of the existence of wild attractors for critical circle coverings with Fibonacci dynamics. This is known to be related to the drift for the corresponding fixed points of renormalization. The fixed point depends only on the order of the critical point ℓ and its drift is a number $\vartheta(\ell)$ which is finite for each finite ℓ. We show that the limit $\vartheta(\infty) := \lim_{\ell\to\infty} \vartheta(\ell)$ exists and is finite. The finiteness of the limit is in a sharp contrast with the case of Fibonacci unimodal maps. Furthermore, $\vartheta(\infty)$ is expressed as a contour integral in terms of the limit of the fixed points of renormalization when $\ell \to \infty$. There is a certain paradox here, since this dynamical limit is a circle homeomorphism with the golden mean rotation number whose own drift is ∞ for topological reasons.

1. *Introduction*

Recall that a wild attractor is a compact forward invariant set K of zero Lebesgue measure such that for each open neighborhood U of K every orbit from a subset of U with positive Lebesgue measure stays forever in U and tends to K, while every orbit from a residual subset of U is outside a certain open neighborhood of K infinitely often under forward iterations. The existence of wild attractors for one-dimensional smooth maps was first established in [**2**] for Fibonacci unimodal maps of the interval. Wild attractors appear when the order ℓ of the critical point becomes sufficiently large. That work introduced a general probabilistic framework for finding a wild attractor once a certain Markov structure has been found in the map. That work was followed by a program of Van Strien and Nowicki for showing the existence of a similar attractor for a complex polynomial, which would imply that the Julia set of such a polynomial has positive measure. While that program has not been followed to successful completion, partial progress was made based on choosing

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sufficiently high criticality and studying limits when it tended to ∞; see [**11**] (cf. [**4**] for complex maps with a positive area Julia set).

Another class of real maps in which a wild attractor was expected was a critical covering map of the circle of Fibonacci type. Note that this problem is equivalent to an apparently different problem of the existence of non-escaping 'fat' Cantor sets: do there exist two branches of increasing C^2-maps of two disjoint subintervals of the unit interval I onto I, so that the set of non-escaping points is a Cantor set of positive length? Necessarily, one of the branches must have a critical point of inflection type and we assume that the itinerary of this point has the Fibonacci combinatorics.

In this paper, we study a certain quantity called the *drift*, which is closely related to the problem of existence of a wild attractor for the fixed point map of renormalizations of the Fibonacci critical circle covers. As we proved in [**6**], such a fixed point map H_ℓ exists and is unique for every odd $\ell \geq 3$, where ℓ is the order at a critical point.

The concept of the drift can be tracked down to [**2**], though the exact definition varies depending on the case under consideration and techniques used. In our case, we introduce a number $\vartheta(\ell)$ (called drift) such that it is positive if and only if the wild attractor exists. The key difference of our approach with [**2**] is that we introduce the drift not with respect of the Lebesgue measure, but with respect to an invariant measure of some induced map with an infinite number of branches. This is crucial for further analysis. Explicitly, a similar definition of the drift appeared in [**10**] in the context of unimodal maps.

We prove that the drift $\vartheta(\ell)$ tends to a limit, which is expressed exclusively through the limit map with infinite criticality (although, as noted, the sign of this limit would determine the existence of wild attractors for maps with big, but *finite*, criticality). In the main step of the proof, we represent the drift $\vartheta(\ell)$ as an integral. This is a completely new step, which allows us to deform the contour of integration to the complex plane and analyze its behavior as the order ℓ tends to infinity.

One should note that the meaning of the limiting drift is not related to the problem of existence of a wild attractor for the limit map of infinite criticality itself—that question is easy because the limit map is a critical circle homeomorphism.

Of course, the key question here is what is the sign of the limit drift. This question is much simpler to answer if the limit is infinite as this is the case for the Fibonacci unimodal maps and, in fact, could be extracted from [**2**]. Details are provided in §5. As we show for the covering maps, the limit is finite. This explains why the problem of wild attractors for unimodal maps is simpler than for critical covering maps.

1.1. *The class of dynamics.* The following maps were studied in [**6**].

Consider open intervals I^0, I^{-1}, I in the following configuration: $\overline{I}^0 \cap \overline{I}^{-1} = \emptyset$, $\overline{I}^0 \cup \overline{I}^{-1} \subset I, 0 \in I^0$. The map g is defined by its branches ψ^0, ψ^{-1} on the corresponding intervals, which are both monotone increasing, C^3, and map onto I. Furthermore, each branch ψ^i, $i = 0, -1$, has exactly one critical point with the local representation as $(\xi^i(x))^\ell$, where ξ^i are diffeomorphisms and ℓ an odd integer bigger then 1. This implies that both branches have the same critical value at 0. The set of such mappings is called $\mathcal{G}_\ell$.

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Maps from $\mathcal{G}_\ell$ can be obtained naturally by inducing from critical circle coverings; see [**6**]. However, to use the results of that, and most other papers, one should consider branches $f(x) = \sqrt[\ell]{g(x^\ell)}$. The effect of this change of normalization is that there is only one critical point at $x = 0$.

A map g from $\mathcal{G}_\ell$ is said to have the Fibonacci combinatorics if there exists a weakly order preserving map from the circle to I with end points identified, which conjugates g on the forward orbit of the critical value to the dynamics of an orbit under the golden mean rotation.

Suppose that g is a map from $\mathcal{G}_\ell$ with Fibonacci combinatorics. We normalize g so that $g(0) = 1$. Let g_1 be the first return map into I^0 with the domains restricted only to those points which return after one or two iterates. It is clear that the new map g_1 still belongs to $\mathcal{G}_\ell$ with the new range which is I^0. Let ζ be a linear rescaling which maps $g_1(0)$ to 1. Then the renormalization of g, $\mathcal{R}(g) = \zeta \circ g_1 \circ \zeta^{-1}$.

From [**6**, Theorem 1], we get the following fact.

Fact 1.1. For each ℓ which is an odd integer greater than 1, there exist exactly one map $H_\ell \in \mathcal{G}_\ell$ with the Fibonacci combinatorics and a constant $\tau = \tau_\ell < -1$ so that if $\phi = \phi^0$, ϕ^{-1} are the branches of H_ℓ, then:

- $\phi(0) = 1$;
- $$\phi^{-1}(x) = \tau\phi\tau^{-1}(x)$$
  for $x \in I^{-1}$;
- the *fixed point equation* holds for all $x \in \tau^{-1}I^0$:
$$\tau^{-1}\phi\tau(x) = \tau\phi\tau^{-1}\phi(x); \tag{1}$$
- for any $g \in \mathcal{G}_\ell$ with the Fibonacci combinatorics, the sequence of renormalizations $\mathcal{R}^n(g)$ converges to H_ℓ uniformly on the domain of H_ℓ.

1.2. *The drift.* The branch $\phi : I^0 \to I$ of H_ℓ extends to a real-analytic homeomorphism $\phi_\ell : (\tau_\ell^2 X_\ell, \tau_\ell x_{0,\ell}) \to (-\infty, \tau_\ell^2)$, where $X_\ell \in I^0$, $X_\ell < 0$, is such that $\phi_\ell(X_\ell) = \tau_\ell X_\ell$ and $x_{0,\ell} \in (\tau_\ell^2 X_\ell, X_\ell)$ is a single critical point of ϕ_ℓ (see §2 for more details). Denote $J = (\tau_\ell^2 X_\ell, X_\ell)$. The tower (cf. [**9**, **6**]) $T : \bigcup_{n\in\mathbb{Z}} \tau_\ell^{-n} J \to \mathbb{R}$ can be defined, where $T|_{\tau_\ell^{-n}J} = \tau_\ell^{-n}\phi_\ell\tau_\ell^n$. One can see from (1) that

$$T|_{\tau_\ell^{-n}(J\cap I^0)} = H_\ell^{S_n}, \quad n = 0, 1, 2, \ldots,$$

where $S_0 = 1$, $S_1 = 2$, $S_n = S_{n-1} + S_{n-2}$, $n \geq 2$, is the Fibonacci sequence.

The *induced dynamics* $\Phi_\ell : J \to J$ is defined with the exception of countably many points and in the form $\Phi_\ell(x) = \tau_\ell^{-m(x)}\phi_\ell(x)$, where $m(x) = m(x, \ell)$ is the unique integer for which $\tau_\ell^{-m(x)}\phi_\ell(x) \in J$. In other words, $m(x) = -k(\phi_\ell(x))$, where $k(x) = n$ for $x \in \tau_\ell^{-n} J$ and $n \in \mathbb{Z}$. Note that the *drift function* $m(x)$ is unbounded:

$$\inf_{x\in J} m(x) = -\infty, \quad \sup_{x\in J} m(x) = \infty,$$

and $m \in L_p(J)$ for every $p > 0$.

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The map $\Phi_\ell : J \to J$ has a unique invariant probability measure μ_ℓ on J which is absolutely continuous with respect to the Lebesgue measure; see §2. The *drift* is

$$\vartheta(\ell) := -\int_J m(x)\, d\mu_\ell(x).$$

Importance of this number comes from the fact that the sign of $\vartheta(\ell)$ is responsible for the existence of wild attractors for the map H_ℓ. Indeed, since Φ_ℓ is ergodic, by the Birkhoff ergodic theorem, for Lebesgue almost every point $x \in J$,

$$-\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(\Phi_\ell^i(x)) = \vartheta(\ell). \tag{2}$$

On the other hand,

$$-\sum_{i=0}^{n} m(\Phi_\ell^i(x)) = k(T^n(x)), \quad n = 0, 1, \ldots.$$

It follows from this and (2) that, if $\vartheta(\ell) < 0$, then Lebesgue almost every point leaves the domain of H_ℓ under the tower dynamics. In particular, H_ℓ has no wild attractor in this case. If $\vartheta(\ell) > 0$, by (2), for Lebesgue almost every point x, $T^n(x) \to 0$ as $n \to \infty$. It follows that there is a set of positive Lebesgue measure E in the domain of H_ℓ such that for every $x \in E$, all forward iterates $T^n(x)$, $n \geq 0$, do not leave this domain and $T^n(x) \to 0$. Every such x must be in the basin of attraction of the orbit of 0 of the map H_ℓ. The proof uses the fact that $H_\ell^{S_n}(0) = \tau_\ell^{-n} \in \tau_\ell^{-n} J$, $n \geq 0$, and repeats the one for the unimodal maps; see [**1**, Lemma 5.1]. Therefore, the set E belongs to the basin of attraction of the orbit of 0.

The case $\vartheta(\ell) = 0$ can be handled with help of the central limit theorem for the map Φ_ℓ (it is proved under some assumptions on the map and the drift function which hold for Φ_ℓ and m). It follows that almost every point must leave the range of H_ℓ; hence, H_ℓ has no wild attractor.

Note also that presumably the fixed point map H_ℓ has a wild attractor if and only if a wild attractor exists for all maps of the class $\mathcal{G}_\ell$ (see [**10**], where a similar claim is proved in the class of Fibonacci unimodal maps).

2. *Induced dynamics and invariant densities*

The following notation is consistent with those used in [**8**]; see also Introduction.

By ℓ, we will denote an odd integer greater than 1 or ∞. For every ℓ, we have the parameters τ_ℓ, X_ℓ, $x_{0,\ell}$ subject to the following inequalities:

$$\tau_\ell < \min(X_\ell \tau_\ell^2, -1) \leq \tau_\ell^2 X_\ell \leq x_{0,\ell} < X_\ell < 0 < \tau_\ell X_\ell < 1.$$

For ℓ finite, the inequality $\tau_\ell^2 X_\ell < x_{0,\ell}$ is sharp, but for $\ell = \infty$ these points are equal.

As finite ℓ tends to ∞, sequences (τ_ℓ), (X_ℓ), $(x_{\ell,0})$ tend to limits equal to their respective values for $\ell = \infty$; see [**8**, Theorem 1, p. 698].

