with
$$\begin{pmatrix} a_1', & b_1', & c_1' \\ a_2', & b_2', & c_2' \\ a_3', & b_3', & c_3' \end{pmatrix} = c_1^{-1} \begin{pmatrix} ib_1, & -ia_1, & 1 \\ a_2, & b_2, & ic_3 \\ a_3, & b_3, & -ic_2 \end{pmatrix}$$

leading to equations of the form of (I) above. Then the ξ, η, ζ, τ are replaced by

$$\xi' = \xi + i\eta + \zeta - i\tau, \quad \eta' = -i(\xi + i\eta - \zeta + i\tau),$$
$$\zeta' = \xi - i\eta + \zeta + i\tau, \quad \tau' = i(\xi - i\eta - \zeta - i\tau)$$

which change the ten quadrics among themselves, leaving four of them unaltered.

And to the ten ways so obtained can be added ten others by the remark that such a set of four tetrahedra is obtained by transposition of the rows and columns of the two matrices which stand on the left in these equations, with the change of $\xi', \eta', \zeta', \tau'$ into $\xi', \eta', \zeta', -\tau'$.

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POINT-CURVE CORRESPONDENCES

II. INDUCED AND EXTENDED CORRESPONDENCES

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In this paper we consider the properties of a point-curve correspondence between two (generally distinct) surfaces F and G, paying special attention to the idea of 'induced' and 'extended' correspondences. We also outline the theory of point-curve correspondences between a curve and a surface, which is not only of interest in itself, but is of importance because such correspondences arise, as do correspondences between curves, as induced correspondences of point-curve correspondences between two surfaces. The results depend chiefly on Lefschetz's theory of correspondences between algebraic curves (1) and the properties of intersections of cycles on an algebraic surface given by the author (2), as well as on the preceding paper of this series (3). Use is also made of a paper by Severi (4) on the self-correspondences on a variable curve of an algebraic surface. The theory of correspondences on a single surface will be further developed in the next paper on this subject.

This work will be regarded throughout as a continuation of (3), and the sections will be numbered accordingly. Thus references in this paper to the results of §§ 1–3 will be understood to refer to (3). We shall also adopt the notational conventions of (2) and (3). In particular, we shall denote by $\hat{\mathbf{a}}$ the complementary matrix of $\mathbf{a}$, i.e. the matrix such that $\hat{\mathbf{a}}\mathbf{a}' = 1$ and shall continue to use the convention which requires summation over every repeated *lower* suffix.

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According to the formula (1·2′) a primal W on the product of two varieties U and V is given by a homology of the form

$$W \approx U \times Y + \xi_{ij} \Gamma_i^{2r-1} \times \Delta_j^{2s-1} + X \times U,$$

and we have pointed out that the point-primal correspondence represented by W is of valency zero if $\boldsymbol{\xi} = \mathbf{0}$. If $\boldsymbol{\xi} \neq \mathbf{0}$ we shall say that the variety W, or the correspondence it represents, is singular‡, and $\boldsymbol{\xi}$ will be referred to as the matrix of W, or of the correspondence represented by W. We also find it convenient to speak not only of a base for the point-primal correspondences between U and V (or for the primals of $U \times V$), but also of a *base for the singular correspondences* (*or singular primals*). This latter is defined to be any set of independent singular correspondences such that every singular correspondence is (weakly) homologous to a sum of multiples of members of the base and, possibly, a correspondence of valency zero.

4. Proper curves of F and G

As in (3), we take on our two surfaces F and G respectively, bases $\gamma_1, \ldots, \gamma_{2p}$ and $\delta_1, \ldots, \delta_{2q}$ for the 1-cycles, together with bases $\Gamma_1, \ldots, \Gamma_{2p}$ and $\Delta_1, \ldots, \Delta_{2q}$ for the 3-cycles, such that

$$(\gamma_i . \Gamma_j) = \delta_{ij}, \quad (\delta_k . \Delta_l) = \delta_{kl},$$

where δ_{ij} is the Kronecker delta. We also take bases $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$ for the simple integrals of the first kind of the surfaces, so that, with this choice of bases for cycles and integrals, the period matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ are reduced to the canonical form (2·1).

