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Examples in non-commutative projective geometry

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0. *Introduction*

Let $A = k \oplus \bigoplus_{n \geqslant 1} A_n$ be a connected graded, Noetherian algebra over a fixed, central field k (formal definitions will be given in Section 1 but, for the most part, are standard). If A were commutative, then the natural way to study A and its representations would be to pass to the associated projective variety and use the power of projective algebraic geometry. It has become clear over the last few years that the same basic idea is powerful for non-commutative algebras; see, for example, [ATV1, 2], [AV], [Sm], [SS] or [TV] for some of the more significant applications. This suggests that it would be profitable to develop a general theory of 'non-commutative projective geometry' and the foundations for such a theory have been laid down in the companion paper [AZ]. The results proved there raise a number of questions and the aim of this paper is to provide negative answers to several of these.

The starting point for these results is a classical result of Serre: Consider a commutative, connected graded k-algebra C with associated projective variety X. Given a finitely generated, graded C-module M then one can form the associated sheaf of modules $\pi(M)$. Conversely, given a coherent sheaf of modules $\mathscr{M}$, then $\dim_k(\mathrm{H}^0(\mathscr{M})) < \infty$ and one can form the associated Noetherian, graded C-module $\Gamma_+(\mathscr{M}) = \bigoplus_{i \geqslant 0} \mathrm{H}^0(s^i(\mathscr{M}))$, where s is the 'shift functor' $\otimes \mathscr{O}_X(1)$. These functors are more or less inverses to each other: if C is generated by C_1 as a k-algebra, then $\pi(\Gamma_+(\mathscr{M})) \cong \mathscr{M}$ while M and $\Gamma_+(\pi(M))$ differ by at most a finite dimensional vector space.

Remarkably, one can generalize this, and many related, results to modules over certain non-commutative, Noetherian, connected graded rings $A = k \oplus \bigoplus_{n \geqslant 1} A_n$. In this case, following [AV] and [AZ], one replaces X (or, more formally the category of coherent sheaves over X) by proj-A. This is defined as follows: Let gr-A denote the category of finitely generated, graded right A-modules, with subcategory tors-A being the full subcategory of finite dimensional modules. Then, define qgr-A to be the quotient category gr-A/tors-A and write proj-A = (qgr-A, $\mathscr{O}_A$) where the distinguished object $\mathscr{O}_A$ is the image of A in qgr-A. [AZ] shows that the natural analogues of Serre's Theorem (and related results on the cohomology of sheaves) do hold for modules over any graded, right Noetherian ring A satisfying a certain condition χ. This is defined as follows: If $i \geqslant 0$, then A is said to satisfy χ_i if $\underline{\mathrm{Ext}}^i_A(A/\bigoplus_{n \geqslant 1} A_n, M)$ is a finite dimensional k-vector space for each finitely generated, graded right A-module M. The ring A satisfies condition χ if it satisfies χ_i for all $i \geqslant 0$. This generalization of Serre's Theorem is particularly useful since proj-A is frequently much nicer than A; to give a simple example, if $U = \mathbb{C}\{x, y\}/(yx - xy - x^2)$ is the

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'Jordan quantum coordinate ring' of the plane, graded by $deg(x) = 1 = deg(y)$, then proj-$U \cong$ proj-$\mathbb{C}[z_1, z_2]$. Some much less trivial examples and applications of non-commutative analogues of Serre's Theorem can be found in [AV].

It is easy to show that any commutative Noetherian ring satisfies χ, and the same is true of many non-commutative rings. For example, [SZ] shows that χ holds for any graded, Noetherian PI algebra. Moreover, most of the results in [AZ] depend upon this condition. This raises the basic question of whether χ always holds.

The aim of this paper is to exhibit rings where χ fails and to show that, for these examples, proj has some rather unpleasant properties:

Example 0·1. Let $U = \mathbb{C}\{x, y\}/(xy - yx - x^2)$. Then $R = \mathbb{C} + Uy$ is a Noetherian, connected graded ring that does not satisfy χ_j for any $j > 0$. (See Theorem 2·3 and Corollary 2·5.)

Example 0·2. The analogue of Serre's Theorem fails for R. Indeed, $\Gamma_+(\mathcal{O}_R) = U$ and U/R is infinite dimensional (see Corollary 2·8).

One should note that proj-$R \cong$ proj-$U \cong$ proj-$\mathbb{C}[z_1, z_2]$ and so has every nice property one could desire. However, with a slight modification this is no longer true:

Example 0·3. Let $T = R[z]$ be a polynomial extension in a commuting indeterminate z. Then $\Gamma_+(\mathcal{O}_T) = T$. However, if $\mathcal{R}$ denotes the image of T/zT in qgr-T, then $\Gamma_+(\mathcal{R})$ is not even finitely generated as a right T-module. Moreover, the shift functor s is not ample on proj-T (see Theorem 2·10 and Corollary 2·11).

Of course, in this last result, $\Gamma_+(\mathcal{O}_T)$ is Noetherian. However, by taking the trivial extension of T by R one obtains a Noetherian ring A for which $\Gamma_+(\mathcal{O}_A)$ is not even Noetherian. The ring R has a number of other curious properties. For example, if C is a graded Noetherian, commutative ring then, for all $d \gg 0$, the Veronese ring $C^{(d)} = \bigoplus_{n \geqslant 0} C_{nd}$ is generated by $C_1^{(d)} = C_d$ (see [Mu, lemma, p. 282]). In contrast, for $d \geqslant 1$, $R^{(d)}$ is a Noetherian ring that is never generated in degree one (see Corollary 3·2). However, one does have an equivalence of categories proj-$R \cong$ proj-$R^{(d)}$ (see Corollary 3·2, again).

Motivated by the examples of this paper, M. Artin and the first author have recently been able to classify non-commutative, Noetherian, connected graded domains of Gelfand-Kirillov dimension two. Roughly speaking, these algebras fall into two families: Those in the first class are close to being twisted coordinate rings in the sense of [AV], they satisfy the condition χ and hence the non-commutative version of Serre's theorem holds for these algebras. Those of the second class look similar to the ring R of this paper; condition χ and Serre's Theorem fail for these algebras and their Veronese rings are never generated in degree one.

The category of graded R-modules also exhibits some peculiarities. If C is a commutative Noetherian, connected graded ring generated by C_1 then the points of the associated projective variety C can be identified with point modules; modules $M = \bigoplus_{i \geqslant 0} M_i$ which are cyclic, 1-critical, and satisfy $\dim_k(M_i) = 1$ for all $i \geqslant 0$. Point modules are also fundamental to the study of various non-commutative rings; see, for example, [ATV1, 2], [SS] or [TV]. Now, for the rings studied in those papers a key result is that, for $m \gg 0$, each point module M is uniquely determined by its truncation $\bigoplus_{i=0}^{m} M_i$. The corresponding result fails for R, thereby answering a version of [ATV1, question 3·16]. The precise result is stated in Corollary 3·7. The structure

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of the point modules for R also answers a question of Artin from [Ar] concerning extensions of point modules by torsion modules (see Theorems 3·5 and 3·6 and the comments thereafter).

It is easy to modify R and obtain a ring S, generated by S_1, that satisfies the conclusions given in (0·1), (0·2) and (0·3); for example take $S = \mathbb{C} + \left(\begin{smallmatrix} Ux+Uy & Uy \\ Ux+Uy & Uy \end{smallmatrix}\right)$ (see Corollary 2·14).

We also consider non-commutative analogues of Serre's Vanishing Theorem ([Ha, theorem III·5·2]). If A is a right Noetherian, graded ring that satisfies χ then this generalizes: [AZ, theorem 7·4] proves that, if $\mathscr{M} \in \text{qgr-}A$, then $\mathrm{H}^i(s^l(\mathscr{M})) = 0$ for all $i > 0$ and $l \gg 0$. Here, s is the natural shift functor. Although we know of no Noetherian ring for which this result falls, right Noetherian examples do exist:

Example 0·4. There exists a right Noetherian, connected graded ring A that satisfies χ_1 but not χ_2. Moreover, $\mathrm{H}^1(s^l(\mathscr{O}_A)) \neq 0$ for infinitely many choices of $l \in \mathbb{N}$ (see Corollary 4·4).

Finally, we note that, if A is a connected graded, right Noetherian ring that satisfies a polynomial identity, then it is easy to determine when A satisfies χ: this happens if and only if A is left Noetherian (see Theorem 4·1).

1. *Proj-A and Serre's Theorem*

In this short section we give the basic definitions and results about proj-A that are relevant to this paper.

Throughout this paper, all rings will be algebras over a fixed, central field k. A k-algebra A is graded, or more strictly $\mathbb{N}$-graded, if $A = \bigoplus_{n \geqslant 0} A_n$ where each A_n is a finite dimensional k vector space and $A_n A_m \subseteq A_{n+m}$ for any n and m. Set

$$A_+ = \bigoplus_{n \geqslant 1} A_n.$$

If $A_0 = k$, then A is called connected graded and we identify A/A_+ with k. An A-module M is called graded if $M = \bigoplus_{n \in \mathbb{Z}} M_n$, with $M_n A_m \subseteq M_{n+m}$, for all n, m. Define $M_{\geqslant m} = \bigoplus_{i \geqslant m} M_i$ and define the shift of M to be the graded module $s(M)$ defined by $(s(M))_n = M_{n+1}$. Let gr-A denote the category of graded, finitely generated, right A-modules with morphisms being graded A-module homomorphisms of degree zero. Note that, for $M, N \in \text{gr-}A$, $\mathrm{Ext}^i_A(M,N)$ denotes the derived group of $\mathrm{Hom}_A(M,N)$ in gr-A. Thus, we denote $\bigoplus_{l \in \mathbb{Z}} \mathrm{Hom}_A(M, s^l(N))$ by $\underline{\mathrm{Hom}}(M,N)$, with derived groups $\underline{\mathrm{Ext}}^i(M,N)$. Under reasonable conditions (for example if M is Noetherian) $\underline{\mathrm{Hom}}(M,N)$ is simply the set of homomorphisms from M to N in the category of ungraded A-modules.

