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BACKGROUND RISK AND PENSIONS

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ABSTRACT

This paper explores the effect of unhedgeable background risks such as mortality and labour risks (such as wages and turnover) on pension funding and finance. We explore the literature on the economics and finance of background risk, and discuss its applicability to pensions. Most of the results in economics apply to the special case of additive background risk, which is part of, but not all of, the background risk faced by pension funds. We develop three illustrative models and show the impact of background risk on pension funding and asset allocation. We find that the asset allocation and funding decisions of pension plans in general change with the introduction of background risk, in some cases significantly. We also explore implications of background risk for fair value calculations.

KEYWORDS

Pension; Background Risk; Equity Allocation

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1. Introduction

1.1 Pension liabilities are sometimes described as 'bond-like', perhaps because, if one was to pick a financial instrument which looked closest to pension liabilities in the asset/liability matching sense of Redington (1952), it may well be a bond. However, at the same time the tail of pension liabilities is of quite long duration, and pensions have many special features which differentiate them significantly from ordinary coupon bonds. Mortality risk is one of the many significant risks which are difficult, if not impossible, to hedge. For example, for both active and deferred members, there is a risk of withdrawal, wage growth uncertainty, as well as a regulatory risk. These unhedgeable risks are called background risks in the economics literature, and it is these risks which are the focus of our paper. Background risks differ from idiosyncratic risks related to the experience or fund, in that they cannot be hedged by constructing a larger portfolio of idiosyncratic risks; no matter how many members are in a fund, background risk does not go away.

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1.2 The actuarial literature on financial economics initiated by Exley *et al.* (1997) abstracts away these risks. To a degree, their focus is on accrued liabilities, on a termination basis, rather than on prospective liabilities, making pensions more amenable to a straight financial economics treatment; but incompleteness of markets does play an important role in practical applications. Consider the application of the Modigliani-Miller theorem to pensions in Sharpe (1976), a model which has a put option (for capturing defaults) against the members. Although a more complete model would also include a call option for improving benefits, this illustrates the concept.

1.3 As equity allocation in the pension fund increases, the discounted value of the liabilities decreases, because the option value of the put rises — a higher equity allocation raises default probabilities, and hence a higher equity allocation reduces the value of the liabilities. Sharpe (1976) shows that, when firms devalue the pension promises, the members have the incentive to bargain up wages, hence creating a situation where the equity investment by the pension fund does not matter for the market value of the firm.

1.4 The problem with this argument is that not all members — and, in particular, deferred members — are in a position to bargain over their pensionable wages. While it might be possible to write a contract such that notional pensionable wages for deferred members and pensioners will rise if a firm invests more in equities, it would require quite a bit of foresight and strong governance in the long-term setting in which pension funds operate. The stakeholder model developed by Chapman *et al.* (2001) helps provide context for decisions in such environments, with conflicting parties to imperfectly specified arrangements.

1.5 At the same time, mortality or other risks faced by pension sponsors could conceivably be securitised, but the degree to which this has happened is quite limited, and, for practical purposes, actuaries need to be able to operate in the context of markets which are imperfect in some areas. This is the starting point of this paper, which explores the effect of unhedgeable risks, such as uncertain wages and mortality, on pension funding. We refer to these unhedgeable risks as 'background' risks in the paper. There is substantial economics literature on background risk and on imperfect markets in general, and we draw upon it extensively, summarising this literature in Section 2. In general, the literature concludes that, with additive background risk, there is typically more saving (due to precautionary motives) and, in many cases, less investment in equity.

1.6 However, in applying these results in a pension context, there are important caveats. In Section 3 we enumerate the types of background risk affecting pension funds and describe securitisation efforts. Many of these risks do not enter pension liabilities in an additive manner, and, as noted in Section 2, the conclusions from the economic literature may differ if the background risk enters multiplicatively. In Section 4 we introduce models for

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pensions incorporating background risks. In Section 5 we present results on optimal asset allocation from a number of simple static models in which funding rates are fixed. In Section 6 we present results on optimal funding with fixed asset allocation. In Section 7 we proceed to look at joint funding and asset allocation decisions. We find that background risk does change asset allocation and funding decisions of pension plans, in some cases significantly, and it may be that the optimal response to background risk is to increase equity exposure. Finally, in Section 8 we discuss the implications of background risk for the valuation of pension liabilities.

2. The Economics of Background Risk

2.1 In traditional financial economics models, the only sources of risk which agents face are from tradable risks, such as in financial markets. However, in practice, economic agents usually face several concomitant risks in markets that are less than complete. Broadly, all sources of risk can be classified as tradable or non-tradable. Examples of tradable risks are risks in foreign exchange and interest rate markets. Examples of risks that may be non-tradable are the risk of labour income shocks and the risk of production losses.

2.2 For instance, as regards labour income risk, some of it may be due to the possibility of disability, if disability insurance is imperfect. A more important source of uninsurable income risk is the possibility of an individual doing worse than expected in his career. This risk is difficult to insure, both for moral hazard reasons and for adverse selection reasons. On top of that, providing insurance also entails marketing and administrative costs. Elmendorf & Kimball (2000) discuss these topics in more detail.

2.3 In the economics literature which has evolved since the mid 1980s, these non-tradable, exogenous risks are referred to as background risks. The interaction between tradable and background risks can affect the willingness to bear either one of them. The economics literature has shown that failing to take the interactions between the types of risk into account can lead to suboptimal decisions in problems such as whether to buy or to sell an asset with uncertain returns, whether to purchase insurance, or how much to pay for other forms of risk protection.

2.4 In the presence of background risk, the theoretical literature has shown (Kimball, 1990) that individuals should save more, a phenomenon known as 'precautionary savings'. The majority of evidence is that this phenomenon is empirically quite important. Ventura & Eisenhauer (2005) find that 20% of saving in Italy is driven by precautionary motives. Cagetti (2003) concludes that a majority of saving among the younger population in the United States of America is for precautionary purposes and not for retirement. Gourinchas & Parker (2002) find significant deviations from the

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life-cycle model for younger individuals, and attribute this to precautionary motives. Fuchs & Schündeln (2005) show that a considerable fraction of wealth in Germany is held for precautionary reasons among some groups of the population. Carroll (2001) also concludes that the precautionary motive is important in the U.S.A. However, there is contradictory evidence as well. Lusardi (1998) finds little evidence of precautionary savings in Italy in one model specification, but the results are significant in another specification.

2.5 The development of the economics and finance literature on background risk has been motivated by some important puzzles. Davis & Willen (2000a; 2000b) find that even moderate covariance between labour income shocks and asset returns can drive large differences between optimal portfolio composition and that implied by a more traditional approach which ignores labour income or other sources of income from non-marketable assets. Weil (1992) argues that prices of risky assets would likely be overestimated and equity premiums be underestimated if background risks are not taken into consideration when calculating optimal portfolio allocations. Gollier & Pratt (1996) define certain utility conditions, under which agents facing, on average, a negative shock on labour incomes will react by reducing their demand for the risky asset, thereby increasing the equilibrium equity premium. In essence, individuals respond to an increase in one risk by reducing their exposure to another independent risk (i.e., risk substitution). This tendency is termed 'temperance', in the sense of moderation in accepting risks, as discussed by Kimball (1991). Recent empirical studies use the idea that adding an independent background risk increases the risk premium for a 'primary' risk, to help explain puzzles about portfolio choice, such as why so few hold equity at all — despite its high expected return — and why equity holdings, as a fraction of wealth, vary so much.

2.6 Another strand in the literature has considered the effect of background risk on the demand for insurance. Eeckhoudt & Kimball (1992) determine the effect of introducing background risk into an insurance demand model. The authors show that, under a straightforward and intuitive restriction on preferences, the introduction of background risk increases the demand for insurance, regardless of whether insurance demand is measured by the coinsurance or the deductible level or not. Meyer & Meyer (1997) assume independently distributed, additive and exogenous background risk, and analyse the effect of several changes in background risk on the demand for insurance.

2.7 Whether and to what extent an individual's preference for risk is affected by background risk carries important implications for the economic analysis. If agents are significantly affected by background risk, economic analyses must move beyond studies of behaviour in single risky situations, which are virtually non-existent. Much has been done over the past 20 years in examining the effects of additive background risks on agents' choices between risky prospects. Typically, additive background risk might represent

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future wage income subject to human/capital risks, or an exogenous pension portfolio provided by the employer. The modern literature on additive background risk stems from the papers of Kimball (1991), Ross (1981) and Nachman (1982). While these early papers focused on interpersonal behaviour comparisons, Doherty & Schlesinger (1983) incorporated the analysis into intrapersonal models of decision making under uncertainty, focusing on differences in optimal behaviour with and without background risk. Interest on this topic was further stimulated in the 1990s, because of the development of new theoretical tools by Pratt & Zeckhauser (1987), Kimball (1993), Eeckhoudt *et al.* (1996) and Gollier & Pratt (1996). These authors derived necessary and sufficient restrictions on utility, such that an addition of, or an increase in, background risk would prompt a utility maximising individual to make more conservative choices in other risky situations. A detailed description of these restrictions can be found in Gollier & Pratt (1996). In contrast, Diamond (1984) investigated conditions under which individuals would find a gamble more attractive when another independent risky gamble was added to the portfolio. Quiggin (2003), on the other hand, showed that aversion to one risk is reduced by an independent background risk for certain classes of non-expected utility preferences consistent with constant risk aversion.

2.8 In line with this lack of theoretical agreement regarding the effect of additive background risk on risk aversion, empirical evidence has also provided conflicting, inconclusive results. Vissing-Jorgensen (2002) found evidence that background risk reduces stock market participation in the U.S.A. Hochguertel (1998) finds that results for the Netherlands are inconclusive, and those of Alessie *et al.* (2001) for the same country did not find a significant effect of income uncertainty on the demand for risky assets. Arrondel & Calvo-Pardo (2002), on the other hand, working with French data, found that, if households are more exposed to risk, they invest a greater proportion of their wealth in risky assets. A comprehensive review of this material can be found in Gollier (2001).

2.9 The literature described so far focuses on the case where background risk is additive to the tradable risky variables (e.g., the outcome variable $y = x + z$, where x is the value of a variable of interest, such as assets in the absence of background risk, and z is the additive background risk). Background risk could be multiplicative as well, and, indeed, Franke *et al.* (2002) assert that multiplicative types of background risk are as prevalent as additive ones. Very little attention has been given in the literature to this type of background risk. Among the examples of multiplicative background risks considered by Franke *et al.* (2002) are the effect of a firm's retention rate net of taxes (where tax rates are random due to tax-legislation uncertainty) on the pre-tax profits of the firm or the return on a mandatory (and exogenously managed) annuity account that uses proceeds from the random wealth in an individual's financial portfolio. Franke *et al.* (2002) is

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one of the first papers to consider how the presence of multiplicative background risk affects risk-taking behaviour. They characterise conditions on preferences which lead to more cautious behaviour, and provide theoretical results for the case of a multiplicative background risk which does not simply 'mirror' those for the case where the background risk is additive.

2.10 Other key contributions in this area are from Sakagami (2005) and Artige (2004). The latter compares the effects of multiplicative and additive background risk on the intertemporal allocation of consumption. Artige (2004) concludes that multiplicative background risk has no effect if preferences are constant relative to risk aversion, and that additive background risk has no effect if preferences are constant relative to absolute risk aversion.

2.11 To summarise, models of background risk have been an evolving area of the economics literature over the past 20 years, and the conclusions for portfolio allocation and saving (e.g. funding) in the presence of background risk can differ appreciably from conclusions when background risk is not modelled.

3. Background Risks Affecting Pensions

3.1 Among background risks, mortality risks have, perhaps, received the most attention from actuaries, but there are also quite a few other non-tradable risks affecting pensions, such as labour market risks (wages, turnover and early retirement), financial risks (incomplete asset markets for long-duration securities) and regulatory risks.

3.2 *Mortality Risk*

3.2.1 There is a degree to which small amounts of mortality exposure are tradable through securitisation, though, in practice, the quantities of risk which can be traded are sufficiently small that, for most schemes, we can regard mortality risk purely as a background risk. Reinsurance is another possibility, but this is not purely market behaviour, and, indeed, background risk methodology could potentially be used in the future to assess how close reinsurance prices are to market clearing rates.

3.2.2 The bulk annuities market (for level and indexed annuities) has remained consistently small in the United Kingdom and elsewhere around the world, and while there have been two major mortality bond issuances, the amounts traded are relatively small, and, even with the bonds in place, there are important residual basis risks.

3.2.3 An important case of mortality securitisation came in December 2003, when Swiss Re and Vita Capital issued US$400m of three-year bonds. This bond linked coupons to a mortality index composed of mortality experience in five countries (France, Italy, Switzerland, the U.K. and the

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U.S.A.). The design of the bond is that the principal would be at risk: "if, during any single calendar year in the risk coverage period, the combined mortality index exceeds 130% of its baseline 2002 level." For hedging a mortality bond in a single country, the bond has the disadvantages of both basis risk (mortality index covers five countries' population mortality instead of insured mortality in one) and its relatively short duration.

3.2.4 A second mortality bond announcement was in November 2004 by BNP Paribus and the European Investment Bank (EIB). The total value of the announced issue was $540 million, but this bond has coupons directly linked to the population mortality in the U.K. Nevertheless, with this bond design there is still basis risk between the coupon payments and the insured mortality experience.