C^* and D^*, of respective genera p^* and q^*, are any two proper curves (cf. (2), p. 415) of F and G. Then ((2), p. 416) we may take a base $\gamma_1^*, \ldots, \gamma_{2p^*}^*$ for the 1-cycles of C^* such that

$$\gamma_i^* \approx \gamma_i \text{ on } F \quad (1 \leqslant i \leqslant 2p), \qquad \gamma_i^* \approx 0 \text{ on } F \quad (2p+1 \leqslant i \leqslant 2p^*). \tag{4·1}$$

We can also take as a base for the integrals of the first kind on C^* the set $u_1, \ldots, u_p$, $u_{p+1}^*, \ldots, u_{p^*}^*$, whose period matrix over the cycles γ_i^* is

$$\boldsymbol{\Omega}^* = \begin{pmatrix} \boldsymbol{\Omega} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}^* \end{pmatrix}. \tag{4·2}$$

Further, if $\mathbf{a}^*$ is the a-matrix of C^* (i.e. $\Gamma_i . C^* \approx a_{ij}^* \gamma_j$ on F), the intersection matrix $\mathbf{b}^*$ of the cycles γ_i^* ($b_{ij}^* = (\gamma_i^* . \gamma_j^*)$) is of the form

$$\mathbf{b}^* = \begin{pmatrix} \mathbf{b}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_2^* \end{pmatrix}, \quad \text{where} \quad \mathbf{b}_1^* = \widehat{\mathbf{a}^*}. \tag{4·3}$$

Similarly, we can define on D^* the cycles $\delta_1^*, \ldots, \delta_{2q^*}^*$, the integrals $v_{q+1}^*, \ldots, v_{q^*}^*$, the period matrices $\boldsymbol{\Psi}^*$ and $\boldsymbol{\psi}^*$, and the intersection matrices $\boldsymbol{\beta}^*$, $\boldsymbol{\beta}_1^*$ ($= \widehat{\boldsymbol{\alpha}^*}$), and $\boldsymbol{\beta}_2^*$.

5. Correspondences induced between C^* and D^*

If U and V are any two varieties and U^* and V^* are subvarieties, any variety W on $U \times V$ determines a correspondence T between U and V, and its intersection W^* with the subvariety $U^* \times V^*$ of $U \times V$ determines a correspondence T^* between U^* and V^*

‡ There is a slight divergence here from normal practice, as on this definition the identical correspondence between a curve and itself is singular.

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which we call the correspondence induced by T. To a point x^* of U^* corresponds the locus $T^*(x^*)$ of V^* consisting of that part of $T(x^*)$ which lies on V^*. It follows that if U^* and V^* are in general position with respect to W, in which case the intersection of W and $U^* \times V^*$ is homologous, on $U \times V$, to $W.U^* \times V^*$, then for a cycle Γ^*, of any dimension, on U^* we shall have

$$T^*(\Gamma^*) \approx T(\Gamma^*).V^* \quad \text{on } V. \tag{5·1}$$

If T is a point-primal correspondence, W is a primal of $U \times V$ and hence either W contains $U^* \times V^*$ or W^* is a primal of it. Thus the irreducible induced correspondences of a point-primal correspondence are either trivial, degenerate or point-primal correspondences, the first possibility arising if W contains $U^* \times V^*$, and one of the other two if it does not. Further, the irreducible non-trivial and non-degenerate correspondences of a point-primal correspondence of valency zero are also of valency zero. For in this case, if x_1^* and x_2^* are any two points of U^*, $T^*(x_1^*)$ and $T^*(x_2^*)$ are primals of V^* (for if one of them were the whole of V^*, T^* would either be degenerate or reducible) and they must be linearly equivalent on V^* for they are the intersections with V^* of the two varieties $T(x_1^*)$ and $T(x_2^*)$ which, since T is of valency zero, are linearly equivalent on V.

Let now T be a point-curve correspondence represented on $F \times G$ by a variety W, where, as in (3·1),

$$W \approx X \times G + F \times Y + \xi_{ij} \Gamma_i \times \Delta_j. \tag{5·2}$$

Let T^* be the correspondence between C^* and D^* represented by the irreducible curve W^* which is the intersection of W and $C^* \times D^*$. Then, as in (3·3),

$$W^* \approx (X.C^*)\, x \times D^* + (Y.D^*)\, C^* \times y + \xi_{ij}^* \gamma_i \times \delta_j \quad \text{on } F \times G$$
$$(i = 1, \ldots, 2p;\ j = 1, \ldots, 2q), \tag{5·3}$$

where

$$\boldsymbol{\xi}^* = (\mathbf{a}^*)' \boldsymbol{\xi} \boldsymbol{\alpha}^* = -\mathbf{a}^* \boldsymbol{\xi} \boldsymbol{\alpha}^*. \tag{5·4}$$

But W^* is also a cycle of $C^* \times D^*$ and we must therefore have a homology of the form

$$W^* \approx mx^* \times D^* + nC^* \times y + \xi_{ij}^* \gamma_i^* \times \delta_j^* \quad \text{on } C^* \times D^*$$
$$(i = 1, \ldots, 2p^*;\ j = 1, \ldots, 2q^*), \tag{5·5}$$

x^* and y^* being generic points of C^* and D^*.