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Also, for every ℓ, a mapping $\phi_\ell : (\tau_\ell^2 X_\ell, \tau_\ell x_{0,\ell}) \to \mathbb{R}$ is given, which is strictly increasing. For every ℓ, we have

$$\lim_{x \searrow x_{0,\ell}} \phi_\ell(x) = 0,$$
$$\phi_\ell(X_\ell) = \tau_\ell X_\ell,$$
$$\phi_\ell(0) = 1,$$
$$\lim_{x \nearrow \tau_\ell x_{0,\ell}} \phi_\ell(x) = \tau_\ell^2.$$

The derivative of $\phi_\ell(x)$ is strictly positive whenever $x_{0,\ell} < x < \tau_\ell x_{0,\ell}$.

Additional information is available for $\ell < \infty$.

Fact 2.1. For every $\ell < \infty$, the representation

$$\phi_\ell(x) = (E(x))^\ell$$

holds, where $E : (\tau_\ell^2 X_\ell, \tau_\ell x_{0,\ell}) \to (-\infty, \sqrt[\ell]{\tau_\ell^2})$ is a diffeomorphism from the Epstein class.

For a proof, see [**8**, Lemma 2.1, p. 703]. In particular, for ℓ finite the range of ϕ_ℓ is $(-\infty, \tau_\ell^2)$, but for $\ell = \infty$ it is only $(0, \tau_\infty^2)$.

2.1. *Convergence statement.* From [**8**], we get the following fact.

Fact 2.2. Transformations $(\log(\phi_\ell))^{-1}$ with inverse branches chosen so that

$$(\log(\phi_\ell))^{-1} \circ \log(\phi_\ell(x)) = x$$

for $x \in (x_{0,\ell}, \tau_\ell x_{0,\ell})$ converge to $(\log(\phi_\infty))^{-1}$ almost uniformly on

$$U := \mathbb{C} \backslash [\log \tau_\infty^2, \infty).$$

This should be understood to include the claim that on every compact subset of U, mappings $(\log(\phi_\ell))^{-1}$ are defined for ℓ large enough.

Theorem 1 in [**8**] gives convergence on a non-trivial segment of the real line. Extensions of ϕ_ℓ^{-1} to $\mathbb{C} \backslash [\tau_\ell^2, \infty)$ exist by Fact 2.1 and form a normal family by Montel's theorem, leading to our statement.

2.2. *Induced dynamics and its properties.* For each ℓ, consider its *inducing domain* $(\tau_\ell^2 X_\ell, X_\ell)$. The image of this interval by ϕ_ℓ is $(-\infty, \tau_\ell X_\ell)$ for ℓ finite and $(0, \tau_\infty^2)$ for $\ell = \infty$. In any case, with the exception of countably many points, it decomposes into a disjoint sum of $\tau_\ell^m(\tau_\ell^2 X_\ell, X_\ell)$ for various integer values of m.

Definition 2.1. The *induced dynamics*

$$\Phi_\ell : (\tau_\ell^2 X_\ell, X_\ell) \to (\tau_\ell^2 X_\ell, X_\ell)$$

is defined with the exception of countably many points and in the form

$$\Phi_\ell(x) = \tau_\ell^{-m(x,\ell)} \phi_\ell(x),$$

where $m(x, \ell)$ is the unique integer for which $\tau_\ell^{-m(x,\ell)} \phi_\ell(x) \in (\tau_\ell^2 X_\ell, X_\ell)$.

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We will talk about inverse branches of Φ_ℓ written as

$$\psi_{m,\ell}(x) = \phi_\ell^{-1}(\tau_\ell^m x). \tag{3}$$

The *domain* of $\psi_{m,\ell}$ means the largest interval on which formula (3) defines a diffeomorphism.

For some combinations of ℓ and m, the domain of $\psi_{m,\ell}$ may be empty. It is non-empty if m is a negative odd integer, or m is any even integer and $\ell < \infty$.

The basic property for us is the following lemma.

LEMMA 2.1. *For every ℓ and m, the domain of $\psi_{m,\ell}$, if non-empty, contains at least the interval $(\tau_\ell^3, 0)$.*

Proof. For the domain of $\psi_{m,\ell}$ to be non-empty, the set $\phi_\ell(\tau_\ell^2 X_\ell, X_\ell)$ must contain the segment $\tau_\ell^m(\tau_\ell^2 X_\ell, X_\ell)$.

If this segment is to the right of 0, then m is an odd negative integer. Then ϕ_ℓ^{-1} defines a diffeomorphism on $(0, \tau_\ell^2)$. After rescaling by τ_ℓ^{-m}, this gives $(\tau_\ell^{-m+2}, 0)$ as the domain, which contains at least $(\tau_\ell^3, 0)$.

If $\tau_\ell^m(\tau_\ell^2 X_\ell, X_\ell)$ is to the left of 0, which may only happen for $\ell < \infty$, then ϕ_ℓ^{-1} can be used to define a diffeomorphism on the entire negative-half line onto $(\tau^\ell X_\ell, x_{0,\ell})$, which is invariant by the rescaling, so it is the domain of $\psi_{m,\ell}$. □

LEMMA 2.2. *Let p_0 be a point chosen so that*

$$\tau_\ell < p_0 < X_\ell \tau_\ell^2$$

for all ℓ sufficiently large or infinite. Then, for every m for which the domain of $\psi_{m,\ell}$ is non-empty, $\psi_{m,\ell}((p_0, 0]) \subset (\tau_\ell^2, \psi_{-1,\ell}(p_0))$.

Proof. Let us first consider the case of m negative and odd. Then $\tau_\ell^m(p_0, 0) \subset (0, p_0\tau_\ell^{-1})$. A further mapping by ϕ_ℓ^{-1} brings this to

$$(x_{\ell,0}, \phi_\ell^{-1}(p_0\tau_\ell^{-1})) = (x_{\ell,0}, \psi_{-1,\ell}(p_0)),$$

so the claim holds in this case.

For m even, the range of every $\psi_{\ell,m}$ is contained in $(\tau_\ell^2 X_\ell, x_{0,\ell})$. □

LEMMA 2.3. *There is a fixed closed segment I_0 contained in the interior of $(p_0, 0)$ such that for every m and ℓ sufficiently large or infinite,*

$$\psi_{m,\ell}(p_0, 0) \subset I_0.$$

Proof. This segment must be contained in $(\tau_\ell^2 X_\ell, \psi_{-1,\ell}(p_0))$ for all ℓ. But the convergence statement, see Fact 2.2, says that

$$q_0 := \lim_{\ell\to\infty} \psi_{-1,\ell}(p_0) = \psi_{-1,\infty}(p_0) < \psi_{-1,\infty}(\tau_\infty) = 0,$$

where the inequalities come from the fact that $\psi_{-1,\ell}$, or generally $\psi_{m,\ell}$ for m odd, are order-reversing. Then, as I_0, we can pick any segment with the right end point in $(p_0, x_{0,\ell})$ and the left end point in $(q_0, 0)$. □

COROLLARY 2.1. *For all ℓ sufficiently large or infinite, Φ_ℓ is uniformly expanding in the Poincaré metric of $\mathbb{C}\backslash\{x \in \mathbb{R} : x \notin [p_0, 0]\}$.*

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2.3. *Perron–Frobenius–Ruelle transfer operators.* The transfer operator is given by the formula

$$P_\ell f(z) = \sum_m (-1)^m \psi'_{m,\ell}(z) f \circ \psi_{m,\ell}(z). \tag{4}$$

The definition is, of course, not complete, without specifying the space to which f belongs.

We will use the term *density* to denote a real-valued non-negative function whose integral over the indicated space with respect to the Lebesgue measure is 1.

Fact 2.3. For every ℓ sufficiently large or infinite, the Perron–Frobenius operator acting on the space $L_1(\tau_\ell^2 X_\ell, X_\ell)$ has unique non-negative invariant density $f_\ell = \lim_{n\to\infty} P_\ell^n \mathbf{e}$, which in fact is positive, where $\mathbf{e}$ is a constant density.

This follows from the folklore theorem in the light of Corollary 2.1.

2.3.1. *Functional spaces.* Let D_0 denote the disk whose diameter is the segment I_0 from Lemma 2.3. We will also need a closed segment I_1 such that

$$I_0 \subset \text{int}\,(I_1) \subset I_1 \subset (p_0, 0).$$

Then D_1 denotes the disk whose diameter is I_1.

The functional space $\mathcal{F}_0$ will consist of all functions bounded and holomorphic on D_0, real on the real line, as well as continuous on $\overline{D_0}$, with the C^0 (supremum) norm. It is clearly a Banach space. We will also use a space $\mathcal{F}_1$ of all continuous complex-valued functions, real on the real line, defined on the disk D_1, with the C^0 norm.

2.3.2. *Properties of the transfer operator.* Looking at formula (4), one can see that the transfer operator is well defined on $\mathcal{F}_1$ as a formal sum, since $\psi_{m,\ell}(D_1) \subset D_0$ by Lemma 2.2 and the Epstein class property expressed in Fact 2.1.

To consider convergence, observe the following simple fact.

LEMMA 2.4. *There exists a constant K_1 such that for all positive integers n, all sequences $m_1, \ldots, m_n$, all ℓ sufficiently large or infinite, and all $z \in D_1$,*

$$|(\tilde{\psi})'(z)| \le K_1 |\tilde{\psi}(\tau_\ell^2 X_\ell, X_\ell)|,$$

where $\tilde{\psi} = \psi_{m_n,\ell} \circ \cdots \circ \psi_{m_1,\ell}$.

Proof. From Lemma 2.1 and Fact 2.1, all $\tilde{\psi}$ have univalent continuations to $\mathbb{C}\backslash\{x \in \mathbb{R} : x \notin [\tau_\ell^3, 0]\}$, so, by Koebe's lemma,

$$|(\tilde{\psi})'(z)| \le L_1 |(\tilde{\psi})'(\hat{z})|$$

for any $\hat{z} \in D_1$, where the constant L_1 is independent of n, (m_n), ℓ, z. But, for a suitably chosen real $\hat{z}$,

$$|(\tilde{\psi})'(\hat{z})| = \frac{|\tilde{\psi}(\tau_\ell^2 X_\ell, X_\ell)|}{|(\tau_\ell^2 X_\ell, X_\ell)|},$$

which yields the lemma, since the denominator is bounded below by

$$2^{-1} |(x_{0,\infty}, \tau_\infty^{-2} x_{0,\infty})|$$

for all ℓ sufficiently large. □

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It is now clear that the sum in formula 4 converges uniformly in the operator norm on $\mathcal{F}_1$ and so the operator is well defined. Furthermore, we have the following lemma.

LEMMA 2.5. *For ℓ sufficiently large or infinite, the operator P_ℓ acting on $\mathcal{F}_0$ is compact.*

Proof. For any f from the unit ball in $\mathcal{F}_0$, formula (4) shows that $P_\ell f$ is a holomorphic element of $\mathcal{F}_1$. From Lemma 2.4, its norm is bounded by $K_1|(\tau_\ell^2 X_\ell, X_\ell)|$. So, $P_\ell B_{\mathcal{F}_0}(0, 1)$ is a normal family on D_1. Then, after restriction to $\overline{D}_0$, every sequence from this family has a uniformly convergent subsequence, but this means compactness in $\mathcal{F}_0$. □

Additionally, we get the following lemma.