3.2.5 Stochastic mortality models have been developed to model aggregate mortality risk. Among the more popular models is the Lee-Carter model (Lee & Carter, 1992). Applications of the model and its variants to the U.K. include work carried out by Haberman & Renshaw (2003a), Haberman & Renshaw (2003b) and Continuous Mortality Investigation Mortality Committee (2005). Tuljapurkur & Boe (1998) provide a useful summary of the literature, while Richards & Jones (2004) look at the financial aspects of longevity risk and, in particular, assess how mortality risk contributes to fluctuations in realised liabilities.

3.2.6 In this context, an important caveat to bear in mind is that background risk from mortality to a pension fund is clearly different to, and could well end up being relatively larger than, the population risk. This is because of the numbers of deaths involved, and the greater likelihood of correlations between them if they share class, geographic or other relevant characteristics. Clearly, the magnitude of these effects will vary as a function of scheme-specific characteristics, such as size and diversity of work profiles.

3.2.7 In Section 2 we noted the distinction between additive and multiplicative risk, without indicating which was more likely to apply to pension schemes. From first principles, we would not expect mortality risk be an additive risk. Consider the formula for an annuity factor:

$$a_x = \sum_{k=1}^{\infty}\left[\prod_{s=1}^{k}\frac{1-q_{x+s-1}}{1+r_{s-1}}\right] \tag{3.1}$$

with interest rates r and mortality q. If q is best modelled with an additive or multiplicative random process, the resulting risk to annuity factors is neither additive nor multiplicative, but more complex.

3.2.8 It is possible to gauge the relative scale of background risk looking at U.K. population mortality data (ONS, 2001). In Figure 3.1 we show mortality in five-year age bands of males aged 70 to 75 years.

3.2.9 Consider the problem of pension plan forecasting mortality 40

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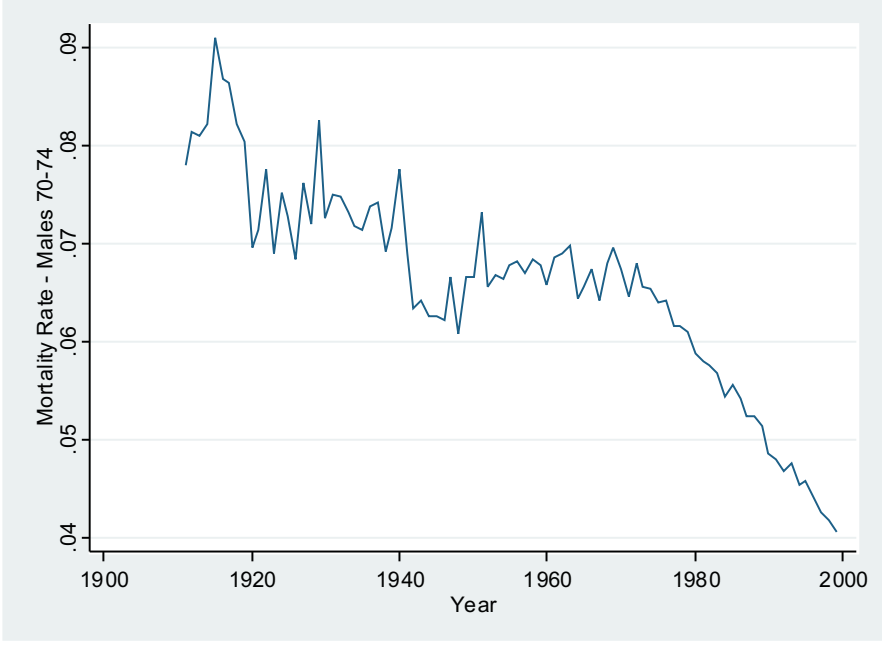


Figure 3.1. Central population mortality rates of males aged 70 to 74 years from 1910 to 1999

years forward. If we consider the ratio of ages 70 to 74 years' mortality after 20 years over the period 1910 to 1999 (1931 is the first year for which we have 20 years of history in our sample), we find a median of 0.894, but 5% and 95% percentiles for the ratio of, respectively, 0.687 and 1.088. This means that, even at 10% confidence intervals, we need to take into account aggregate forecast errors of over 40%. Table 3.1 repeats this calculation for

Table 3.1. Mortality risk (70 to 74 and 75 to 79 years old men)

Horizon	Width of 90% confidence bands (% of median) for males aged 70 to 74	Width of 90% confidence bands (% of median) for males aged 75 to 79
1 year	20.0%	21.4%
5 years	25.3%	30.5%
10 years	30.0%	28.5%
15 years	35.6%	33.4%
20 years	44.8%	35.1%
30 years	43.6%	38.8%
40 years	38.4%	37.9%

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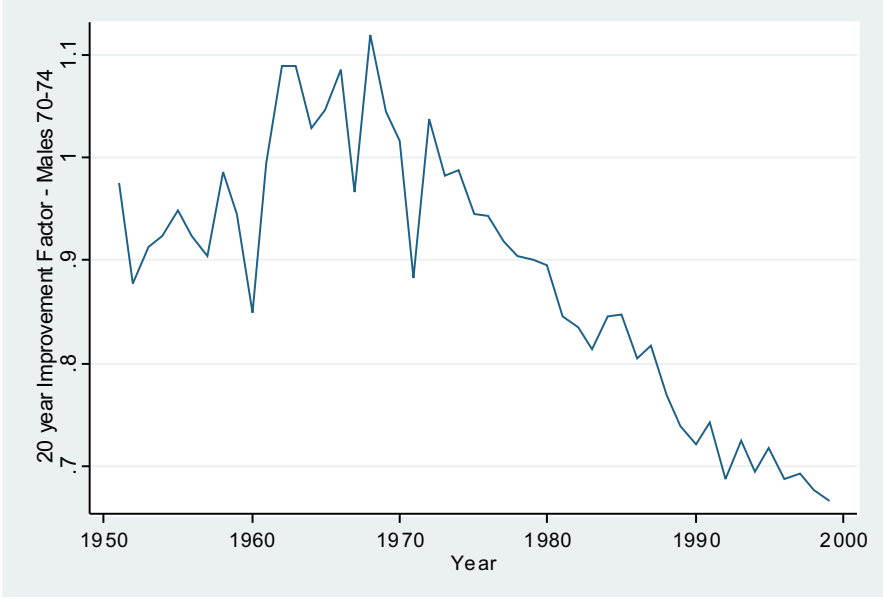


Figure 3.2. Improvement factors for males aged 70 to 74 years

different horizons from one year up to 40 years, using the available data for men aged 70 to 74 years as well as men aged 75 to 79 years. The table calculates the width of 90% confidence bands as the difference between the 95th and the fifth percentile divided by the median.

3.2.10 In Figure 3.2 we plot 20-year historical improvement factors in a population mortality of males aged 70 to 74 years, and, indeed, in ten years the improvement factor would have implied worsening mortality. This same phenomenon is witnessed internationally with mortality forecasts, with some countries and regions exhibiting rapid improvements (the Middle East is a good example, as is Asia), and other countries even having a worsening mortality pattern.

3.3 *Labour Market Risks*

3.3.1 Defined benefit pension schemes face a wide range of labour market risks, many, but not all, of which are from their active members. Wage risk is an important risk, as, while employers do have some control over wages, they operate in labour markets in which they do have to pay wages related to aggregate market levels, and aggregate wage levels are not insurable, even though there have been some proposals to issue securities

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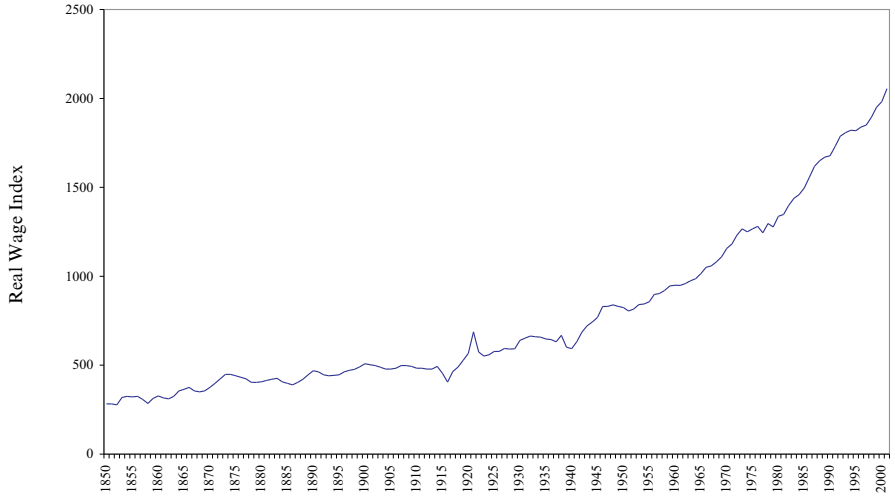


Figure 3.3. Real wages from 1850 to 2000 in the U.K.

indexed to average wages. These are discussed by Valdés-Prieto (2005) and Shiller (1994).

3.3.2 In Figure 3.3 we plot log real wages since 1850. Historical U.K. wage data used in this context are described in Cardinale (2004). In Figure 3.4 we plot the corresponding real annual growth rates of wages. While, in recent years, shocks to real wages have been relatively mild, there have been significant shocks in the past, with years in which the real growth rates have been over 10% in absolute value, and, indeed, at a 90% percentile over the 151 years in the sample, the real annualised wage growth rate is more than 4% above the sample median.

3.3.3 Aggregate real wages are a relevant background risk for pensions, not just because there can be significant shocks, but because of the duration of exposure which a pension fund has to real wage shocks in the economy. Consider someone who joins a defined benefit pension scheme 40 years prior to retirement — in this case the sponsor is exposed to 40 years of aggregate risk. If we consider the ratio of real wages after 40 years over the period 1890 to 2001 (1890 is the first year for which we have 40 years of history in our sample), we find a median of 1.66, but 5% and 95% percentiles for the ratio of respectively 1.13 and 2.15. This means that, even at 10% confidence intervals, we need to take into account forecast errors of slightly over 30% in either direction. Table 3.2 repeats this calculation for different horizons from one year up to 40 years using available data from 1850 to 2001. The table calculates the width of the 90% confidence interval as the difference

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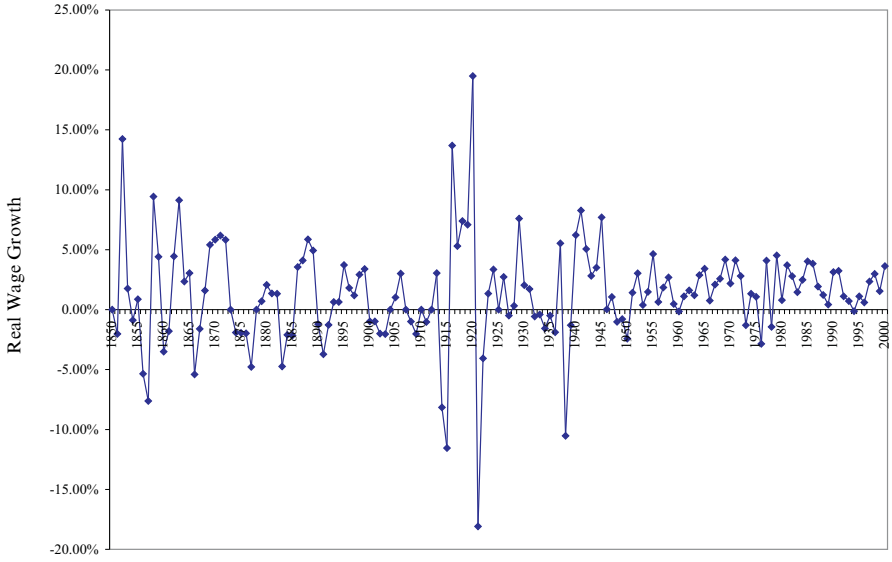


Figure 3.4. Real wage growth

between the 95th and the fifth percentiles divided by the median, and the width of the 98th confidence interval as the difference between the 99th and the first percentiles.

3.3.4 Other important labour market risks which arise include movements in turnover rates and early retirement. The degree to which these labour market risks are correlated with financial market variables is unclear, particularly in the short run, but, in the longer run, some degree of hedging may be possible (see Cardinale, 2004, for a discussion). Another potential mitigating factor refers to hedging of wage increases through rises in contribution revenues, but, whilst this is an important theoretical

Table 3.2. Real wage risk

Horizon	Width of 90% confidence bands (% of median)	Width of 98% confidence bands (% of median)
1 year	12.4%	25.9%
5 years	29.1%	44.3%
10 years	31.1%	45.3%
15 years	35.9%	54.6%
20 years	38.2%	50.6%
30 years	48.8%	60.9%
40 years	61.5%	71.2%

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argument, member contribution rates in most pension funds are not set at the right level to offset wage growth shocks (and there are still non-contributory schemes). Still, labour market risks are, in our view, largely background risks.