Let

$$\boldsymbol{\zeta}^* = \begin{pmatrix} \boldsymbol{\zeta}^1 & \boldsymbol{\zeta}^2 \\ \boldsymbol{\zeta}^3 & \boldsymbol{\zeta}^4 \end{pmatrix},$$

where $\boldsymbol{\zeta}^1$ is a $(2p, 2q)$-matrix, and let

$$T^*(\gamma_i^*) \approx \epsilon_{ij}^* \delta^* \quad \text{on } D^*, \quad (T^*)^{-1}(\delta_j^*) \approx \eta_{ji}^* \gamma_i^* \quad \text{on } C^*$$
$$(i = 1, \ldots, 2p^*;\ j = 1, \ldots, 2q^*).$$

Then by Lefschetz's formulae ((1), p. 349)

$$\boldsymbol{\epsilon}^* = -\mathbf{b}^* \boldsymbol{\zeta}^*, \quad \boldsymbol{\eta}^* = \boldsymbol{\beta}^* (\boldsymbol{\zeta}^*)'; \tag{5·6}$$

i.e.

$$\boldsymbol{\epsilon}^* = -\begin{pmatrix} \mathbf{b}_1^* & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_2^* \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta}^1 & \boldsymbol{\zeta}^2 \\ \boldsymbol{\zeta}^3 & \boldsymbol{\zeta}^4 \end{pmatrix} = -\begin{pmatrix} \mathbf{b}_1^* \boldsymbol{\zeta}^1 & \mathbf{b}_1^* \boldsymbol{\zeta}^2 \\ \mathbf{b}_2^* \boldsymbol{\zeta}^3 & \mathbf{b}_2^* \boldsymbol{\zeta}^4 \end{pmatrix}, \tag{5·7}$$

with a similar expression for $\boldsymbol{\eta}^*$.

Every cycle of C^* is transformed into an invariant cycle of D^*, as the transformed cycle depends only on C^* and D^* and must therefore not be altered if we circulate D^* in its pencil *while keeping C^* fixed*. (This last proviso is vital. The results obtained here

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can be carried over to the case where G is a copy of F and D^* a copy of C^*, but only as a special case of the correspondence between two distinct curves C^* and D^* belonging to the linear system $|C^*|$.) Since every $T^*(\gamma_i^*)$ is an invariant cycle we must now have $\epsilon_{ij}^* = 0$ for $2q+1 \leqslant j \leqslant 2q^*$, and since $\mathbf{b}^*$, and hence also $\mathbf{b}_1^*$ and $\mathbf{b}_2^*$, is non-singular, it follows from (5·7) that $\boldsymbol{\zeta}^2 = \mathbf{0}$ and $\boldsymbol{\zeta}^4 = \mathbf{0}$. Similarly, by considering the reverse correspondence, $\boldsymbol{\zeta}^3 = \mathbf{0}$. Hence we have

$$\boldsymbol{\zeta}^* = \begin{pmatrix} \boldsymbol{\zeta}^1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{5·8}$$

$\boldsymbol{\zeta}^1$ being a $(2p, 2q)$-matrix.

If we substitute this value of $\boldsymbol{\zeta}^*$ in (5·5) we obtain

$$W^* \approx mx^* \times D^* + nC^* \times y^* + \zeta_{ij}^1 \gamma_i^* \times \delta_j^* \quad \text{on } C^* \times D^*,$$

where now $i = 1, \ldots, 2p$ and $j = 1, \ldots, 2q$ only. Combining this expression with (4·1) we obtain

$$W^* \approx mx \times D^* + nC^* \times y + \zeta_{ij}^1 \gamma_i \times \delta_j \quad \text{on } F \times G,$$

and comparing this with (5·3) we find that

$$m = (X.C^*), \quad n = (Y.D^*), \quad \boldsymbol{\zeta}^1 = \boldsymbol{\xi}^*. \tag{5·9}$$

Thus T^* is an (m, n) correspondence represented by a curve W^* which satisfies the homologies

$$\left.\begin{aligned} W^* &\approx mx \times D^* + nC^* \times y + \xi_{ij}^* \gamma_i \times \delta_j \quad \text{on } F \times G, \\ W^* &\approx mx^* \times D^* + nC^* \times y^* + \xi_{ij}^* \gamma_i^* \times \delta_j^* \quad \text{on } C^* \times D^*, \end{aligned}\right\} \tag{5·10}$$

where in each case $i = 1, \ldots, 2p$; $j = 1, \ldots, 2q$. Applying now the Lefschetz formulae quoted above, (5·6), we have

$$\boldsymbol{\epsilon}^* = -\begin{pmatrix} \mathbf{b}_1^* \boldsymbol{\xi}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\eta}^* = \begin{pmatrix} \boldsymbol{\beta}^* (\boldsymbol{\xi}^*)' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