LEMMA 2.6. *There is a constant K_2 such that for the transfer operator acting on $\mathcal{F}_0$, for all ℓ and n, $\|P_\ell^n\| \le K_2$.*

Proof. The transfer operator for iterates is given by

$$P_\ell^n f(z) = \sum_{m_1,\dots,m_n} (-1)^{\sum m_k} (\psi_{m_n,\ell} \circ \cdots \circ \psi_{m_1,\ell})'(z) \cdot (f \circ \psi_{m_n,\ell} \circ \cdots \circ \psi_{m_1,\ell}(z)).$$

By Lemma 2.4,

$$\|P_\ell^n f\| \le K_1 \|f\| \sum_{m_1,\dots,m_n} |\psi_{m_n,\ell} \circ \cdots \circ \psi_{m_1,\ell}(\tau_\ell^2 X_\ell, X_\ell)|.$$

Segments $\psi_{m_n,\ell} \circ \cdots \circ \psi_{m_1,\ell}(\tau_\ell^2 X_\ell, X_\ell)$ are domains of branches of Φ_ℓ^n and hence disjoint and contained in $(\tau_\ell^2 X_\ell, X_\ell)$. Their sum is uniformly bounded by $|p_0|$ and so we get the claim. □

2.3.3. *Convergence of densities.* We are ready to prove our main result for this section, namely the following proposition.

PROPOSITION 1. *As $\ell \to \infty$, invariant densities f_ℓ tend to f_∞ almost uniformly on $(x_{0,\infty}, \tau_\infty^{-2} x_{0,\infty})$.*

LEMMA 2.7. *For every $f \in \mathcal{F}_0$,*

$$\lim_{\ell \to \infty} P_\ell f = P_\infty f.$$

Proof. Since the sum in (4) converges in the operator norm, it is sufficient to show that for each term $\psi'_{m,\ell} f \circ \psi_{m,\ell}$ converge in $\mathcal{F}_0$. This is so since $\psi_{m,\ell}$ converge to $\psi_{m,\infty}$ analytically and hence in C^1, by Fact 2.2. □

LEMMA 2.8. *For every ℓ sufficiently large or infinite, there is $f_{0,\ell} \in \mathcal{F}_0$ which is a fixed point of P_ℓ and, when restricted to $(\tau_\ell^2 X_\ell, X_\ell)$, it is equal to f_ℓ introduced by Fact 2.3.*

Proof. Consider the constant density $\mathbf{e}$ for which

$$P_\ell^n \mathbf{e} \to f_\ell$$

in $L_1(\tau_\ell^2 X_\ell, X_\ell)$. In $\mathcal{F}_0$, the sequence $(P_\ell^n \mathbf{e})$ is bounded by Lemma 2.6 and so belongs to a compact set by Lemma 2.5. Let $f_{0,\ell}$ be an accumulation point of this sequence. On

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the segment $(\tau_\ell^2 X_\ell, X_\ell)$, it is equal to f_ℓ in L_1. So, any two accumulation points are simply equal on this segment and so equal in $\mathcal{F}_0$ as holomorphic functions. It develops that $P_\ell^n \mathbf{e} \to f_{0,\ell}$ and so $f_{0,\ell}$ is a fixed point of P_ℓ. □

Proof of Proposition 1. The sequence $(f_{0,\ell}) = (P_\ell f_{0,\ell})$ has a convergent subsequence in $\mathcal{F}_0$. Let us call its limit $\tilde{f}$. Observe that $P_\infty \tilde{f} = \tilde{f}$. Indeed, for any ℓ sufficiently large,

$$\|P_\infty \tilde{f} - \tilde{f}\| \le \|P_\infty \tilde{f} - P_\ell \tilde{f}\| + \|P_\ell \tilde{f} - P_\ell f_{0,\ell}\| + \|P_\ell f_{0,\ell} - \tilde{f}\|.$$

As ℓ tends to ∞, all three terms tend to 0. For the first one, this follows by Lemma 2.7. The second one can be estimated by $K_2\|\tilde{f} - f_{0,\ell}\|$ by Lemma 2.6 and the final one is just $\|\tilde{f} - f_{0,\ell}\|$, again since $f_{0,\ell}$ is a fixed point of P_ℓ.

But now $\tilde{f}$ restricted to I_0 is a real-valued function and a uniform limit of similarly restricted $f_{0,\ell}$. By Lemma 2.8, $f_{0,\ell}$ is a density on $(\tau_\ell^2 X_\ell, X_\ell)$. It follows that $\tilde{f}$ is a density on $(x_{0,\infty}, \tau_\infty^{-2} x_{0,\infty})$. Furthermore, since $\tilde{f}$ was a fixed point of P and all $\psi_{m,\infty}$ map $(x_{0,\infty}, \tau_\infty^{-2} x_{0,\infty})$ into itself, this density is invariant. By the uniqueness claim in Fact 2.3, it is equal to $f_{0,\infty}$. Thus, $\tilde{f}$ and $f_{0,\infty}$ obtained from Lemma 2.8 are equal on $(x_{0,\infty}, \tau_\infty^{-2} x_{0,\infty})$ and hence equal in $\mathcal{F}_0$.

Recall that $\tilde{f}$ was the limit in $\mathcal{F}_0$ of an arbitrary convergent subsequence of $(f_{0,\ell})$. Hence, $f_{0,\ell} \to f_{0,\infty}$ in $\mathcal{F}_0$, which implies almost uniform convergence of f_ℓ to f_∞ on $(x_{0,\infty}, \tau_\infty^{-2} x_{0,\infty})$.

This ends the proof of Proposition 1. □

3. *The drift*

Definition 3.1. Recall Definition 2.1 of the induced dynamics Φ and in particular the function $m(x, \ell)$. For any $\ell < \infty$, the *drift* is

$$\vartheta(\ell) := -\int_{\tau_\ell^2 X_\ell}^{X_\ell} m(x, \ell) f_\ell(x)\, dx.$$

Recall Fact 2.2 and the set U defined there. Also, we get a holomorphic function $f_{0,\infty} \in \mathcal{F}_0$ from Lemma 2.8. The function $f_{0,\infty}$ is defined on the disk D_0 which contains $(x_{0,\infty}, \tau_\infty^{-2} x_{0,\infty})$ in its interior.

Here is the main theorem of this paper.

THEOREM 1. *There is* $\theta : [0, 1) \to (\log \phi_\infty)^{-1}(U) \cap D_0$, *an arc which is piecewise smooth on every compact subset of* $[0, 1)$, *and satisfies:*

(1) $\theta(0) = \tau_\infty^{-2} x_{0,\infty}$, $\lim_{t\to 1} \theta(1) = x_{0,\infty}$, *and* $\Im\theta(t) > 0$ *for all* $t \in (0, 1)$;

(2) *the image of* θ *and the segment* $(x_{0,\infty}, \tau_\infty^{-2} x_{0,\infty}]$ *are contained in a simply connected open subset of* $(\log \phi_\infty)^{-1}(U) \cap D_0$;

(3)
$$\limsup_{t\to 1} \frac{|\Re\theta(t) - \Re x_{0,\infty}|}{|\theta(t) - x_{0,\infty}|^3 \log |\theta(t) - x_{0,\infty}|^{-1}} < \infty.$$

Additionally, the formula

$$\lim_{\ell\to\infty} \vartheta(\ell) = (\log|\tau_\infty|)^{-1} \lim_{\epsilon\to 0} \Re\left[\int_0^{1-\epsilon} \log \frac{\phi_\infty(\theta(t))}{\theta(t)} f_{0,\infty}(\theta(t)) \frac{d\theta}{dt}\, dt\right]$$

holds.

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COROLLARY 3.1. *Let W be any open neighborhood of $[x_{0,\infty}, \tau_\infty^{-2}x_{0,\infty}]$. When ϵ_0 is small enough, the arc $[x_{0,\infty}, \tau_\infty^{-2}x_{0,\infty}] + \theta$ is a Jordan curve which is homotopically trivial in $(\log \phi_\infty)^{-1}(U) \cap W$.*

Proof. Clearly, the closed curve mentioned here is a Jordan curve by Theorem 1. For ϵ_0 small enough, the bounded connected component of its complement is contained in W, but then it is also contained in $(\log \phi_\infty)^{-1}(U)$, which is simply connected. □

3.1. *Integrated density as a coordinate.* Since $f_{0,\ell}$ is a positive density on $(x_{0,\ell}, \tau_\ell^{-2}x_{0,\ell})$,

$$\mu(x) = \int_{x_{0,\ell}}^{x} f_{0,\infty}(z)\, dz$$

defines a univalent map from a neighborhood of the interval $[x_{0,\ell}, \tau_\infty^{-2}x_{0,\ell}]$ contained in D_0 onto a neighborhood of $[0, 1]$. This coordinate not only normalizes that inducing domain in a way independent of ℓ, but also internalizes that invariant density factor in integration.

Fix a neighborhood W' of $[0, 1]$. Then, for ℓ large enough or infinite, μ_ℓ^{-1} is defined on W'. Let $W'_\ell = \mu_\ell^{-1}(W_0)$. Since μ_ℓ is defined on W'_ℓ, we must have $W'_\ell \subset D_0$.

Definition 3.2. Let ℓ be finite and large enough or infinite, as specified above. Introduce a function $\hat{\phi}_\ell := \phi_\ell \circ \mu_\ell^{-1}$. Also, define the set W_ℓ to be the connected component of the intersection of $\mu_\ell \circ (\log \phi_\ell)^{-1}(U) \cap W'$.

Surely, W_ℓ is contained in the domain of $\hat{\phi}_\ell$. It is also simply connected, because connected components of the intersection of two simply connected sets W'_ℓ and $(\log \phi_\ell)^{-1}(U)$ are simply connected.

Theorem 1 preserves its meaning in the new coordinate and the main integral formula becomes

$$\lim_{\ell\to\infty} \vartheta(\ell) = \lim_{\epsilon\to 0} (\log|\tau_\infty|)^{-1} \Re\left[\int_0^{1-\epsilon} \log \frac{\hat{\phi}_\infty(\theta(t))}{\mu^{-1}(\theta(t))} \frac{d\theta}{dt}\, dt\right]. \tag{5}$$

3.2. *A few identities.* The proof of Theorem 1 is based among other things on several identities, which will be given now.

3.2.1. *Associated dynamics.* For every ℓ, we define *associated maps*

$$\begin{aligned} G_\ell &= \tau_\ell^{-1}\phi_\ell\tau_\ell^{-1}, \\ \Gamma_\ell &= \tau_\ell\phi_\ell\tau_\ell^{-2}. \end{aligned} \tag{6}$$

Their relevant properties will now be listed.

Fact 3.1.

- For each ℓ, $G_\ell = \Gamma_\ell \circ \Gamma_\ell$ and Γ_ℓ fixes $\tau_\ell^2 X_\ell$ with a negative derivative. Also, the functional equation
$$\phi_\ell \circ G_\ell = \tau_\ell^{-2}\phi_\ell$$
holds.

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- For any finite ℓ, the point $x_{0,\ell}$ is an attracting fixed point of G_ℓ and $\tau_\ell^2 X_\ell$ is a repelling fixed point.
- For $\ell = \infty$, $x_{0,\ell}$ is a neutral point for G_∞ with the expansion

$$G_\infty(z) = z - c(z - x_{0,\infty})^3 + O(|z|^4)$$

for $c > 0$.

The identities follow from the functional equation $\phi_\ell = \tau_\ell^2 \phi_\ell \tau_\ell^{-1} \phi_\ell \tau_\ell^{-1}$. The other claims are contained in [**8**, Theorem 3 and Lemma 2.1].

We will use the associated dynamics in the integrated density coordinate; for example,

$$\hat{\Gamma}_\ell := \mu_\ell \circ \Gamma_\ell \circ \mu_\ell^{-1}. \tag{7}$$

Claims of Fact 3.1 have obvious reformulations in the new coordinate. We will just quote one:

$$\hat{\phi}_\ell \circ \hat{G}_\ell = \tau_\ell^{-2} \hat{\phi}_\ell, \tag{8}$$

where $\hat{\phi}_\ell$ is introduced by Definition 3.2.