3.4 *Financial Market Risks*

Although financial markets are very advanced, the lack of availability of sufficient long-dated bonds in some jurisdictions means that there are elements of financial market risks which act in a manner similar to background risks. In the U.K., the introduction of 50-year bonds will mean that pension sponsors effectively need not bear reinvestment risk, as the proportion of liabilities with over 50 years to go will be very small indeed. Yet, in many other countries without such developed bond markets, reinvestment risks may pose significant challenges for pension funds. This is especially the case, given that long-run interest rates appear to be somewhat volatile relative to what one would expect if long-run interest rates were an average of expected short-term rates (Shiller, 1979).

3.5 *Regulatory, Operational and Policy Risks*

Further pension schemes also face background risks in the form of regulatory, operational and policy risks. Throughout the world, governments change the tax treatment of pensions as well as the definitions of liabilities in ways which can be quite costly for pension scheme sponsors. In addition, litigation, such as that in the U.S.A. on cash balance plans, imposes significant uncertainty for pension plan sponsors. Examples abound in the U.K. alone: (1) the introduction of the PPF; (2) the change in dividend taxation in 1997; (3) the introduction of mandatory indexation; and (4) changes in contracting-out terms. Each of these events changes the nature of pension liabilities, often in ways which are not small, and almost invariably in ways which cannot be hedged. At the same time, operational risk in areas such as how pension funds are managed and administered is a further source of uncertainty. Hence, these regulatory, operational and policy risks should be factored into the analysis which actuaries undertake, and background risk provides a conceptual framework to do this.

4. Models for Pensions with Background Risk

4.1 *Economic Models of Background Risk*

4.1.1 The standard economic model of background risk at first sight appears significantly different to that faced by a pension fund. We follow here broadly the approach of Poon & Stapleton (2005, pp32-33) and Franke *et al.* (1998), which rely on the maximisation of expected utility:

$$\mathrm{E}[U(x+\varepsilon)] \tag{4.1}$$

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where x is a random cash flow for which there is a complete market and ε is background risk, which is, by definition, not hedgeable. In most economic problems x is consumption and equation 4.2 is maximised subject to the budget constraint:

$$\sum_{i=1}^{N} x_i q_i = Bw \tag{4.2}$$

where there are N states indexed by i with state prices q_i (price of a unit of x in state i), w is the initial level of wealth and B represents the capital plus return on a coupon bond.

4.1.2 Optimal decisions in this model (as with most economic models) are determined by marginal utility, and Kimball (1990) has analysed how the case of independent background risks change marginal utility. Specifically, he defines the precautionary premium ψ as:

$$\underset{\varepsilon}{\mathrm{E}}[U'(x+\varepsilon)] = U'(x-\psi). \tag{4.3}$$

4.1.3 Poon & Stapleton (2005) show that, under the realistic assumption of decreasing absolute risk aversion, ψ is positive and also typically decreasing in x (wealthier individuals are generally less susceptible to additive risks). A positive precautionary premium means that independent background risks lead to more saving, as discussed in Kimball (1990).

4.1.4 To explore pension fund models with background risk, we only focus on simple models. We stick to simple models because so much depends on the structure of preferences, and we want to present models which only illustrate the key issues. Nevertheless, we are aware that necessary and sufficient technical conditions in the literature, which are summarised in Gollier (2001), can be very complex, and there are distinctions, such as prudence, standard risk aversion, proper risk aversion and risk vulnerability, which rely on detailed properties of utility functions (up to the fourth derivative), but we felt that in-depth discussion of these issues would lie outside the scope of this paper. We start with a discussion of utility functions for pensions, touch briefly upon some intuition using results from the economics literature applied to pensions, and then present numerical results (which is our main focus, because of the complexity of the analytical problem).

4.2 *Objective Functions for Pension Decision Making*

4.2.1 The focus of the economic literature is a useful starting point for an analysis of pension funds, but a key issue is how to use utility functions for pensions. The literature on pension funds uses expected utility in different ways. In Exley *et al.* (1997) there are no utility functions, because the

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corporate standpoint is taken and markets are complete. Whilst this is clearly one possible approach, one could argue that members have utility functions, and that pension fund trustees or plan fiduciaries, as their delegated decision makers, will also have preferences, and these preferences matter. We think that the pension trustee standpoint is, in many cases, more consistent, though, as we are dealing with incomplete markets, companies will have utility functions as well.

4.2.2 Some papers, such as Sherris (1992), define utility over surpluses. Haberman *et al.* (2003) argue that loss measures are better than utility functions; Bordley & LiCalzi (2000), however, show, in general, that target-related measures have similar properties to utility functions. It is slightly more natural to define utility over funding (e.g. contributions made), as negative funding is essentially the same as consumption in economic models. Just as consumers generally prefer a smooth consumption path, neither trustees nor companies will want to have sudden jumps in the financing required by the pension plan.

4.2.3 At the same time, however, deficits seem to matter in practice, both for companies as well as for members of schemes, so we believe the traditional approach also has merit. There is also scope, in our view, to include both surpluses (equivalently deficits) and funding (e.g. contributions made) in utility functions. In such a case, surpluses (equivalently deficits) represent the interests of members, and funding contributions represent the interests in companies, so, by including both in a utility function, we can capture the strength of the different parties in bargaining in how the utility function is parameterised.

4.3 *Prudence and Pension Funds*

4.3.1 It is helpful for intuitive purposes to begin with a simple model in which there is an objective function, which depends on the funding decision — among other things which are independent of funding — and no other decisions are taken. Even simple models rapidly become complex in this area, so our main focus is on numerical examples.

4.3.2 For the time being, this objective function can be one either of the sponsors or of the pension fund trustees. In the absence of background risk, we let the optimised value of this objective function be $V(f^*, \bar{l})$, where $\bar{l}$ is the liability level. Assuming an interior solution and $V(f^*, \bar{l})$ concave, and $\bar{l}$ is not influenced by the funding decision, f^* solves $V_f(f^*, \bar{l}) = 0$. If we add zero mean background uncertainty to liabilities, and define V^b as simply V with background noise added, we have $V(f^*, \bar{l}) \geq \mathrm{E}[V^b(f^*, l)] = G^b(f^*, \bar{l})$, as long as V is concave in l (it is also reasonable to assume background risk makes pension funds worse off). It could be either the case that background risk puts the function with background risk G^b in a region with positive slope (in which case desired funding levels should increase in the presence of background risk), or negative (in which case the opposite occurs).

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4.3.3 A simple model along the lines of Leland (1968) is that the trustee or sponsor maximises a simple dynamic model $G^b = V^1(f, \bar{l}) + \mathrm{E}[V^2(f, l)]$, where V^1 is the value of funding today and V^2 is the uncertain impact of today's funding decisions in the future. The uncertainty here arises from uncertain future liabilities. In the absence of zero mean background risk, the optimal funding level f^* solves:

$$V_f^1(f^*, \bar{l}) = -V_f^2(f^*, \bar{l}). \tag{4.4}$$

Recall that, for precautionary funding, we need $G_f^b(f^*, \bar{l}) > 0$, or using equation (4.4) that:

$$\mathrm{E}[V_f^2(f^*, l)] - V_f^2(f^*, \bar{l}) > 0 \tag{4.5}$$

or that V_f^2 is convex in l. This is a standard result whose derivation largely follows that of Gollier (2001, p236).

4.3.4 Risk aversion does not, therefore, imply that background risk leads to more funding. In fact, an additional condition is required: the convexity of the value of future funding, which is a higher order property of the objective function; and a term which Leland (1968) and others have referred to as prudence.

4.3.5 Most objective functions in economics exhibit positive prudence. In an actuarial context, though, things may be different, and we need to think through the intuition of the marginal value of future funding carefully. Clearly, the considerations from a sponsor perspective will be different from those of the trustees, and, indeed, it may be that, if a sponsor wants to engage in precautionary funding, it may not actually make a contribution, but otherwise set aside a provision to prepare prudently for future risks.

4.3.6 For members and trustees, it is reasonable that a marginal increment of future funding should be declining; after all, extra funding has only benefit up to a point; but should it really be decreasing at a decreasing rate? There is a direct link between prudence (or the convexity of the marginal objective function) and risk aversion (or the convexity of the objective function). Prudence depends on how risk aversion changes with the level of funding. As long as risk aversion is decreasing in funding (a reasonable assumption, we think), we will observe prudence, and hence a response of increased funding in the presence of background risk.

4.4 *Background Risks and Asset Allocation*

4.4.1 It is quite simply false that background risk does not affect asset allocation or that, if it does affect asset allocation, it leads one to hold more in bonds. Background risks can increase the demand for risky assets. To use an example from Gollier (2001, pp126-127), suppose that a

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pension fund has an objective function in the solvency level x (in percentage terms) of:

$$V(x) = \begin{cases} x & \text{if } x \leq 100 \\ 50 + 0.5x & \text{if } x > 100. \end{cases} \tag{4.6}$$

4.4.2 This is entirely plausible, as there are decreasing benefits to solvency after achieving 100% solvency. In this example, the risky asset has equally probable returns of −100% and 190%, and the pension fund can have a sure solvency level of 101%. In this case, the optimal asset allocation without background risk is an exposure of $y = 1$ unit of the risky asset. Now (and this example differs a bit in terms of specific numbers from the one in Gollier (2001), but not in the overall result) we consider a zero mean uncorrelated risk, so that there are four states of the world:

(1) State 1: 111% solvency minus y;
(2) State 2: 111% solvency plus y^*190%;
(3) State 3: 91% solvency minus y; and
(4) State 4: 91% solvency plus y^*190%.

In this example, it can be verified using an Excel solver that the optimal portfolio has 11 units of the risky asset y, a significant increase in the holdings of the risky asset in the presence of background risk over one unit in the absence of background risk.

4.4.3 The general economic problem can be applied in a pension context, with wealth being the difference between assets and liabilities. A pension fund decision maker can choose a level λ of risk to take, and this influences the subsequent level of assets $a = s + \lambda x$, where s is a fixed-return asset and x is the excess return on a risky asset (which has at least some probability of being negative to avoid arbitrage opportunities). The objective function depends on the ultimate difference between assets and liabilities $U(s + \lambda x - l)$.

4.4.4 The conditions on background risk affecting asset allocation are very complex, and depend, in general, on higher order curvature of the objective function, as discussed by Gollier (2001, p128). However, a sufficient condition for safer investing in the presence of background risk is that absolute risk aversion is convex and decreasing. We do not have good intuition why absolute risk aversion for pension decision makers should be convex, and, even if it were, the conditions apply to additive background noise, and pension funds face more complex types of noise, and there are also simultaneous decisions over funding and asset allocation. Hence, as the example above shows, it *can* be the case that, in the presence of background noise, there is more investment in equity; but the considerations are complex, and we think that it is essential to focus our attention on numerical examples.

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4.5 *Cobb-Douglas Utility Function*

4.5.1 We now turn to numerical simulations. The use of Cobb-Douglas utility functions (Cobb & Douglas, 1928) to capture the interests of different parties is a common approach in economics, and we therefore start with this case, while sticking as closely as possible to the framework described so far. The solution to the problem of optimising a Cobb-Douglas function $g_1^{\alpha} g_2^{1-\alpha}$ is the generalised Nash solution for bargaining problems, where g_1 and g_2 are the gains to the two parties to the bargain and α and $1-\alpha$ are their bargaining strengths, as discussed by Cahuc & Zylberberg (2004, p383). Because of the uncertainty about the correct modelling framework to use, we explore a range of different models in the following sections.

4.5.2 The model which we present in this section is a stylised model of a pension scheme, where the fund makes a decision over how much to fund by contributing to the scheme and how to invest pension assets, subject to uncertainty over the rate of return on equities, which affect the value of the assets and realisations of background risk affecting the value of liabilities. As an extension, the case of uncertainty over the rate of return on bonds, which affect both assets and liabilities, is also considered.

4.5.3 Pension assets in the model are given by:

$$A_{K,i} = N_{K-1} B_{K,i} + M_{K-1} X_{K,i} + F_K \tag{4.7}$$

where:

— i corresponds to the ith state of the world;
— $A_{K,i}$ is the value of the assets at time K and for the ith state of the world;
— N_{K-1} is expressed as a quantity (e.g. number) of bonds set in period $K-1$ (we assume that no trading of the bond portfolio occurs between the two periods);
— M_{K-1} is expressed as a quantity (e.g. number) of equities set in period $K-1$ (we assume that no trading of the equity portfolio occurs between the two periods);
— $B_{K,i}$ is the level of the total return bond index at time K for the ith state of the world;
— $X_{K,i}$ is the level of the total return equity index at time K; and
— F_K is the value of the new cash inflows (or contributions net of outflows) at time K. We will often refer to these funding contributions simply as 'funding' decisions in our analysis. We also assume that F_K is set before resolving the uncertainty over which of the states of the world will prevail at time K.

4.5.4 In the baseline specification for the pension assets equation, the main parameter assumptions can be summarised as follows:

— $A_{K-1} = 0.5$, $B_{K-1} = 1$ and $X_{K-1} = 1$.

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— Short positions and leverage are ruled out by the restrictions $N_{K-1} \geq 0$, $M_{K-1} \geq 0$, $N_{K-1} \leq A_{K-1}$ and $M_{K-1} \leq A_{K-1}$.
— $X_{i,K}$ is defined using three alternative equally probable states of the world designed to illustrate the volatility of equity investments. These are defined as follows:
  $i = 1$: $X_{K,1} = (1.09)X_{K-1}$ with $X_{K-1} = 1$ (9% assumed return on equity investments);
  $i = 2$: $X_{K,2} = (0.94)X_{K-1}$ with $X_{K-1} = 1$ (−6% assumed return on equity investments); and
  $i = 3$: $X_{K,3} = (1.24)X_{K-1}$ with $X_{K-1} = 1$ (24% assumed return on equity investments).

