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Parameter rigid actions of the Heisenberg groups

NATHAN M. DOS SANTOS†

Instituto de Matemática, Universidade Federal Fluminense, 24020-005, Niterói, RJ, Brazil

(e-mail: santosnathan@bol.com.br)

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Abstract. A locally free action of a Lie group is parameter rigid if for each other action with the same orbit foliation there exists a C^∞ orbit-preserving diffeomorphism which conjugates the action to a reparametrization of the other by an automorphism of the Lie group. We show that for actions of the Heisenberg groups, if the first leafwise cohomology group of the orbit foliation is isomorphic to the first cohomology of the Lie algebra of the group, then the action is parameter rigid. Using this, we give examples of parameter rigid actions for all of the Heisenberg groups.

0. *Introduction*

Let G be a contractible Lie group and A be a right locally free (LF) action of G on a closed, connected manifold M. All of the objects we consider here are C^∞. The action A gives a G-module structure on the ring $C^\infty(M)$ of the smooth real functions on M, denoted by $C^\infty_A(M)$ and defined by $g \cdot h = h \circ A_g$ for $g \in G$ and $h \in C^\infty(M)$. The *cohomology of* A is by definition the cohomology $H^*(G, C^\infty_A(M))$ of the G-module $C^\infty_A(M)$. Let $\mathcal{F}$ be the foliation given by the orbits of A and $H^1(\mathcal{F})$ be the first leafwise cohomology group of the foliated manifold $(M, \mathcal{F})$ [**1**]. We prove in §1 (Proposition 1.2) that the first cohomology of A is isomorphic to $H^1(\mathcal{F})$.

We say that the action A is *cohomologically rigid in dimension 1* (CR1) if the first cohomology of A is isomorphic to the first cohomology $H^1(\mathcal{G}^*)$ of the Lie algebra $\mathcal{G}$ of G.

The action A is *parameter rigid* (PR) if given any other G-action B on M with the same orbit foliation $\mathcal{F}$, there exist an automorphism Φ of G and a leaf-preserving diffeomorphism F of M isotopic to the identity through leaf-preserving diffeomorphisms such that

$$F(A(x, g)) = B(F(x), \Phi(g)) \quad \text{for } x \in M \text{ and } g \in G.$$

† Correspondence address: Rua Lopes Quintas, 255 ap. 401-A, Jardim Botânico, Rio de Janeiro, CEP 22460-010, Brazil.

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For $\mathbb{R}^n$-actions, CR1 and PR are equivalent properties [**8**]. We show in §1 (Proposition 1.3) that a CR1 G-action A leaves invariant a smooth volume form Ω, i.e.

$$A_g^*\Omega = (\deg A_g)\Omega \quad \text{for all } g \in G,$$

where $\deg A_g$ is the degree of A_g.

In §§2 and 3, we study actions of the Heisenberg groups H_n (see [**4**]). We give a sufficient cohomological condition for a LF action A of H_n to be PR.

THEOREM 2.4. *Let A be a smooth LF action of the Heisenberg group H_n, $n \geq 1$ on a closed manifold M. Suppose that A is CR1. Then A is PR.*

Using this, we show in Theorem 3.2 the existence of PR actions of the Heisenberg groups H_n for all $n \geq 1$.

1. *Cohomology and locally free actions of Lie groups*

Let G be a contractible Lie group and M be a closed, orientable, connected m-manifold and A be a right LF smooth action of G on M. The action A gives a homomorphism of G into the group of automorphisms of the ring $C^\infty(M)$ of the smooth functions on M given by $\gamma \cdot f = f \circ A_\gamma$ for $\gamma \in G$ and $f \in C^\infty(M)$, defining a G-module structure $C_A^\infty(M)$ on $C^\infty(M)$. The *cohomology of the action* A is by definition the cohomology $H^*(G, C_A^\infty(M))$ of the G-module $C_A^\infty(M)$ [**2**]. The first cohomology group plays a fundamental role in the study of LF actions, and we recall the definition of this group. A 1-cocycle over the action $A(x, g) = xg$ is a smooth mapping $a\colon M \times G \to \mathbb{R}$ which satisfies the equations

$$a(x, gh) = a(x, g) + a(xg, h) \quad \text{for } x \in M \text{ and } g, h \in G.$$

We call a a 1-*coboundary* if there exists a smooth mapping $b\colon M \to \mathbb{R}$ such that

$$a(x, g) = -b(x) + b(xg) \quad \text{for } x \in M \text{ and } g \in G.$$

The set $Z^1(G, C_A^\infty(M))$ of all 1-cocycles over A is a vector space and the set $B^1(G, C_A^\infty(M))$ of all 1-coboundaries is a vector subspace of the vector space of all 1-cocycles. The first cohomology group $H^1(G, C_A^\infty(M))$ is the corresponding quotient space.

Definition 1.1. A LF action $A\colon M \times G \to M$ is CR1 if every 1-cocycle $a\colon M \times G \to \mathbb{R}$ over A is cohomologous to a homomorphism $\Phi\colon G \to \mathbb{R}$, i.e. there exists a smooth function $b\colon M \to \mathbb{R}$ such that

$$a(x, g) = -b(x) + \Phi(g) + b(xg). \tag{1.1}$$

We note that the action A is CR1 if and only if $H^1(G, C_A^\infty(M))$ is isomorphic to the first cohomology $H^1(\mathcal{G}^*)$ of the Lie algebra $\mathcal{G}$ of G. In fact, $\mathrm{Hom}(G, \mathbb{R}) \simeq \mathrm{Hom}(\mathcal{G}, \mathbb{R})$ since G is contractible and $\mathrm{Hom}(\mathcal{G}, \mathbb{R}) \simeq \mathrm{Hom}(\mathcal{G}/[\mathcal{G}, \mathcal{G}], R) \simeq H^1(\mathcal{G}^*)$. Let $\mathcal{F}$ be the foliation given by the orbits of A. In [**10**] we introduced the dual description of LF actions in terms of differential forms. This will play an important role in this paper. Associated to A we have a non-singular homomorphism $\mathcal{G} \to \chi(\mathcal{F})$ of the Lie algebra $\mathcal{G}$ of G into the Lie algebra $\chi(\mathcal{F})$ of all smooth vector fields tangent to $\mathcal{F}$, given by $\overline{E}(x) = DA_x(e) \cdot E$