LEMMA 3.1. *For every finite ℓ, there exist positive constants $K(\ell)$ and $\epsilon(\ell)$ such that*

$$\begin{aligned} &\textit{for all } |z - x_{0,\ell}| < \epsilon(\ell), \quad \textit{then } |\log \hat{\phi}_\ell(z)| \le K(\ell) \log |z - x_{0,\ell}|^{-1} \\ &\textit{for all } |z - \tau_\ell^2 X_\ell| < \epsilon(\ell), \quad \textit{then } |\log \hat{\phi}_\ell(z)| \le K(\ell) \log |z - \tau_\ell^2 X_\ell|^{-1}. \end{aligned}$$

The branch of the logarithm in $\log \hat{\phi}_\ell$ is chosen so that the value is real on $(0, 1)$.

Proof. Both $x_{0,\ell}$ and $\tau_\ell^2 X_\ell$ are hyperbolic fixed points for $\hat{G}_\ell$. For example, for z in a neighborhood of $\tau_\ell^2 X_\ell$,

$$\log \hat{\phi}_\ell(z) = n \log \tau_\ell^2 + \log \hat{\phi}_\ell(\hat{G}_\ell^n(z)).$$

Choosing $G_\ell^n(z)$ in some fixed ring of outer radius r and picking $1 < \rho$ so that it is smaller than $|\hat{G}_\ell'(z)|$ inside this ring, we get

$$|\log \phi_\ell(z)| \le \frac{\log r/|z|}{\log \rho} \log \tau_\ell^2 + L(\ell),$$

where $L(\ell)$ is a estimate on the ring. The claim of the lemma follows and we proceed in the same way near $x_{0,\ell}$ using $\hat{G}_\ell^{-1}$ instead of $\hat{G}_\ell$. □

We have the following corollary.

COROLLARY 3.2. *For any $\ell < \infty$, the function $\log \hat{\phi}_\ell$ is integrable on any piecewise Lipschitz arc in a neighborhood of* $[0, 1]$ *intersected with the domain of* $\log \hat{\phi}_\ell$.

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3.2.2. *Formula for the drift.*

LEMMA 3.2. *For every finite ℓ,*

$$\vartheta(\ell) = -(\log|\tau_\ell|)^{-1}\Re\int_0^1 \log\frac{\hat\phi_\ell(x)}{\mu_\ell^{-1}(x)}\,dx.$$

Proof. Since $d\mu_\ell(x)$ is an invariant measure of the induced map Φ_ℓ, see Definition 2.1, for any complex-valued integrable function f, one gets

$$\int_{\tau_\ell^2 X_\ell}^{X_\ell} f\circ\Phi_\ell(x)\,d\mu_\ell(x) = \int_{\tau_\ell^2 X_\ell}^{X_\ell} f(x)\,d\mu_\ell(x).$$

Picking $f(x)=\log|x|$ and recalling that $\Phi_\ell(x)=\tau^{-m_\ell(x)}\phi_\ell(x)$ leads to

$$\int_{\tau_\ell^2 X_\ell}^{X_\ell}[\log|\phi_\ell(x)|-\log|\tau_\ell|m(x)-\log|x|]\,d\mu_\ell(x)=0,$$

which results in

$$\log|\tau_\ell|\int_{\tau_\ell^2 X_\ell}^{X_\ell} m_\ell(x)\,d\mu_\ell(x) = \int_{\tau_\ell^2 X_\ell}^{X_\ell}\log\frac{|\phi_\ell(x)|}{|x|}\,d\mu_\ell(x).$$

Given Definition 3.1, the left-hand side is just $-\log|\tau_\ell|\vartheta(\ell)$. The right-hand side can be expressed as

$$\Re\int_0^1\log\frac{\hat\phi_\ell(x)}{\mu_\ell^{-1}(x)}\,dx$$

in the integrated density coordinate, which leads to the formula of (3.2). □

3.2.3. *An identity for the invariant density f_ℓ.* Observe that for every ℓ, the mapping $\tau^{-2}\Gamma_\ell$ transforms the interval $(\tau_\ell^2 X_\ell, X_\ell)$ onto a smaller interval. Since $\tau_\ell^2 X_\ell$ is fixed by Γ_ℓ by Fact 3.1, $\tau_\ell^{-2}\Gamma_\ell(\tau_\ell^2 X_\ell)=X_\ell$. Then, since Γ_ℓ reverses the orientation, $\tau_\ell^{-2}\Gamma_\ell(X_\ell)<X_\ell$. Also, $\tau_\ell^{-2}\Gamma_\ell(X_\ell)=\tau_\ell^{-1}\phi_\ell(\tau_\ell^{-2}X_\ell)>\tau_\ell^{-1}\phi_\ell(\tau_\ell X_\ell)=\tau_\ell^2 X_\ell$.

LEMMA 3.3. *There exists $\epsilon_3>0$ such that for all ℓ sufficiently large and finite and $x\in(\tau_\ell^2 x_\ell,\ \tau_\ell^2 X_\ell+\epsilon_3)$, one gets*

$$f_\ell(\tau_\ell^{-2}\Gamma_\ell(x))(\tau_\ell^{-2}\Gamma_\ell)'(x) - f_\ell(\Gamma_\ell(x))\Gamma_\ell'(x) = \tau_\ell^{-2} f_\ell(\tau_\ell^{-2}(x)).$$

Proof. We have $P_\ell f_\ell = f_\ell$, which expands to

$$f_\ell(x)=\sum_m(-1)^m(\psi'_{m,\ell}(x))\,f_\ell(\psi_{m,\ell}(x))$$

by formula (4). So, for any differentiable function $g(x)$,

$$f_\ell(g(x))g'(x)=\sum_m(-1)^m(\psi_{m,\ell}\circ g)'(x)\,f_\ell(\psi_{m,\ell}\circ g)(x).$$

We see that

$$\psi_{m,\ell}\circ(\tau_\ell^{-2}\Gamma_\ell)=\psi_{m-2,\ell}\circ\Gamma_\ell$$

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by formula (3). So,

$$f_\ell(\tau_\ell^{-2}\Gamma_\ell(x))(\tau_\ell^{-2}\Gamma_\ell)'(x) = \sum_m (-1)^m (\psi_{m-2,\ell} \circ \Gamma_\ell)'(x) f_\ell((\psi_{m-2,\ell} \circ \Gamma_\ell)(x)).$$

Naturally, $f_\ell(\Gamma_\ell(x))\Gamma_\ell'(x)$ is the same sum, but with m instead of $m-2$. Indices m range over the set which contains all even integers and negative odd integers. The sums will therefore cancel, leaving only the term corresponding to $m=-1$ in the second one,

$$f_\ell(\tau_\ell^{-2}\Gamma_\ell(x))(\tau_\ell^{-2}\Gamma_\ell)'(x) - f_\ell(\Gamma(x))\Gamma'(x) = (\psi_{-1,\ell}\Gamma_\ell)'(x) f_\ell((\psi_{-1,\ell} \circ \Gamma_\ell)(x)).$$

But

$$\psi_{-1,\ell} \circ \Gamma_\ell(x) = \phi_\ell^{-1}\tau_\ell^{-1}\tau_\ell\phi_\ell\tau_\ell^{-2}(x) = \tau_\ell^{-2}(x),$$

and the claim of the lemma follows. □

3.2.4. *A functional equation for* $\hat{\Gamma}_\ell$. By analytic continuation, the formula of Lemma 3.3 is valid for $f_{\ell,0}$ on a neighborhood of the interval $(\tau_\ell^2 x_\ell, X_\ell)$.

Recall $\hat{\Gamma}_\ell$ introduced by formula (7) and similarly define $\hat{T}_\ell := \mu_\ell \circ \tau_\ell^{-2} \circ \mu_\ell^{-1}$. We obtain a functional equation in the form of the following lemma.

LEMMA 3.4. *For z in a fixed simply connected neighborhood of 0 and all ℓ sufficiently large,*

$$\hat{T}_\ell(z) = (\hat{T}_\ell \circ \hat{\Gamma}_\ell)(z) - \hat{\Gamma}_\ell(z).$$

Proof. Since $\hat{\Gamma}_\ell$ fixes 0, the equality holds at $z=0$. We will write $w = \mu_\ell^{-1}(z)$ for short. Then the derivative of the right-hand side is

$$\begin{aligned} &f_{0,\ell}((\tau_\ell^{-2}\Gamma_\ell)(w)) \cdot (\tau_\ell^{-2}\Gamma_\ell)'(w) \cdot f_{0,\ell}^{-1}(z) - f_{0,\ell}((\Gamma_\ell)(w)) \cdot (\Gamma_\ell)'(w) \cdot f_{0,\ell}^{-1}(z) \\ &\quad = \tau_\ell^{-2} f_{0,\ell}(\tau_\ell^{-2} w) f_{0,\ell}^{-1}(z), \end{aligned}$$

which is seen to be the derivative of the left-hand side. □

3.3. *A conditional proof of Theorem 1.* We will show how to reduce Theorem 1 to a certain simpler and more technical statement.

Choose a point z_0 in the domain of W_∞, see Definition 3.2, and therefore in W_ℓ for all ℓ large enough. Moreover, we can choose it so that the entire disk $D(0, |z_0|)$ is in W'.

From Fact 3.1, $\hat{G}_\ell(z) = a_\ell z - c_\ell z^3 + O(|z|^4)$, where $c_\ell > 0$ for ℓ large enough or infinite, while $a_\ell \geq 1$. It follows that when $|z|$ and $|\arg z - \pi/2|$ are both suitably small, then for ℓ sufficiently large $|(\hat{G}_\ell)^{-n}(z)|$ is a monotone decreasing sequence for $n \geq 0$.

Let $\mathcal{C}_{0,\ell}$ for any ℓ denote the straight-line arc from 1 to z_0. It is still in W_ℓ for all ℓ sufficiently large or infinite by the remark above.

Suppose that z_0 is in the domain of $\hat{G}_\ell^{-1}$ for all ℓ sufficiently large or infinite. Then, for any such ℓ, we can define $z_{1,\ell} = \hat{G}_\ell^{-1}(z_0)$ still in W_ℓ and pick an arc $\mathcal{C}_{1,\ell}$ to be the straight-line segment joining z_0 to $z_{1,\ell}$. The arc $\mathcal{C}_\ell$ is still in the domain of $\log \hat{\phi}_\ell$ for all ℓ large enough or infinite. Moreover, they are all in W' and hence in W_ℓ.

Suppose that $\mathcal{C}_{1,\ell}$ is in the domain of $\hat{G}_\ell^{-n}$ for all n and ℓ large or infinite. Then we can define points $z_{n,\ell} := \hat{G}_\ell^{-n+1}(z_{1,\ell})$ and arcs $\mathcal{C}_{n,\ell} := \hat{G}_\ell^{-n+1}(\mathcal{C}_1)$ joining them. They are all

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in the corresponding $\log \hat{\phi}_\ell$ by the functional equation (8) and in W_ℓ since the points tend to 0.

Define

$$s_{n,\ell} = \Re \int_{\mathcal{C}_{n,\ell}} \log \frac{\hat{\phi}_\ell(z)}{\mu_\ell^{-1}(z)}\, dz. \tag{9}$$

We will rely on the following statement, to be proved later.

PROPOSITION 2. *There is a choice of z_0 arbitrarily close to 0 with $\arg z_0$ less than $\pi/2$ but arbitrarily close to $\pi/2$ so that for all ℓ sufficiently large or infinite:*

- *arcs $\mathcal{C}_{n,\ell}$ belong to the domains of the corresponding $\hat{G}_\ell^{-n}$ for all positive n;*
- *there is a constant Q_4 independent of ℓ such that for all $n > 1$, $|s_{n,\ell}| \le Q_4((\log n)/n^{3/2})$;*
- $$\sup_{t\to 1} \frac{|\Re z - \Re x_{0,\infty}|}{|z - x_{0,\infty}|^3 \log |z - x_{0,\infty}|^{-1}} < \infty$$

  for $z \in \theta_\infty$.

