An inequality for the permanent function

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The *permanent function* on the set of $n \times n$ matrices is defined by the equation

$$\operatorname{per} A = \sum_{\pi} a_{1,\pi 1} \dots a_{n,\pi n},$$

where $A = (a_{rs})$ is any $n \times n$ matrix and π ranges over all $n!$ permutations of the symbols $1, 2, \dots, n$. The behaviour of the permanent function has been the subject of numerous recent investigations, especially at the hands of M. Marcus and several collaborators (see e.g. (1, 2, 3)). Most of this work has been inspired by van der Waerden's conjecture and is consequently concerned with the permanents of *doubly stochastic* matrices, i.e. real square matrices with non-negative elements in which all row-sums and column-sums are equal to 1. The present note too deals with a question of this type.

It is fairly natural to surmise that the permanent function is convex on the set of doubly stochastic matrices. However, this is not in fact the case as may be seen, for example, by considering the matrices

$$A = \tfrac{1}{15}\begin{pmatrix} 0 & 6 & 9 \\ 8 & 7 & 0 \\ 7 & 2 & 6 \end{pmatrix}, \quad B = \tfrac{1}{15}\begin{pmatrix} 6 & 0 & 9 \\ 7 & 8 & 0 \\ 2 & 7 & 6 \end{pmatrix}$$

for which $$\operatorname{per}\{\tfrac{1}{2}(A+B)\} = \tfrac{195}{750} > \tfrac{194}{750} = \tfrac{1}{2}(\operatorname{per} A + \operatorname{per} B).$$

(I am indebted for this information to Prof. M. Marcus.)

Our object is to establish a relation weaker than that of convexity. We shall show that the inequality

$$\operatorname{per}\{\tfrac{1}{2}(I+A)\} \leqslant \tfrac{1}{2}(1 + \operatorname{per} A)$$

holds for every doubly-stochastic matrix A. Indeed, it will be seen that this relation is valid for the wider class of *substochastic* matrices, i.e. matrices with non-negative elements in which all row-sums are *at most* equal to 1.

We shall denote by N_r the collection of the $\binom{n}{r}$ unordered sets of r numbers taken from $\{1, 2, \dots, n\}$. If $\sigma \in N_r$, then $\overline{\sigma}$ will denote the set complementary to σ (so that $\overline{\sigma} \in N_{n-r}$). Further, for any matrix A, we shall denote by A_σ the principal submatrix formed by the elements in rows and columns with indices in σ.

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By a straightforward expansion, we have

$$\operatorname{per}\{\tfrac{1}{2}(I+A)\} = 2^{-n}\{1+\sum_{\sigma\in N_1}\operatorname{per}A_\sigma+\sum_{\sigma\in N_2}\operatorname{per}A_\sigma+\ldots+\sum_{\sigma\in N_{n-1}}\operatorname{per}A_\sigma+\operatorname{per}A\}$$

$$= 2^{-n-1}\{(1+\operatorname{per}A)+(\sum_{\sigma\in N_1}\operatorname{per}A_\sigma+\sum_{\sigma\in N_{n-1}}\operatorname{per}A_\sigma)+\ldots+(\sum_{\sigma\in N_{n-1}}\operatorname{per}A_\sigma + \sum_{\sigma\in N_1}\operatorname{per}A_\sigma)+(\operatorname{per}A+1)\}. \quad (1)$$

Also, for $1 \leqslant r \leqslant n-1$,

$$\sum_{\sigma\in N_r}\operatorname{per}A_\sigma+\sum_{\sigma\in N_{n-r}}\operatorname{per}A_\sigma = \sum_{\sigma\in N_r}(\operatorname{per}A_\sigma+\operatorname{per}A_{\bar{\sigma}}). \quad (2)$$

Now the permanent of any substochastic $n\times n$ matrix $X=(x_{rs})$ is not greater than 1, for every term of the permanent is found among the terms in the expansion of

$$\prod_{r=1}^{n}(x_{r1}+x_{r2}+\ldots+x_{rn}).$$

If A is substochastic, then so are A_σ, $A_{\bar{\sigma}}$ $(\sigma\in N_r)$, and consequently we have

$$0\leqslant\operatorname{per}A_\sigma,\quad \operatorname{per}A_\sigma\leqslant 1.$$

Hence
$$\operatorname{per}A_\sigma+\operatorname{per}A_{\bar{\sigma}}\leqslant 1+\operatorname{per}A_\sigma.\operatorname{per}A_{\bar{\sigma}}$$

and, since every term of $\operatorname{per}A_\sigma.\operatorname{per}A_{\bar{\sigma}}$ occurs among the terms of $\operatorname{per}A$, we have

$$\operatorname{per}A_\sigma+\operatorname{per}A_{\bar{\sigma}}\leqslant 1+\operatorname{per}A. \quad (3)$$

It follows, by (2) and (3), that, for $1\leqslant r\leqslant n-1$,

$$\sum_{\sigma\in N_r}\operatorname{per}A_\sigma+\sum_{\sigma\in N_{n-r}}\operatorname{per}A_\sigma\leqslant\binom{n}{r}(1+\operatorname{per}A).$$

Hence, by (1),

$$\operatorname{per}\left(\frac{I+A}{2}\right)\leqslant 2^{-n-1}\left\{\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}\right\}(1+\operatorname{per}A)=\tfrac{1}{2}(1+\operatorname{per}A),$$

and the assertion is therefore established.

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