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where $A_x\colon G \to M$, $A_x(g) = xg$, and $E \in \mathcal{G}$. *Non-singular* means that $\overline{E}(x) = 0$ for some $x \in M$ implies $E = 0$. Dually, we have a natural non-singular homomorphism i_A of the exterior algebra $\Lambda_R(\mathcal{G}^*)$ of the invariant forms on G into the exterior algebra $\Lambda(\mathcal{F})$ of the leafwise forms of the foliated manifold $(M, \mathcal{F})$. This homomorphism commutes with the differentials, i.e. $i_A(dw) = d_f i_A(w)$ and is given by $i_A w(\overline{E}(x)) = w(E)$. *Non-singular* means that $(i_A w)_x = 0$ for some $x \in M$ implies $w = 0$ (see [**1**, Example 1.5]). Let $H^*(\mathcal{F})$ be the leafwise cohomology of the foliated manifold $(M, \mathcal{F})$ (see [**1**]). Since $i_A\colon \Lambda_R(\mathcal{G}^*) \to \Lambda(\mathcal{F})$ commutes with the differentials then it induces an injective linear mapping $i_A\colon H^*(\mathcal{G}^*) \to H^*(\mathcal{F})$.

If μ is any A-invariant probability measure on M (such measures exist if G has polynomial growth [**6**]) then we have linear operators $P_A\colon \Lambda^r(\mathcal{F}) \to \Lambda^r_R(\mathcal{G}^*)$ defined by

$$P_A w(E_1, \ldots, E_r) = \int_M w(\overline{E}_1, \ldots, \overline{E}_r)\, d\mu, \quad E_j \in \mathcal{G},\ 1 \le j \le r, \tag{1.2}$$

which commutes with the differentials and we have a split short exact sequence of differential complexes

$$0 \to K \to \Lambda(\mathcal{F}) \underset{i_A}{\overset{P_A}{\rightleftarrows}} \Lambda_R(\mathcal{G}^*) \to 0$$

giving

$$H^r(\mathcal{F}) = i_A H^r(\mathcal{G}^*) \oplus H^r(K), \quad 0 \le r \le p, \tag{1.3}$$

where p is the dimension of G (see [**10**]).

PROPOSITION 1.2. *Let $A\colon M \times G \to M$ be a right LF smooth action and $\mathcal{F}$ be the orbit foliation of A. If G is contractible, then $H^1(G, C^\infty_A(M))$ is isomorphic to $H^1(\mathcal{F})$.*

Proof. We consider the linear transformation $T\colon Z^1(G, C^\infty_A(M)) \to Z^1(\mathcal{F})$ which sends each cocycle a into the d_f-closed form w, $w_x = i_A\, da_x(e)$ where $a_x(g) = a(x, g)$ for $x \in M$ and $g \in G$. We note that if σ is a smooth path on G, $\sigma(0) = e$, then differentiation of the cocycle equation

$$a(x, g\sigma(t)) = a(x, g) + a(xg, \sigma(t))$$

gives

$$da_x(g) \cdot \sigma'(0) = da_{xg}(e) \cdot \sigma'(0). \tag{1.4}$$

Thus,

$$w_{xg} = i_A\, da_x(g), \quad x \in M,\ g \in G. \tag{1.5}$$

Therefore, $d_f w = 0$. The transformation T is clearly injective. To show it is surjective, given $w \in Z^1(\mathcal{F})$ we consider the mapping $a\colon M \times G \to \mathbb{R}$ defined by

$$a(x, g) = \int_{x\alpha} w, \tag{1.6}$$

where $\alpha\colon [0, 1] \to G$ is a smooth path such that $\alpha(0) = e$ and $\alpha(1) = g$. Since G is contractible it is easy to see that a does not depend on the choice of α and satisfies the cocycle equation. Moreover,

$$da_x(e) \cdot E = \lim_{s \to 0} \frac{1}{s} \int_0^s (A_x^* w(\alpha(t)))\alpha'(t)\, dt = w_x(\overline{E}) = i_A\, da_x(e) \cdot \overline{E}(x)$$

proving that T is surjective. □

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PROPOSITION 1.3. *Let G be a contractible Lie group and $A\colon M \times G \to M$ be a smooth LF CR1 right action on a closed orientable smooth manifold. Then there exists an A-invariant smooth volume form Ω on M, i.e.*

$$A_g^*\Omega = \Omega \quad \textit{for all } g \in G.$$

Proof. Choose a smooth volume form Ω_0 on M and define $\det DA_g$ by

$$A_g^*\Omega_0(x) = \det DA_g(x)\Omega_0(x) \quad \text{for } x \in M \text{ and } g \in G.$$

It is easy to see that $a(x, g) = \log|\det DA_g(x)|$ is a cocycle. Since, by assumption, the action A is CR1, then there exist a homomorphism $\Phi\colon G \to \mathbb{R}$ and a smooth function $h\colon M \to \mathbb{R}$ such that

$$a(x, g) - \Phi(g) = h(x) - h(xg) \tag{1.7}$$

and we may assume that $\int_M e^h\Omega_0 = 1$.

We claim that $\Omega = e^h\Omega_0$ is A-invariant. From (1.7) we get

$$e^{h(xg)} \det DA_g(x) = e^{h(x)}e^{\phi(g)} \deg A_g$$

and, thus,

$$A_g^*\Omega(x) = \deg A_g e^{\phi(g)}\Omega(x). \tag{1.8}$$

Observing that $\deg A_g = \int_M A_g^*\Omega$ and $\int_M \Omega = 1$ we see from (1.8) that $e^{\Phi(g)} = 1$ for all $g \in G$. Thus, a is a coboundary and Ω is A-invariant. □

2. *Parameter rigid actions of the Heisenberg groups*

A LF smooth right action $A\colon M \times G \to M$ is called PR if given any other G-action B with the same orbit foliation $\mathcal{F}$, there exist an automorphism Φ of G and a leaf-preserving diffeomorphism $F\colon (M, \mathcal{F}) \to (M, \mathcal{F})$ isotopic to the identity through leaf-preserving diffeomorphisms such that

$$F(A(x, g)) = B(F(x), \Phi(g)) \tag{2.1}$$

for all $x \in M$ and $g \in G$.