Since 0 is a repelling fixed point of $\hat{G}_\ell$ for ℓ finite, arcs $\mathcal{C}_\ell := \bigcup_{n=0}^\infty \mathcal{C}_{n,\ell}$ are Lipschitz arcs in the corresponding simply connected sets W_ℓ. By Lemma 3.2 and Corollary 3.2, for every ℓ sufficiently large but finite,

$$\sum_{n=0}^\infty s_{n,\ell} = \log |\tau_\ell| \vartheta(\ell).$$

From the convergence statement, see Fact 2.2, for any fixed n $\lim_{\ell\to\infty} s_{n,\ell} = s_{n,\infty}$. This convergence is dominated by a summable series by Proposition 2 and so by the theorem of Lebesgue

$$\lim_{\ell\to\infty} \log |\tau_\ell| \vartheta(\ell) = \sum_{n=0}^\infty s_{n,\infty};$$

hence,

$$\lim_{\ell\to\infty} \vartheta(\ell) = (\log |\tau_\infty|)^{-1} \sum_{n=0}^\infty s_{n,\infty}. \tag{10}$$

Proof of Theorem 1. The first claim follows by the construction. The second one is true since the simply connected set in question is just W_∞. The third claim follows directly from Proposition 2. □

As for the formula,

$$\lim_{\epsilon\to 0} \Re\left[\int_0^{1-\epsilon} \log \frac{\phi_\infty(\theta(t))}{\theta(t)} f_{0,\infty}(\theta(t)) \frac{d\theta}{dt}\, dt\right] = \lim_{N\to\infty} \sum_{n=0}^\infty s_{n,\infty} = (\log |\tau_\infty|) \lim_{\ell\to\infty} \vartheta(\ell)$$

from formula (10).

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3.4. *Proof of Proposition 2.* Proposition 2 will be derived from Theorem 2 stated in the last section of the paper. The function $g(z, \eta)$ mentioned there is $-i\hat{G}_\ell^{-1}(iz)$, where $\eta := \eta(\ell)$ has to be related to ℓ. Let us write $\rho(\ell) = (\hat{\Gamma}_\ell^{-1})'(0)$. Then $dg(z, \eta)/dz = \rho(\ell)^2$ by Fact 3.1, so that

$$\eta(\ell) = \rho(\ell)^{-4} - 1.$$

By Facts 2.2 and 3.1, $\rho(\ell)$ is negative and greater than -1 for all ℓ and its limit is -1 as $\ell \to \infty$. This leads to $\eta(\ell) > 0$ and $\lim_{\ell\to\infty} \eta(\ell) = 0$.

The third coefficient $c(\eta(\ell))$ is real, since the coefficients of $\hat{G}_\ell^{-1}$ are all real and $c(0)$ is positive from Fact 3.1 again. It is also easy to see that the second and fourth coefficients of $g(z, \eta(\ell))$ are imaginary, but it is less clear that they are small with $\eta(\ell)$, as needed in the setting of Theorem 2.

3.4.1. *Calculation of* $\hat{\Gamma}^{-1}$ *at* 0. Our basic tool will be Lemma 3.4, which will be used in the form

$$\hat{T}_\ell \circ \hat{\Gamma}_\ell^{-1}(z) = \hat{T}_\ell(z) - z.$$

Taking the second derivative at $z = 0$, we derive

$$(\hat{\Gamma}_\ell^{-1})''(0) = \frac{(\hat{T}_\ell)''(0)}{(\hat{T}_\ell)'(0)}(1 - \rho(\ell)^2).$$

The nonlinearity of $\hat{T}_\ell$ at 0 tends to its limit value for $\hat{T}_\infty$ and is bounded for ℓ large enough. Then

$$0 < 1 - \rho(\ell)^2 < 1 - \rho(\ell)^4 = \frac{\eta(\ell)}{1 + \eta(\ell)} < \eta(\ell).$$

Hence,

$$(\hat{\Gamma}_\ell)^{-1}(z) = \rho(\ell)z + r(\ell)\eta(\ell)z^2 + s(\ell)z^3 + t(\ell)z^4 + O(|z|^5).$$

Computing the expansion of $\hat{G}_\ell^{-1}(z)$ is easy up to the third-order terms:

$$\begin{aligned}(\hat{G}_\ell)^{-1}(z) &= \rho(\ell)^2 z + [(\rho(\ell) + 1)r(\ell)\rho(\ell)\eta(\ell)]z^2 + s'(\ell)z^3 + O(|z|^4)\\ &= \rho(\ell)^2 z + [k_1(\ell)r(\ell)\rho(\ell)\eta^2(\ell)]z^2 + s'(\ell)z^3 + O(|z|^4)\end{aligned}$$

with the substitution $1 + \rho(\ell) = k_1(\ell)\eta(\ell)$, where $k_1(\ell)$ is bounded for all ℓ.

To determine the fourth coefficient, observe that

$$(S\hat{G}_\ell^{-1})'(0) = (\hat{G}_\ell^{-1})^{(4)}(0)/(\hat{G}_\ell^{-1})'(0) + k_2(\ell)\eta(\ell)$$

with $k_2(\ell)$ similarly bounded for all ℓ. On the other hand, from the formula

$$S\hat{G}_\ell^{-1} = (S\hat{\Gamma}_\ell^{-1}) \circ \Gamma_\ell^{-1} \cdot [(\Gamma_\ell^{-1})']^2 + S\hat{\Gamma}_\ell^{-1}$$

one gets

$$(S\hat{G}_\ell^{-1})'(0) = (S\hat{\Gamma}_\ell^{-1})'(0)(\rho(\ell)^3 + 1) + 2(S\hat{\Gamma}_\ell^{-1})(0) \cdot (\hat{\Gamma}_\ell^{-1})''(0) \cdot (\hat{\Gamma}_\ell^{-1})'(0) = k_3(\ell)\eta(\ell)$$

with $k_3(\ell)$ bounded for ℓ since $\rho(\ell)^3 + 1$ and $(\Gamma_\ell^{-1})''(0)$ are both $O(\eta(\ell))$. Hence,

$$(\hat{G}_\ell^{-1})^{(4)}(0) = \rho(\ell)^2(k_3(\ell) - k_2(\ell))\eta(\ell),$$

which leads to the form of the coefficients required by Theorem 2.

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3.4.2. *Derivation from Theorem 2.* Arcs $\mathcal{C}_{1,\ell}$ are straight-line segments which join z_0 to $z_{1,\ell}$, which corresponds to $z_1(\eta(\ell))$ in Theorem 2. From the first claim of that theorem, they belong to the domain of $\hat{G}_\ell^{-n}$ for all n. Moreover, since they can be enclosed in a fixed open set, for all $\eta(\ell)$ sufficiently small the distortion of $\hat{G}_\ell^{-n}$ on $\mathcal{C}_{1,\ell}$ is bounded independently of n and ℓ, leading to

$$|\mathcal{C}_{n,\ell}| \le L_1 |z_{n+1,\ell} - z_{n,\ell}| \le L_1 Q_2 n^{-3/2}, \tag{11}$$

where the last estimate and constant Q_2 come from Theorem 2. From formula (9),

$$|s_{n,\ell}| \le \left| \Re \int_{\mathcal{C}_{n,\ell}} \log \hat{\phi}_\ell(z)\, dz \right| + \int_{\mathcal{C}_{n,\ell}} |\log \mu^{-1}(z)|\, |dz|.$$

The function $\mu^{-1}(z)$ tends to $\tau_\ell^2 X_\ell$ as z tends to 0, so the second integral is bounded:

$$\int_{\mathcal{C}_{n,\ell}} |\log \mu^{-1}(z)|\, |dz| \le L_2 |\mathcal{C}_{n,\ell}| \le L_2 L_1 Q_2 n^{-3/2}$$

from estimate (11).

By the functional equation (8), for $z \in \mathcal{C}_{n,\ell}$,

$$\log \hat{\phi}_\ell(z) = 2(n-1) \log |\tau_\ell| + \log \hat{\phi}_\ell(\hat{G}^{n-1}(z)).$$

But $\hat{G}^{n-1}(z)$ belongs to the initial arc $\mathcal{C}_{1,\ell}$, where $\log \hat{\phi}_\ell$ is bounded independently of n and ℓ. Hence,

$$\int_{\mathcal{C}_{n,\ell}} |\log \hat{\phi}_\ell(\hat{G}^{n-1}(z))|\, |dz| \le L_3 n^{-3/2},$$

as for the previous term.

We are left with

$$\left| \Re \int_{\mathcal{C}_{n,\ell}} 2(n-1) \log \left| \tau_\ell \right| dz \right| \le L_4 n \left| \Re \int_{\mathcal{C}_{n,\ell}} dz \right| = L_4 n |\Re(z_{n+1,\ell} - z_{n,\ell})| \le L_4 Q_3 n^{-3/2}$$

from the final claim of Theorem 2, where we have to remember that the theorem is stated in a system of coordinates which swaps the real and imaginary axes and therefore real and imaginary parts.

As regards the third claim of Proposition 2, observe that points $z_{n,\infty}$ are all on the same horizontal line in the repelling Fatou coordinate and so by the bounded distortion the entire arc θ_∞ is contained in a horizontal strip of fixed width in the repelling Fatou coordinate. Since $-i\hat{G}_\infty^{-1}(iz)$ lacks the fourth-order coefficient, see formula (12), its Fatou coordinate has the form

$$w = \Psi(z) = Cz^{-2} + O(\log |z|^{-1}).$$

The inverse is

$$z = \Psi^{-1}(w) = \sqrt{C/w} + O(\log |z|^{-1} |z|^3).$$

By an elementary geometric argument, $\sqrt{C/w}$ will map a horizontal line to a curve that is third-order tangent to the real axis at 0, so that actually the $O(\log |z|^{-1} |z|^3)$ dominates the imaginary value of the corresponding z. This leads to the third claim of Proposition 2 and, since the fourth one has already been checked, the proof of the proposition is now finished.

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4. *Uniform estimates*

4.1. *Hypotheses of the mapping.* We consider a family of maps in the form

$$g(z, \eta) = \frac{1}{\sqrt{1+\eta}} z + b(\eta)\eta^2 z^2 - c(\eta)z^3 + d(\eta)\eta z^4 + z^5 R(z, \eta). \tag{12}$$

Here z is a complex variable and η a positive parameter from some set which has 0 as an accumulation point. All $g(z, \eta)$ are assumed to be holomorphic for z in some neighborhood of 0, which is independent of η. Coefficients $b(\eta)$, $c(\eta)$, $d(\eta)$ are complex and assumed to have limits $b(0)$, $c(0)$, $d(0)$, respectively. Functions $R(z, \eta)$ also converge uniformly to $R(z, 0)$ as $\eta \to 0$. Additionally, assume that for all η, coefficients $b(\eta)$ are imaginary, $c(\eta)$ are real, and the limit $c(0)$ is strictly positive.

From the dynamical point of view, we observe that for all η the function $g(z, \eta)$ has an attracting fixed point at 0. As $\eta \to 0$, the function $g(z, \eta)$ converges uniformly to $g(z, 0)$, which as a neutral point is of the third order and still topologically attracting along the positive real axis. The second and fourth coefficients are assumed to vanish for the limiting map and at definite rates as $\eta \to 0$.

If z_0 is a given point, we will write $z_n(\eta) := g^n(z_0, \eta)$, where the composition here and elsewhere is on the first coordinate.