With the aid of (4·3) and (5·4) these reduce to

$$\boldsymbol{\epsilon}^* = \begin{pmatrix} -\boldsymbol{\xi}\boldsymbol{\alpha}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\eta}^* = \begin{pmatrix} \boldsymbol{\xi}' \mathbf{a}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{5·11}$$

6. Transformations of cycles in a point-curve correspondence between two surfaces

We can use the formulae (5·11) to provide an alternative derivation of the formulae (1·8) for surfaces. For any invariant cycle γ_i^* of C^* we have, from (5·1),

$$T^*(\gamma_i^*) \approx T(\gamma_i^*).D^* \quad \text{on } G. \tag{6·1}$$

But by (5·11)

$$\begin{aligned} T^*(\gamma_i^*) &\approx -\xi_{ij} \alpha_{jk}^* \delta_k^* \quad \text{on } D^* \quad (i = 1, \ldots, 2p;\ j, k = 1, \ldots, 2q) \\ &\approx -\xi_{ij} \alpha_{jk}^* \delta_k \quad \text{on } G \quad (i, j, k \text{ as above}). \end{aligned} \tag{6·2}$$

Also as $\gamma_i \approx \gamma_i^*$ on F (4·1)

$$T(\gamma_i^*) \approx T(\gamma_i) \approx \epsilon_{ij} \Delta_j \quad \text{on } G, \tag{6·3}$$

the matrix $\boldsymbol{\epsilon}$ being defined, as in (1·3′), by the second of these equations. Substituting (6·2) and (6·3) in (6·1), and recalling the definition of $\boldsymbol{\alpha}^*$ ($\Delta_j.D^* \approx \alpha_{jk}^* \delta_k$ on G), we have

$$-\xi_{ij} \alpha_{jk}^* \delta_k \approx \epsilon_{ij} \Delta_j.D^* \approx \epsilon_{ij} \alpha_{jk}^* \delta_k^*.$$

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But α^* is non-singular ((2), p. 418), so that

$$\epsilon = -\xi,$$

and similarly, by considering the reverse correspondence, we should have

$$\eta = \xi'.$$

7. Extended correspondences

In § 5 we have considered the idea of the correspondence T^* induced between the subvarieties U^* and V^* by a correspondence T between U and V. This leads us naturally to the idea of an *extended correspondence* of a correspondence S^* between U^* and V^*, which it would be natural to define as a correspondence T, if such exists, such that S^* is its induced correspondence T^*. This definition is, however, rather restrictive, as in this kind of work we are concerned not so much with a particular correspondence, as with classes of homologous correspondences, so that it would be equally natural to consider the idea of a *weak extension* of S^*, which we define as a correspondence T such that T^* is homologous to S^* *as a correspondence between* U^* *and* V^*. (This means that S^* and T^* must be represented by cycles homologous not merely on $U \times V$ but also on $U^* \times V^*$.) We shall also introduce the idea of a *fractional extension*, defined as a correspondence T between U and V such that T^* plus a suitable correspondence of valency zero between U^* and V^* is homologous on $U^* \times V^*$ to a multiple of S^*. This simply means that the matrix of T^* is a multiple of the matrix of S^*, regarding each as a cycle of $C^* \times D^*$.

It is important to note that, although every correspondence T between U and V induces a correspondence (which may, of course, be trivial or degenerate) between U^* and V^*, it is not necessary that every correspondence between U^* and V^* have an extension, or that an extension, if it does exist, be unique. To take a simple example, if U and V are both copies of the same surface F, and U^* and V^* are both copies of the same proper curve C, of genus greater than the irregularity of F, then the identical correspondence I_C on C has not even a fractional extension to a point-curve correspondence between F and itself. This is because I_C is not an 'invariant correspondence' in the sense we shall define below. But the result is intuitively plausible since the identical correspondence on F is a natural extension of I_C and this is a point-point, and not a point-curve, correspondence. We shall, however, show that any correspondence between two sufficiently general curves (in a sense we shall explain later) of two (possibly identical) surfaces has at least a fractional extension. If U^* and V^* are 'proper' primals of U and V, and a point-primal correspondence S^* between them has a strict extension T, it means that, as we vary U^* and V^* independently in pencils, we can deform S^* continuously so that the locus of the variety which represents it is the primal of $U \times V$ which represents the correspondence T.