Assume, now, that A is right Noetherian and let proj-A = (qgr-A, $\mathscr{O}_A$) be defined as in the introduction. Given $M \in \text{gr-}A$, let $\tau(M)$ denote the unique, largest finite dimensional submodule of M and let $\pi(M)$ denote the image of M in qgr-A. One important object is the shift functor $s = s_A$ on proj-A, which is simply the autoequivalence of proj-A induced by the shift functor s on gr-A. If $M \in \text{gr-}A$ or $M \in \text{qgr-}A$, we will frequently write $s^r(M) = M[r]$. The morphisms in qgr-A are those induced from the morphisms in gr-A. Explicitly, if $M, N \in \text{gr-}A$, then

$$\mathrm{Hom}_{\text{qgr-}A}(\pi(M), \pi(N)) = \lim_{i \to \infty} \mathrm{Hom}_{\text{gr-}A}(M_{\geqslant i}, N). \tag{1·0·1}$$

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More details concerning these observations can be found in [AZ] or [Po, chapter 4].

Consider a triple $(\mathscr{C}, \mathscr{O}, s)$, where $\mathscr{C}$ is a k-linear, abelian category, $\mathscr{O}$ is an object of $\mathscr{C}$, and s is an autoequivalence of $\mathscr{C}$. The global section functor H^0 is defined by $\mathrm{H}^0(\mathscr{M}) = \mathrm{Hom}_{\mathscr{C}}(\mathscr{O}, \mathscr{M})$ for $\mathscr{M} \in \mathrm{Ob}(\mathscr{C})$, with derived functors $\mathrm{H}^i(\mathscr{M})$. Set $\Gamma_+(\mathscr{M}) = \bigoplus_{n \geqslant 0} \mathrm{H}^0(s^n(\mathscr{M}))$. Observe that $\Gamma_+(\mathscr{O})$ has a natural graded ring structure and that $\Gamma_+(\mathscr{M})$ has a natural structure as a graded right $\Gamma_+(\mathscr{O})$-module. If $M \in \mathrm{gr}\text{-}A$, for some ring A, then

$$\Gamma_+(\pi(M)) = \{\lim_{i \to \infty} \underline{\mathrm{Hom}}_A(A_{\geqslant i}, M/\tau(M))\}_{\geqslant 0}. \tag{1·0·2}$$

This provides a natural embedding of $M/\tau(M)$ into $\Gamma_+(\pi(M))$. Indeed:

LEMMA 1·1. *Suppose that $M \in \mathrm{gr}\text{-}A$, with $\tau(M) = 0$ and $M_i = 0$ for $i < 0$. Then $\Gamma_+(\pi(M))$ is the unique, largest, $\mathbb{N}$-graded submodule L of the (graded) injective hull $E(M)$ of M with the property that L/M is torsion.*

Proof. The natural map $\mathrm{Hom}(A_{\geqslant n}, M[r]) \to \mathrm{Hom}(A_{\geqslant n+1}, M[r])$ is an injection for all n and r. Now, let $\theta \in \mathrm{Hom}(A_{\geqslant n}, M[r])$ and set $J = \mathrm{Ker}(\theta)$. Clearly, $\tau(A_{\geqslant n}/J) = 0$. If $K/J = \tau(A/J)$, we may factor θ as $\theta: A_{\geqslant n} \twoheadrightarrow (A_{\geqslant n} + K)/K \hookrightarrow M[r]$. Since A/K is an essential extension of $(A_{\geqslant n} + K)/K$ the second map extends to a graded injection $A/K \hookrightarrow E(M[r])$ and so θ extends to a homomorphism $\theta_1: A \to E(M[r])_{\geqslant 0}$. If θ extends to two such homomorphisms, say θ_1 and θ_2, then $(\theta_1 - \theta_2)(A_{\geqslant n}) = 0$. Thus,

$$(\theta_1 - \theta_2)(A) \in \tau(E(M[r])) = 0$$

and θ_1 is unique. Conversely, if $x \in E(M[r])_0$ is such that $(xA + M[r])/M[r]$ is finite dimensional, then the graded homomorphism $\phi: A \to E(M[r])$ defined by $1 \mapsto x$ clearly satisfies $\phi(A_{\geqslant m}) \in M[r]$ for all $m \gg 0$. Thus, the map $\theta \to \theta_1$ does define the required isomorphism. ∎

The shift operator s in a triple $(\mathscr{C}, \mathscr{O}, s)$ is called ample if:

1·1·1. For each $\mathscr{M} \in \mathrm{Ob}(\mathscr{C})$, there exists an epimorphism $\bigoplus_{i=1}^{p} s^{-n_i}(\mathscr{O}) \twoheadrightarrow \mathscr{M}$, for some $n_j \in \mathbb{N}$; and

1·1·2. Given $\mathscr{M}, \mathscr{N} \in \mathrm{Ob}(\mathscr{C})$ and an epimorphism $f: \mathscr{M} \twoheadrightarrow \mathscr{N}$, then the maps $\mathrm{H}^0(s^n(f))$ are surjective for all $n \gg 0$.

Given two such triples $(\mathscr{C}_i, \mathscr{O}_i, s_i)$, define $(\mathscr{C}_1, \mathscr{O}_1, s_1)$ to be isomorphic to $(\mathscr{C}_2, \mathscr{O}_2, s_2)$ if there exists a category equivalence $\mathscr{F}: \mathscr{C}_1 \to \mathscr{C}_2$ such that $\mathscr{F}(\mathscr{O}_1) = \mathscr{O}_2$ and there exists a natural isomorphism between $\mathscr{F} \circ s_1$ and $s_2 \circ \mathscr{F}$. The desired non-commutative analogue of Serre's Theorem is provided by the next two theorems and is proved in [AZ, corollary 4·6].

THEOREM 1·2. *Suppose that a triple $(\mathscr{C}, \mathscr{O}, s)$ satisfies: (H1) $\mathscr{O}$ is a Noetherian object, (H2) $\mathrm{H}^0(\mathscr{M})$ is a finite dimensional over k for all objects $\mathscr{M}$ of $\mathscr{C}$, and (H3) s is ample. Then, $A = \Gamma_+(\mathscr{O})$ is a right Noetherian $\mathbb{N}$-graded k-algebra that satisfies χ_1. Moreover, $(\mathscr{C}, \mathscr{O}, s) \cong (\mathrm{qgr}\text{-}A, \mathscr{O}_A, s_A)$.*

THEOREM 1·3. *Conversely, suppose that B is a right Noetherian graded algebra satisfying χ_1. Then the triple $(\mathrm{qgr}\text{-}B, \mathscr{O}_B, s_B)$ satisfies the conditions (H1), (H2), and (H3). If $M \in \mathrm{gr}\text{-}B$, then the natural ring homomorphism $\iota: B \to \Gamma_+(\mathscr{O}_B)$ and the natural B-module homomorphism $\iota: M_{\geqslant 0} \to \Gamma_+(\pi(M))$ have finite dimensional kernels and cokernels.*

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2. *The basic example*

The aim of this section is to construct and study the basic properties of the example $R = k + yk\{x, y\}/(yx - xy - x^2)$ mentioned in the introduction. That example will only have the required properties in characteristic zero, and so, since we prefer to give a construction that works in all characteristics, we will use a 'quantum' version of that ring.

Notation 2·1. Let $q \in k \backslash \{0\}$. Throughout this paper, $U = U_q$ will denote the k-algebra $k\{x, y\}/(yx - qxy - x^2)$, graded by defining $\deg(x) = \deg(y) = 1$. Set $R = R_q = k + Uy$.

We will denote the images of x, y in U again as x and y. If $q \neq 1$, then

$$U \cong k\{u, v\}/(uv - qvu),$$

but the given presentation is more convenient for our purposes. Using the fact that U is an Ore extension of $k[x]$, it is readily checked that U is a Noetherian graded domain with (Rentshler–Gabriel) Krull dimension, Gelfand–Kirillov and global homological dimensions all equal to 2. Moreover, U has standard k-basis $\{x^i y^j : 0 \leqslant i, j < \infty\}$ or, if one prefers, $\{y^i x^j : 0 \leqslant i, j < \infty\}$.

Given $p \in k$ and $n \in \mathbb{N}$, define $[n]_p = \sum_{i=1}^{n} p^{i-1}$. Then, an induction beginning with the relation $yx - qxy - x^2$ yields the following useful identity:

$$x^n y = q^{-n} y x^n - q^{-n}[n]_q x^{n+1}. \tag{2·1·1}$$

Some of the elementary properties of the rings U_q and R_q are provided by the following lemma:

LEMMA 2·2. (i) $U = R \oplus xk[x]$. (ii) *As a subalgebra of* U, $R = k\langle y, xy\rangle$. *Moreover,* $U = R + Rx$. (iii) *If* $R' = R'_q = k + yU$, *then* $R \cong R'$ *via the inner automorphism* $a \to yay^{-1}$ *of the quotient division ring* $Q(R) = Q(U)$. (iv) $U = U_q$ *satisfies* χ *and* $\underline{\mathrm{Ext}}^1_U(k, U) = 0$.

Proof. (i) This is immediate from the fact that U has basis $\{x^i y^j\}$. In order to prove part (ii), note that R has basis $\{1, x^i y^j : i \geqslant 0, j \geqslant 1\}$. By induction, the formula $x^n y \,.\, xy - qx^{n+1} y \,.\, y = x^{n+2} y$ ensures that $k\langle y, xy\rangle \supset \{x^m y : m \geqslant 0\}$. Multiplying these elements on the right by appropriate powers of y ensures that $R = k\langle y, yx\rangle$. Similarly, the formula $x^n y \,.\, x - qx^{n+1} y = x^{n+1}$ implies that $U = R + Rx$. Part (iii) follows easily from part (ii). Finally, part (iv) follows from [LS, lemma] combined with [AZ, theorem 8·1].

THEOREM 2·3. (i) *Assume that* $[n]_q \neq 0$, *for any* $n \geqslant 1$. *Then*: (*a*) $R = R_q$ *is a Noetherian domain of infinite global dimension.* (*b*) *Moreover,* $\dim_k(\underline{\mathrm{Ext}}^1_R(k_R, R)) = \dim_k(\underline{\mathrm{Ext}}^1_R({}_R k, R)) = \infty$; *indeed,* $\underline{\mathrm{Ext}}^1_R(k_R, R) \cong U/R$ *as graded left* R*-modules. Consequently,* R *does not satisfy the condition* χ_1 *on either side.* (*c*) *If* R *is a homomorphic image of a Noetherian, graded ring* S, *then* S *also fails to satisfy* χ_1.