We also say that A is *isotopic* to a reparametrization of the action B if (2.1) holds.

In this section we show that CR1 actions of Heisenberg groups are PR.

A *Heisenberg group* H_n is the semi-direct product $H_n = \mathbb{R}^{n+1} \underset{\sigma}{\times} \mathbb{R}^n$, $n \geq 1$ where $\sigma\colon \mathbb{R}^n \to \operatorname{Aut}\mathbb{R}^{n+1}$ is the homomorphism given by

$$\sigma(t)(z, u) = \begin{bmatrix} 1 & t_1 & \cdots & t_n \\ & 1\ 0 & \cdots & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} z \\ u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} z + \langle t, u\rangle \\ u_1 \\ \vdots \\ u_n \end{bmatrix} \tag{2.2}$$

for $u, t \in \mathbb{R}^n$ and $z \in \mathbb{R}$ where $\langle t, u\rangle = t_1u_1 + \cdots + t_nu_n$ is the standard inner product.

Thus, the product in H_n is given by

$$\begin{aligned}(z, u, t) \cdot (w, v, s) &= ((z, u) + \sigma(t)(w, v), t + s) \\ &= (z + w + \langle t, v\rangle, u + v, t + s).\end{aligned} \tag{2.3}$$

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We consider the standard matrix representation [**4**]

$$g = \begin{bmatrix} 1 & t_1 & \cdots & t_n & z \\ & \ddots & 0 \cdots 0 & & u_1 \\ & & \ddots & & u_n \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}, \quad g = (z, u, t) \in H_n. \tag{2.4}$$

The Lie algebra $\mathcal{H}_n$ of H_n has the representation

$$E = \begin{bmatrix} 0 & u_1 & \cdots & u_n & w \\ & \ddots & \cdots & 0 & v_1 \\ & & \ddots & & \vdots \\ & & & \ddots & v_n \\ 0 & & & & 0 \end{bmatrix}, \quad E \in \mathcal{H}_n. \tag{2.5}$$

The standard basis of $\mathcal{H}_n$ is $\mathcal{B} = \{U_1, \ldots, U_n, V_1, \ldots, V_n, W\}$ where U_j is the matrix in which $u_j = 1$ and all of the other entries are zero, V_j is the matrix where $v_j = 1$ and all of the other entries are zero and W is the matrix where $w = 1$ and all of the other elements are zero. The pairwise brackets are equal to zero except for

$$[U_j, V_j] = U_j V_j = W, \quad 1 \le j \le n. \tag{2.6}$$

If $\mathcal{B}^* = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma\}$ is the dual basis of $\mathcal{B}$ then

$$d\alpha_j = d\beta_j = 0, \quad 1 \le j \le n$$

and

$$d\gamma = -\sum_{j=1}^{n} \alpha_j \wedge \beta_j. \tag{2.7}$$

Let $A\colon M \times H_n \to M$ be a smooth LF right action of H_n and $\mathcal{F}$ be the orbit foliation of A. We denote by $\mathcal{A}(H_n, \mathcal{F})$ the set of all smooth LF right actions with the same orbit foliation $\mathcal{F}$.

PROPOSITION 2.1. *If the action A is CR1 with orbit foliation $\mathcal{F}$, then for each action B with the same orbit foliation $\mathcal{F}$ there exists an automorphism Φ of H_n such that if*

$$C(x, g) = B(x, \Phi^{-1}(g)) \quad \textit{for } x \in M, g \in H_n$$

is the reparametrized action then there exist functions h_j and k_j in $C^\infty(M)$ such that

$$i_C\alpha_j = i_A\alpha_j + d_f h_j \quad \textit{and} \quad i_C\beta_j = i_A\beta_j + d_f k_j, \quad 1 \le j \le n.$$

Proof. Let μ be any probability B-invariant measure on M and $P_B\colon \Lambda(\mathcal{F}) \to \Lambda(\mathcal{H}_n^*)$ be the linear transformation as in (1.2), then $L^\wedge = P_B \circ i_A$ is a linear operator of $\Lambda(\mathcal{H}_n^*)$ which commutes with the differentials. We observe that since A is CR1, then the restriction L_1^* of $L^\wedge$ to $H^1(\mathcal{H}_n^*)$ is an isomorphism. If $L_1^*\theta = P_B i_A \theta = 0$ for a closed form θ, since

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by (1.3) and Proposition 1.2 $H^1(\mathcal{F}) = i_A H^1(\mathcal{H}_n^*)$, then $i_A\theta = d_f h$. Thus, $i_A\theta(x_0) = 0$ for some $x_0 \in M$ and as i_A is non-singular, then $\theta = 0$. We note that $L^\wedge$ is not a homomorphism of the exterior algebra $\Lambda(\mathcal{H}_n^*)$. So we extend L_1^* to a homomorphism L^* of $\Lambda(\mathcal{H}_n^*)$ commuting with the differential, i.e. we search for $L_1^*\gamma$ such that $L_2^*\, d\gamma = dL_1^*\gamma$ where $L_2^* = \Lambda^2 L_1^*$. By (2.7) this is equivalent to

$$L_2^*\, d\gamma = -\sum_{j=1}^{n} L_1^*\alpha_j \wedge L_1^*\beta_j = dL_1^*\gamma. \tag{2.8}$$

Now since A is CR1, then $i_B \circ P_B$ is the identity of $H^1(\mathcal{F})$, thus there exist functions h_j and k_j in $C^\infty(M)$, $1 \le j \le n$ such that

$$i_B \circ L_1^*\alpha_j = i_A\alpha_j + d_f h_j \quad \text{and} \quad i_B \circ L_1^*\beta_j = i_A\beta_j + d_f k_j. \tag{2.9}$$

From (2.8) and (2.9) we get

$$i_B L_2^*\, d\gamma = -i_A\left(\sum_{j=1}^{n} \alpha_j \wedge \beta_j\right) - d_f w, \tag{2.10}$$

where

$$w = \sum_{j=1}^{n} (k_j i_A\alpha_j - h_j i_A\beta_j - h_j d_f k_j).$$

Applying P_B to (2.10) and recalling that $P_B \circ i_B$ is equal to the identity on $\Lambda(\mathcal{H}_n^*)$ we see that

$$L_2^*\, d\gamma = d(L^\wedge\gamma - P_B w) \tag{2.11}$$

since $L^\wedge$ and P_B commute with the differentials.