  Jointly, the three scenarios imply a distribution of equity returns characterised by a 9% mean return and a 12.2% standard deviation:
— We set $B_{K,i} = (1.05)B_{K-1}$ $\forall i$ with $B_{K-1} = 1$, where 5% is the rate of return on bond investments (assumed to be a riskless asset).
— We set $F_K \leq A_{K-1} \Rightarrow F_K \leq 0.5$, to rule out contributions in excess of assets in the previous period.

4.5.5 Pension liabilities in this model are defined as follows:

$$L_{K,i} = \lambda c w_{K-1} B_{K,i} \mu_{K,i} + (1 - \lambda) c B_{K,i} \mu_{K,i} \tag{4.8}$$

where:
— $L_{K,i}$ is the value of liabilities at time K for the ith state of the world;
— $0 \leq \lambda \leq 1$ is the proportion of active members and $(1 - \lambda)$ is the proportion of non-actives (deferred and pensioners);
— c is a constant;
— w_K is the level of the relevant wage index (because in a final salary scheme active liabilities are in function of members' final salaries), which is assumed to be constant across the three states of the world;
— $B_{K,i}$ enters both the assets and the liabilities equations (we assume that the bond-like component of pension liabilities is perfectly hedged by matching fixed income investments on the asset side); and
— $\mu_{K,i}$ represents multiplicative residual risk, which is non-hedgeable, through a bond portfolio (e.g. unexpected changes in mortality, turnover rates, etc.). Liabilities are defined in two alternative ways according to whether $\mu_{K,i}$ is assumed to be constant or variable (without or with background risk).

4.5.6 In the baseline specification of the liabilities equation, the main parameter assumptions are as follows:
— $B_{K,i}$ is as defined above for the asset equation.
— $\mu_{K,i}$ is defined according to three states of the world, which correspond to the three equally probable scenarios defined for the total return

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stock index. The states of the world are defined in two alternative ways: (a) without background risk (constant $\mu_{K,i}$); and (b) with multiplicative background risk (variable $\mu_{K,i}$). With constant background noise, the relationship $\mu_{K,i} = 1$ holds across all states of the world, while, in the case with background risk, the values of $\mu_{K,i}$ are given by the following equally probable states of the world, which correspond to those defined for equities and bonds:
$i = 1$: $\mu_{K,1} = 1.1$;
$i = 2$: $\mu_{K,2} = 0.95$; and
$i = 3$: $\mu_{K,3} = 0.95$.

Jointly, these scenarios characterise stochastic background noise as uncorrelated with equity returns.

— We set $\lambda = 1$ and $w_K = 1$, which means that all liabilities accrue to active members, but wages are kept constant in order to focus on one single class of background risk (which, in turn, could be related to the stochastic component of wage growth).
— We set $c = 1$ for convenience, but this has no impact on the results, as it is a constant factor which multiplies liabilities.
— $\mu_{K-1} = 1$.

4.5.7 Jointly, these assumptions, together with $B_{K-1} = 1$ discussed in the assets equation, imply:

$$L_{K-1} = \lambda c w_{K-1} B_{K-1} \mu_{K-1} + (1 - \lambda) c B_{K-1} \mu_{K-1} = 1 \tag{4.9}$$

which means that the scheme initial funding (or assets over liabilities) ratio is 50%, given the assumption $A_{K-1} = 0.5$.

4.5.8 The objective function is based on the economics Cobb-Douglas utility function, and is defined at time K and for the ith scenario as follows:

$$G_{K,i} = \frac{((\overline{F} - F_K)^{\alpha}(-L_{K,i} + A_{K,i} + \overline{D})^{1-\alpha})^{(1-\gamma)} - 1}{1 - \gamma} \tag{4.10}$$

where:
— $G_{K,i}$ is the value of the objective function at time K for the ith state of the world;
— $\overline{F} \geq 0$ is the net contributions' threshold;
— $\overline{D} \geq 0$ is the pension fund deficit threshold;
— $0 \leq \alpha \leq 1$ is the standard Cobb-Douglas parameter which measures the relative weighting of the two 'goods': net contributions and deficit; and
— γ is the risk aversion parameter (higher values of γ represent higher aversion to risk).

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4.5.9 The main parameter assumptions can be summarised as follows:

— $\overline{F} = 0.6$;
— $\overline{D} = 0.6$;
— $\alpha = 0.5$ (indifference between the two 'goods'); and
— in the initial run we set $\gamma = 7$, but the results are then compared for different levels of risk aversion.

4.5.10 According to the objective function, funding inflows above a minimum threshold reduce the pension fund's utility, because paying contributions is a cost to the sponsor, while, at the same time, deficit above a minimum threshold also reduces utility, because it increases the risk faced by the sponsor (and the member). In this framework, whilst paying money into the scheme is undesirable *per se* because of the negative impact on companies' cash flows (and, in the presence of transparent accounting standards, on bottom line earnings as well), new inflows may lead to increased utility under certain conditions, as they reduce the plan deficit, which, under FRS 17, has a direct impact on companies' net assets.

4.5.11 Clearly, the relative preference between funding inflows and deficits depends on specific company and plan fundamentals, such as the strength of the firm's cash flows from operating activities, the size of the plan liabilities relative to enterprise value, etc. Both funding inflows and deficit thresholds can be thought of as minimum levels below which either contributions or deficits are not economically significant, and whose magnitude varies, depending on the size of the company and its pension scheme.

4.5.12 We compute the maximum of the sum of the objective function defined under each of the three scenarios and for different values of the parameters. In correspondence to each optimal point, we compute two ratios: funding relative to initial deficit f; and equity allocation e, defined as follows:

$$f = \frac{F_K}{L_{K-1} - A_{K-1}} \tag{4.11}$$

$$e = \frac{M_{K-1}X_{K-1}}{M_{K-1}X_{K-1} + N_{K-1}B_{K-1}}. \tag{4.12}$$

4.5.13 Formally, the problem can be written down as a maximisation with two decision variables (equity allocation and funding), and subject to four constraints:

$$\max_{F_K, N_{K-1}} \sum_{i=1}^{3} G_{K,i}$$

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such that:

$$\begin{aligned} F_K &\leq A_{K-1} \\ F_K &\geq 0 \\ N_{K-1} &\leq A_{K-1} \\ N_{K-1} &\geq 0. \end{aligned} \tag{4.13}$$

4.5.14 The first and the second constraints state that the value of new funding inflows cannot be greater than total assets in the previous period, and cannot be lower than zero. These jointly imply that $0 \leq f \leq 1$.

4.5.15 The third and fourth constraints, as discussed at the beginning of this section, rule out short positions and leverage. Similar restrictions hold for M_{K-1}, but do not need to enter the maximisation problem, as they would be redundant. These restrictions jointly imply that $0 \leq e \leq 1$.

4.6 *Exponential Utility Function*

4.6.1 We investigate here whether our results are robust to the choice of utility function, and introduce another standard function in the economics literature: exponential utility.

4.6.2 In this model, pension assets are given by equation (4.7), pension liabilities by equation (4.8), funding relative to initial deficit by equation (4.11), and equity allocation by equation (4.12). The only difference in the assumptions is the initial level of assets ($A_{K-1} = 0.85$, which is now higher, in order to bring the initial asset/liability ratio (85% instead of 50%) more in line with a typical U.K. pension scheme.

4.6.3 The new objective function defined on funding and deficit is given by:

$$\begin{aligned} G_{K,i} &= e^{\phi\left(-\alpha F_k^2 - \chi \max(L_{K,i} - A_{K,i}, 0)^2\right)} \quad \text{if } \phi \succ 0 \\ G_{K,i} &= -e^{\phi\left(-\alpha F_k^2 - \chi \max(L_{K,i} - A_{K,i}, 0)^2\right)} \quad \text{if } \phi \prec 0 \end{aligned} \tag{4.14}$$

where:

— ϕ is the risk aversion coefficient (or the reciprocal of risk tolerance); and
— α and χ are two parameters representing the importance attached by the pension fund to the two 'negative goods' (funding inflows and deficit) over which the function is defined.

4.6.4 Parameter assumptions are set as follows:

— $\phi = 1$;
— $\chi = 1$; and
— $\alpha = 1$.

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4.6.5 The maximisation problem here is defined by equation (4.13), using the objective function defined in equation (4.14) instead of the Cobb-Douglas one defined by equation (4.10).

4.7 *A Simple Two-Period Model*

4.7.1 The pension fund here is modelled over two periods (period 1 and period 2), and uncertainty over asset returns and background risk is captured by nine states of the world. Assets in the two periods are defined as:

$$A_{1,i} = A_0(1 + B_{1,i} + w_1 V_{1,i}) + F_1 \tag{4.15}$$

$$A_{2,i} = A_{1,i}(1 + B_{2,i} + w_2 V_{2,i}) \tag{4.16}$$

where:

— $A_{1,i}$ is the value of assets at time 1 for the ith state of the world;
— $A_{2,i}$ is the value of assets at time 2 for the ith state of the world;
— $B_{1,i}$ is the level of the bond index at time 1;
— $V_{1,i}$ is the level of the excess equity return index at time 1;
— B_{2i} is the level of the bond index at time 2;
— $V_{2,i}$ is the level of the excess equity return index at time 2;
— F_1 is the value of new funding inflows in period 1;
— w_1 is the equity allocation in period 1;
— w_2 is the equity allocation in period 2;
— A_0 is the level of assets at time zero;
— A_1 is the level of assets at time 1; and
— A_2 is the level of assets at the beginning of time 2 (before new inflows are accumulated in the fund).

4.7.2 The return on the asset portfolio is expressed here in terms of excess returns. Let the return R_1 on a portfolio with a share w_1 invested in equities be:

$$R_1 = 1 + (1 - w_1)B_1 + w_1 X_1. \tag{4.17}$$

This can be rewritten as:

$$R_1 = 1 + B_1 + w_1(X_1 - B_1) \quad \text{or } R_1 = 1 + B_1 + w_1 V_1 \quad \text{with } V_1 = X_1 - B_1.$$

4.7.3 The main parameter assumptions for the asset equation can be summarised as follows:

— We set $A_0 = 0.85$,$B_0 = 1$ and $X_0 = 1$, to make the results more easily readable.
— Short positions and leverage are ruled out by the restrictions $w_1 \geq 0$, $w_2 \geq 0$, $w_1 \leq 1$ and $w_2 \leq 1$.
— $V_{1,i}$ and $V_{2,i}$ are defined using nine alternative equally probable states of

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Table 4.1. Distribution of excess equity returns under the two-period model

Scenario	V_1	V_2
$i = 1$	0	0
$i = 2$	0	0.15
$i = 3$	0	−0.1
$i = 4$	0.15	0
$i = 5$	0.15	0.15
$i = 6$	0.15	−0.1
$i = 7$	−0.1	0
$i = 8$	−0.1	0.15
$i = 9$	−0.1	−0.1

the world designed to illustrate the volatility of equity investments. These scenarios define the distribution of excess equity returns, and have no relationship with those defined in Section 4.4. These scenarios jointly imply a distribution of excess equity returns with mean 1.7% and standard deviation of 10.3% in both periods. Table 4.1 illustrates the assumed distribution of excess equity return.

— $B_{1,i}$ and $B_{2,i}$ are assumed to be riskless and with a rate of return equal to 5%.
— We set $F_1 \leq A_0 \Rightarrow F_1 \leq 0.85$, to rule out funding in excess of assets in the previous period.

Liabilities under the two-period model are defined as follows:

$$L_{1,i} = \frac{(1 + B_{1,i})B_{AVE}h_{1,i}}{(1 + B_{1,i})} = B_{AVE}h_{1,i} \tag{4.18}$$

$$L_{2,i} = \frac{(1 + B_{1,i})(1 + B_{2,i})B_{AVE}h_{2,i}}{(1 + B_{2,i})} = (1 + B_{1,i})B_{AVE}h_{2,i} \tag{4.19}$$

where:

— $L_{1,i}$ is the value of liabilities at time 1 for the ith state of the world.
— $L_{2,i}$ is the value of liabilities at time 2 for the ith state of the world.
— B_{AVE} is a parameter representing the average level of the bond index (it can also be interpreted as the value of the index at time zero). Even here $B_{i,K}$, decomposed into B_{AVE} and respectively period 1 return$(1 + B_{1,i})$ and period 2 return $(1 + B_{2,i})$, enters both the assets and the liabilities equations (we assume that the bond-like component of pension liabilities is perfectly hedged by matching fixed-income investments on the asset side).
— $h_{1,i}$ is a factor representing the background risk at time 1.
— $h_{2,i}$ is a factor representing the background risk at time 2.