(ii) *If* $[n]_q = 0$ *for some* $n \geqslant 1$, *then* R_q *is not Noetherian.*

Proof. (i*a*) The proof that R is Noetherian is very similar to that of [St2, theorem 3·3] but, since it is easy, we will give it here. Let I be a left ideal of R. Then UI and UyI are (finitely generated) left ideals of the Noetherian ring U. Since $U = R + Rx$, this implies that UI and UyI are finitely generated left R-modules. Thus, in order to prove that I is a finitely generated left R-module, it suffices to prove that UI/UyI is

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a Noetherian left R-module. As a U-module (or R-module) UI/UyI is a homomorphic image of a finite direct sum of copies of U/yU and so it is suffices to show that U/Uy, or even U/R, is a Noetherian left R-module.

Let B/R be a non-zero R-submodule of U/R and pick $a \in B \backslash R$. Since $U = R \oplus xk[x]$, we may assume that $a \in xk[x]$. By hypothesis, $[m]_q \neq 0$, for any $m \geqslant 1$. Thus, if a has degree m, then (2·1·1) implies that $q^{-m}ya - ay = \lambda x^{m+1} + f_1(x) + f_2(x)y$, where $\lambda \in k^*$ and $f_i \in k[x]$, with $deg(f_1) \leqslant m$. Thus, by induction, $B/Ra + R$ is at most m-dimensional and so is certainly a finitely generated left R-module. Thus, U/Uy and I are finitely generated left R-modules and R is left Noetherian. A dual argument implies that $R' = k + yU$ is right Noetherian. Thus, Lemma 2·2(iii) implies that R is Noetherian.

The Hilbert series $\bigoplus_{i \geqslant 0} \dim_k (M_i) t^i$ of a graded module $M = \bigoplus_{i \geqslant 0} M_i$ is denoted by $H_M(t)$, while the projective homological dimension of M is written $\mathrm{hd}\,(M)$. A standard fact is that, if M is finitely generated left R-module with $\mathrm{hd}\,(M) < \infty$, then there exists a polynomial $p(t) \in \mathbb{Z}[t]$ such that $H_M(t) = p(t) H_R(t)$ (see [ATV2, p. 340]). The standard bases of U and R ensure that $H_U(t) = 1/(1-t)^2$ and $H_R(t) = 1 + t/(1-t)^2$. Thus, $\mathrm{hd}\,({}_R U) = \infty$. A more explicit proof can be obtained as follows: A straightforward computation implies that

$$U(y^2 - (1+q)xy)x = U(qyx - qx^2)y = R \cap Rx.$$

Hence one has a short exact sequence

$$0 \to Uhx \to R \oplus Rx \to U \to 0, \quad \text{for} \quad h = (y^2 - (1+q)xy). \tag{2·3·1}$$

Since U is a graded, uniform left R-module that is not cyclic, it cannot be projective. Thus, (2·3·1) implies that $\mathrm{hd}\,({}_R U) = \infty$.

(ib) It remains to identify $\underline{\mathrm{Ext}}^1(k_R, R)$. From the short exact sequence (of right R-modules) $0 \to Uy \to R \to k \to 0$, one sees that $\underline{\mathrm{Ext}}^1(k_R, R) \cong (Uy)^*_R / R$, where

$$(Uy)^*_R = \{q \in Q(R) : qUy \subseteq Uy\}.$$

Trivially, $(Uy)^* \supseteq U$. Conversely, suppose that $\theta \in (Uy)^*$. Then, $\theta UyU \subseteq U$. But, $UyU \supseteq Ux^2 + Uy \supseteq U_{\geqslant 2}$ and so $\theta U_{\geqslant 2} \subseteq U$. By Lemma 2·2(iv), this forces $\theta \in U$ and so $(Uy)^* = U$. In other words, $\underline{\mathrm{Ext}}^1(k_R, R) \cong U/R \cong k[x]$ is infinite dimensional. Once again, Lemma 2·2(iii) can be used to show that $\underline{\mathrm{Ext}}^1({}_R k, R)$ is infinite dimensional.

(ic) This follows from part (ib) combined with [AZ, theorem 8·3].

(ii) Let $Z(S)$ denote the centre of a ring S. Since $[n]_q = 0$, (2·1·1) implies that $C = k[x^n] \subseteq Z(R_q)$. Thus, for any $r \in R_+ = Uy$ and $s \in \mathbb{N}$, $rx^{ns} \in Uy \subseteq R$. Hence, $\sum_{s \geqslant 0} kx^{ns} + R$ is an infinitely generated left R-module. Since U is a finitely generated left R-module, this implies that R is not left Noetherian. Similarly, using the isomorphism $R \cong R'$, R is not right Noetherian.

Remark 2·4. The reader should note the distinction in Theorem 2·3 between the cases $q \neq 1$ and $q = 1$. Indeed: (i) If $q \neq 1$, then $[n]_q = 0$ for some $n \geqslant 1$ if and only if q is a root of unity. Thus, if $q \neq 1$ then $R = R_q$ is Noetherian if and only if q is not a root of unity. (ii) If $q = 1$, then $[n]_q = 0$, for some $n \geqslant 1$ if and only if $\mathrm{char}\,(k) > 0$. Thus, if $q = 1$, then $R = R_1$ is Noetherian if and only if $\mathrm{char}\,(k) = 0$.

COROLLARY 2·5. *If* $[n]_q \neq 0$ *for* $n \geqslant 1$, *then* $\dim_k (\underline{\mathrm{Ext}}^i_R (k, R)) = \infty$ *for all* $i > 0$.

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Proof. The exact sequence (2·3·1) implies that $\underline{\mathrm{Ext}}^i(U,R) = \underline{\mathrm{Ext}}^1(U,R)$, for all $i > 1$. Similarly, the fact that $U \cong Uy = R_+$ implies that

$$\underline{\mathrm{Ext}}^i(k,R) \cong \underline{\mathrm{Ext}}^{i-1}(U,R) \cong \underline{\mathrm{Ext}}^{i+1}(k,R),$$

for all $i \geqslant 2$. Thus, it remains to compute $\underline{\mathrm{Ext}}^2(k,R) \cong \underline{\mathrm{Ext}}^1(U,R)$. From the sequence (2·3·1) we obtain the exact sequence

$$R \oplus x^{-1}R \xrightarrow{\theta} x^{-1}h^{-1}Uy \to \underline{\mathrm{Ext}}^1(U,R) \to 0,$$

where θ is the homomorphism defined by inclusion on each component. This quickly implies that $\underline{\mathrm{Ext}}^1(U,R)$ maps onto the infinite dimensional vector space $Uy/(hUy + h.k + hx.k)$. ∎

Although condition χ does not hold for $R = R_q$, the space proj-R is very pleasant; indeed, proj-$R \cong$ proj-$U \cong$ proj-$k[z_1, z_2]$. This is easy to prove directly but also follows from the next proposition.

LEMMA 2·6. (i) *Let L be a finitely generated, right module over a Noetherian ring A and N a finite dimensional left A-module. Then, for all i, $\mathrm{Tor}_i^A(L,N)$ is a finite dimensional k-vector space.*

(ii) *Let $\theta: N_1 \to N_2$ be a homomorphism of finitely generated, left A-modules for which $\mathrm{Ker}(\theta)$ and $\mathrm{Coker}(\theta)$ are finite dimensional. Then both $\mathrm{Ker}(L \otimes \theta)$ and $\mathrm{Coker}(L \otimes \theta)$ are finite dimensional.*

Proof. (i) Pick a finitely generated, graded free resolution $F^\cdot \to L$. Then $\mathrm{Tor}_i^A(L,N)$ is a subfactor of $F^i \otimes N$ and so is finite dimensional.

(ii) This is a routine consequence of part (i).

PROPOSITION 2·7. *Let $B \subseteq A$ be graded, Noetherian rings. Assume that A/B is a torsion right B-module and a finitely generated left B-module. Then*

$$(\mathrm{qgr}\text{-}A, \mathcal{O}_A, s) \cong (\mathrm{qgr}\text{-}B, \mathcal{O}_B, s).$$

Proof. Let $A = \sum_{i=1}^m Ba_i$. Since A/B is a torsion right B-module, there exists n such that $a_i B_{\geqslant n} \subseteq B$, for each i. Therefore, $a_i(B_{\geqslant n}) \subseteq A_{\geqslant n} \cap B = B_{\geqslant n}$. Thus,

$$A(B_{\geqslant n}) = \sum Ba_i(B_{\geqslant n}) \subseteq B_{\geqslant n}$$

and $B_{\geqslant n}$ is an (A,B)-bimodule. The functors we will use to induce the equivalence of categories are induced from the functors:

$$\mathcal{F}: \mathrm{gr}\text{-}A \to \mathrm{gr}\text{-}B \quad \text{given by} \quad M \mapsto M \otimes_A B_{\geqslant n}$$

and

$$\mathcal{G}: \mathrm{gr}\text{-}B \to \mathrm{gr}\text{-}A \quad \text{given by} \quad N \mapsto N \otimes_B A.$$

If $M \in \mathrm{tors}\text{-}A$, then Lemma 2·6 implies that $M \otimes_A B_{\geqslant n}$ is finite dimensional and hence lies in tors-B. Thus, $\mathcal{F}$ induces a functor $\tilde{\mathcal{F}}: \mathrm{qgr}\text{-}A \to \mathrm{qgr}\text{-}B$. Similarly, as ${}_BA$ is finitely generated, $\mathcal{G}(\mathrm{tors}\text{-}B) \subseteq \mathrm{tors}\text{-}A$ and so $\mathcal{G}$ induces a functor $\tilde{\mathcal{G}}: \mathrm{qgr}\text{-}B \to \mathrm{qgr}\text{-}A$.

We first show that $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ are inverse functors. Let $M \in \mathrm{gr}\text{-}A$. Then,

$$\mathcal{G} \circ \mathcal{F}(M) = M \otimes_A B_{\geqslant n} \otimes_B A.$$

As $B/B_{\geqslant n}$ is finite dimensional, Lemma 2·6 implies that the kernel and cokernel of

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the natural map $B_{\geqslant n} \otimes_B A \to B \otimes_B A = A$ are finite dimensional. Therefore, the same is true of the map $M \otimes_A B_{\geqslant n} \otimes_B A \to A \otimes A = M$. Thus, $\tilde{\mathscr{G}} \circ \tilde{\mathscr{F}}(\pi(M)) = \pi(M)$. If $N \in \text{gr-}B$, then $\mathscr{F} \circ \mathscr{G}(N) = N \otimes_B B_{\geqslant n}$ and a similar argument implies that $\tilde{\mathscr{F}} \circ \tilde{\mathscr{G}}(\pi(N)) = \pi(N)$. Thus, $\tilde{\mathscr{F}}$ is a category equivalence qgr-$A \to$ qgr-B with inverse equivalence $\tilde{\mathscr{G}}$. Finally, it is clear that $\tilde{\mathscr{F}}(\mathscr{O}_A) = \mathscr{O}_B$ and that $\tilde{\mathscr{F}}$ commutes with s. ∎

COROLLARY 2·8. *Assume that* $[n]_q \neq 0$ *for* $n \geqslant 1$ *and set* $R = R_q$. *Then* (i) *the triples* (qgr-$R, \mathscr{O}_R, s$) *and* (qgr-$U, \mathscr{O}_U, s$) *are equivalent. Consequently,* proj-R *is isomorphic to* proj-$k[x, y]$; (ii) $\Gamma_+(\mathscr{O}_R) = \Gamma_+(\mathscr{O}_U) = U$.