Choosing $L_1^*\gamma = L^\wedge\gamma - P_B w$, (2.11) gives

$$L_2^*\, d\gamma = dL_1^*\gamma. \tag{2.12}$$

To show that $L_1^*\colon \mathcal{H}_n^* \to \mathcal{H}_n^*$ is an isomorphism it suffices to show that $dL_1^*\gamma \neq 0$. For this observe that

$$L_{2n}^*(\alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n) = \Delta(\alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n), \tag{2.13}$$

where Δ is the determinant of the restriction of $L^\wedge$ to $H^1(\mathcal{H}_n^*)$. Now noting that

$$(d\gamma)^n = n(-1)^n(\alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n), \tag{2.14}$$

then from (2.12), (2.13) and (2.14) we get

$$L_{2n}^*(d\gamma)^n = (dL_1^*\gamma)^n = \Delta(d\gamma)^n \neq 0$$

since $\Delta \neq 0$ because $L^\wedge$ is an isomorphism on $H^1(\mathcal{H}_n^*)$.

Now since L^* is an automorphism of the exterior algebra $\Lambda(\mathcal{H}_n^*)$ and commutes with the differential, the dual $L\colon \mathcal{H}_n \to \mathcal{H}_n$ of L_1^* is an automorphism of the Lie algebra $\mathcal{H}_n$ of H_n and as H_n is contractible, then there is a unique automorphism Φ of H_n such that $D\Phi = \Phi_* = L$ and $\Phi^* = L_1^*$. It is easy to see that

$$\overline{E}(x) = DC_x(e)\cdot E = DB_x(e)\cdot \Phi_*^{-1}E \tag{2.15}$$

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for all $E \in \mathcal{H}_n$ and from (2.15) we get

$$i_B \Phi^* \theta(\overline{E}) = \theta(E) = i_C \theta(\overline{E}).$$

Thus,

$$i_C \theta = i_B \circ \Phi^* \theta. \tag{2.16}$$

Finally from (2.16) we see that

$$i_C \theta = i_B \circ L^\wedge \theta = (i_B \circ P_B) i_A \theta \tag{2.17}$$

and (2.17) gives $\ell \in C^\infty(M)$ such that

$$i_C \theta = i_A \theta + d_f \ell$$

for all $\theta \in H^1(\mathcal{H}_n^*)$ since, by assumption, A is CR1, proving the proposition. □

Remark 2.2. It is easy to see that for LF actions of the Heisenberg groups CR1 is a weaker hypothesis than *cocycle rigidity*, a notion introduced in [**8**], i.e. cocycle rigidity implies CR1 but not conversely. In fact, we choose homomorphisms $j\colon \mathbb{R} \to H_n$ and $p\colon H_n \to \mathbb{R}$ such that $p \circ j$ is the identity of $\mathbb{R}$. Let $a\colon M \times H_n \to \mathbb{R}$ be any cocycle over a CR1 action $A\colon M \times H_n \to M$. Then $j \circ a$ is a cocycle over A with values in H_n. Now if A is cocycle rigid, then there exist an endomorphism ϕ of H_n and a smooth function $b\colon M \to H_n$ such that

$$(j \circ a)(x, g) = b^{-1}(x) \Phi(g) b(xg) \tag{2.18}$$

for all $x \in M$ and $g \in H_n$.

Applying p to (2.18) we get

$$a(x, g) = -b_0(x) + \Phi_0(g) + b_0(xg) \tag{2.19}$$

where

$$b_0 = p \circ b \quad \text{and} \quad \phi_0 = p \circ \Phi.$$

Thus, A is CR1. □

If $\theta\colon TG \to \mathcal{G}$, $\theta_g(E(g)) = E$, $g \in G$ and $E \in \mathcal{G}$ is the Maurer–Cartan form of G, then it satisfies the structural equation

$$d\theta + [\theta \wedge \theta] = 0, \tag{2.20}$$

where

$$[\theta \wedge \theta](E_1, E_2) = [\theta(E_1), \theta(E_2)] \quad \text{for } E_j \in \mathcal{G},\ i \le j \le 2.$$

The *canonical form* of a LF smooth action A of a Lie group G on M with orbit foliation $\mathcal{F}$ is, by definition, the leafwise 1-form $w\colon T\mathcal{F} \to \mathcal{G}$ given by

$$w = i_A \theta, \tag{2.21}$$

where θ is the Maurer–Cartan form of G and $\mathcal{G}$ is the Lie algebra of G. It satisfies the equation

$$d_f w + [w \wedge w] = 0. \tag{2.22}$$

We now adapt to our needs some results of Matsumoto and Mitsumatsu in [**8**] to prove the following.

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PROPOSITION 2.3. *Let A and B be smooth LF actions of a contractible Lie group G on a smooth manifold M, with the same orbit foliation $\mathcal{F}$. Let w^0 and w be the canonical forms of A and B, respectively. Suppose that there is a smooth function $b\colon M \to G$ and an automorphism Φ of G such that*

$$w = b^{-1}\Phi^* w^0 b + b^{-1} d_f b, \tag{2.23}$$

then the action A is isotopic to the reparametrization

$$C(x, g) = B(x, \phi(g)), \quad x \in M, g \in G$$

of B.