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Table 4.2. Distribution of background risk under the two period model

	Without background risk		With background risk	
Scenario	$h_{1,i}$	$h_{2,i}$	$h_{1,i}$	$h_{2,i}$
$i = 1$	1	1	1	1
$i = 2$	1	1	1.03	1
$i = 3$	1	1	0.97	1
$i = 4$	1	1	1	1.03
$i = 5$	1	1	1.03	1.03
$i = 6$	1	1	0.97	1.03
$i = 7$	1	1	1	0.97
$i = 8$	1	1	1.03	0.97
$i = 9$	1	1	0.97	0.97

4.7.4 The main parameter assumptions for the liabilities equation can be summarised as follows:

— $B_{AVE} = 1.05$;
— $h_0 = 1$; and
— $h_{1,i}$ and $h_{2,i}$ are defined under the same nine scenarios of bond and equity returns in two alternative ways, according to whether background risk is included or not in the model. Table 4.2 illustrates the assumed distribution of background risk The background risk case corresponds to a distribution with mean equal to one and standard deviation of 2.6% in both periods, with zero correlation with both bond and equity returns.

4.7.5 Jointly, these assumptions, together with $B_0 = 1$ from the assets equation, imply that:

$$L_0 = \frac{B_{AVE} h_0}{1 + B_0} = 1 \tag{4.20}$$

which implies an initial funding or asset/liability ratio equal to 85% (similarly to that in Section 3.3).

4.7.6 Because the pension funds' planning horizon ends at time 2, we assume that funding inflows at time 2 under all scenarios will close the gap between assets and liabilities. This reduces a four-variable to a three-variable problem. In other words, we assume:

$$F_{2,i} = L_{2,i} - A_{2,i}. \tag{4.21}$$

4.7.7 The objective function in this model is defined as a standard negative exponential utility over two 'goods': funding inflows in period 1 ($F_{1,i}$); and deficit in period 2 (equal to $F_{2,i}$, given that we assume that the

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shortfall is closed at the end of the planning horizon). This can be written as:

$$G_i = \sum_{i=1}^{9} e^{(-\alpha_1 F_1^2 - \alpha_2 F_{2,i}^2)} \tag{4.22}$$

where α_1 and α_2 are the two parameters of the exponential utility function.

In the baseline case, parameters are set as follows:

— $\alpha_1 = 1.5$; and
— $\alpha_2 = 10.6$.

4.7.8 To keep the analysis simple, we assume that the asset allocation decision for both periods is made at the same time. We could, alternatively, recast this as a dynamic programming problem in which asset allocation in the second period is variable, but that would be more complex. Hence, given assets, liabilities and the objective function, the maximisation problem can be written as:

$$\max_{w_1, w_2, F_1} \sum_{i=1}^{9} G_{K,i}$$

such that:

$$\begin{aligned} F_1 &\leq A_0 \\ F_1 &\geq 0 \\ w_1 &\leq 1 \\ w_2 &\leq 1 \\ w_1 &\geq 0 \\ w_2 &\geq 0. \end{aligned} \tag{4.23}$$

4.8 *Dynamic Models of Background Risk for Pensions*

4.8.1 Although we do not solve complex dynamic models of background risk for pensions, it is useful to comment briefly on how economic models in this area could be adapted for pension modelling. The standard dynamic economic model of Merton (1969) and Merton (1971) was adapted by Bodie *et al.* (1992) to include labour market risks. In fact, in Bodie *et al.* (1992) this labour market risk is perfectly correlated with the stock market, whereas in Merton (1971) it is akin to a riskless bond. Bodie *et al.* (1992) conclude that risky human capital leads to more investment in risky assets than would otherwise occur. Viceira (2001) examines the case where labour income is uncorrelated with risky assets, and shows that there is also more investment in risky assets among those in the working life.

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4.8.2 The dynamic pension fund problem is strikingly similar to the optimal consumption problem faced by individuals with uncertain labour income, with the principal difference being that, instead of facing random positive income, pension funds face random negative costs; and, in addition, instead of optimising over consumption as in economic models, pension funds choose funding optimally as in the models above.

4.8.3 Hence, if positive human capital leads to more equity investment in an economic model, negative human capital which arises in pension models (due to pension benefits tied to human capital) would presumably lead to less equity investment relative to the unconstrained case; and, if in an economic model, more human capital risk leads to a change to or from equity investment, we would expect the same to occur in a pension model. These areas clearly need to be investigated further, but, as the effect of human capital risk in economic models is far from unambiguous, one would not expect intuitively liability risk to work to have unambiguous effects in pension models either.

4.8.4 In the following sections, we will explore the solutions of our static models with and without background risk, to assess how the introduction of background risk affects the results. We will start with the analysis of cases of optimal asset allocation, given fixed funding as a proportion of initial deficit (fixed funding), then proceed to look at optimal funding inflows (optimal funding) for fixed asset allocation. We will finally proceed to look at how joint solutions for funding inflows and asset allocation depend on background risk.

5. Results on Asset Allocation with Fixed Funding

5.1 This section computes optimal asset allocation with fixed funding for the three models presented in Section 4. We find that the effect of background risk on asset allocation can go in either direction, and the effects can be non-negligible.

5.2 *Cobb-Douglas Model*

5.2.1 Figures 5.1 to 5.3 (all the figures for this section are in Appendix A) review the results, with and without background risk, under the assumption of a Cobb-Douglas utility function for the case of fixed funding assumed to be equal to 0.1 (or 20% of initial deficit). This appears to be a realistic baseline scenario for a pension fund with a relatively high proportion of active members and a large proportion of unfunded liabilities at the outset.

5.2.2 Figure 5.1 presents the results as a function of the risk aversion parameter γ. Here, background risk to liabilities raises the optimal equity allocation to 100%, independent of the level of risk aversion. However, there

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are other forms of uncorrelated risk with which we experimented, in which the asset allocation in equities declined. So, the results are not universal, and the primary issue here is to illustrate a particular case where the results are not in line with expectations.

5.2.3 As one might expect, equity allocation with no background risk is a decreasing function of the risk aversion parameter. The optimal share invested in equities goes down from 100%, with γ below one to around 25%, with levels of γ in the range of seven.

5.2.4 Figure 5.2 displays the sensitivity of the asset mix results to changes in the parameter α of the Cobb-Douglas utility function, which represents the relative importance of the two 'negative goods' (funding and deficit) entering the objective function (baseline assumption was indifference between the two, or $\alpha = 0.5$). As α goes down, the relative importance of deficit vs. funding increases. As shown in the figures, optimal equity allocation is still 100%, independent of the value of the parameter α in the case with background risk, while it is an increasing function of α when there is no background risk. Intuitively, this implies that, with Cobb-Douglas preferences defined over funding and deficit, the pension plan will invest more conservatively as α decreases, because the penalty for running deficits is higher.

5.2.5 Figure 5.3 replicates the analysis in a three-dimensional space, and lets both parameters, α and γ, vary. Because equity allocation is 100%, independent of parameter values when background risk is allowed for, we only display here the results referred to as the no background risk case:

— For all values of the parameter α, optimal equity allocation decreases with risk aversion, and the higher the value of α the steeper is the relationship (equity allocation declines faster as γ increases). For example, for $\alpha = 0.1$, equity allocation goes down by around 70% from 48.46% ($\gamma = 2.15$) to 14.95% ($\gamma = 7.10$), while for $\alpha = 0.5$ it goes down by just over 60%, for the same levels of γ, from 62.83% to 24.09%.

— For all values of the parameter γ, optimal equity allocation increases with relative importance of funding vs. deficit, and the higher the value of γ the steeper is the relationship. For example, for $\gamma = 2.15$ optimal allocation to equities increases by around 30% from 48.46% ($\alpha = 0.1$) to 62.83% ($\alpha = 0.5$), while for $\gamma = 7.1$ the rise is much steeper (around 60%) for the same values of α, from 14.95% to 24.09%.

5.3 *Exponential Model*

5.3.1 Figure 5.4 presents the results referred to as the exponential model, described in Section 4.6, for the case of fixed funding and variable asset allocation. In order to solve for the optimal asset allocation, we assume here funding inflows in period 1 to be equal to zero (no contributions to the pension fund). Moreover, as discussed in the previous section, we assume in this model that the initial asset/liability ratio is equal to 85%.

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5.3.2 The figure presents the results, with and without background risk, as a function of the parameter χ, which is a measure of the importance attached by the pension fund to scheme deficits. Unlike the case of a Cobb-Douglas function, it is worth remembering that the deficit parameter χ is not a measure of relative preference, given that the two parameters entering the utility function do not need to sum up to one (see Section 4.6).

5.3.3 Background risk in this framework has the opposite effect of the Cobb-Douglas model discussed in the previous section, as it lowers the optimal allocation to equities. This confirms that the inclusion of background risk, even in a simplified setting, has an ambiguous impact on pension funds' optimal investment strategy, and the results depend on how the fund preferences over funding and scheme deficits are modelled.

5.3.4 When background risk is absent, the equity allocation increases, with χ rising from 0.05 up to 1.0, while, if background risk is allowed for, equity allocation becomes a decreasing function of the parameter χ.

5.3.5 Given the assumption of zero funding, the results on optimal asset allocation are clearly insensitive to changes in the parameter α, which measures the penalty attached to funding. Because of this, we do not present the results as a function of α in this context.

5.4 *Two-Period Model*

5.4.1 Figure 5.5 presents the results referred to as the two-period model with exponential preferences, described in Section 4.7 for the case of fixed funding and variable asset allocation. As with the exponential model discussed in the previous section, we assume an 85% funding level at the outset and zero funding inflows in the first period.

5.4.2 In the figures, we express the optimal level of equity allocation as the mean of the optimal allocation in each of the two periods (w_1 and w_2) resulting from the optimisation problem. This is justified, on the grounds that it is reasonable to assume that most pension funds do not set targets for their strategic asset allocation on an annual basis.

5.4.3 As shown in Figure 5.5, equity allocation is an increasing function of α_2, and this implies that, when the penalty attached to a deficit at the end of the second period (or, alternatively, to funding in the second period, given the assumption that any shortfall must be covered at the end of the planning horizon) is higher, the optimal strategy, given an 85% funded scheme at the outset, is to invest in the highest expected return asset class to close the gap.

5.4.4 Unlike with the one-period model with exponential preferences, non-additive background risk here increases the average allocation to equities, and the difference with the no background risk model is inversely proportional to the importance attached to deficits captured by the deficit parameter α_2. With $\alpha_2 = 0.1$, background risk leads to a shift in the range of

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50% (41.34% against 27.66%), while the increase comes down to around 25% for $\alpha_2 = 10.6$ (55.07% against 43.68%).

5.4.5 Even here, because of the assumption of zero funding, the results are insensitive to changes in the parameter α_1, which represent the penalty associated with funding, and are not displayed.

6. Results on Funding with Fixed Asset Allocation

6.1 This section computes optimal funding with fixed asset allocation for the three models presented in Section 4. We find that, in general, background risk affects the level of optimal funding, and the effects can be non-negligible.

6.2 *Cobb-Douglas Model*

6.2.1 Figures 6.1 to 6.3 (all figures for this section are in Appendix B) review the results with and without background risk under the assumption of a Cobb-Douglas utility function for the case of fixed asset allocation with a 50-50 bond/equity split, which is a conservative assumption for most U.K. pension schemes. In fact, from the Watson Wyatt U.K. Pension Risk Indicators database based on FRS 17 disclosures, the average equity allocation for FTSE 350 pension schemes appears to be over 60%, although it has come down slightly in recent years (the mean equity share of total pension assets excluding post-retirement medical obligations went down from around 69% in 2001 to 64% in 2003).

6.2.2 Figure 6.1 presents the results as a function of the risk aversion parameter γ. With Cobb-Douglas preferences, the introduction of uncorrelated and closer to multiplicative background risk unambiguously raises funding. This is in line with a precautionary savings explanation, where funding can be interpreted as isomorphic to individual savings.

6.2.3 Although not very steep in absolute terms (for instance in the case of no background risk, optimal funding goes up only from 51.71% with $\gamma = 0.5$ to 52.56% with $\gamma = 7$), the resulting function does increase with the risk aversion parameter, and more so when background risk is allowed for.

6.2.4 Optimal funding under the assumption of a 50% shortfall at the outset is above 50% under all assumptions and parameter values. This is a result which depends on the one-period framework of the modelling exercise, where the objective function only considers a deficit in period 1 and does not incorporate any intertemporal dimension.

6.2.5 Figure 6.2 displays the sensitivity of optimal funding as a percentage of deficit results to changes in the parameter α of the Cobb-Douglas utility function defined in the previous section. As shown in the charts, optimal funding is always a steeply increasing function of the relative importance of scheme deficit (or a decreasing function of the relative importance of

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funding). This pattern holds independently of whether background risk is included or not, but optimal funding is slightly higher when background risk is allowed for. Moreover, independent of the inclusion of background risk, optimal funding is 100% for α equal to 0.10, and goes down to below 60% when α goes up to 0.5. In conclusion, if the objective function places higher emphasis on deficits as a result of bargaining between the firm and workers, the outcome is a higher level of optimal funding of the scheme deficit.

6.2.6 Figure 6.3 replicates the analysis in a three-dimensional space, by letting both parameters α and γ vary. The figure confirms the pattern described so far:

— Background risk increases the funding of the scheme deficit for high levels of risk aversion measured by the parameter γ.
— Funding is an increasing function of risk aversion, but the steepness of the relationship is very low.
— Funding increases steeply if the objective function places higher emphasis on deficits (measured by the parameter α).