Proof. (i) Since $U.R_+ = Uy \subseteq R$, Theorem 2·3(i) and Lemma 2·2(ii) imply that the hypotheses of Proposition 2·7 are satisfied. The first assertion therefore follows from that proposition. The second assertion then follows from [Zh, theorem 3·7].

(ii) Lemmas 2·2(iv) and 1·1 combine to show that $\Gamma_+(\mathscr{O}_U) = U$, while Proposition 2·7 implies that $\Gamma_+(\mathscr{O}_R) = \Gamma_+(\mathscr{O}_U)$. ∎

In summary, Corollary 2·8 shows that, although R does not satisfy χ, the projective scheme proj-R does satisfy all the properties one could desire. However, by replacing R by the polynomial extension $T = R[z]$ we obtain a ring for which both proj-T and the associated global section functor are badly behaved. (Given a graded ring $A = \bigoplus_{i \geqslant 0} A_i$ we will always let $A[z]$ denote the polynomial extension of A in a commutative indeterminate z and grade $A[z]$ by $A[z]_i = \bigoplus_{j=0}^{i} A_j z^{i-j}$.)

LEMMA 2·9. *Let* A *be a graded, right Noetherian ring and* $M, N \in \text{gr-}A$. *Then, for all* n *and* l, *one has* $\dim_k (\underline{\text{Ext}}^n_A(N, M))_l < \infty$.

Proof. Take a finitely generated, graded free resolution $F^\cdot \to N \to 0$; thus,

$$X = \underline{\text{Ext}}^n_A(N, M)$$

is a (graded) subfactor of the graded group $L = \underline{\text{Hom}}_A(F^n, M)$. Now, $L = \bigoplus_{i \in \mathbb{Z}} L_i$, where $L_i = \text{Hom}_{\text{gr-}A}(F^n, M[i])$. Since each L_i is finite dimensional, certainly $\dim_k (X_i) < \infty$ for each i.

THEOREM 2·10. *Let* A *be a right Noetherian, connected graded ring that does not satisfy* χ_1 (*for example, take* $A = R_q$ *for any* q *such that* $[n]_q \neq 0$ *for* $n > 0$). *Then* (1·1·2) *fails for the polynomial extension ring* $A[z]$.

Proof. Identify A with $A[z]/(z)$ and note that this identifies $k = A/A_+$ with $A[z]/A[z]_+$. Let $M \in \text{gr-}A$ be such that $\dim_k (\underline{\text{Ext}}^1_A(k, M)) = \infty$. By induction, we may assume that M is a cyclic A-module. Since $\underline{\text{Ext}}^i_A(k, k)$ is finite dimensional for all i, we may replace M by $M/\tau(M)$ and also assume that $\tau(M) = 0$. Thus, if $\bar{A} = A/\tau(A)$, there exists a surjection $f: \bar{A}[z] \to \bar{A} \to M$. Since $\underline{\text{Ext}}^1_A(k, M)$ is infinite dimensional, Lemma 2·9 implies that $\underline{\text{Ext}}^1_A(k, M)_n \neq 0$ for infinitely many $n \geqslant 0$. Thus, for any $r_0 \geqslant 0$, there exists $r \geqslant r_0$ and a graded, essential extension (of A-modules and therefore of $A[z]$-modules) $0 \to M \to L \to k \to 0$ for which $L_r \neq M_r$. Pick $l \in L_r \backslash M_r$ and let $g: A[z] \to s^r(L)$ be defined by $g(1) = l$. Clearly, $g(A[z]_+) \in s^r(M)$ and so g induces a morphism $\pi(g) \in H^0(s^r \pi(M))$.

Suppose that $\pi(g) \in \text{Im}(H^0(s^r \pi(f))$. By the definition of morphisms in qgr-$A[z]$ (see (1·0·1)) this is equivalent to the following assertion: for some $t \gg 0$, there exists $h \in \text{Hom}(A[z]_{\geqslant t}, s^r(\bar{A}[z]))$ such that, upon restriction to $A[z]_{\geqslant t}$, one has $g = s^r(f) \circ h$.

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As in the proof of Lemma 1·1, h extends to a homomorphism $\tilde{h}: A[z] \to s^r(E_{A[z]}(\bar{A}[z]))$, where $E(\bar{A}[z])$ denotes the graded injective hull of $\bar{A}[z]$. Note that

$$(\tilde{h}(A[z]) + s^r\bar{A}[z]))/s^r(\bar{A}[z])$$

is finite dimensional. Now, [Ro, corollary 11·68] implies that $\underline{\mathrm{Ext}}^1_{A[z]}(k, \bar{A}[z]) = \underline{\mathrm{Hom}}_A(k, \bar{A}) = 0$. Thus $\tilde{h}(A[z]) \in s^r(\bar{A}[z])$. Thus, g extends to a homomorphism $\tilde{g} = s^r(f)\tilde{h}: A[z] \to s^r(M) \subset s^r(L)$. Finally, since $\tilde{g}$ and g coincide on $A[z]_{\geqslant t}$, $\mathrm{Im}(\tilde{g} - g)$ is finite dimensional. As $\tau(L) = \tau(M) = 0$, this implies that $\tilde{g} = g$. But, as $g(1) \notin M$ and $\tilde{g}(1) \in M$, this is absurd. ∎

Consider the application of Theorem 2·10 to $R = R_q$ and $T = R[z]$, where $[n]_q \neq 0$ for any $n \geqslant 1$. In this case, we may take the module M to be R itself, and take f to be the natural surjection $T \to R$. Theorem 2·3(i) implies that, under the identification $\underline{\mathrm{Ext}}^1_R(k, R) = U/R$, one has $x^r \in \underline{\mathrm{Ext}}^1(k, R)_r$, for each r. Thus, the proof of Theorem 2·10 implies that $\mathrm{H}^0(s^r(f))$ is not surjective for any $r > 0$. In contrast to Corollary 2·8, the functor $\Gamma_+(\pi(__))$ behaves badly for T:

Corollary 2·11. *Assume that* $[n]_q \neq 0$ *for* $n > 0$ *and set* $R = R_q$ *and* $T = R[z]$. *Then,* $\Gamma_+(\mathcal{O}_T) = T$ *but* (1·1·2) *fails for* proj-T. *Moreover, as a right* T*-module,* $\Gamma_+(\pi(T/zT)) = U$ *is not even finitely generated.*

Proof. By [Ro, corollary 11·68], $\underline{\mathrm{Ext}}^1_T(k, T) = \underline{\mathrm{Hom}}_R(k, R) = 0$. Thus, by Lemma 1·1, $\Gamma_+(\mathcal{O}_T) = T$ and, by Theorem 2·10, (1·1·2) fails for T. Let $\theta \in \underline{\mathrm{Hom}}_T(T_{\geqslant m}, R)$. Then, for all $t \in T_{\geqslant m-1}$, one has $\theta(zt)T_+ = \theta(tT_+)z = 0$. Since $\tau_T(R) = 0$, this implies that $\theta(zT_{\geqslant m-1}) = 0$. Hence, by Corollary 2·8(ii),

$$\theta \in \underline{\mathrm{Hom}}_T((T/zT)_{\geqslant m}, R) = \underline{\mathrm{Hom}}_R(R_{\geqslant m}, R) = U. \quad ∎$$

By taking the trivial extension of T by R, one can obtain a Noetherian ring A for which $\Gamma_+(\mathcal{O}_A)$ is not right Noetherian.

Corollary 2·12. *Let* $R = R_q$ *and* $T = R[z]$, *where* $[n]_q \neq 0$ *for* $n > 0$, and set

$$A = \begin{pmatrix} T & R \\ 0 & T \end{pmatrix} \quad \text{and} \quad \tilde{A} = \begin{pmatrix} T & U \\ 0 & T \end{pmatrix}.$$

Then, $\Gamma_+(\mathcal{O}_A) = \tilde{A}$ *and* $\Gamma_+(\mathcal{O}_A)$ *is not right Noetherian.*

Proof. As usual, we regard R and U as T-modules via the identification $R = T/zT$. Let $N(A)$ denote the nil-radical of A and observe that $A/N(A) \cong T \oplus T$, while $N(A) \cong R$. Moreover, $A_+N(A) = N(A)A_+$. Given these observations, the proof becomes very similar to that of Corollary 2·11 and is left to the reader. ∎

One should remark that the idea for the construction of R_q from U_q comes from the examples of idealizers in [St2]: in particular it is easy to show that, if $[n]_q \neq 0$ for $n \geqslant 1$, then $R_q = \mathbb{I}_U(Uy) = \{u \in U: Uyu \subseteq Uy\}$. There is nothing particularly special about our choice of U; rather it is the simplest ring for which the proof of Theorem 2·3 (and hence the subsequent results of this section) will work. Thus, one should expect that the same basic technique of taking an idealizer inside a graded ring U ought to be useful in providing other examples of graded rings with peculiar properties. In particular, the proof that R_q does not satisfy χ_1 generalizes considerably:

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LEMMA 2·13. *Let $B \subset A$ be Noetherian rings such that A/B is an infinite dimensional k-vector space that is finitely generated as a left B-module but torsion as a right B-module. Then B does not satisfy the condition χ_1.*

Proof. As in the proof of Proposition 2·7, $A(B_{\geqslant m}) \subseteq B_{\geqslant m}$, for any sufficiently large m. By Lemma 2·6, $A/(B_{\geqslant m}A) \cong B/B_{\geqslant m} \otimes_B A$ is finite dimensional and so $B_{\geqslant m}A \supseteq A_{\geqslant n}$, for some n. Thus, if $J = \{a \in A : aB_{\geqslant m} = 0\}$, then $JA_{\geqslant n} = 0$ and, as A is Noetherian, J is finite dimensional. Now, we have a natural exact sequence

$$0 \to (B/B_{\geqslant m})^* \to B \xrightarrow{\phi} (B_{\geqslant m})^* \to \underline{\mathrm{Ext}}^1_B(B/B_{\geqslant m}, B) \to 0,$$

where $X^* = \mathrm{Hom}_B(X, B)$. Left multiplication by elements of A provides a natural injection $A/J \hookrightarrow (B_{\geqslant m})^*$. Thus,

$$A/(B+J) \hookrightarrow (B_{\geqslant m})^*/\mathrm{Im}(\phi) \cong \underline{\mathrm{Ext}}^1_B(B/B_{\geqslant m}, B).$$

Thus, $\dim_k(\underline{\mathrm{Ext}}^1_B(B/B_{\geqslant m}, B)) = \infty$. By [AZ, proposition 3·8(2)], B does not satisfy χ_1. ∎

An alternative proof of Lemma 2·13 can be obtained by using Theorem 1·3: We may assume that $\tau(A) = \tau(B) = 0$. Proposition 2·7 now implies that

$$\Gamma_+(\mathcal{O}_B) = \Gamma_+(\mathcal{O}_A) \supseteq A$$

and so $\dim_k(\Gamma_+(\mathcal{O}_B)/B) = \infty$. Thus, Theorem 1·3 implies that B does not satisfy χ_1.