THEOREM 2. *For a family of functions $g(z, \eta)$ given by formula (12) with the hypotheses underneath, there exists positive parameters α_0, η_0, δ_0, together with positive constants Q_1, Q_2, Q_3 and an open set U_Q, such that the following claims hold whenever $|z_0| < \delta_0$, $|\arg z_0| < \alpha_0$ for all $0 < \eta < \eta_0$, and all integers $n > 1$:*

- *U_Q contains z_0, $z_1(\ell)$ and the straight-line segment which connects them and, for every $z \in U_Q$, iterates $g^n(z, \eta)$ are defined;*
- $$|z_n(\eta)| \le Q_1 \sqrt{\max(n^{-1}, \eta)}\, \exp(-n\eta/8);$$
- $$|z_{n+1}(\eta) - z_n(\eta)| \le Q_2 n^{-3/2};$$
- $$|\Im z_{n+1}(\eta) - \Im z_n(\eta)| \le Q_3 n^{-5/2} \log n.$$

4.2. *Reduction to the case of $b(\eta) = 0$.* By a change of coordinates, we can eliminate the 'small' second-order term completely.

Consider the change of coordinates $z(u, \eta) = u - a(\eta)u^2$, where

$$a(\eta) = b(\eta)\eta(1+\eta)(\sqrt{1+\eta} + 1),$$

which is $O(\eta)$ for η small. By a direct calculation, one verifies that this change of coordinates removes the second-order coefficient from $g(z, \eta)$. At the same time, since every coefficient of $z(u, \eta)$ and the inverse function, except for the first-order ones, is $0(\eta)$, $c(\eta)$, $d(\eta)\eta$, and $R(z, \eta)$ change only by $O(\eta)$. In particular, their limits as $\eta \to 0$ do not change and one can still factor out η from the fourth-order coefficients.

To see that the third coefficient remains real, observe that $a(\eta)$ is imaginary and hence the Schwarzian derivative of $z(u, \eta)$ at $u = 0$ is real. By the chain rule for Schwarzian

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derivatives, the same is true for the inverse map. As regards $g(z, \eta)$, its Schwarzian derivative at 0 is also real under our assumptions. Then, by the chain rule again, the Schwarzian derivative at 0 for the conjugated map is real and, since its first coefficient is real and the second one is missing, that means that the third one is real.

So, the change of coordinates leaves $g(z, \eta)$ in the form postulated at the beginning and additionally sets $b(\eta)$ to 0.

Looking now at Theorem 2, we observe that the hypotheses on z_0 preserve their meaning in the new coordinate for the point $u_0 = u(z_0, \eta)$. Its claims are now for points $u_n(\eta)$ and have to verified for $z_n(\eta) = z(u_n(\eta), \eta)$. The first claim is satisfied for η sufficiently small, since the distortion of the change of coordinates tends to 0 with η. Since $\lim_{u\to 0} (z(u, \eta)/u) = 1$ uniformly for all bounded η, the second claim is satisfied.

For the third one, we estimate

$$|z(u_{n+1}(\eta), \eta) - z(u_n(\eta), \eta)| \le |u_{n+1}(\eta) - u_n(\eta)|[1 + O(\eta)|u_{n+1}(\eta) + u_n(\eta)].$$

The expression in the square parentheses is bounded by the second claim, so the third claim follows.

4.2.1. *Verification of the fourth claim.* We state a simple fact, which will be useful in several estimates.

LEMMA 4.1. *For $\alpha, \beta > 0$, the function $x^\beta e^{-\alpha x}$ considered on the set $x > 0$ has the maximum equal to $(\beta/\alpha)^\beta e^{-\beta}$.*

Proof. This is an elementary exercise. □

Regarding the fourth claim, we estimate

$$\begin{aligned}
&|\Im z(u_{n+1}(\eta), \eta) - \Im z(u_n(\eta), \eta)| \\
&\quad \le |\Im u_{n+1}(\eta) - \Im u_n(\eta)| + L_1\eta(|u_{n+1}(\eta)| + |u_n(\eta)|)|u_{n+1}(\eta) - u_n(\eta)| \\
&\quad \le |\Im u_{n+1}(\eta) - \Im u_n(\eta)| + 2L_2\eta \exp(-n\eta/8)\sqrt{\max(n^{-1}, \eta)}|u_{n+1}(\eta) - u_n(\eta)|
\end{aligned}$$

from the second claim, where L_1, L_2 are constants independent of η. Let us concentrate on the expression which follows L_2 and take first $n \le \eta^{-1}$. Then

$$\eta \exp(-n\eta/8)\sqrt{\max(n^{-1}, \eta)} \le n^{-3/2}.$$

For $n > \eta^{-1}$, use Lemma 4.1 with $\beta = 1$ to get

$$\exp(-n\eta/8) \le (8/e)\eta^{-1}n^{-1},$$

which then leads to

$$\eta \exp(-n\eta/8)\sqrt{\max(n^{-1}, \eta)} \le (8/e)n^{-1}\sqrt{\eta} < (8/e)n^{-3/2}.$$

Now, using the third and fourth claims of the theorem for u, we get

$$|\Im z(u_{n+1}(\eta), \eta) - \Im z(u_n(\eta), \eta)| \le Q_{4,u}n^{-5/2}\log n + L_3 Q_{3,u}n^{-3}.$$

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4.3. *Pre-Fatou coordinate and the proof of the first claim.* Inspired by the Fatou coordinate for the limiting map $g(z, 0)$, we change the variable to $w(z) = z^{-2}$ on the set $\{z : \Re z > 0\}$. The map corresponding to g is then

$$\gamma(w, \eta) = (1 + \eta)w + \hat{c}(\eta) + \frac{\hat{d}(\eta)\eta}{\sqrt{w}} + \frac{\hat{R}(w, \eta)}{w}, \tag{13}$$

where

$$\hat{c}(\eta) = 2c(\eta)(\sqrt{1 + \eta})^3,$$
$$\hat{d}(\eta) = -2d(\eta)(\sqrt{1 + \eta})^3,$$

and $\hat{R}(w, \eta)$ are holomorphic functions of $\sqrt{w}$. Coefficients $\hat{c}(\eta)$, $\hat{d}(\eta)$ and functions $\hat{R}(w, \eta)$ still have limits as $\eta \to 0$ and $\hat{c}(0) = 2c(0) > 0$.

LEMMA 4.2. *There is $W_0 \geq 1$ such that whenever $\Re w \geq W_0$, then, for all η sufficiently small,*

$$\Re\gamma(w, \eta) > \Re w + c(0).$$

Proof. From the form of (13) for η small, the nonlinear part of $\gamma(w, \eta)$ tends to 0 as $|w| \to \infty$. Also, for η small enough, $c(0) \leq \frac{2}{3}\hat{c}(\eta)$. Then

$$\Re\gamma(w, \eta) \geq \Re w(1 + \eta) + c(0)$$

and the claim of the lemma follows immediately. □

We will write $w_n(\eta) = w(z_n(\eta))$. By setting δ_0 small enough in the hypothesis of Theorem 2, we can ensure that $\Re w_o(\eta) \geq W_0 + 1$ and then $\Re w_n(\eta) \geq nc(0)$. The region $\{w : \Re w > W_0\}$ is invariant under $\gamma(w, \eta)$ and $z_n(\eta) = 1/\sqrt{w_n(\eta)}$.

Then the first claim of Theorem 2 follows together with an estimate

$$|z_n|(\eta) \leq (\Re w_n(\eta))^{-1/2} \leq (c(0))^{-1/2} n^{-1/2} \tag{14}$$

for all n. The open set can be taken to be the image on $\{w : \Re w > W_0\}$ by $z = w^{-1/2}$. The distortion of this map can be made arbitrarily small by taking W_0 large and then it will contain the straight-line segment from z_0 to $z_1(\eta)$.

4.3.1. *Estimates of derivatives of γ.* From formula (13), we conclude that with some $K > 0$,

$$\left|\frac{d\gamma}{dw}(w, \eta)(1 + \eta)^{-1} - 1\right| \leq K|w|^{-3/2} \tag{15}$$

for all η sufficiently small and all $\Re w \geq W_0$.

LEMMA 4.3. *The exist $W_1 \geq W_0$ and $K_1 > 0$ such that for all η small enough, all w which satisfy $\Re w \geq W_1$, and all $n > 0$,*

$$\left|\log \frac{d\gamma^n(w, \eta)}{dw}(w, \eta)(1 + \eta)^{-n}\right| \leq \frac{K_1}{\sqrt{\Re w}}.$$

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Proof. Start with estimate (15) and choose W_1 so that $\Re w \geq W_1$ implies that

$$\left|\log \frac{d\gamma(w,\eta)}{dw}(w,\eta)(1+\eta)^{-1}\right| \leq 2K|w|^{-3/2}.$$

Since $W_1 \geq W_0$, we have $\Re g^n(w,\eta) \geq \Re w + nc(0)$. From the chain rule,

$$\left|\log \frac{d\gamma^n(w,\eta)}{dw}(w,\eta)(1+\eta)^{-n}\right| \leq \sum_{k=0}^{n-1} \frac{2K}{(\Re w + c(0)k)^{3/2}} \leq \frac{K_1}{\sqrt{\Re w}},$$

by comparing the sum with an integral. □

COROLLARY 4.1. *There is a constant K_2 such that for all η small enough and, if $\Re w \geq W_0$, one gets the estimate*

$$\left|\log \frac{d\gamma^n(w,\eta)}{dw}(w,\eta)(1+\eta)^{-n}\right| \leq K_2.$$

This follows from Lemma 4.3, since the derivative dg/dw is bounded uniformly for all $\Re w \geq W_0$ and η small enough and since $\Re\gamma^k(w,\eta) \geq \Re w + c(0)k$ so that after $(W_1 - W_0)/c(0)$ steps the hypothesis of Lemma 4.3 is satisfied.

4.4. *Linear approximation and the second claim.* The main tool for our handling of the estimates for the function $\gamma(w,\eta)$ will be its approximation by the linear part $\gamma_0(w,\eta) = (1+\eta)w + c(0)$.

LEMMA 4.4. *There are positive constants K_3, K_4 such that for all $\Re w \geq W_0$ and η sufficiently small and for all positive integers n, the estimate*

$$(1+\eta)^{-n}|\gamma^n(w,\eta) - \gamma_0^n(w,\eta)| \leq K_3 + K_4 \min(\log n, -\log\eta)$$

holds.

Proof. Start with an elementary identity:

$$\gamma^n(w,\eta) = \gamma_0^n(w,\eta) + \sum_{k=1}^{n}[\gamma^{n-k}(\gamma(\gamma_0^{k-1}(w,\eta),\eta),\eta) - \gamma^{n-k}(\gamma_0(\gamma_0^{k-1}(w,\eta),\eta),\eta)].$$

This yields an estimate

$$\begin{aligned}&(1+\eta)^{-n}|\gamma^n(w,\eta) - \gamma_0^n(w,\eta)| \\ &\quad\leq \sum_{k=1}^{n}(1+\eta)^{-n}\left|\frac{d\gamma^{n-k}(\tilde{w}_k,\eta)}{dw}\right||\gamma(\gamma_0^{k-1}(w,\eta),\eta) - \gamma_0(\gamma_0^{k-1}(w,\eta),\eta)|,\end{aligned}$$

where $\tilde{w}_k$ is a point on the segment joining points

$$\gamma(\gamma_0^{k-1}(w,\eta),\eta)$$

and

$$\gamma_0(\gamma_0^{k-1}(w,\eta),\eta).$$

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Those derivatives can be estimated by Corollary 4.1, which results in

$$(1+\eta)^{-n}|\gamma^n(w,\eta)-\gamma_0^n(w,\eta)| \le K_2\sum_{k=1}^{n}(1+\eta)^{-k}|\gamma(\gamma_0^{k-1}(w,\eta),\eta)-\gamma_0(\gamma_0^{k-1}(w,\eta),\eta)|.$$

From (13), we obtain

$$|\gamma(\gamma_0^{k-1}(w,\eta),\eta)-\gamma_0(\gamma_0^{k-1}(w,\eta),\eta)| \le \frac{2|\hat{d}(0)|\eta}{\sqrt{|\gamma_0^{k-1}(w,\eta)|}}+\frac{Q}{|\gamma_0^{k-1}(w,\eta)|}.$$