Proof. Let $a\colon M \times G \to G$ be the cocycle given by

$$A(x, g) = B(x, a(x, g)), \quad x \in M, g \in G. \tag{2.24}$$

As in the proof of [**8**, Proposition 2.7] the correspondence between the cocycles and canonical forms of actions is bijective, then (2.23) gives

$$a(x, g) = b^{-1}(x)\Phi(g)b(A(x, g)), \quad x \in M, g \in G. \tag{2.25}$$

Now as in the proof of [**8**, Proposition 2.3], (2.25) gives a leaf-preserving diffeomorphism F of the foliated manifold $(M, \mathcal{F})$ isotopic to the identity through such diffeomorphisms and satisfying

$$F(A(x, g)) = C(F(x), g), \quad x \in M, g \in G. \qquad \square$$

We now prove a theorem that will allow us to show, in the next section, the existence of PR actions of the Heisenberg groups.

THEOREM 2.4. *Let A be a smooth LF action of the Heisenberg group H_n, $n \geq 1$ on a closed manifold M. Suppose that A is CR1. Then A is parameter rigid.*

Proof. Let $B_0 = \{X_1^0, \ldots, X_n^0, Y_1^0, \ldots, Y_n^0, Z^0\}$ where

$$X_j^0(x) = DA_x(e) \cdot U_j, \quad Y_j^0(x) = DA_x(e) \cdot V_j \quad \text{and} \quad Z^0(x) = DA_x(e) \cdot W$$

for $x \in M$ and $1 \leq j \leq n$ be the frame of generators of A and

$$B_0^* = \{\xi_1^0, \ldots, \xi_n^0, \eta_1^0, \ldots, \eta_n^0, \zeta^0\}$$

be the coframe dual to B_0. We note that

$$\xi_j^0 = i_A\alpha_j, \quad \eta_j^0 = i_A\beta_j \quad \text{and} \quad \zeta^0 = i_A\gamma, \quad 1 \leq j \leq n.$$

Let $B \in \mathcal{A}(\mathcal{H}_n, \mathcal{F})$ where $\mathcal{F}$ is the orbit foliation of A. Since, by assumption, A is CR1, then by Proposition 2.1 we may assume that, up to a reparametrization by an automorphism of H_n, we have

$$\xi_j = \xi_j^0 + d_f h_j^0 \quad \text{and} \quad \eta_j = \eta_j^0 + d_f k_j^0 \quad \text{where } h_j^0 \text{ and } k_j^0 \tag{2.26}$$

are in $C^\infty(M)$ and

$$\xi_j = i_B\alpha_j, \quad \eta_j = i_B\beta_j, \quad \zeta = i_B\gamma, \quad 1 \leq j \leq n.$$

So $B^* = \{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n, \zeta\}$ is the coframe of B.

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Now consider the smooth mapping $b_0\colon M \to H_n$ given by

$$b_0 = \begin{bmatrix} 1 & h_1^0 & \cdots & h_n^0 & \ell^0 \\ & 1 & 0\cdots 0 & & k_1 \\ & & \ddots & & \vdots \\ & & & \ddots & k_n^0 \\ 0 & & & & 1 \end{bmatrix}, \tag{2.27}$$

where the smooth function $\ell^0\colon M \to \mathbb{R}$ is to be determined. Computing we see that

$$b_0^{-1} = \begin{bmatrix} 1, & -h_1^0 & , \ldots , & -h_n^0 & \ell^* \\ & 1 & 0\ldots 0 & & -k_1^0 \\ & & \ddots & & \vdots \\ & & & \ddots & -k_n^0 \\ 0 & & & & 1 \end{bmatrix},$$

where $\ell^* = \sum_{j=1}^n (-1)^{j+1} h_j^0 k_j^0 + (-1)^{n+1}\ell^0$.

Let $w^0 = i_A\theta$ and $w = i_B\theta$ as in (2.21) be the canonical forms of the actions A and B, respectively. By Proposition 2.3 to prove Theorem 2.4 it suffices to show the existence of a smooth function $b\colon M \to H_n$ satisfying (2.23). Let us consider first the $\mathcal{H}_n$-valued form

$$\Omega^0 = b_0^{-1} w^0 b_0 + b_0^{-1} d_f b_0. \tag{2.28}$$

A straightforward computation using (2.26) and (2.27) gives

$$\Omega^0_{1,j+1} = \xi_j^0 + d_f h_j^0,$$
$$\Omega^0_{j+1,n+2} = \eta_j^0 + d_f k_j^0$$

for $1 \le j \le n$ and

$$\Omega^0_{1,n+2} = \zeta^0 + \sum_{j=1}^n (k_j^0 \xi_j^0 - h_j^0 \eta_j^0 - h_j^0 d_f k_j^0) + d\ell^0.$$

Now observing that

$$d_f \zeta^0 = -\sum_{j=1}^n \xi_j^0 \wedge \eta_j^0$$

and using (2.26) we obtain

$$\begin{aligned} d_f \zeta &= -\sum_{j=1}^n \xi_j \wedge \eta_j \\ &= d_f \zeta^0 + \sum_{j=1}^n d_f (k_j^0 \xi_j^0 - h_j^0 \eta_j^0 - h_j^0 d_f k_j^0). \end{aligned} \tag{2.29}$$

From (2.29) we see that there exists $\tau \in Z^1(\mathcal{F})$

$$\zeta = \zeta^0 + \sum_{j=1}^n (k_j^0 \xi_j^0 - h_j^0 \eta_j^0 - h_j^0 d_f k_j^0) + \tau. \tag{2.30}$$

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Since A is CR1, then

$$\tau = \sum_{j=1}^{n}(a_j\xi_j^0 + b_j\eta_j^0) + d_f\ell' \tag{2.31}$$

where a_j and b_j are real numbers and $\ell' \in C^\infty(M)$.