Although the ring R_q is a finitely generated k-algebra, it is not generated in degree one. If one is willing to work with prime rings that are not domains, then one can easily adjust R_q in order to add this extra hypothesis:

COROLLARY 2·14. *Assume that $[n]_q \neq 0$ for $n \geqslant 1$ and let $R = R_q$. Set*

$$S = S_q = k + \begin{pmatrix} Uy & Ux+Uy \\ Uy & Ux+Uy \end{pmatrix} \subset S' = \begin{pmatrix} R & U \\ Uy & U \end{pmatrix} \subset M_2(U).$$

Then S is a connected graded Noetherian ring, generated by S_1, which does not satisfy χ_1.

Proof. The ring $M_2(U)$ of 2×2 matrices over U is graded by $M_2(U)_j = M_2(U_j)$. Clearly, S is a connected graded subring of $M_2(U)$. Since the modules ${}_RU$ and $(Uy)_R$ are finitely generated, S' is finitely generated as a left or right module over the Noetherian subring $\left(\begin{smallmatrix} R & 0 \\ 0 & U \end{smallmatrix}\right)$. Thus, S' is Noetherian. Since S'/S is finite dimensional, [St1, lemma 1·4] implies that S is Noetherian. Let e_{ij} denote the standard matrix units in $M_2(U)$. Then, as a k-algebra, S is generated by $\{ye_{11}, ye_{12}, xe_{12}, ye_{21}, ye_{22}, xe_{22}\}$, and so S is indeed generated by S_1. Finally, Lemma 2·13, with $B = S \subset A = M_2(U)$, implies that S does not satisfy condition χ_1. ∎

Remark. If one replaces R by S in (2·10–2·12), then the conclusions of those results still hold, but one may now add the extra conclusion that the ring in question is generated in degree one.

3. *Further properties of R_q*

Let $R = R_q$ be the ring defined by Notation 2·1. There are two main aims in this section. First, we wish to describe R in terms of generators and relations. This has the consequence that no Veronese ring of R is generated in degree one; this is in stark

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contrast to the commutative situation. Secondly, we will compute the point modules for R (the definitions will be given later) as this enables us to answer some questions raised by Artin.

Recall that, by Lemma 2·2(ii), $R = k\langle y, w\rangle$, where $w = xy$. An easy induction on m provides the identity

$$y^m x - q^m x y^m - [m]_q x y^{m-1} x = 0. \tag{3·0·1}$$

Set $r_m = y^m w - q^m w y^m - [m]_q w y^{m-2} w$. Then, multiplying (3·0·1) on the right by y shows that $r_m = 0$ holds in R for all $m \geqslant 2$. Let $k\{Y, W\}$ denote the free algebra, graded by $deg(Y) = 1$ and $deg(W) = 2$ and set $I_m = Y^m W - q^m W Y^m - [m]_q W Y^{m-2} W$ for $m \geqslant 2$.

PROPOSITION 3·1. (i) $R \cong k\{Y, W\}/(I_2, I_3)$.

(ii) *If* $[r]_q \neq 0$ *for any* $r > 0$, *then a vector space basis of* R_n, *for* $n \geqslant 0$, *is given by*

$$\{y^n, y^i w y^j : i, j \geqslant 0, i+j+2 = n\}. \tag{3·1·1}$$

(iii) *If* $[p]_q = 0$ *and* $[r]_q \neq 0$ *for all* $0 < r < p$, *then a vector space basis of* R_n *for* $n \geqslant 0$ *is given by* $\{y^n, y^i (w y^{p-2})^s w y^j : p > i \geqslant 0, s, j \geqslant 0; i+sp+j+2 = n\}$.

Proof. We will only prove the result in the case when $[r]_q \neq 0$ for all r. The other case is similar and is left to the reader.

An easy computation proves that, in the free algebra $k\{Y, W\}$,

$$I_{m+2} = Y^2 I_m + q^m I_2 Y^m - [2]_q W I_m + [m]_q I_2 Y^{m-2} W \tag{3·1·2}$$

for all $m \geqslant 4$. Set $A := k\{Y, W\}/(I_2, I_3)$. Since $r_2 = r_3 = 0$ holds in R, there is a unique surjection $\phi : A \twoheadrightarrow R$, given by $\phi(Y) = y$ and $\phi(W) = w$. To prove that ϕ is an isomorphism, it suffices to prove that $\dim_k(A_n) \leqslant \dim_k(R_n)$ for all $n \geqslant 2$.

By (3·1·2), $I_m = 0$ in A for all $m \geqslant 2$. Thus A_n is spanned by the elements

$$\{Y^n, Y^i W Y^j : i, j \geqslant 0, i+j+2 = n\}.$$

Therefore, $\dim_k(A_n) \leqslant 1 + (n-1) = \dim_k(R_n)$. Consequently, ϕ is an isomorphism and the elements given in (3·1·1) do form a basis of R. ∎

Given a graded ring A and an integer $d \geqslant 2$, let $A^{(d)}$ denote the Veronese subring $\oplus_{n \geqslant 0} A_{nd}$ of A. We give $A^{(d)}$ a graded structure by defining $A^{(d)}_n = A_{nd}$. The shift functor for $A^{(d)}$ is simply s^d. If C is a graded, commutative Noetherian ring, then it is standard that some Veronese ring $C^{(d)}$ will be generated in degree one (see, for example, [Mu, lemma, p. 282]). The situation for R_q is rather different:

COROLLARY 3·2. *Assume that* $[n]_q \neq 0$ *for* $n \geqslant 1$ *and set* $R = R_q$. Then:

(i) *For each* $d \geqslant 1$, *the Veronese ring* $R^{(d)}$ *is a Noetherian, finitely generated* k*-algebra that is not generated in degree* 1.

(ii) *For each* $d \geqslant 1$, *there is an equivalence of categories* $(\text{proj-}R, s_R^d) \cong (\text{proj-}R^{(d)}, s_R^d)$.

Remark. If $[n]_q = 0$ for some $n > 0$, it is still the case that no Veronese ring $(R_q)^{(d)}$ is generated in degree one. However, this is of less interest since one can show that $(R_q)^{(n)}$ is then a commutative ring that is not Noetherian. Thus, it is certainly not finitely generated as a k-algebra.

Proof. (i) As R is Noetherian and graded, [AZ, proposition 5·10] implies that $R^{(d)}$ is also Noetherian and hence is finitely generated. Fix $d \geqslant 2$. By Proposition 3·1, the

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elements $\{y^d, y^i w y^j : i,j \geqslant 0, i+j+2 = d\}$ form a basis of $R_1^{(d)}$. Consequently, $(R_1^{(d)})^2$ is spanned by the elements

$$y^{2d}, \quad y^{d+i} w y^j, \quad y^i w y^{d+j}$$

and (using the identities $r_n = 0$)

$$y^i w y^j . y^k w y^l = a y^{i+j+k+2} w y^l + b y^i w y^{j+k+l+2},$$

for some $a, b \in k$. Here, i,j,k,l run through all elements of $\mathbb{N}$ satisfying $i+j = k+l = d-2$. Since $i, l \leqslant d-2$, this implies that the basis element $y^{d-1} w y^{d-1}$ of $R_2^{(d)}$ does not lie in $(R_1^{(d)})^2$. Therefore $R^{(d)}$ is not generated in degree 1.

(ii) Since $U.R_+ \subseteq R_+$, it follows easily that $U^{(d)}.R_+^{(d)} \subseteq R_+^{(d)}$. As $R^{(d)}$ is Noetherian, $\bigoplus_{n \geqslant 0} R_{nd+k} y^{d-k}$ is a finitely generated left $R^{(d)}$-module for all $1 \leqslant k < d$. Thus both R and U are finitely generated left $R^{(d)}$-modules. Therefore, by Proposition 2·7 and [AZ, theorem 5·10] $(\text{proj-}R, s_R^d) \cong (\text{proj-}U, s_U^d) \cong (\text{proj-}U^{(d)}, s_U^d) \cong (\text{proj-}R^{(d)}, s_R^d)$. ∎

Part (ii) of Corollary 3·2 is perhaps slightly surprising since one does not expect the projective schemes of a ring and its Veronese ring to be isomorphic, unless the ring is generated in degree one. To give a simple illustration, take $R = k[x^2]$, graded by $\deg(x^2) = 2$. Then, whatever the shift functor, proj-R has two non-isomorphic simple objects, R and xR, whereas proj-$R^{(2)}$ has just one.

The aim of the rest of this section is to classify the 1-critical right R-modules. Here, a finitely generated, graded R-module M is called (Gelfand–Kirillov) 1-critical if M is not finite dimensional but every proper factor of M is finite dimensional. We will actually do this for $R' = R'_q = k + yU$ rather than the isomorphic ring R_q. The reason for this is purely technical; we want to use the fact that U is a finitely generated right R'-module. The analogous overring of R is $y^{-1}Uy$ and so, by using R', we avoid having to deal with fractions. If A is generated in degree one, then a point module for A is a 1-critical, graded right A-module M with Hilbert series $1/(1-t)$. Such a module is automatically cyclic, generated by M_0. These have proved very useful to the study of graded, non-commutative rings; see, in particular, [ATV2]. Since R' is not generated in degree one, this definition is inappropriate and we will define a *point module* to be a graded, 1-critical right A-module $M = \bigoplus_{i \geqslant 0} M_i$, generated by M_0, for which $\dim_k(M_i) \leqslant 1$ for all i. If A is generated by A_1 this does coincide with the earlier definition.