Let us now consider an (α, β) correspondence S^* between proper curves C^* and D^* of F and G respectively, represented on $C^* \times D^*$ by a curve Z^* where, as in (5·5),

$$Z^* \approx \alpha x^* \times D^* + \beta C^* \times y^* + \zeta^*_{ij}\, \gamma^*_i \times \delta^*_j \quad \text{on } C^* \times D^* \tag{7·1}$$

and

$$\zeta^* = \begin{pmatrix} \zeta^1 & \zeta^2 \\ \zeta^3 & \zeta^4 \end{pmatrix}, \quad (\text{where } \zeta^1 \text{ is a } (2p, 2q) \text{ matrix}), \tag{7·2}$$

is a mutual multiplier of Ω^* and Ψ^*.

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It follows from the results of § 5 that if S^* has an extension of any kind we must have $\zeta^2 = 0$, $\zeta^3 = 0$, $\zeta^4 = 0$, so that in this case

$$\zeta^* = \begin{pmatrix} \zeta^1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{7·3}$$

where ζ^1 is a mutual multiplier of Ω and Ψ.

A correspondence between C^* and D^* whose matrix is of the form (7·3) is represented by a cycle of the form (7·1) in which only the invariant cycles of C^* and D^* are involved. Such a correspondence we shall refer to in future as an *invariant correspondence*. The class of invariant correspondences is not necessarily the same as the class of correspondences represented by invariant cycles of $C^* \times D^*$, but is possibly less extensive. For instance, if F and G are copies of the same surface, and D^* is a copy of C^*, the identical correspondence on C^* (which is represented by an invariant cycle) is not, as we have already remarked, invariant in our sense if the genus of C^* is greater than the irregularity of F. We have already shown, (5·8), that every induced correspondence is invariant. It follows that no non-invariant correspondence S^* can have even a fractional extension. For if T is such a correspondence it induces, by definition, a correspondence T^* whose matrix (regarding T^* as a cycle of $C^* \times D^*$) is a multiple of the matrix of S^*, so that T^* cannot be invariant. We can, however, go further than this, and we shall prove that *every invariant correspondence has (at least) a fractional extension*.

Consider an arbitrary invariant correspondence S^* with matrix ζ^* given by (7·3), so that ζ^1 is a mutual multiplier of Ω and Ψ. We have proved in § 2 that $\zeta = -(\mathfrak{a}^*)^{-1}\zeta^1(\alpha^*)^{-1}$ must therefore be a mutual anti-multiplier of Ω and Ψ, so that, by the results of § 3, there will be an integer μ such that $\xi = \mu\zeta$ is the matrix of a (possibly virtual) algebraic threefold Z on $F \times G$. Then for a sufficiently large integer κ the linear system $|\kappa K + Z|$ (K being the 'prime section' of $F \times G$ introduced in § 3) will contain an effective variety W, and because K is non-singular W will have the same matrix as Z. Hence, if T is the correspondence represented by W, its induced correspondence T^* will, by (5·10) and (5·4), have the matrix

$$\begin{pmatrix} -\mathfrak{a}^*\xi\alpha^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mu\zeta^1 & 0 \\ 0 & 0 \end{pmatrix},$$

so that T is a fractional extension of S^*. Put in a slightly different form this means that, given any matrix ζ^* of the form (7·3), there is a correspondence, with a multiple of ζ^* as its matrix, which has a strict extension. Thus we have shown that, given any base for the singular invariant correspondences between C^* and D^*, we can, by replacing each curve of the base by a new curve whose matrix is a multiple of the matrix of the original curve, construct a new base‡ such that every member of the second base has a strict extension. If ζ^*, given by (7·3), is the matrix of one of these correspondences, the matrix of its extension is $-(\mathfrak{a}^*)^{-1}\zeta^1(\alpha^*)^{-1}$, and as the matrices ζ^1, of correspondences forming a base for the invariant correspondences, form a base for the mutual multipliers of Ω and Ψ, it follows, from Lemma 2·3, that the matrices of the extended correspondences form a base for the mutual anti-multipliers of Ω and Ψ, and hence

‡ This base need by no means be fundamental. Cf. (2), p. 415.