Now if we choose

$$h_j = h_j^0 - b_j, \quad k_j = k_j^0 + a_j, \quad 1 \le j \le n \tag{2.32}$$

and

$$\ell = \ell' + \sum_{j=1}^{n}(a_j h_j - b_j k_j)$$

we obtain

$$\zeta = \zeta^0 + \sum_{j=1}^{n}(k_j\xi_j^0 - h_j\eta_j^0 - h_j d_f k_j) + d_f\ell. \tag{2.33}$$

Defining $b\colon M \to H_n$ by

$$b = \begin{bmatrix} 1 & h_1 & \cdots & h_n & \ell \\ & 1 & 0\cdots 0 & & k_1 \\ & & \ddots & & \vdots \\ & & & \ddots & k_n \\ 0 & & & & 1 \end{bmatrix}$$

and using (2.28) and (2.33) we see that

$$w = b^{-1}w^0 b + b^{-1}d_f b$$

proving Theorem 2.4. □

We do not know whether the converse of Theorem 2.4 is true.

3. *Existence of parameter rigid actions of the Heisenberg groups*

We consider the cocompact discrete subgroups $\Gamma = \mathbb{Z}^{n+1} \underset{\sigma}{\times} \mathbb{Z}^n$ of the Heisenberg groups H_n as in (2.3) and the nilmanifolds $M = \Gamma \setminus H_n$. We identify the smooth functions $f\colon M \to \mathbb{R}$ with the Γ-periodic functions on H_n, i.e.

$$f(\gamma g) = f(g) \quad \text{for } \gamma \in \Gamma \text{ and } g \in H_n. \tag{3.1}$$

For each fixed $t \in \mathbb{R}^n$ we have a $\mathbb{Z}^{n+1}$-periodic function

$$f_t(n+u) = f_t(u) = f(u,t), \quad u \in \mathbb{R}^{n+1},\ n \in \mathbb{Z}^{n+1},\ n = (n_0, n_1, \ldots, n_n) \tag{3.2}$$

since $(n,0)\cdot(u,t) = (n+u,t)$ and f is Γ-invariant. Thus, f is given by the following Fourier series which converges in the C^∞ topology

$$f(u,t) = \sum_{k\in\mathbb{Z}^{n+1}} \hat{f}_k(t)e^{2\pi i\langle k,u\rangle}. \tag{3.3}$$

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The Fourier coefficients are the smooth functions given by

$$\hat{f}_k(t) = \int_{T^{n+1}} f(u, t) e^{-2\pi i \langle k, u \rangle} \, du \tag{3.4}$$

and the functions $\hat{f}_k(t)$ are bounded, i.e. $|\hat{f}_k|_0 \leq |f|_0$, where $|\cdot|_0$ is the C^0 norm. We observe that

$$\hat{f}_k(t + \ell) = \hat{f}_{k\sigma(\ell)}(t) \quad \text{for } k \in \mathbb{Z}^{n+1}, \ell \in \mathbb{Z}^n, t \in \mathbb{R}^n, \tag{3.5}$$

because [**3**]

$$f((0, \ell) \cdot (u, t)) = f(\sigma(\ell)u, t + \ell) = f(u, t),$$

and

$$\hat{f}_{k\sigma(\ell)}(t) = \int_{T^{n+1}} f(u, t) e^{-2\pi i \langle k\sigma(\ell), u \rangle} \, du = \hat{f}_k(t + \ell).$$

A canonical set of generators of Γ is given by

$$B = \{\gamma_0, \ldots, \gamma_n, \delta_1, \ldots, \delta_n\},$$

where

$$\gamma_j = (e_{j+1}, 0), \quad e_{j+1} \in \mathbb{R}^{n+1} \text{ for } 0 \leq j \leq n \tag{3.6}$$

and

$$\delta_i = (0, e_i), \quad e_i \in \mathbb{R}^n, 1 \leq i \leq n$$

satisfying the equations

$$\begin{gathered} \gamma_i \gamma_j = \gamma_j \gamma_i, \quad 0 \leq i, j \leq n \\ \delta_r \delta_s = \delta_s \delta_r, \quad 1 \leq r, s \leq n \\ \gamma_0 \delta_r = \delta_r \gamma_0, \quad 1 \leq r \leq n \\ \delta_i \gamma_j = \gamma_j \delta_i, \quad i \neq j \end{gathered} \tag{3.7}$$

and

$$\delta_j \gamma_j = \gamma_j \delta_j \gamma_0, \quad 1 \leq j \leq n. \tag{3.8}$$

The right translation $H_n \times H_n \xrightarrow{R} H_n$ gives the right LF action

$$R^0 \colon M \times H_n \to M \tag{3.9}$$

which will be used to define CR1 actions

$$A \colon M \times \Gamma \to M \tag{3.10}$$

whose suspensions will (by [**8**]) give CR1 actions

$$\tilde{A} \colon \tilde{M} \times H_n \to \tilde{M}, \tag{3.11}$$

where $M \to (\tilde{M}, \mathcal{F}) \xrightarrow{\tau} M$ is the suspension foliated bundle of A and $\tilde{A}$ is the canonical action of τ (see [**1**]). Moreover, since H_n is contractible then by [**9**, Theorem 2.1]

$$H^j(\Gamma, C^\infty_A(M)) \simeq H^j(\mathcal{F}), \quad 0 \leq j \leq 2n + 1. \tag{3.12}$$

We define an action A of Γ on M by

$$A(\gamma_j) = R^0_{\alpha_j}, \quad \alpha_j = (a_j e_{j+1}, 0) \quad \text{for } 0 \leq j \leq n \tag{3.13}$$

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and

$$A(\delta_i) = R^0_{\beta_i}, \quad \beta_i = (0, b_i e_i) \quad \text{for } 1 \le i \le n$$

From (3.8) to define an action of Γ we must have

$$a_j b_j = a_0 \quad \text{for } 1 \le j \le n. \tag{3.14}$$

We will need the following.

PROPOSITION 3.1. *For each positive integer n there exist Diophantine numbers a_j, $0 \le j \le n$ and b_j, $1 \le j \le n$ such that:*
(i) $a_j b_j = a_0$, *for* $1 \le j \le n$;
(ii) $\{1, a_0, a_1, \ldots, a_n, b_1, \ldots, b_n\}$ *are linearly independent over* $\mathbb{Q}$.