The 1-critical modules over the ring U are easily classified:

LEMMA 3·3. *Assume that k is algebraically closed and set $U = U_q$, for any $q \in k^*$. The non-isomorphic, 1-critical, graded right U-modules are:*

$$\{{}_pC = U/(ax+by)U : p = (a,b) \in \mathbb{P}^1\},$$

together with their shifts $s^i({}_pC)$. Moreover, each ${}_pC$ is a point module.

Proof. Clearly $U/(ax+by)U = U/(acx+bcy)U$, for any $0 \neq c \in k$ and so the definition of ${}_pC$ does make sense. By [Zh, example 3·6], U is a twist of the commutative polynomial ring $k[x,y]$ corresponding to the automorphism $\alpha : x \mapsto x, y \mapsto q^{-1}y - q^{-1}x$. In other words, define a new multiplication $\circ$ on $k[x,y]$ by $a \circ b = ab^{\alpha^n}$, for $a \in k[x,y]_n$ and $b \in k[x,y]_m$. Then U may be identified with $(k[x,y], +, \circ)$. By [Zh, theorem 3·1], $\text{gr-}k[x,y] \cong \text{gr-}U$. This equivalence sends a $k[x,y]$-module $M = \bigoplus M_i$ to the module M', which is defined to be the same graded vector space as

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M, but whose U-module structure is given by $m \circ a = ma^n$, for $m \in M_n$ and $a \in U_r = k[x, y]_r$.

This equivalence clearly sends critical modules to critical modules and point modules to point modules. Thus, it suffices to classify the 1-critical modules over $k[x, y]$, where the proposition is standard. ∎

Natural candidates for point modules over R' are the modules

$$_{(a,b)}M = R'/(y(ax+by))\,U$$

since these modules do have Hilbert series $1/(1-t)$. The parametrization of these modules by $\mathbb{P}^1$ becomes a little cumbersome and so we will write $_\infty M = R'/vU$, for $v = yx$ and $_a M = R'/y(y+ax)\,U$ for $a \in k$. Observe that, if we write $[r]$ for the image in $_a M$ of $r \in R'$, then $_a M$ has basis $\{z_0 = [1], z_{i+1} = [yx^i] : i \geqslant 0\}$. Similarly, $_\infty M$ has basis $\{z_i = [y^i] : i \in \mathbb{N}\}$. Given $a \in k$, set $\sigma(a) = q^{-1}(a+1)$. By induction, $\sigma^n(a) = q^{-n}(a+[n]_q)$ and $\sigma^{-n}(a) = q^n a - [n]_q$ for all $n \geqslant 0$.

LEMMA 3·4. *Assume that $[n]_q \neq 0$ for $n > 0$ and write $R' = R'_q$. Then:*

(i) *$_\infty M$ is a cyclic, 1-critical right R'-module and $(_\infty M)_{\geqslant n} = {}_\infty M[-n]$ for all $n > 0$. Set $_\infty N = {}_\infty M$.*

(ii) *If $a \in k$ and $a \neq \sigma^{-n}(0)$ for $n \geqslant 0$, then $_a N = {}_a M$ is a cyclic, 1-critical right R'-module.*

(iii) *Assume that $a = \sigma^{-s}(0)$ for some $s \geqslant 0$. Then $z^{s+1}.k = [yx^s]\,k$ is a 1-dimensional R'-submodule of $_a M$. Moreover, $_a N = {}_a M/z^{s+1}k$ is a cyclic, 1-critical right R'-module.*

(iv) *Assume that $a \in k \cup \{\infty\}$. Then $_a N$ has Hilbert series $1/(1-t)$, unless $a = \sigma^{-s}(0)$, for some $s \geqslant 0$, in which case $_a N$ has Hilbert series $(1/(1-t)) - t^{s+1}$. Moreover, the only non-zero, graded submodules of $_a N$ are the $z^i R'$, for $i \geqslant 0$.*

(v) *If $a \neq 0$, then $(_a N)_{\geqslant 1} = (_{\sigma(a)}N)[-1]$. However, $(_0 N)_1 = 0$ and $(_0 N)_{\geqslant 1} = (_0 N)_{\geqslant 2} = (_{\sigma^2(0)}N)[-2]$.*

Proof. (i) The equation $xU \cap R' = wU$ induces a ring isomorphism $k[y] \cong U/xU \cong R'/wU$ and the result follows.

(ii) By Lemma 2·2, $R' = k\langle y, v\rangle$, for $v = yx$. Also, as $_a M \subset {}_a\tilde{M} = U/y(y+ax)\,U$, one sees that $z_i x$ is a well-defined element of $_a\tilde{M}$. Thus the structure of $_a M$ as an R'-module is determined by the formulae: $z_0 y = z_1, z_0 v = z_2$ and, for $i \geqslant 0$,

$$\begin{aligned} z_{i+1} y = z_1 x^i y &= z_1(q^{-i} y x^i - q^{-i}[i]_q x^{i+1}) \\ &= z_1(q^{-i}(-axx^i) - q^{-i}[i]_q x^{i+1}) = -\sigma^i(a)\, z_{i+2} \end{aligned} \tag{3·4·1a}$$

and

$$z_{i+1} v = z_{i+1} yx = -\sigma^i(a)\, z_{i+3}. \tag{3·4·1b}$$

Since σ is a set-theoretic automorphism of k, $\sigma^i(a) = 0$ if and only if $a = \sigma^{-i}(0)$ which, by hypothesis, is forbidden. Thus (3·4·1) implies that the only (graded) submodules of $_a M$ are 0 and $(_a M)_{\geqslant n} = z_n R'$ for $n \geqslant 0$. Hence $_a M$ is a cyclic 1-critical right R'-module.

(iii) Equations (3·4·1) now imply that $z_{s+1}.y = z_{s+1}.v = 0$ and so $z_{s+1}k$ is a 1-dimensional R'-submodule. Similarly, as $\sigma^n(a) \neq 0$ for $n \neq s$, these equations imply that $_a M/z_{s+1}k$ is a cyclic 1-critical right R'-module.

(iv) This is an immediate consequence of the proofs of parts (i)–(iii).

(v) We may assume that $a \in k$. Equations (3·4·1) imply that

$$z_n y(y + \sigma^n(a)\, x) = z_n y^2 + \sigma^n(a)\, z_n v = 0 \quad \text{for all} \quad n > 0.$$

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As an element of the U-module $U/y(y+ax)U$, this imples that

$$\operatorname{ann}(z_n) \supseteq y(y+\sigma^n(a)x)U.$$

Thus, $z_n(y(y+\sigma^n(a)x)U) = 0$ also holds in the R'-modules ${}_aM$ and ${}_aN$. Therefore, if $z_n \neq 0$ in ${}_aN$, then, up to a shift, z_nR' is a (necessarily 1-critical) factor of ${}_{\sigma^n(a)}M$. Hence $z_nR' = ({}_{\sigma^n(a)}N)[-n-1]$.

Consequently, if $a \neq 0$ then $({}_aN)_{\geqslant 1} = ({}_{\sigma(a)}N)[-1]$. If $a = 0$, then ${}_0N_1 = 0$ but $z_2 \neq 0$ and so ${}_0N_{\geqslant 1} = {}_0N_{\geqslant 2} = ({}_{\sigma^2(0)}N)[-2]$. ▮

THEOREM 3·5. *Assume that k is algebraically closed and that $[n]_q \neq 0$ for $n > 0$. Set $R' = R'_q$. Then*:

(i) *The non-isomorphic, cyclic, 1-critical graded right R'-modules are precisely the ${}_aN$, for $a \in k \cup \{\infty\}$, together with their shifts $s^i({}_aN)$, for $i \in \mathbb{Z}\backslash\{0\}$. The ${}_aN$ are all point modules.*

(ii) *No cyclic, 1-critical, graded right R'-module properly contains ${}_{\sigma(0)}N$.*

Proof. (i) In order to prove that these modules are non-isomorphic, it suffices to prove that the ${}_aN$ are non-isomorphic. Combined with the description of the ${}_aN$ given by Lemma 3·4, the non-isomorphism follows immediately from the fact that, if there exists a graded isomorphism of right modules $R'/I \to R'/J$, then $I = J$. By Lemma 3·4, each ${}_aN$ is a point module.

Let $P = R'/I$ be a cyclic, 1-critical, graded right R'-module, generated by P_0. We need to prove that $P \cong {}_aN$, for some $a \in k \cup \{\infty\}$. Now, $I \subset R'_+ = yU$ and I/IyU is finite dimensional. Let yJ/IyU be the maximal U-submodule of yU/IyU for which yJ/IyU is finite dimensional. Since yU/I is a 1-critical R'-module, certainly $yJ \subseteq I$. As an R'-module, yU/yJ has a (possibly zero) finite dimensional submodule, with the factor being 1-critical. Thus, the same assertion holds for yU/yJ regarded as a U-module. The definition of yJ therefore implies that yU/yJ is a 1-critical U-module. Thus, Lemma 3·3 implies that $J = (cx+by)U$, for some $c, b \in k$. Therefore, $P = R'/I$ is a factor of $R'/y(cx+by)U$ and so, by Lemma 3·4, $P = {}_aN$, for the appropriate $a \in k \cup \{\infty\}$.