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from the results of § 3, that these extended correspondences form a base for the singular point-curve correspondences between F and G. We have thus

THEOREM 7·1. *There is a base for the singular invariant correspondences between C^* and D^* such that each member of the base has an extension to a point-curve correspondence between F and G. These extended correspondences form a base for the singular point-curve correspondences between F and G.*

This result can be extended further, because sufficiently general curves C^* and D^* have only invariant correspondences between them. This follows from a result of Severi ((4), p. 544), which is equivalent to the assertion that if C^* is a generic curve of a linear system, which is at least ∞^2, then the matrix $\boldsymbol{\omega}^*$ of (4·2) is pure and without complex multiplication. If then C^* and D^* each belong to a system of this type, it is impossible for the matrices $\boldsymbol{\omega}^*$ and $\boldsymbol{\psi}^*$ to be related unless the genera p^* and q^*, of C^* and D^* respectively, are such that p^*-p is equal to q^*-q, because two related pure matrices are isomorphic. Further, $\boldsymbol{\omega}^*$ cannot be related to $\boldsymbol{\Psi}$ unless p^*-p is equal to the genus of some pure sub-matrix of $\boldsymbol{\Psi}$, and $\boldsymbol{\psi}^*$ cannot be related to $\boldsymbol{\Omega}$ unless q^*-q is equal to the genus of some pure sub-matrix of $\boldsymbol{\Omega}$. Of course even if one of these equalities holds there need be no relationship, but what we are concerned with is pointing out that 'in general' these matrices are not related. Thus, in general, any matrix $\boldsymbol{\zeta}^*$ which is a mutual multiplier of $\boldsymbol{\Omega}^*$ and $\boldsymbol{\Psi}^*$ must be of the form (7·3), so that every correspondence between C^* and D^* is invariant. Hence, if C^* and D^* are a pair of sufficiently general curves (in the above sense), Theorem 7·1 can be strengthened by the omission of the word 'invariant'.

The results for singular correspondences are, in one way, simpler than those for non-singular ones, for there are only two independent non-singular correspondences between C^* and D^*, whereas there are $\rho+\sigma$ non-singular point-curve correspondences between F and G, ρ and σ being the respective base-numbers for the algebraic curves. So that the extensions of a base for the non-singular correspondences between C^* and D^* (if these extensions exist) cannot possibly, in general, form a base for the non-singular correspondences between F and G. It is, however, easy to find two non-singular correspondences between F and G which induce independent correspondences between C^* and D^*. Adding these latter to the base for the singular correspondences between C^* and D^* we obtain

THEOREM 7·2. *There exists a base for the correspondences between two general curves (in the sense defined above) C^* and D^* of F and G respectively, such that each member of the base can be extended to a point-curve correspondence between F and G. These extended correspondences, together with a further $\rho+\sigma-2$ correspondences of valency zero, form a base for the point-curve correspondences between F and G.*

8. CORRESPONDENCES BETWEEN A CURVE AND A SURFACE

So far the only induced correspondences we have discussed are those between two curves. But we can easily consider the correspondence between one surface and a curve of the other induced by a point-curve correspondence between the surfaces. It is

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therefore of interest to consider the theory of point-primal correspondences between a curve and a surface, which we shall also call point-curve correspondences, and which are the only type of non-trivial and non-degenerate correspondences possible.

Let C be a curve of genus p and G a surface. Then by a suitable choice of bases for the 1-cycles $g_1, \ldots, g_{2p}$ of C and $\delta_1, \ldots, \delta_{2q}$ of G, as well as of the simple integrals of the first kind $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$, we can take the period matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ in the form (2·1). A correspondence T between C and G in which a point of C corresponds to a curve of the continuous system $\{Y\}$ of index m (i.e. m distinct curves of $\{Y\}$ pass through a generic point) on G, and a point of G therefore corresponds to a set of m points of C, will be represented by a surface W satisfying the homology

$$W \approx mx \times G + C \times Y + \xi_{ij} g_i \times \Delta_j \quad \text{on } C \times G \quad (i = 1, \ldots, 2p;\ j = 1, \ldots, 2q). \quad (8{\cdot}1)$$

We shall state and outline the proofs of the properties of such correspondences which parallel the results of § 3 for correspondences between two surfaces. They are summed up, as before, in three theorems.

THEOREM 8·1. *A correspondence represented by a cycle W of the form* (8·1) *is algebraic if and only if $W \,.\, C \times D^*$ is algebraic as a cycle of $C \times G$, where D^* is any proper curve of G.*

The proof of this is derived from the simpler

THEOREM 8·2. *If D^* and $D^\dagger$ are any two proper curves of G, then $W \,.\, C \times D^*$ is algebraic if and only if $W \,.\, C \times D^\dagger$ is algebraic.*

This is exactly analogous to Theorem 3·2 and depends on the property that $W \,.\, C \times D^*$ is algebraic if, and only if, $\boldsymbol{\xi\alpha}^*$ is a mutual multiplier of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$.