6.3 *Exponential Model*

6.3.1 Figures 6.4 to 6.5 present the results referred to as the exponential model, described in Section 4.6 for the case of fixed asset allocation. The assumption here is 50-50 bond/equity split, and an 85% funded scheme at the outset.

6.3.2 Figure 6.4 presents the results, with and without background risk, as a function of the parameter χ, which is a measure of the importance attached by the pension fund to scheme deficits. Consistent with the Cobb-Douglas results and the precautionary savings interpretation, funding here increases with the inclusion of background risk, but, due to the 85% initial funding assumption, it is not surprising that optimal funding levels as a proportion of deficit are well below those reported in the previous sub-section, which assumes a 50% funded scheme. In all cases, funding is a steeply increasing function of the parameter χ, because, the higher the importance attached to shortfalls, the more important it is to fund existing deficits.

6.3.3 Figure 6.5 investigates how optimal funding varies with α, which represents the penalty associated with funding resulting from the assumed objective function. Clearly, as the penalty associated with having to fund more decreases, the optimal level of funding goes up steeply, until reaching 100% of deficit at the outset. Given that χ here is fixed at one, the objective function becomes more sensitive to deficits as α comes down, leading to additional funding inflows to cover for any uncertainty surrounding future shortfalls.

6.3.4 In this context, background risk increases the level of optimal funding for all levels of α greater than 0.1.

6.3.5 To summarise the patterns highlighted so far:

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— Background risk unambiguously increases optimal funding expressed as a proportion of scheme deficit at the outset.
— Optimal funding is increasing as a function of χ and decreasing as a function of α.

6.4 *Two-Period Model*

6.4.1 Figures 6.6 and 6.7 present the results referred to as the two-period model with exponential preferences described in Section 4.7 for the case of fixed equity allocation (assumed to be equal to 50% similar to the other two models discussed in the two previous sub-sections) and variable funding. As with the exponential model discussed in the previous section, we assume here an 85% funded scheme at the outset.

6.4.2 Figure 6.6 presents the results as a function of the funding parameter α_1. As one would expect, optimal funding is a decreasing function of the parameter, but precautionary saving does not prevail in this model, and background risk marginally reduces optimal funding for values of the parameter α_1 greater than one.

6.4.3 Figure 6.7 investigates the sensitivity of optimal funding to changes in the deficit (or period 2 funding) parameter α_2. Funding here is a steeply increasing function of the parameter α_2, given the restriction that any remaining shortfall must be covered in period 2 (optimal funding goes up from below 10% of initial deficit for $\alpha_2 = 0.1$ to around 100% for $\alpha_2 = 10.6$). Background risk appears to reduce funding, but the shift is very small (for instance, with $\alpha_2 = 10.6$ the difference is in the range of 0.8%).

6.4.4 Key implications can be summarised as follows:
— Background risk marginally reduces optimal funding, but the shifts are not very large.
— Optimal funding is decreasing in the funding parameter.
— Optimal funding is a steeply increasing function of the deficit parameter.

7. Results on Variable Funding and Asset Allocation

7.1 This section computes optimal funding and asset allocation, both variable, for the three models presented in Section 4. We find that, in general, background risk affects both funding and equity allocation, and the effects can be non-negligible.

7.2 *Cobb-Douglas Model*

7.2.1 Figures 7.1 to 7.2 (all figures for this section are in Appendix C) review the results with and without background risk under the assumption of a Cobb-Douglas utility function when both asset allocation and funding are variable and are given as outputs by the solution of the optimisation problem.

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7.2.2 Figure 7.1 presents the results as a function of the risk aversion parameter γ. With Cobb-Douglas preferences, uncorrelated and closer to multiplicative background risk increases funding for all levels of γ and raises the equity allocation to 100% for high levels of risk aversion. Conversely, the optimisation algorithm gives an interior solution for equity allocation, and high levels of risk aversion if background risk is absent (for instance, with $\gamma = 7$, the optimal share invested in equities is 47.06%).

7.2.3 Figure 7.2 displays the sensitivity of the optimisation results to changes in the parameter α of the Cobb-Douglas utility function, defined in the previous sections. Consistent with the case of fixed asset allocation, optimal funding here becomes a steeply increasing function of the relative importance of deficits. Conversely, unlike with fixed funding, equity allocation in the case of no background risk becomes a non-linear function of α, which is increasing for low levels of the parameter (up to 0.15), and decreasing thereafter. Intuitively, if the plan dislikes deficits, but funding is fixed, the model predicts that it will invest less in equities, as it fears the volatility of the stock market, whereas, with variable funding, the implication is that the scheme would generally invest more in equities and increase funding, because the penalty associated with more funding becomes lower. However, when the penalty associated with deficits becomes very large, the volatility effect seems to prevail again, and the plan reduces the equity allocation.

7.2.4 When background risk is included, results on funding are similar, except that the resulting optimal funding is slightly above the no background risk solution. Conversely, background risk in this model leads to a 'corner' solution for asset allocation (100% equities), independent of the value of α, confirming the pattern highlighted in Section 5.

7.2.5 We also replicated the analysis, letting both parameters α and γ vary at the same time. This is not shown here, but the analysis substantially confirms the patterns described so far:

— Background risk leads to higher funding and higher equity allocation, and the optimal share in equities is always 100% when background risk is allowed for.
— A rise in the risk aversion parameter raises funding and lowers the optimal equity allocation.
— A rise in the parameter α, measuring the relative importance of funding vs. deficit (the two arguments entering the objective function), lowers optimal funding, and, unlike with fixed funding, decreases the optimal equity share, except for low values of the parameter.

7.3 *Exponential Model*

7.3.1 Figures 7.3 to 7.4 present the results referred to as the exponential model, described in Section 4.6 for the case of variable funding and asset allocation.

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7.3.2 Figure 7.3 presents the results with and without background risk as a function of the parameter χ, which is a measure of the importance attached by the pension fund to scheme deficits. As pension funds dislike deficits more, the model predicts a rise in funding, while equity allocation moves in the opposite direction. Background risk marginally increases funding and equity allocation for values of χ greater than 0.75.

7.3.3 Figure 7.4 displays the sensitivity of the optimal solution to changes in the value of the funding parameter α. Funding here is a decreasing function of α, as in the case with fixed asset allocation, and background risk leads to additional funding, especially for low values of the funding parameter. Conversely, equity allocation is increasing in α with no background risk, while it becomes a decreasing function of the funding parameter when background risk is incorporated. Background risk leads to a significantly higher equity share, and the shifts are very large for low values of the funding parameter (for $\alpha = 0.1$ the optimal equity share goes up from below 5% to over 80%). Intuitively, given additional uncertainty over liabilities (and therefore deficits), the scheme not only raises funding, but increases the equity weight as well.

7.3.4 To summarise the pattern described so far:

— Background risk increases funding and equity allocation for most parameter values.
— Funding is increasing in the deficit parameter and decreasing in the funding parameter.
— For low values of the funding parameter, the optimal equity allocation with background risk is substantially higher than the no background risk solution.

7.4 *Two-Period Model*

7.4.1 Figures 7.5 to 7.6 present the results referred to as the two-period model with exponential preferences, described in Section 4.7 for the case of variable funding and asset allocation. As with the exponential model discussed in the previous section, we assume an 85% funding level for the scheme at the outset.

7.4.2 Figure 7.5 displays the results as a function of the funding parameter α_1. Background risk, when both equity allocation and funding are variable, marginally decreases funding while increasing optimal equity allocation for all values of the funding parameter. Funding is again a decreasing function of the funding parameter, while the equity allocation in this context increases with α_1.

7.4.3 Figure 7.6 displays the results as a function of the deficit parameter α_2. Average equity allocation here is a decreasing function of the deficit parameter because of underlying volatility, and funding is increasing with the penalty associated with deficits in period 2. Background risk in this context does unambiguously raise the equity allocation, while again marginally

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reducing funding. One possible economic interpretation of this model is that, when background risk is accounted for, portfolio choice considerations prevail over precautionary savings.

7.4.4 To summarise the patterns highlighted so far:

— Background risk appears to increase equity allocation and reduce funding for most values of parameters, but shifts are not very large compared to the benchmark solution with no background risk.
— Funding decreases with the funding parameter and increases with the deficit parameter.
— Equity allocation is increasing in the funding parameter and decreasing in the deficit parameter.

8. Implications for Market Valuation

8.1 In this section, we assess how state prices (which are the q introduced in equation (4.2)) depend on background risk. As the market value of a security is determined with respect to state prices and its payoffs, an analysis of how state prices change is sufficient to assess how market values of pension liabilities would change with changes in background risk. In imperfect markets, such as the one with which we are working, it is generically the case that state prices are not unique (Magill & Quinzii, 1996), so there is no unique market price which we can assign to pension liabilities (this is, in a sense, not surprising, because the market does not exist!).

8.2 However, we can examine the case of the specific utility functions with which we are working, assuming that these are representative of the market as a whole. Using the framework of Section 4.1, Poon *et al.* (2005) show that state prices satisfy the formula:

$$\frac{\mathop{\mathrm{E}}_{\varepsilon}\left[U'(x_i+\varepsilon)\right]}{\mathop{\mathrm{E}}_{x}\mathop{\mathrm{E}}_{\varepsilon}[U'(x+\varepsilon)]} = \frac{q_i}{p_i} \tag{8.1}$$

where p_i are the probabilities of a state occurring. For the case of pension funds, assuming only one state variable x_i, the theoretical market value of pension liabilities is then:

$$p_L = \sum_{i=1}^{N} q_i L_i. \tag{8.2}$$

Equation (8.1) is true more generally, but with more than one state variable the computation of the state prices is more involved than it will be in the case below. We can also express equation (8.1) in terms of the precautionary premium ψ:

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Table 8.1. Changes in fair value liability due to background risk (optimal asset allocation with 20% of deficit funded)

Gamma	Background risk
0.0001	1.67%
0.5	2.51%
1.0001	3.33%
1.5	4.11%
2	4.84%
2.5	5.51%
3	6.11%
3.5	6.65%
4	7.12%
4.5	7.53%
5	7.89%
5.5	8.20%
6	8.46%
6.5	8.69%
7	8.89%

$$\frac{U'(x_i-\psi)}{\underset{x}{\mathrm{E}}[U'(x-\psi)]} = \frac{q_i}{p_i} \tag{8.3}$$

which means that, in general, the precautionary premium is positive and affects the value of liabilities.

8.3 We return to the Cobb-Douglas model, with the parameterisation adopted earlier, and let deficits be the state variable for pricing purposes. We apply equation (8.1) with and without background risk. In Table 8.1 we show how valuations change.

8.4 Of course, the incompleteness of the market does mean there is no unique market price, but, at the same time, the calculations here suggest that background risk induces a risk premium.

9. Conclusion

9.1 This paper has explored the effects of unhedgeable background risks such as mortality and labour risks (such as wages and turnover) on pension funding and financing. Background risks are very common in the pension arena, and hence are important for practicing actuaries to consider.

9.2 Our starting point was the extensive economics and finance literature on background risk. While aspects of this are applicable to pensions, there are differences from standard cases considered by economists. We then developed three simple, illustrative pension models, and showed the impact of background risk on pension funding and asset allocation. Just as in

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economic models, background risk affects savings and asset allocation decisions. In our pension models it affects funding and asset allocation decisions. In particular, uncorrelated background risk can lead to a higher level of optimal funding, but, at the same time, it can lead to an optimal asset allocation with a higher equity share.

9.3 We also explore implications of background risk for fair market valuation. Where background risk translates into precautionary funding, there is a prudence margin which will be reflected in fair value calculations. It would also be interesting to look at implied precautionary premia as a basis for looking at how assumptions used in funding valuations can incorporate margins for background risk.

9.4 The area of background risk is clearly an important one, and one in which much more work, particularly on dynamic models, is called for in the future.