(ii) If ${}_{\sigma(0)}N \subseteq ({}_aN)[r]$, for some a and $r > 0$, then Lemma 3·4(iv) implies that ${}_aN_{\geqslant r} = ({}_{\sigma(0)}N)[-r]$. This contradicts Lemma 3·4(v). ▮

By Lemma 3·4, $\dim_k(\underline{\operatorname{Ext}}^1_{R'}(k, {}_{\sigma^2(0)}N)) \geqslant 2$, the corresponding extensions being ${}_{\sigma(0)}N[1]$ and ${}_0N[2]$. More generally, ${}_{\sigma^2(0)}N$ has essential extensions ${}_{\sigma^{-r}(0)}N[2+r]$, for $r \geqslant 0$ and ${}_{\sigma(0)}N[1]$. Since neither of these embeds in the other this implies that the injective hull $E({}_{\sigma^2(0)}N)[-r]$ contains a module, say $L = {}_{\sigma^{-r}(0)}L$, that is the sum of two submodules that are isomorphic to ${}_{\sigma^{-r}(0)}N$ and ${}_{\sigma(0)}N[-r-1]$, respectively. It is easy to write L down explicitly:

$${}_{\sigma^{-r}(0)}L \cong \frac{{}_{\sigma^{-r}(0)}N \oplus {}_{\sigma(0)}N[-r-1]}{\Delta},$$

where $\Delta = \{(r, r) : r \in ({}_{\sigma^2(0)}N)[-r-2]\}$, under the natural embeddings of $({}_{\sigma^2(0)}N)[-r-2]$ into ${}_{\sigma^{-r}(0)}N$ and $({}_{\sigma(0)}N)[-r-1]$. Note that ${}_{\sigma^{-r}(0)}L$ has Hilbert series $1/(1-t)$.

Clearly, ${}_{\sigma^{-r}(0)}L$ is a 1-critical R'-module and so Theorem 3·5 needs to be modified in order to classify the (possibly non-cyclic) 1-critical R'-modules:

THEOREM 3·6. *Suppose k is algebraically closed and that $[n]_q \neq 0$ for all $n > 0$. Set $R' = R'_q$. Then*:

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(i) *The non-isomorphic, finitely generated, graded* 1*-critical right* R'*-modules are precisely the* ${}_aN$*, for* $n \in k \cup \{\infty\}$ *and* ${}_{\sigma^{-r}(0)}L$*, for* $r \geqslant 0$*, together with their shifts.*

(ii) *Let* $\tilde{L} = \lim_{r\to\infty}({}_{\sigma^{-r}(0)}L[r+1])$*. Then every finitely generated submodule of* $\tilde{L}$ *is* 1*-critical. Moreover,* $\underline{\mathrm{Ext}}^1_{R'}(k, \tilde{L}) = 0$*, yet* $\tilde{L}R'_+ \neq \tilde{L}$*.*

Proof. (i) Any 1-critical, graded right R'-module P contains a cyclic submodule and hence contains some ${}_aN$. Thus, the result follows by induction if one can prove: Let Q be a graded essential extension of some ${}_aN$ or ${}_bL$ by k. Then $Q = {}_cN$ or $Q = {}_cL$ for some c. The proof of this assertion follows from a routine but tedious case by case analysis and will be omitted.

(ii) This follows from part (i). Note that $\tilde{L}R'_+ = \lim_{r\to\infty}({}_{\sigma^{-r}(0)}N[r+1])$, under the natural inclusion. ∎

There are a number of remarks that should be made about Theorems 3·5 and 3·6. First, the existence of the modules given by part (ii) of each theorem answers a question raised by M. Artin in his lecture [Ar]. The point of this question was that, if (among other things) such modules do not exist, then he would be able to classify Noetherian, graded prime rings of Gelfand–Kirillov dimension two. Since the ring R' does not appear in his classification, the existence of these modules is to be expected. Actually, Artin's question was raised for rings generated in degree one, and so these example do not, formally, answer his question. However, if one uses the ring S of Corollary 2·14 in place of R', then one will obtain the desired counterexample. We have not described the 1-critical modules over that ring, since the classification is considerably messier than it is for R'.

Secondly, Theorem 3·5 answers a variant of [ATV1, question 3·16] concerning the variety of truncated point modules Γ_d. The first problem is one of definition. Let A be a connected graded ring, generated in degree one. Then $M \in \mathrm{gr}\text{-}A$ is defined to be a truncated point module of length d if $M = \bigoplus_{i=0}^{d} M_i$ is generated by M_0 and $\dim_k(M_i) = 1$ for $0 \leqslant i \leqslant d$ (see [ATV1, definition 3·8]). However if A is not generated in degree one, it is unreasonable – even when A is commutative – to demand that $M_i \neq 0$ for each $i \leqslant d$. Given that one must allow some M_i to be zero, the natural definition is:

Definition. *Let* A *be a connected graded ring. Then a graded right* A*-module* M *is defined to be a truncated point module of length* d *if* $M = \bigoplus_{i=0}^{e} M_i$ *is generated by* M_0*,* $\dim_k(M) = d+1$ *and* $\dim_k(M_i) \leqslant 1$ *for* $0 \leqslant i \leqslant e$*. Let* Γ'_d *denote the set of isomorphism classes of truncated point modules of length* $< d$ *over* A*.*

If A is generated by A_1, and $M \in \Gamma'_d$, then $\dim_k(M_i) = 1$ for $0 \leqslant i \leqslant d$ and so our definition does coincide with that from [ATV1, section 3]. Note that the map $M \mapsto M/M_f$, where f is the largest integer such that $M_f \neq 0$, defines a map $\phi_d : \Gamma'_d \to \Gamma'_{d-1}$. The generalization of [ATV1, question 3·16] to rings that are not necessarily generated in degree one asks: *Let* A *be a connected graded, Noetherian* k*-algebra that is finitely presented. Then, is* ϕ_d *an isomorphism for* $d \gg 0$*?*

Corollary 3·7. *Assume that* k *is algebraically closed and* $[n]_q \neq 0$ *for all* $n > 0$*. If* $A = R'_q$*, then* A *is a connected graded, Noetherian* k*-algebra that is finitely presented. However, the map* ϕ_d *is not injective, for any* $d \geqslant 2$*.*

Proof. A is Noetherian by Theorem 2·3 and finitely presented by Proposition 3·1.

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In the notation of Lemma 3·4, set $M = {}_{\sigma^{-d+1}(0)}M$ and $N = {}_{\sigma^{-d+1}(0)}N$. Consider the modules $X = M/M_{\geqslant d+1}$ and $Y = N/N_{\geqslant d+2}$. Then, $Y_d = 0$ and so $X, Y \in \Gamma'_d$. However, $\phi_d(X) = M/M_{\geqslant d} = \phi_d(Y)$ and so ϕ_d is not injective. ∎

There are a number of other definitions of Γ'_d that generalize that from [ATV1]; for example, one could define Γ'_d to be the set of all cyclic, graded modules $M = \bigoplus_{i=0}^{d} M_i$ for which $M_0 \neq 0, M_d \neq 0$ and $\dim_k(M_i) \leqslant 1$ for $1 \leqslant i \leqslant d$. In this case, the truncation map $\phi_d: \Gamma'_d \to \Gamma'_{d-1}$ is not even defined. Indeed, for any d, let $N = {}_{\sigma^{-d+1}(0)}N$ and $Z = N/N_{\geqslant d+2}$. Then, $Z \in \Gamma_{d+1}$, but $Z/Z_{d+1} \in \Gamma_{d-1}$.

Obviously, our solution of [ATV1, question 3·16] would be much more satisfactory if one could replace R' by a ring that is generated in degree one, but we know of no such example. In particular, the ring S from Corollary 2·14 does not provide a counterexample, since it has no point modules.

4. *Right noetherian examples*

Most of the results in [AZ], in particular Theorems 1·2 and 1·3, are proved for rings that are assumed to be right Noetherian rather than two-sided Noetherian. It is therefore natural to study the various questions raised by [AZ] for rings that are just Noetherian on one side, particularly since it is usually easier to find counterexamples in this case. This we do in this section and, among other things, we exhibit a right Noetherian ring for which the non-commutative analogue of Serre's Vanishing Theorem will fail. As we first show, condition χ will never hold in PI rings that are only Noetherian on one side:

THEOREM 4·1. *Let A be a graded, right Noetherian PI ring. Then the following are equivalent*: (i) *A is left Noetherian*; (ii) *A satisfies χ_1*; (iii) *A satisfies χ.*

Remark. We emphasize that a ring A is defined to satisfy χ_1, respectively χ, if the appropriate Ext condition holds for graded right A-modules.

Proof. (iii) ⇒ (ii) is a triviality while (i) ⇒ (iii) follows from [SZ, theorem 3·5]. Let A be a graded, right Noetherian, PI ring that is not left Noetherian. It remains to show that χ_1 fails for some right A-module.

The nilradical N of A is graded and nilpotent while the factor ring A/N is left and right Noetherian [MR, theorem 13·6·15]. Since A is not left Noetherian, there therefore exists i such that N^i/N^{i+1} is not Noetherian, and hence not finitely generated, as a left A/N-module. Thus, there exists a graded ideal I, maximal with respect to ${}_AI$ not being finitely generated. Set $B = A/I$ and note that, by the same argument applied to B, the ring B is left and right Noetherian. Consider the right B-module $Y = \underline{\mathrm{Tor}}_1^A(A/A_+, B)$. By [Ro, corollary 11·27 (iii)], $Y \cong A_+ \cap I/A_+I$. Since ${}_AI$ is not finitely generated, $\dim_k(I/A_+I) = \infty$, whereas $I/A_+ \cap I \cong (I+A_+)/A_+$ is finite dimensional. Thus, $\dim_k(Y) = \infty$. Since A is right Noetherian, Y is a right Noetherian B-module and so has a graded, 1-critical factor module, say M. By [MR, corollary 8·3·16 and theorem 6·8·26], $ann_B M$ is a prime ideal. Thus, [MR, corollary 13·6·7 (i)] implies that there is a central element x of positive degree in the ring $B/ann(M)$. Hence multiplication by x^i, for $i \geqslant 0$, provides infinitely many linearly independent elements in $\underline{\mathrm{Hom}}_B(M, M)$. Consequently, $\underline{\mathrm{Hom}}_B(\underline{\mathrm{Tor}}_1^A(A/A_+, B), M)$ is infinite dimensional.