Proof. Let $\alpha \in R$ be a root of an irreducible polynomial $p(x) \in \mathbb{Q}[x]$ of degree greater than $n(n+1)/2 + 2$. Therefore,

$$\{1, \alpha, \alpha^2, \ldots, \alpha^{n(n+1)/2+2}\}$$

is linearly independent over $\mathbb{Q}$. Choose

$$a_0 = \alpha^{n(n+1)/2+2}, \quad a_1 = \alpha, \ldots, a_n = \alpha^n \tag{3.15}$$

and

$$b_j = a_0 a_j^{-1} \quad \text{for } 1 \le j \le n.$$

Now, by a well-known theorem of Liouville, every irrational algebraic number is Diophantine, finishing the proof. □

THEOREM 3.2. *Let $A\colon M \times \Gamma \to M$ be the action defined by (3.13). Suppose that the numbers in (3.14) are given by Proposition 3.1. Then A is CR1, i.e.*

$$H^1(\Gamma, C^\infty_A(M)) \simeq H^1(\mathcal{H}^*_n).$$

Proof. For each smooth cocycle $c\colon M \times \Gamma \to \mathbb{R}$ over A consider the homomorphism $\Phi\colon \Gamma \to \mathbb{R}$ given by

$$\Phi(g) = \int_M c(x, g)\, d\mu(x), \tag{3.16}$$

where μ is the measure given by the R^0-invariant volume form on M. To prove Theorem 3.2 we have to show that c is cohomologous to Φ, i.e. to show the existence of a smooth function $h\colon M \to \mathbb{R}$ such that

$$c_0(x, \gamma) = c(x, \gamma) - \Phi(\gamma) = h(x) - h(x \cdot \gamma) \tag{3.17}$$

for all $x \in M$ and $\gamma \in \Gamma$ where $x \cdot \gamma = A(x, \gamma)$.

We note that since c_0 is a cocycle, it suffices to show that (3.17) is true on the generators $\gamma \in B$.

Let

$$F^j(x) = c_0(x, \gamma_j), \quad 0 \le j \le n \tag{3.18}$$

and

$$G^r(x) = c_0(x, \delta_r), \quad 1 \le r \le n$$

where $x = (u, t) \in H_n$.

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We will show the existence of a smooth function $h\colon M \to \mathbb{R}$ satisfying the equations

$$h(x) - h(x \cdot \gamma_j) = F^j(x), \quad 0 \le j \le n \tag{3.19}$$

and

$$h(x) - h(x \cdot \delta_i) = G^i(x), \quad 1 \le i \le n. \tag{3.20}$$

From the cocycle equations of c_0 we obtain the equations

$$F^j(x) - F^j(x \cdot \gamma_i) = F^i(x) - F^i(x \cdot \gamma_j), \quad 0 \le i,\, j \le n, \tag{3.21}$$

which give the following equations on the Fourier coefficients,

$$(1 - e^{2\pi i \langle k, \sigma(t) a_i e_{i+1}\rangle}) F_k^j(t) = (1 - e^{2\pi i \langle k, \sigma(t) a_j e_{j+1}\rangle}) F_k^i(t) \tag{3.22}$$

for $0 \le i,\, j \le n$.

We see from (3.22) that for $k = (k_0, k_1, \ldots, k_n) \in \mathbb{Z}^{n+1}$ we have

$$F_k^0(t) = 0 \quad \text{for } k \ne 0,\, k_0 = 0 \tag{3.23}$$

and

$$F_k^j(t) = 0 \quad \text{for } k \ne 0,\, k_0 = k_j = 0$$

for all $t \in \mathbb{R}^n$.

The Fourier coefficients of a solution of the cohomological equations (3.19) must satisfy

$$(1 - e^{2\pi i \langle k, \sigma(t)\alpha_j\rangle}) h_k(t) = F_k^j(t) \tag{3.24}$$

for $0 \le j \le n$ and all $t \in \mathbb{R}^n$.

We note from (3.22) that the value of $h_k(t)$ does not depend on $0 \le j \le n$. Thus, for $k \ne 0$ we have

$$h_k(t) = \frac{F_k^j(t)}{1 - e^{2\pi i k_j a_j}} \quad \text{if } k_0 = 0,\, k_j \ne 0 \tag{3.25}$$

and

$$h_k(t) = \frac{F_k^0(t)}{1 - e^{2\pi i k_0 a_0}}$$

if $k_0 \ne 0$, and all $t \in \mathbb{R}^n$.

Since the numbers a_j, $0 \le j \le n$ are Diophantine, (3.25) gives

$$|h_k(t)| \le |F_k^j(t)||k|^p \tag{3.26}$$

for all $k \in \mathbb{Z}^{n+1}$, $k \ne 0$, $0 \le j \le n$.

Since, by (3.4),

$$|F_k^j|_0 \le |F^j|, \quad 0 \le j \le n, \tag{3.27}$$

then

$$\lim_{|k| \to \infty} |k|^p |h_k(t)| = 0 \tag{3.28}$$

since F^j are smooth functions.