We deduced Theorem 3·1 from Theorem 3·2 ((3), p. 143) with the aid of a system of 'prime sections'. If p is a set of points cut off by a prime section of C and Q is a prime section of G, we can carry out the same procedure as in § 3 with the system of 'prime sections' $K \equiv |\, p \times G + C \times Q \,|$ of $C \times G$, thus deducing Theorem 8·1 from Theorem 8·2. These theorems imply that W is algebraic if and only if $\boldsymbol{\xi\alpha}$ is a mutual multiplier of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$, for some non-singular multiplier $\boldsymbol{\alpha}$ of $\boldsymbol{\Psi}$. (It is a consequence of Lemma 2·2 that if $\boldsymbol{\xi\alpha}$ is a mutual multiplier, so is $\boldsymbol{\xi\alpha}_1$ where $\boldsymbol{\alpha}_1$ is any other non-singular multiplier of $\boldsymbol{\Psi}$.) The condition just stated ensures that there are exactly λ independent matrices $\boldsymbol{\xi}$ (λ being as usual the simultaneity index of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$) giving rise to singular algebraic correspondences. These matrices are obtained by replacing every $\mathbf{c}_{\beta\gamma}^{\alpha}$ of (2·4) by an element (possibly zero) of the division algebra $\mathfrak{D}_\alpha$ of rational $(2p_\alpha, 2p_\alpha)$ matrices associated with the multiplications of $\boldsymbol{\sigma}_\alpha$. The conditions imposed on $\boldsymbol{\xi}$ are equivalent to the existence of a relation

$$\boldsymbol{\Omega\xi} = \mathbf{X\Psi}. \quad (8{\cdot}2)$$

The sufficiency of the existence of such a relation is obvious if we multiply each side of (8·2) by $\boldsymbol{\alpha\Psi}'$. Its necessity, which can also be proved by algebraic considerations, is easily demonstrated by considering the integrals of C and G. For, if $x_1, \ldots, x_m$ are the points of C corresponding to the point y of G, we must have a relation of the form

$$u_i(x_1) + u_i(x_2) + \ldots + u_i(x_m) = X_{ij} v_j(y) + X_i \quad (i = 1, \ldots, p;\ j = 1, \ldots, q). \quad (8{\cdot}3)$$

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Now, as y describes the cycle δ_k, the points $x_1, \ldots, x_m$ describe (in the notation of § 1) the cycle $\eta_{kl} g_l$. So that integrating the v_j round the cycle δ_k we have the relation

$$\eta_{kl} \Omega_{il} = X_{ij} \Psi_{jk},$$

or

$$\boldsymbol{\Omega \eta'} = \mathbf{X \Psi}.$$

But by (1·7) $\boldsymbol{\eta} = \boldsymbol{\xi}'$, so that the necessity of (8·2) is established, and the matrix $\mathbf{X}$ occurring in it satisfies (8·3):

We thus have the following analogue of Theorem 3·3:

THEOREM 8·3. *The correspondence given by* (8·1) *is algebraic if and only if the matrix* $\boldsymbol{\xi}$ *satisfies an equation of the form* (8·2). *The number of independent correspondences between* C *and* G *is* $\lambda + \sigma + 1$, *where* λ *is the simultaneity index of the period matrices of the simple integrals of the first kind of* C *and* G, *and* σ *is the base number for the algebraic curves of* G.

It is worth noting that the proof that $\boldsymbol{\xi}$ necessarily satisfies a relation of the form (8·2) can be carried over to a correspondence between a curve and any algebraic variety of dimensions greater than one. The proof of the sufficiency is only possible for a surface, because for varieties of higher dimensions we have, as yet, no analogue to the results of (2) on the intersections of the 3-cycles and algebraic 2-cycles of a surface.

9. THE GENERAL NOTION OF INDUCED, EXTENDED AND INVARIANT CORRESPONDENCES FOR TWO SURFACES

Given a pair of surfaces F and G we can consider induced correspondences of three distinct types. *Type one* are the correspondences induced between a pair of curves C and D by a point-curve correspondence between F and G. *Type two* are the point-curve correspondences between C and G (or between F and D) induced by the point-curve correspondences between F and G. *Type three* are the correspondences induced between C and D by a point-curve correspondence between C and G (or between F and D). If this latter correspondence, in its turn, is an induced correspondence of type two, the correspondence between C and D is also of type one. The type of such a correspondence is thus determined by our point of view and not by any intrinsic property. We may also consider correspondences of type three without any assumption as to the existence of a surface on which C lies. (In fact we can carry out for these correspondences an analysis exactly parallel to the discussion of §§ 5 and 6. The methods used are exactly the same as before, and the results seem of no very great interest.)