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Now consider the convergent spectral sequence of right B-modules:

$$\underline{\mathrm{Ext}}^p_B(\underline{\mathrm{Tor}}^A_q(A/A_+,B),M) \Rightarrow \mathrm{Ext}^n_A(A/A_+,M)$$

(see [SZ, equation (2·2)]). By [Ro, theorem 11·2] the five-term exact sequence of this spectral sequence is

$$0 \to \underline{\mathrm{Ext}}^1_B(A/A_+ \otimes_A B,M) \to \underline{\mathrm{Ext}}^1_A(A/A_+,M_A) \to \underline{\mathrm{Hom}}_B(\underline{\mathrm{Tor}}^A_1(A/A_+,{}_AB),M)$$
$$\to \underline{\mathrm{Ext}}^2_B(A/A_+ \otimes_A B,M) \to \underline{\mathrm{Ext}}^2_A(A/A_+,M_A). \quad (4\cdot1\cdot1)$$

Since B is a Noetherian left A-module, $A/A_+ \otimes_A B$ is a finite dimensional module. Thus, [SZ, theorem 3·5] implies that $\underline{\mathrm{Ext}}^i_B(A/A_+ \otimes_A B,M)$ is finite dimensional for each i. But, by the last paragraph, $\underline{\mathrm{Hom}}_B(\underline{\mathrm{Tor}}^A_1(A/A_+,B),M)$ is infinite dimensional. Thus, (4·1·1) implies that $\dim_k(\underline{\mathrm{Ext}}^1_A(A/A_+,M_A)) = \infty$ and so χ_1 does not hold. ∎

Corollary 4·2. *Let A be any graded PI ring that is right Noetherian but not left Noetherian. Then* (1·1·2) *fails for the polynomial extension $A[z]$.*

Proof. Combine Theorems 4·1 and 2·10. ∎

A typical example of a right but not left Noetherian, graded PI ring is $\left(\begin{smallmatrix} k & k[x] \\ 0 & k[x] \end{smallmatrix}\right)$ or, if one desires the ring to be connected graded,

$$\left\{\begin{pmatrix} \lambda & xf \\ 0 & \lambda + xg \end{pmatrix} : \lambda \in k, f, g \in k[x]\right\}.$$

If A is either of these rings, then Corollary 4·2 implies that (1·1·2) fails for right modules over the ring $A[z]$.

Let A be a graded, right Noetherian ring in which χ_1 holds. Then Theorem 1·3 applies. However, one would also like A to satisfy χ_i, for $i > 1$, since this would imply that the functors H^i behave well on proj-A. For example, one would have the following analogue of Serre's Vanishing Theorem: Suppose that A is a right Noetherian ring that satisfies χ. Then, for every $\mathscr{M} \in \mathrm{Ob}(\mathrm{qgr}\text{-}A)$ and $i > 0$, there exists an integer l_0 such that $\mathrm{H}^i(\mathscr{M}[l]) = 0$, for $l > l_0$ (see [AZ, theorem 7·4]).

We know of no graded, Noetherian ring A for which χ_1 holds but χ fails. However, we can construct a right Noetherian example and, for this example, Serre's Vanishing Theorem also fails:

Example 4·3. Assume that $[n]_q \neq 0$ for $n \geqslant 1$ and let $U = U_q$ be defined by (2·1). Set

$$W = \left\{\begin{pmatrix} \lambda + u & 0 \\ m & \lambda \end{pmatrix} : \lambda \in k, u \in U_+ \quad \text{and} \quad m \in yU/y^2U\right\} \subset \begin{pmatrix} U & 0 \\ U & k \end{pmatrix} \Big/ \begin{pmatrix} 0 & 0 \\ y^2U & 0 \end{pmatrix}.$$

Then, W is a right Noetherian, connected graded ring. Moreover, W satisfies χ_1 but does not satisfy χ_2.

Proof. As usual, $M_2(U)$ is graded by $(M_2(U))_r = M_2(U_r)$ and so, under the induced grading, W is connected graded. Also, W is a finitely generated right module over the subring

$$U' = \left\{\begin{pmatrix} \lambda + u & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in k \quad \text{and} \quad u \in U_+\right\} \cong U,$$

and so W is a right Noetherian ring. Let $N = N(W)$ denote the nilradical of W. Then

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we may identify $W/N = U$ and so, as a right U-module, $N \cong yU/y^2U$. However, note that $W_+N = 0$.

We first prove that W satisfies χ_1. To begin with, consider yU/y^2U as a right U-module. Then, using (2·1·1),

$$(yU/y^2U)_{\geqslant n+1} = (yx^nU + y^2U)/y^2U \cong U/(y + q^{-n}[n]_q\, x)\, U,$$

up to a shift. Since $[n]_q$ is never zero, this implies that yU/y^2U is isomorphic to no shift of a proper submodule of itself. Hence, $\underline{\mathrm{Hom}}_U(yU/y^2U, yU/y^2U) = k$. It follows that $\dim_k(\underline{\mathrm{Hom}}_U(yU/y^2U, M)) < \infty$ for every $M \in \mathrm{gr}\text{-}A$. To see this, note that, by the obvious inductions, one may assume that M is a cyclic, critical U-module. Now, as U has Gelfand–Kirillov dimension two, the only non-trivial case is when M is 1-critical. In this case, lemma 3·3 implies that M can contain at most one graded submodule isomorphic to a shift of yU/y^2U. Therefore, $\dim_k(\underline{\mathrm{Hom}}_U(yU/y^2U, M)) \leqslant 1$.

Since $W_+ \supset N$, and $W_+N = 0$, [Ro, corollary 11·27(iii)] implies that

$$\underline{\mathrm{Tor}}_1^W(W/W_+, U) \cong (W_+ \cap N)/W_+N = N \cong yU/y^2U.$$

Thus, by the last paragraph, $\dim_k(\underline{\mathrm{Hom}}_U(\underline{\mathrm{Tor}}_1^W(W/W_+, U), M)) < \infty$, for any finitely generated right U-module M. Since $(W/W_+) \otimes_W U = U/U_+ = k$, Lemma 2·2(iv) implies that $\underline{\mathrm{Ext}}_U^1((W/W_+) \otimes_W U, M)$ is finite dimensional. Now combine these observations with the exact sequence (4·1·1) in the case where $A = W$ and $B = W/N = U$. Then, (4·1·1) implies that $\dim_k(\underline{\mathrm{Ext}}_W^1(W/W_+, M)) < \infty$. Therefore, W satisfies χ_1.

It remains to prove that W does not satisfy χ_2 and to do this, we will prove that $\dim_k(\underline{\mathrm{Ext}}_W^2(k, W)) = \infty$. Note that, for any right Noetherian, connected graded ring A and $M \in \mathrm{gr}\text{-}A$, $\dim_k(\underline{\mathrm{Ext}}_A^i(M, k)) < \infty$. Hence, for any j,

$$\dim_k(\underline{\mathrm{Ext}}_A^j(M, A)) < \infty \Leftrightarrow \dim_k(\underline{\mathrm{Ext}}_A^j(M, A_+)) < \infty \qquad (4\cdot3\cdot1)$$

Now, certainly $\underline{\mathrm{Ext}}_U^1(yU/y^2U, U) \cong Uy^{-2}/Uy^{-1}$ is infinite dimensional. Thus, (4·3·1) implies that

$$\dim_k(\underline{\mathrm{Ext}}_U^1(yU/y^2U, U_+)) = \infty.$$

As any non-split extension of (W/N)-modules is a non-split extension of W-modules, this implies that $\dim_k(\underline{\mathrm{Ext}}_W^1(yU/y^2U, U_+)) = \infty$. However,

$$W_+ \cong U_+ \oplus (yU/y^2U),$$

as right W-modules. Thus, $\dim_k(\underline{\mathrm{Ext}}_W^1(W_+, W_+)) = \infty$. Finally, using (4·3·1), this implies that $\underline{\mathrm{Ext}}_W^2(k, W) \cong \underline{\mathrm{Ext}}_W^1(W_+, W)$ is infinite dimensional. ∎

COROLLARY 4·4. *Let W be defined by Example* 4·3 *and consider* $(\mathrm{qgr}\text{-}W, \mathcal{O}_W, s)$. *Then, for any $l_0 \geqslant 0$, there exists $l > l_0$ such that $H^1(\mathcal{O}_W[l]) \neq 0$.*

Proof. To begin with, consider $X = \underline{\mathrm{Ext}}_W^2(W/W_+, W)$. By the proof of Example 4·3, $\dim_k(X) = \infty$. Thus, Lemma 2·9 implies that $X_j \neq 0$ for infinitely many $j \in \mathbb{N}$.

For $n \in \mathbb{N}$ consider the exact sequence

$$\underline{\mathrm{Ext}}_W^1(W_+/W_{\geqslant n}, W) \to X \xrightarrow{\phi} \underline{\mathrm{Ext}}_W^2(W/W_{\geqslant n}, W).$$

As W satisfies χ_1, the first term in this sequence is finite dimensional; indeed, [AZ,

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proposition 3·5] implies that there exists l such that $\underline{\mathrm{Ext}}^1_W(W_+/W_{\geqslant n}, M)_{\geqslant l} = 0$ for all n. Hence, for all $n \geqslant 1$, there is an injection $X_{\geqslant l} \hookrightarrow \underline{\mathrm{Ext}}^2_W(W/W_{\geqslant n}, W)_{\geqslant l}$. Consequently,

$$X_{\geqslant l} \hookrightarrow \lim_{n\to\infty} \underline{\mathrm{Ext}}^2_W(W/W_{\geqslant n}, W)_{\geqslant l} = \bigoplus_{j\geqslant l} \mathrm{H}^1(\mathcal{O}_W[j]).$$

By the first paragraph, this implies that $\mathrm{H}^1(\mathcal{O}_W[j]) \neq 0$ for infinitely many values of j. ∎

The proof of Example 4·3 can be extended to show that $\dim_k(\underline{\mathrm{Ext}}^n(W/W_+, W)) = \infty$ for all $n > 0$. In contrast, for every $n > 1$ and $\mathcal{M} \in \mathrm{Ob}(\mathrm{qgr}\text{-}W)$, one can prove that $\mathrm{H}^n(\mathcal{M}) = 0$. In outline, the argument is as follows: Since $\mathrm{gldim}(U) = 2$, it is easy to see that $\mathrm{H}^2 = 0$ in $(\mathrm{qgr}\text{-}U, \mathcal{O}_U, s)$. The techniques of [AZ, proposition 6·11] can be used to prove that $\mathrm{qgr}\text{-}W \approx \mathrm{qgr}\text{-}U$. Although the two distinguished objects $\mathcal{O}_W$, respectively $\mathcal{O}_U$, are not identified under this equivalence, one can still use it to prove that $\mathrm{H}^2 = 0$ also holds in $(\mathrm{qgr}\text{-}W, \mathcal{O}_W, s)$.

It would be interesting to know if there exists a Noetherian (or even right Noetherian) graded ring W with the property that, for every $n > 1$, there exists $\mathcal{M} \in \mathrm{Ob}(\mathrm{qgr}\text{-}W)$ with $\mathrm{H}^n(\mathcal{M}) \neq 0$.

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