Since $|\gamma_0^{k-1}(w,\eta)| \ge W_0+(k-1)c(0)$, we get

$$(1+\eta)^{-n}|\gamma^n(w,\eta)-\gamma_0^n(w,\eta)| \le K_2\sum_{k=0}^{n-1}(1+\eta)^{-k-1}\left[\frac{2|\hat{d}(0)|\eta}{\sqrt{W_0+c(0)k}}+\frac{Q}{W_0+c(0)k}\right].$$

Applying Abel's transformation yields

$$\sum_{k=0}^{n-1}(1+\eta)^{-k-1}\frac{2|\hat{d}(0)|\eta}{\sqrt{W_0+c(0)k}} \le L_1+L_2\frac{1}{\eta}\sum_{k=0}^{n-1}(1+\eta)^{-k-1}\frac{\eta}{(W_0+c(0)k)^{3/2}}$$

with constants L_1, L_2 independent of η. The factor η in the numerator came from the form of the fourth coefficient and it cancels with η in the denominator, which arises from the summation of $(1+\eta)^{-k}$. This is clearly uniformly bounded for all n and can be included in K_3 in the claim of Lemma 4.4. We have

$$\sum_{k=0}^{n-1}(1+\eta)^{-k-1}\frac{Q}{W_0+c(0)k} \le L_1+L_2\log n$$

trivially from a comparison with the integral. To get an alternative estimate claimed in the lemma, introduce an integer-valued function

$$N(\eta)=\lfloor 1/\eta \rfloor. \tag{16}$$

Then, for $n \ge N(\eta)$, we split

$$\begin{aligned}
&\sum_{k=0}^{n-1}(1+\eta)^{-k-1}\frac{Q}{W_0+c(0)k}\\
&\quad=\sum_{k=0}^{N(\eta)-1}(1+\eta)^{-k-1}\frac{Q}{W_0+c(0)k}+\sum_{k=N(\eta)}^{n-1}(1+\eta)^{-k-1}\frac{Q}{W_0+c(0)k}\\
&\quad\le L_1+L_2\log N(\eta)+L_3+\frac{L_4}{\eta}\sum_{k=N(\eta)}^{n-1}(1+\eta)^{-k-1}(W_0+kc(0))^{-2},
\end{aligned}$$

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where we used the trivial estimate for the first sum and Abel's transformation for the second one. The term which follows L_4 is estimated by

$$\eta^{-1}\sum_{k=N(\eta)}^{n-1}(1+\eta)^{-k-1}(W_0+kc(0))^{-2}\le\eta^{-1}\sum_{k=N(\eta)}^{n-1}(W_0+kc(0))^{-2}$$
$$\le\eta^{-1}\int_{N(\eta)-1}^{\infty}\frac{dx}{(W_0+xc(0))^2}\le\frac{1}{\eta c(0)(W_0+c(0)(N(\eta)-1))}\le\frac{L_5}{\eta(N(\eta)+1)}$$

for η small enough.

From the definition of $N(\eta)$, we get $\eta(N(\eta)+1)>1$ and so this term is uniformly bounded. Also, $\log N(\eta)\le-\log\eta$. Thus, finally,

$$\sum_{k=0}^{n-1}(1+\eta)^{-k-1}\frac{Q}{W_0+c(0)k}\le K_4\log\eta^{-1}. \qquad \square$$

Proof of the second claim in Theorem 2. As an elementary exercise, we obtain

$$\gamma_0^n(w,\eta)=(1+\eta)^n w+c(0)\frac{(1+\eta)^n-1}{\eta}. \tag{17}$$

From Lemma 4.4,

$$\Re w_n(\eta)=\Re\gamma^n(w_0,\eta)\ge(1+\eta)^n\Re w_0$$
$$+(1+\eta)^n(c(0)\eta^{-1}-K_3+K_4\log\eta)-c(0)\eta^{-1}. \tag{18}$$

The second claim of Theorem 2 is easy for $n\le N(\eta)$; recall formula (16). Then the exponential term is bounded below by e^{-1} and the second claim says only the same as estimate (14).

So, let $n>N(\eta)$. For $n\ge N(\eta)$ and η sufficiently small, we have:

- $c(0)\eta^{-1}-K_3+K_4\log\eta>c(0)\eta^{-1}\exp(-0.1)$;
- $(1+\eta)^n\ge\exp(n\eta-0.1)$.

This leads to

$$\Re w_n(\eta)\ge\Re w_0+c(0)[\exp(n\eta-0.2)-1]\eta^{-1}.$$

Since $n>N(\eta)$, we have $n\eta>1$ and note an elementary inequality $e^x-1>e^{x/4}$ for $x\ge0.8$. Then

$$\Re w_n(\eta)\ge\Re w_0+c(0)\exp(n\eta/4)\eta^{-1}. \tag{19}$$

Then

$$|z_n(\eta)|=|w_n(\eta)|^{-1/2}\le\exp(-n\eta/8)\sqrt{\eta/c(0)},$$

and the second claim of Theorem 2 is satisfied. $\square$

4.5. *Proofs of the remaining claims.* We start with a lemma which extracts some estimates from the discussion of the preceding section.

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LEMMA 4.5. *There are constants K_5, K_6 such that for all η sufficiently small and $n > 1$, the following estimates hold:*

$$\Re w_n(\eta) \geq \Re w_0 + K_5 \min(n, \eta^{-1}) \exp(n\eta/4),$$
$$|\Im w_n(\eta)| \leq K_6 |\Re w_n(\eta)| \max\left(\frac{\log n}{n}, -\eta \log \eta\right).$$

Proof. The first estimate is just the estimate (19) in the case when $n > N(\eta)$ and the minimum comes from using the estimate of Lemma 4.2 for smaller n.

To prove the second one, we need to return to Lemma 4.4 and formula (17). Observe that γ_0 preserves the imaginary part and so

$$|\Im w_n(\eta)| \leq (1+\eta)^n [K_3 + |\Im w_0| + K_4 \min(\log n, -\log \eta)].$$

Since w_0 is the fixed initial point, it does not depend on η. So, we can rewrite the estimate as for $n > 1$ and η sufficiently small:

$$|\Im w_n(\eta)| \leq L_1 (1+\eta)^n \min(\log n, -\log \eta).$$

For $n \leq N(\eta)$, the growth of $\Re w_n(\eta)$ is at least linear by Lemma 4.1, while the exponential term in the preceding estimate is bounded by e and so the second estimate will hold with $(\log n)/n$ picked in the maximum.

For $n > N(\eta)$, we have to come back to estimate (18), which yields

$$\Re w_n(\eta) \geq (1+\eta)^n c(0) \eta^{-1} (0.9 - (1+\eta)^{-n}) \geq (1+\eta)^n c(0) \eta^{-1}/2$$

for η sufficiently small. Now the second estimate also follows with the second possibility in the maximum. □

LEMMA 4.6. *There is a constant K_7 such that for every η sufficiently small and $n > 1$,*

$$|\Im z_n(\eta)| \leq K_7 \max(n^{-3/2} \log n, -\eta^{3/2} \log \eta) \exp(-n\eta/8).$$

Proof. Clearly,

$$|\Im z_n(\eta)| = \left| \Im \left(\frac{1}{\sqrt{\Re w_n(\eta) + i \Im w_n(\eta)}} \right) \right| = (\Re w_n)^{-1/2} \left| \Im \left(1 + i \frac{\Im w_n(\eta)}{\Re w_n(\eta)} \right)^{-1/2} \right|.$$

From Lemma 4.5, the ratio of the imaginary part to the real part is small for small η and large n and so for all $n \leq L_1$, where L_1 does not depend on η, because the estimate on the ratio is uniform, and η sufficiently small,

$$\left| \Im \left(1 + i \frac{\Im w_n(\eta)}{\Re w_n(\eta)} \right)^{-1/2} \right| \leq \left| \frac{\Im w_n(\eta)}{\Re w_n(\eta)} \right|.$$

So,

$$|\Im z_n(\eta)| \leq \left| (\Re w_n)^{-1/2} \frac{\Im w_n(\eta)}{\Re w_n(\eta)} \right|.$$

From Lemma 4.5,

$$|\Im z_n(\eta)| \leq K_6/\sqrt{K_5} \max\left(\frac{\log n}{n}, -\eta \log \eta\right) \sqrt{\max(n^{-1}, \eta)} \exp(-n\eta/8)$$
$$= K_7 \max(n^{-3/2} \log n, -\eta^{3/2} \log \eta) \exp(-n\eta/8)$$

since both maxima pick up their first or second value simultaneously depending on whether $n \leq N(\eta)$ or not. □

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Proof of the third claim. From the form of $\gamma(z, \eta)$, we conclude that

$$|z_{n+1}(\eta) - z_n(\eta)| \le L_1|z_n(\eta)| \max(\eta, |z_n(\eta)|^2) \le L_2|z_n(\eta)| \max(\eta, n^{-1})$$

using estimate (14). Here L as usual denotes a constant independent of n and η.

We see that for $n \le N(\eta)$, we get

$$|z_{n+1}(\eta) - z_n(\eta)| \le L_3 n^{-3/2}.$$

For $n > N(\eta)$, the maximum will be η and, from the second claim of the theorem, we get

$$|z_{n+1}(\eta) - z_n(\eta)| \le L_4 \eta^{3/2} \exp(-n\eta/8).$$

By Lemma 4.1, with $\beta = 3/2$, for any positive n,

$$\exp(-n\eta/8) \le n^{-3/2}\eta^{-3/2}(12)^{3/2}$$

and in conclusion

$$|z_{n+1}(\eta) - z_n(\eta)| \le L_5 n^{-3/2}. \qquad \square$$

Proof of the fourth claim. Taking into account that η and $\hat{c}(\eta)$ are real, we obtain an estimate

$$\begin{aligned}&|\Im z_{n+1}(\eta) - \Im z_n(\eta)|\\ &\quad \le \eta|\Im z_n(\eta)| + 3\hat{c}(\eta)|\Im z_n(\eta)||z_n(\eta)|^2 + |\hat{d}(\eta)|\eta|z_n(\eta)|^4 + L_1|z_n(\eta)|^5.\end{aligned}$$

Taking into account the estimates of the second claim of Theorem 2 and of Lemma 4.6 gives

$$\begin{aligned}&L_2^{-1}|\Im z_{n+1}(\eta) - \Im z_n(\eta)|\\ &\quad \le \max(n^{-3/2}\log n, -\eta^{3/2}\log\eta)\exp(-n\eta/8)[\eta + \max(n^{-1}, \eta)\exp(-n\eta/4)]\\ &\qquad + \max(n^{-2}, \eta^2)\exp(-n\eta/2)\left[\eta + \sqrt{\max(n^{-1}, \eta)}\right].\end{aligned}$$

For $n \le N(\eta)$, we can continue by

$$L_2^{-1}|\Im z_{n+1}(\eta) - \Im z_n(\eta)| \le L_3 n^{-5/2}\log n.$$

For $n > N(\eta)$, we get

$$\begin{aligned}&L_2^{-1}|\Im z_{n+1}(\eta) - \Im z_n(\eta)|\\ &\quad \le L_4\eta^{5/2}\log\eta^{-1}\exp(-n\eta/8) + \eta^{5/2}\exp(-n\eta/2) \le 2L_4\eta^{5/2}\log\eta^{-1}\exp(-n\eta/8)\end{aligned}$$

for η sufficiently small. Lemma 4.1 leads to $\exp(-n\eta/8) \le (20/e)^{5/2}\eta^{-5/2}n^{-5/2}$ and hence to

$$L_2^{-1}|\Im z_{n+1}(\eta) - \Im z_n(\eta)| \le L_5\log\eta^{-1}n^{-5/2} < L_5 n^{-5/2}\log n,$$

which gives the fourth claim. $\square$

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5. *Unimodal Fibonacci maps*

We prove that the limiting drift is infinite in this case, following the scheme which is used for the Fibonacci covering maps. This result is not new and could be extracted from [**2**].