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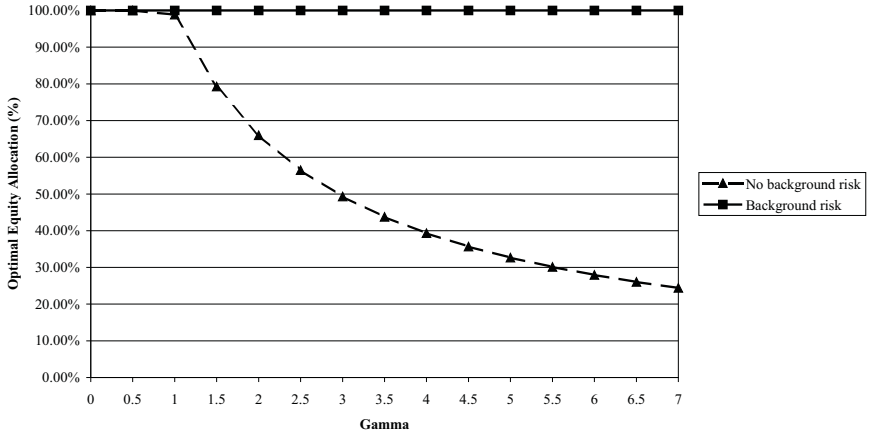
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APPENDIX A

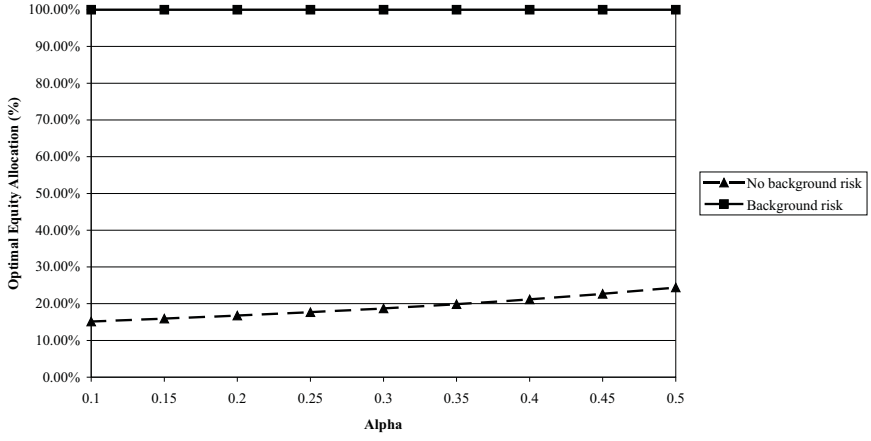
SECTION 5 FIGURES



Notes:
1. $\overline{D} = 0.6$
2. $\overline{F} = 0.6$
3. $\alpha = 0.5$
4. $F_K = 0.1$

Figure 5.1. Optimal equity allocation with fixed funding for different values of parameter Gamma (Cobb-Douglas utility)

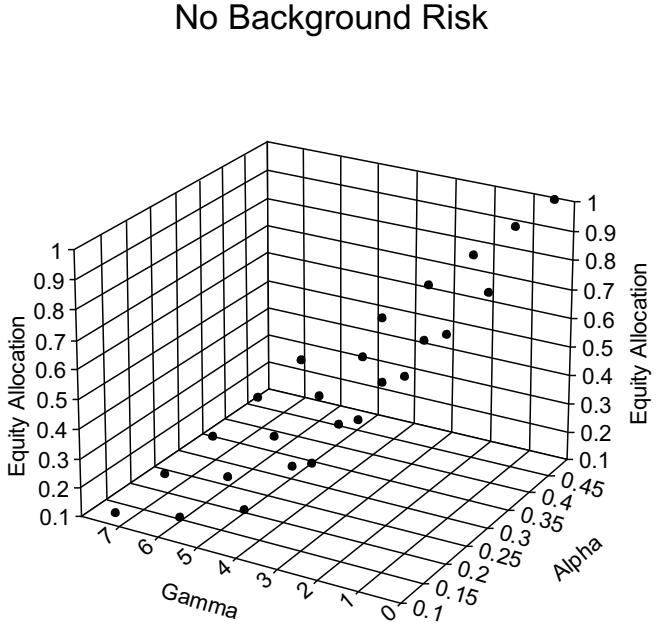
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Notes:
1. $\overline{D} = 0.6$
2. $\overline{F} = 0.6$
3. $\gamma = 7$
4. $F_K = 0.1$

Figure 5.2. Optimal equity allocation with fixed funding for different values of parameter Alpha (Cobb-Douglas utility)

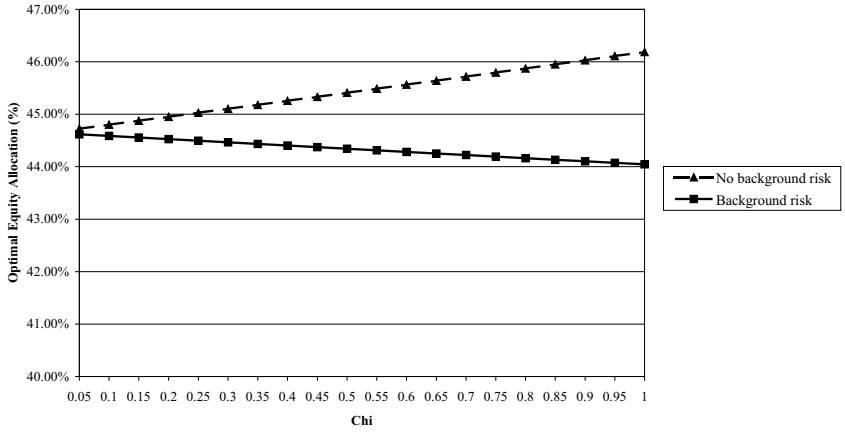
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Notes:
1. $\overline{D} = 0.6$
2. $\overline{F} = 0.6$
3. $F_K = 0.1$

Figure 5.3. Results for asset allocation with fixed funding for different values of parameters α and γ (Cobb-Douglas function)

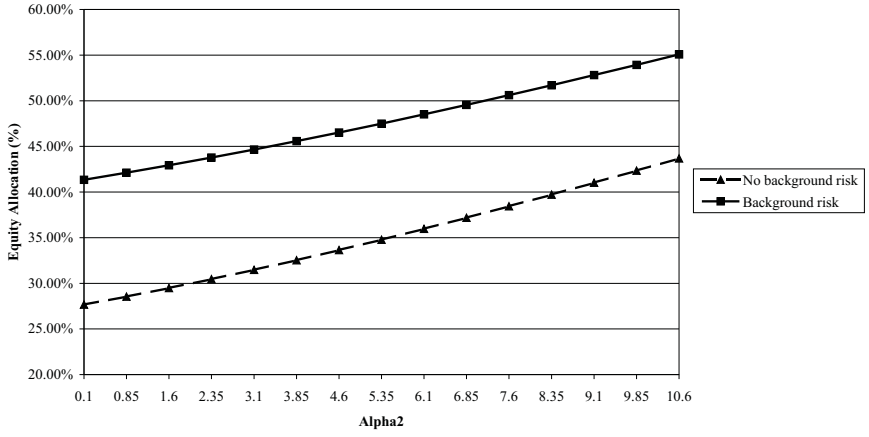
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Notes:
1. $\alpha = 1$
2. $\phi = 1$
3. $F_K = 0\%$

Figure 5.4. Optimal equity allocation with fixed funding for different values of parameter Chi (exponential utility)

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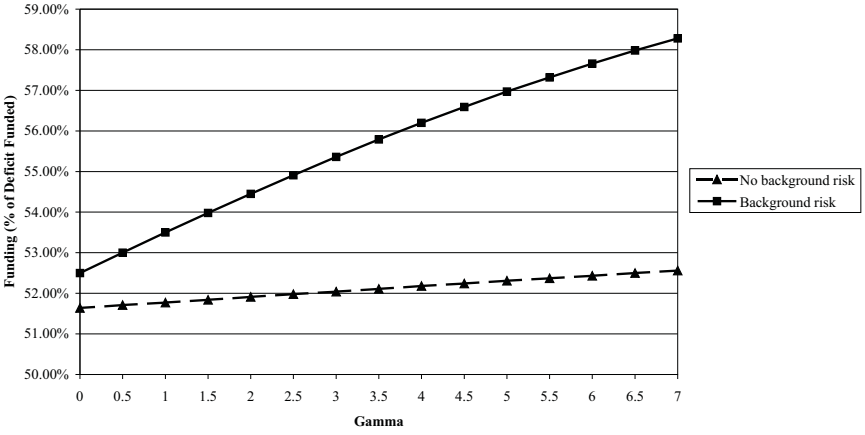
Notes:
1. $\alpha_1 = 1.5$
2. $F_K = 0$

Figure 5.5. Optimal equity allocation with fixed funding for different values of parameter Alpha2 (two-period model)

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APPENDIX B

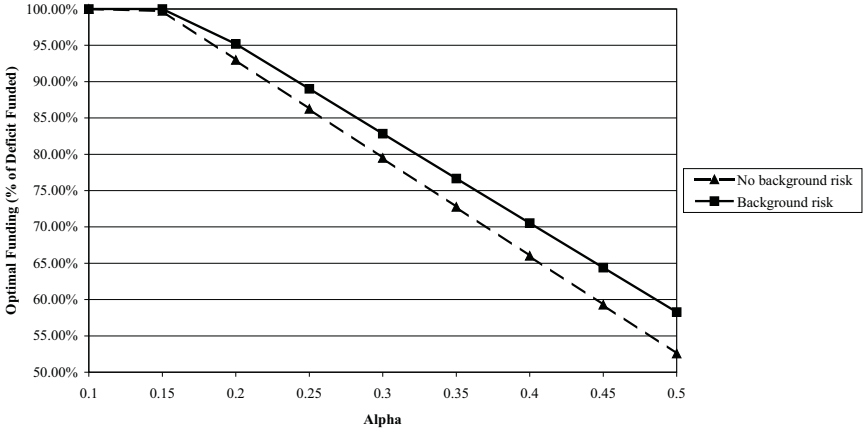
SECTION 6 FIGURES



Notes:
1. $\overline{D} = 0.6$
2. $\overline{F} = 0.6$
3. $\alpha = 0.5$
4. 50/50 bond-equity split

Figure 6.1. Optimal funding with fixed asset allocation for different values of parameter Gamma (Cobb-Douglas utility)

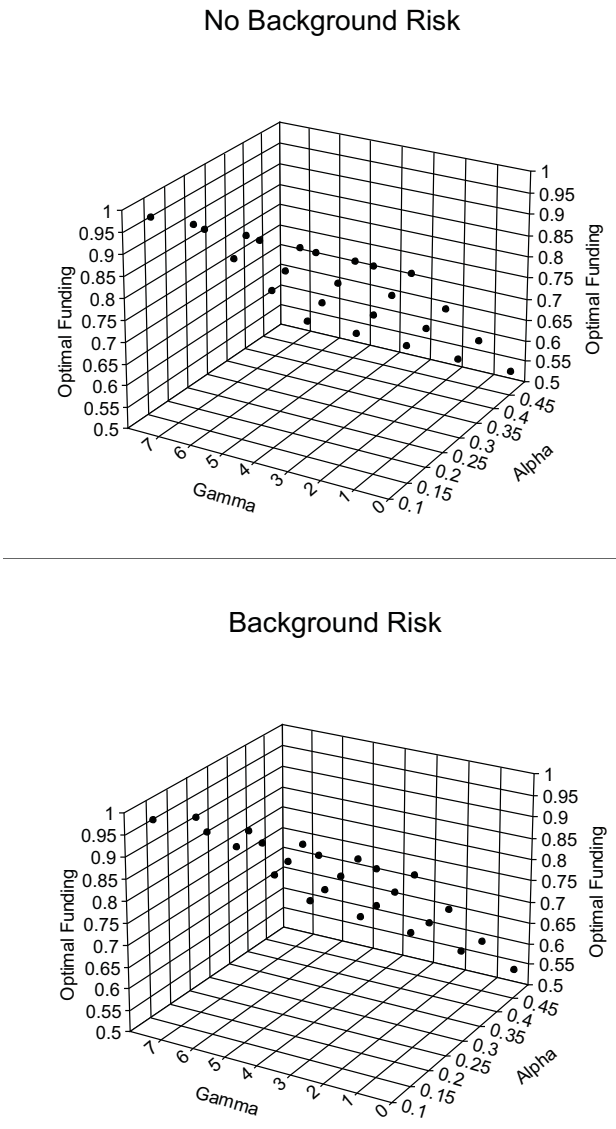
https://doi.org/10.1017/S1357321700004712 Published online by Cambridge University Press



Notes:
1. $\overline{D} = 0.6$
2. $\overline{F} = 0.6$
3. $\gamma = 7$
4. 50/50 bond-equity split

Figure 6.2. Optimal funding with fixed asset allocation for different values of parameter Alpha (Cobb-Douglas utility)

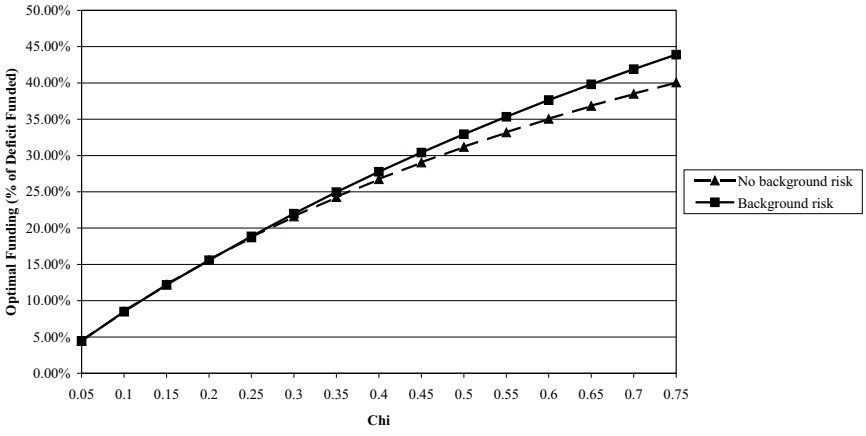
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Notes:
1. $\overline{D} = 0.6$
2. $\overline{F} = 0.6$
3. 50/50 bond-equity split

Figure 6.3. Results of funding with fixed asset allocation for different values of parameters α and γ (Cobb-Douglas function)