With each type of induced correspondence we may consider corresponding types of extended correspondences, just as we have already done for type one in § 7. The relationship between induced and extended correspondences of types two and three is exactly parallel to what we have found for type one, the key concept again being that of an invariant correspondence. We find, however, that we shall wish to regard the same correspondence as invariant in one context and not in another, so that we shall have to modify the definition. We shall accordingly say that a correspondence S, represented by a curve Z, between a pair of curves C and D of F and G respectively, is F-invariant if C is a proper curve of F and the homology for Z, in terms of a base for $C \times D$, does not

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involve the vanishing cycles of C. G-invariance is defined similarly, and if a correspondence is both F- and G-invariant we shall call it (F, G)-invariant. The invariance we have considered so far is of this last kind. In the notation of § 7 the correspondence S^* of (7·1) between the proper curves C^* and D^*, is F-invariant if $\zeta^2 = 0$, $\zeta^4 = 0$, G-invariant if $\zeta^3 = 0$, $\zeta^4 = 0$ and (F, G)-invariant if $\zeta^2 = 0$, $\zeta^3 = 0$, $\zeta^4 = 0$. We shall also say that a point-curve correspondence between C and G is F-invariant if C is a proper curve of F and the surface representing the correspondence is expressed, in terms of the base for the cycles of $C \times G$, by a homology not involving the vanishing cycles of C. The theorems of § 7 extend immediately to correspondences of types two and three; the proofs follow the lines of the argument given previously. The main results for type three are the following:

THEOREM 9·1. *There is a base for the singular G-invariant correspondences between C and D^* such that each member of the base has an extension to a point-curve correspondence between C and G. These extended correspondences form a base for the singular point-curve correspondences between C and G.*

THEOREM 9·2. *There exists a base for the correspondences between C and a sufficiently general proper curve D^* of G, such that each member of the base has an extension to a point-curve correspondence between C and G. These extended correspondences, together with a further $\sigma - 1$ correspondences of valency zero, form a base for the point-curve correspondences between C and G.*

Similar results can be obtained for the extensions of type two correspondences between C and G using this time the concept of F-invariance.

If D^* is a sufficiently general curve of G, every correspondence between C and D^* is G-invariant and has a (fractional) extension to a point-curve correspondence between C and G (cf. Theorem 9·2). This extension in its turn induces a correspondence between C and any curve D of G. Even if C is a proper curve of F, it may not be a 'general' curve in the sense we require, so that there may be a singular correspondence between C and D^* which is not F-invariant, and therefore cannot be extended to a correspondence between F and G. The same must also be true of the correspondence between C and D induced by the extension of our correspondence to one between C and G. It follows, therefore, that if there is a correspondence between C and a general curve D^* of G which is not extensible to a point-curve correspondence between F and G, there is also such a correspondence between C and every curve of G. Further, if we take D to coincide with D^* we have a correspondence between C and D^* (whose matrix is a multiple of the correspondence we started with) which is strictly extensible to a correspondence between C and G, but not extensible to one between F and G.

It is also worth noticing that the type one extension of an (F, G)-invariant correspondence between C^* and D^* to a correspondence between F and G can be carried out in two stages (and in two ways). First an extension of type three and then one of type two. One of these extensions requires F-invariance and the other G-invariance.

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THE LARGE-SAMPLE THEORY OF SEQUENTIAL TESTS

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1. The general theory underlying sampling inspection methods introduced during the war under the name of sequential analysis, applicable in cases where the sampling units can be taken serially, has mainly been developed by Wald and is now published (5). The original purpose of the present investigation was to exhibit the main structure of the distributional theory relating to the size of sample required by noting its relationship with the classical 'random-walk' problem. Thus in Part I of this paper are derived the distribution and characteristic functions of the absorption time for one-dimensional random-walk theory with constant drift and absorbing barriers. Attention is confined to the asymptotic case of numerous independent displacements, for which it is well known that the unrestricted total displacement becomes Gaussian; the motion may also be treated as continuous. Since these results were obtained Wald has independently given an alternative and more general discussion of this problem ((4); see also Tweedie (3)), but the extension here of the method of images used by Chandrasekhar (1) still appears of interest, and has advantages over alternative direct methods of solution for the distribution function involving Fourier expansion (cf. 2).

In Part II these results are applied to sequential analysis, and in particular is shown the relation of this large-sample theory to Fisher's information function.

PART I

One-dimensional random-walk theory with constant drift and absorbing barriers

2. For one-dimensional diffusion from a point source with a constant drift in the random displacement X at rate m (>0) per unit time, and a growth in variance at rate σ^2, the differential equation for the cumulant function $K\{iu\} \equiv \log E\{e^{ixu}\}$ (where E denotes average value) will be under the conditions noted in § 1 above

$$\frac{\partial K}{\partial t} = miu - \tfrac{1}{2}\sigma^2 u^2, \tag{1}$$

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