We will use the same notation for analogous notions for the Fibonacci covering and unimodal maps. By ℓ, we will denote any *even* integer greater than 2. The following facts can be obtained from [**2**, **11**, **5**, **3**].

Fact 5.1. For every ℓ, there are parameters $\tau_\ell > 1$, $x_{0,\ell} \in (0, 1)$, and a map ϕ_ℓ with a representation

$$\phi_\ell(x) = (E_\ell(x))^\ell$$

as follows:

- $E_\ell : I \to \left(-\sqrt[\ell]{\tau_\ell^2}, \sqrt[\ell]{\tau_\ell^4}\right)$ is a decreasing diffeomorphism from the Epstein class defined on a real interval $I \supset [0, 1]$;
- $E_\ell(x_{0,\ell}) = 0$, $E_\ell(0) = 1$;
- the *fixed point equation* holds for all x where both sides are defined:

$$\tau_\ell^{-1}\phi_\ell\tau_\ell(x) = (\tau_\ell\phi_\ell\tau_\ell^{-1}) \circ \phi_\ell(x); \tag{20}$$

- $T_1 \le \tau_\ell \le T_2$, $c_1 \le x_{0,\ell} < \tau_\ell x_{0,\ell} < c_2$ for all ℓ and some $1 < T_1 < T_2 < \infty$, $0 < c_1 < c_2 < 1$.

5.1. *Induced dynamics.* For each ℓ, there is a unique point $X_\ell \in (0, x_{0,\ell})$ so that $\phi_\ell(X_\ell) = \tau_\ell X_\ell$. It follows from the fixed point equation (20) that $\phi_\ell(\tau_\ell X_\ell) = \tau_\ell^3 X_\ell$. Note also that $\phi_\ell(X_\ell) = \tau_\ell X_\ell < \tau_\ell x_{0,\ell} = \phi_\ell(x_{0,\ell}/\tau_\ell)$, where the latter equality follows from the equation (20). Hence, $x_{0,\ell} < \tau_\ell X_\ell$. Using Fact 5.1, $0 < c_1/T_2 < X_\ell < x_{0,\ell} < \tau_\ell X_\ell < \tau_\ell x_{0,\ell} < c_2 < 1$ for all ℓ.

For each ϕ_ℓ, consider its *inducing domain* $J_\ell = (X_\ell, \tau_\ell X_\ell)$.

Definition 5.1. The *induced dynamics*

$$\Phi_\ell : J_\ell \to J_\ell$$

is defined with the exception of countably many points and in the form

$$\Phi_\ell(x) = \tau_\ell^{-m(x)}\phi_\ell(x),$$

where $m(x) = m(x, \ell)$ is the unique integer for which $\tau_\ell^{-m(x)}\phi_\ell(x) \in J_\ell$.

The function $m(x)$ takes values $0, -1, -2, \ldots$ on $(X_\ell, x_{0,\ell})$ and $2, 1, 0, -1, -2, \ldots$ on $(x_{0,\ell}, \tau_\ell X_\ell)$. A crucial difference with the drift function for the Fibonacci covering maps is that $-m(x)$ is bounded from below. The inverse branches of Φ_ℓ can be written as

$$\psi_{m,\ell}^{\pm}(x) = (\phi_\ell^{-1})_\pm(\tau_\ell^m x). \tag{21}$$

Here $(\phi_\ell^{-1})_\pm(x) = E_\ell^{-1}(\pm x^{1/\ell})$ and we choose $-$ if $\phi_\ell^{-1}(x) \in (x_{0,\ell}, \tau_\ell X_\ell)$ and $+$ if $\phi_\ell^{-1}(x) \in (X_\ell, x_{0,\ell})$, so that $(\phi_\ell^{-1})'_+ < 0$, $(\phi_\ell^{-1})'_- > 0$.

It follows from Fact 5.1 that $(\phi_\ell^{-1})_+$ extends to a univalent map in $\mathbb{C}\backslash\{x \in \mathbb{R} : x \notin [0, 1]\}$ and $(\phi_\ell^{-1})_-$ extends to a univalent map in $\mathbb{C}\backslash\{x \in \mathbb{R} : x \notin [0, \tau_\ell^2]\}$. Therefore, we have the following corollary.

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COROLLARY 5.1. *For all ℓ sufficiently large, the induced map Φ_ℓ is uniformly expanding in the Poincaré metric of $\mathbb{C}\setminus\{x \in \mathbb{R} : x \notin [0, 1]\}$.*

Let I_0, I_1 be fixed closed segments such that

$$I_0 \subset \text{int}\,(I_1) \subset I_1 \subset (0, 1). \tag{22}$$

The real bounds from Fact 5.1 imply that if I_0 is close enough to $(0, 1)$, then, for every m and ℓ sufficiently large, $\psi^{\pm}_{m,\ell}(I_1) \subset I_0$.

5.2. *Transfer operators and invariant densities.* The transfer operator is given by the formula

$$P_\ell f(z) = \sum_{m\le 2} (\psi^-_{m,\ell})'(z) f \circ \psi^-_{m,\ell}(z) - \sum_{m\le 0} (\psi^+_{m,\ell})'(z) f \circ \psi^+_{m,\ell}(z). \tag{23}$$

Fact 5.2. For every ℓ sufficiently large, the transfer operator acting on the space $L_1(J_\ell)$ has unique non-negative invariant density $f_\ell = \lim_{n\to\infty} P^n_\ell \mathbf{e}$, which in fact is positive, where $\mathbf{e}$ is a constant density.

This follows from the folklore theorem in the light of Corollary 5.1.

Let D_0, D_1 be disks with diameters I_0, I_1, respectively. The functions $\psi^{\pm}_{m,\ell}$ admit holomorphic extensions to the disk whose diameter is $(0, 1)$. Based on Fact 5.1 and (22), $\psi_{m,\ell}(D_1) \subset D_0$ for all m and ℓ sufficiently large. The functional space $\mathcal{F}_0$ will consist of all functions bounded and holomorphic on D_0, real on the real line, as well as continuous on $\overline{D_0}$, with the C^0 norm. Repeating considerations from the proofs of Lemmas 2.4, 2.5, and 2.6, we conclude that the operators P_ℓ are well defined and compact in the space $\mathcal{F}_0$, the densities f_ℓ admit holomorphic continuations to $f_{0,\ell} \in \mathcal{F}_0$, and the sequence $(f_{0,\ell})$ is uniformly bounded in this space.

5.3. *The drift.* For every ℓ, we define the *drift* as before:

$$\vartheta(\ell) := -\int_{J_\ell} m(x, \ell) f_\ell(x)\, dx.$$

We need an *associated map*

$$G_\ell = \tau_\ell^{-1} \phi_\ell \tau_\ell^{-1}. \tag{24}$$

It follows from the fixed point equation (see also [**5**, **3**]).

Fact 5.3. For any finite ℓ,

$$G_\ell(x_{0,\ell}) = x_{0,\ell}, \quad G'_\ell(x_{0,\ell}) = -\frac{1}{\tau_\ell^{2/\ell}}.$$

This means that the point $x_{0,\ell}$ is an attracting orientation reversing fixed point of G_ℓ and $G_\ell(x_{0,\ell}) \to -1$ as $\ell \to \infty$.

As in Lemma 3.1, we then get that for any ℓ function, $\log \phi_\ell$ is integrable on the interval J_ℓ. This allows us to repeat the proof of Lemma 3.2, which gives the identical *formula for the drift*:

$$\theta(\ell) = -(\log|\tau_\ell|)^{-1} \int_{J_\ell} \log \frac{|\phi_\ell(x)|}{|x|} f_\ell(x)\, dx. \tag{25}$$

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5.4. *Passing to the limit.* In order to show that $\theta(\ell) \to +\infty$, we prove that for any sequence $\ell_i \to \infty$ there is a subsequence $\ell_{i(j)}$ such that $\theta(\ell_{i(j)}) \to +\infty$. For the proof, it is enough that the sequence (ϕ_ℓ) is pre-compact (which is easier to show than the convergence).

LEMMA 5.1. *Given $\ell_i \to \infty$, there is a subsequence $\ell_{i(j)}$ such that there exist limits $x_0 = \lim x_{0,\ell_{i(j)}} \in (0, 1)$, $X = \lim X_{\ell_{i(j)}}$, $\tau = \lim \tau_{\ell_{i(j)}} > 1$, and $\phi_{\ell_{i(j)}}$ converge on $(0, \tau x_0)$ uniformly on compacts to a function ϕ_∞ such that $\phi_\infty(x) \geq 0$, $\phi_\infty(x) = 0$ if and only if $x = x_0$. Moreover, the map ϕ_∞ is real analytic on $(0, x_0) \cup (x_0, \tau x_0)$. Furthermore, the function $G_\infty = \tau^{-1}\phi_\infty\tau^{-1}$ is real analytic on $(0, \tau x_0)$, $G_\infty(x_0) = x_0$, $G'_\infty(x_0) = -1$, and $G^2_\infty(x) = x - c(x - x_0)^3 + O(|x - x_0|^4)$, where $c > 0$. Finally,*

$$\tau^{-2}\phi_\infty(x) = \phi_\infty G_\infty(x) \tag{26}$$

for x where both sides are defined.

The proof is based on Fact 5.1 and then proceeds as in [**7**, Theorem 2]. One takes into account that $\phi_\ell(1/\tau_\ell) = 1/\tau_\ell^2$, $\phi'_\ell(1/\tau_\ell) = 1$, and $\phi(x_{0,\ell}/\tau_\ell) = \tau_\ell x_{0,\ell}$.

Lemma 5.1 allows us to define the induced map $\Phi_\infty : J_\infty \to J_\infty$ ($J_\infty = (X, \tau X)$) and the transfer operator P_∞ using the limit map ϕ_∞ and then conclude that Corollary 5.1 and Fact 5.2 hold for $\ell = \infty$. In particular, there is a unique invariant density f_∞ for P_∞ which is real analytic and strictly positive on J_∞. Moreover, repeating the proof of Proposition 1, we have that the sequence of densities $f_{\ell_{i(j)}}$ tends to f_∞ almost uniformly on J_∞.

On the other hand, by (26), the function $-(\log \tau^4)^{-1} \log \phi_\infty$ is a Fatou coordinate for G^2_∞ at the fixed point x_0. Since G^2_∞ has two attracting petals, we have that necessarily as $x \to x_0$,

$$\log \phi_\infty(x) = C_0(x - x_0)^{-2} + C_1(x - x_0)^{-1} + O(|\log |x - x_0||),$$

where $C_0 < 0$. There is $K > 0$ such that $\log \phi_\ell(x) f_\ell(x) < K$ for all ℓ large or $\ell = \infty$ and all $x \in J_\ell$. Hence,

$$\int_{J_\infty} \log \phi_\infty(x) f_\infty(x)\, dx = -\infty.$$

Coupled with the facts that $\log \phi_{\ell_{i(j)}}(x) f_{\ell_{i(j)}}(x) \to \log \phi_\infty(x) f_\infty(x)$ and $J_{\ell_{i(j)}} \to J_\infty$, this implies that

$$\theta(\ell) = -(\log |\tau_\ell|)^{-1} \int_{J_\ell} \log \frac{|\phi_\ell(x)|}{|x|} f_\ell(x)\, dx \to +\infty$$

along the subsequence $\ell_{i(j)}$.

Acknowledgements. The second author acknowledges that the work on the problem of the drift started as a joint project with Edson Vargas. The work is supported in part by an ISF grant 1378/13 and by a grant 2012/05/B/ST1/00551 funded by Narodowe Centrum Nauki.

https://doi.org/10.1017/etds.2016.10 Published online by Cambridge University Press

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https://doi.org/10.1017/etds.2016.10 Published online by Cambridge University Press