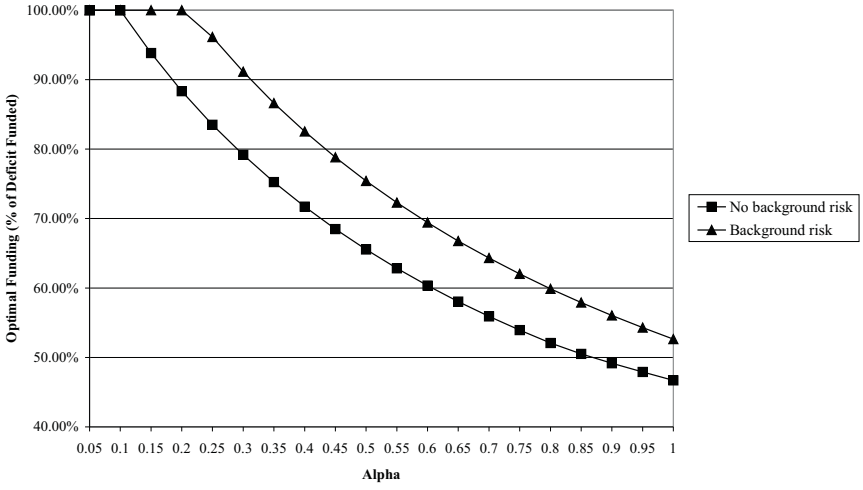
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Notes:
1. $\alpha = 1$
2. $\phi = 1$
3. 50/50 bond-equity split

Figure 6.4. Optimal funding with fixed asset allocation for different values of parameter Chi (exponential utility)

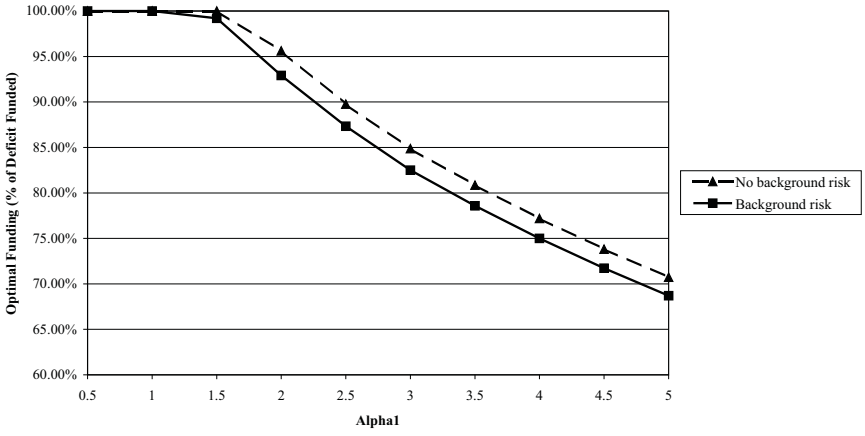
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Notes:
1. $\chi = 1$
2. $\phi = 1$
3. 50/50 bond-equity split

Figure 6.5. Optimal funding with fixed asset allocation for different values of parameter Alpha (exponential utility)

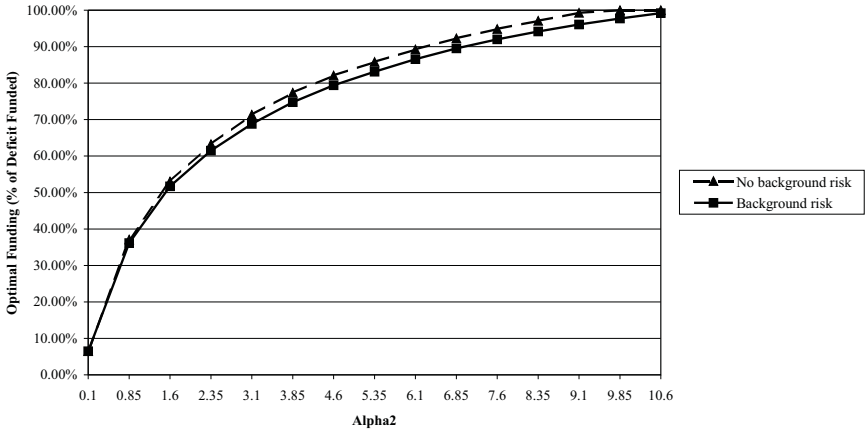
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Notes:
1. $\alpha_2 = 10.6$
2. 50/50 bond-equity split

Figure 6.6. Optimal funding with fixed asset allocation for different values of parameter Alpha1 (two-period model)

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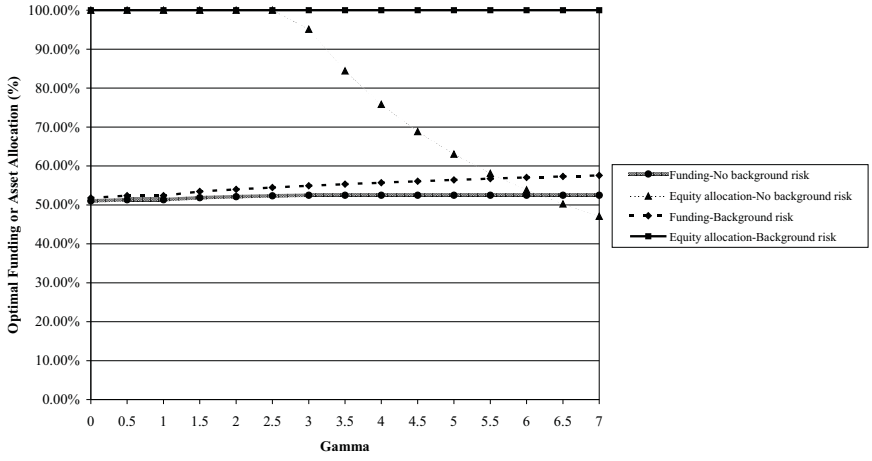
Notes:
1. $\alpha_1 = 1.5$
2. 50/50 bond-equity split

Figure 6.7. Optimal funding with fixed asset allocation for different values of parameter Alpha2 (two-period model)

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APPENDIX C

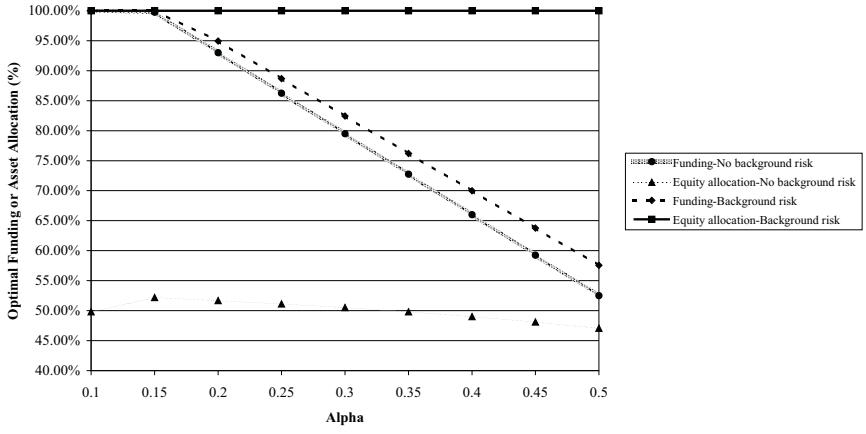
SECTION 7 FIGURES



Notes:
1. $\overline{D} = 0.6$
2. $\overline{F} = 0.6$
3. $\alpha = 0.5$

Figure 7.1. Optimal variable funding and asset allocation for different values of parameter Gamma (Cobb-Douglas utility)

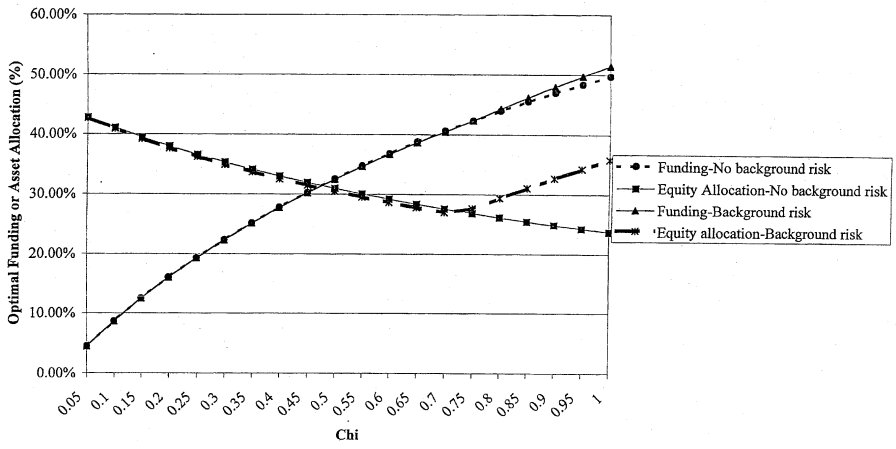
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Notes:
1. $\overline{D} = 0.6$
2. $\overline{F} = 0.6$
3. $\gamma = 7$

Figure 7.2. Optimal variable funding and asset allocation for different values of parameter Alpha (Cobb-Douglas utility)

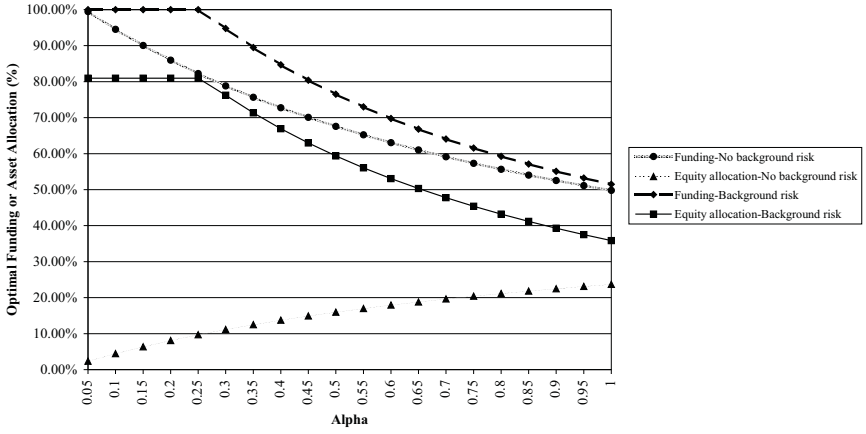
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Notes:
1. $\alpha = 1$
2. $\phi = 1$

Figure 7.3. Optimal variable funding and asset allocation for different values of parameter Chi (exponential utility)

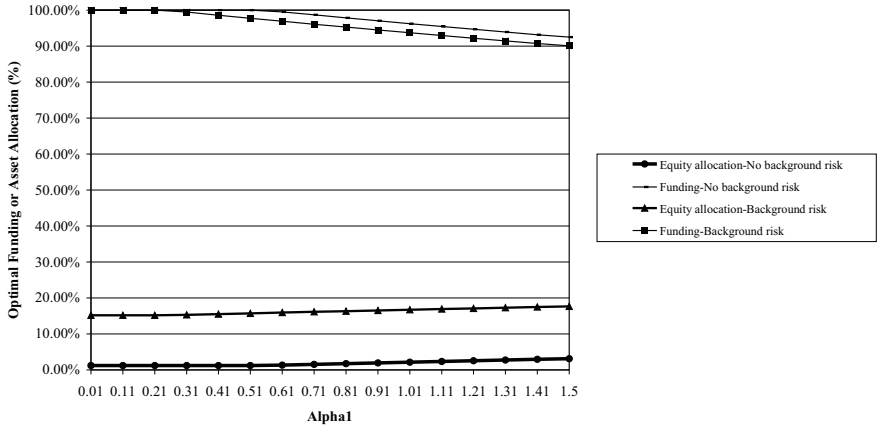
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Notes:
1. $\chi = 1$
2. $\phi = 1$

Figure 7.4. Optimal variable funding and asset allocation for different values of parameter Alpha (exponential utility)

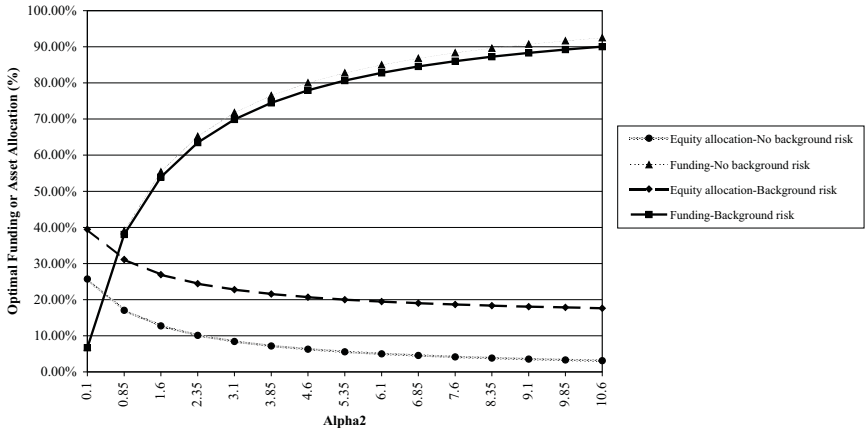
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Note:
1. $\alpha_2 = 10.6$

Figure 7.5. Optimal variable funding and asset allocation for different values of parameter Alpha1 (two-period model)

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Note:
1. $\alpha_1 = 1.5$

Figure 7.6. Optimal variable funding and asset allocation for different values of parameter Alpha2 (two-period model)

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