We now show that

$$F_0^j(t) = 0 \quad \text{for all } t \in \mathbb{R}^n \text{ and } 0 \le j \le n. \tag{3.29}$$

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In fact, from $\delta_i \gamma_j = \gamma_j \delta_i$, $i \neq j$ we obtain

$$G^i(u, t) + F^j((u, t) \cdot (0, b_i e_i)) = F^j(u, t) + G^i(u + \sigma(t) a_j e_{j+1}, t). \tag{3.30}$$

From the exact sequences

$$0 \to \mathbb{R}^{n+1} \xrightarrow{j} H_n \xrightarrow{\pi} \mathbb{R}^n \to 0$$

and

$$0 \to \mathbb{Z}^{n+1} \to \Gamma \to \mathbb{Z}^n \to 0,$$

where $j(u) = (u, 0)$, $\pi(u, t) = t$ we obtain the fibre bundle

$$\mathbb{T}^{n+1} \to M \xrightarrow{\pi} \mathbb{T}^n \tag{3.31}$$

and integrating (3.30) over the fibre of π (see [**5**]) we obtain

$$F_0^j(t + b_i e_i) = F_0^j(t) \quad \text{for } t \in \mathbb{R}^n,\ 0 \leq j \leq n,\ i \neq j \tag{3.32}$$

where

$$F_0^j(t) = \fint_{\mathbb{T}^{n+1}} F^j(u, t)\, du.$$

In particular,

$$F_0^0(t + b_i e_i) = F_0^0(t) \quad \text{for } 1 \leq i \leq n,\ t \in \mathbb{R}^n. \tag{3.33}$$

Since, by assumption, $\{1, b_1, \ldots, b_n\}$ are linearly independent over $\mathbb{Q}$, then the translation of $\mathbb{T}^n$ given on the cover $\mathbb{R}^n$ by

$$T : \mathbb{R}^n \to \mathbb{R}^n, \quad T(x) = x + b, \tag{3.34}$$

where $b = (b_1, \ldots, b_n)$, is ergodic [**7**].

Now from (3.17) and (3.18) and Fubini's theorem we get

$$0 = \int_M F^j(u, t)\, du = \int_{\mathbb{T}^n} \left(\fint_{\mathbb{T}^{n+1}} F^j(u, t)\, du \right) dt. \tag{3.35}$$

Thus, from (3.33), (3.34) and (3.35) we see that

$$F_0^0(t) = 0 \quad \text{for all } t \in \mathbb{R}^n. \tag{3.36}$$

From (3.8) and the cocycle equations for c_0 we obtain

$$c_0(x \cdot \delta_j, \gamma_j) - c_0(x, \gamma_j) = c_0(x \cdot \gamma_j, \delta_j) - c_0(x, \delta_j) + c_0(x \cdot \gamma_j \delta_j, \gamma_0). \tag{3.37}$$

Integrating (3.37) over the fibre of π and using (3.36) we obtain

$$F_0^j(t + b_j e_j) = F_0^j(t), \quad 1 \leq j \leq n,\ t \in \mathbb{R}^n. \tag{3.38}$$

Now from (3.32), (3.36) and (3.38) we get

$$F_0^j(t + b) = F_0^j(t), \quad 0 \leq j \leq n,\ t \in \mathbb{R}^n \tag{3.39}$$

and from (3.34) and (3.39) we get (3.29). From (3.29) and (3.24) we are free to choose any smooth function $h_0 : \mathbb{T}^n \to \mathbb{R}$ and the Fourier series

$$h(u, t) = \sum_{k \in \mathbb{Z}^{n+1}} h_k(t) e^{2\pi i \langle k, u \rangle} \tag{3.40}$$

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will by (3.28) converge in the C^∞ topology giving a smooth solution to the cohomological equations (3.19).

We will now show that we can choose the function h_0 such that h given in (3.40) also satisfies the cohomological equation (3.20).

In fact, from $\delta_i\gamma_j = \gamma_j\delta_i$, for $i \neq j$ we get

$$G^i(x) - G^i(x\cdot\gamma_j) = F^j(x) - F^j(x\cdot\delta_i), \quad i \neq j \tag{3.41}$$

or on Fourier coefficients

$$(1 - e^{2\pi i\langle k,\sigma(t)\alpha_j\rangle})G^i_k(t) = F^j_k(t) - F^j_k(t + b_ie_i) \quad \text{for } i \neq j. \tag{3.42}$$

Now (3.24) and (3.42) give

$$\begin{aligned}(1 - e^{2\pi i\langle k,\sigma(t)\alpha_j\rangle})G^i_k(t) = {} & (1 - e^{2\pi i\langle k,\sigma(t)\alpha_j\rangle})h_k(t)\\ & - (1 - e^{2\pi i\langle k,\sigma(t+b_ie_i)\alpha_j\rangle})h_k(t + b_ie_i) \quad \text{for } i \neq j.\end{aligned} \tag{3.43}$$

We claim that

$$h_k(t) - h_k(t + b_ie_i) = G^i_k(t) \tag{3.44}$$

for all $k \in \mathbb{Z}^{n+1}$, $k \neq 0$, $t \in \mathbb{R}^n$ and $1 \leq i \leq n$. This is because:

(i) if $k_0 \neq 0$ take $j = 0$, then $\sigma(t + b_ie_i)\alpha_0 = \alpha_0$, $\langle k, \sigma(t)\alpha_0\rangle = k_0\alpha_0 \notin \mathbb{Z}$ and (3.44) follows from (3.43);

(ii) if $k_0 = 0$ and $k \neq k_ie_i$, $1 \leq i \leq n$, then

$$\langle k, \sigma(t)\alpha_j\rangle = \langle k, \sigma(t + b_ie_i)\alpha_j\rangle = k_j\alpha_j \notin \mathbb{Z}$$

for some $j \neq i$ and (3.44) follows from (3.43);

(iii) $k_0 = 0$ and $k = k_je_j$ for some $1 \leq j \leq n$.

From (3.8) we obtain

$$G^j(x) - G^j(x\cdot\gamma_j) = F^j(x) - F^j(x\cdot\delta_j) + F^0(x\cdot\gamma_j\delta_j) \tag{3.45}$$

or on Fourier coefficients

$$(1 - e^{2\pi i\langle k,\sigma(t)\alpha_j\rangle})G^j_k(t) = F^j_k(t) - F^j_k(t + b_je_j) \tag{3.46}$$

since by (3.23) $F^0_k(t) = 0$ if $k_0 = 0$ for $t \in \mathbb{R}^n$.

Now (3.24) and (3.46) give

$$G^j_k(t) = h_k(t) - h_k(t + b_je_j) \tag{3.47}$$

since $\langle k, \sigma(t)\alpha_j\rangle = k_ja_j \notin \mathbb{Z}$.

Finally, we have to find a smooth solution for the cohomological equations

$$h_0(t) - h_0(t + b_ie_i) = G^i_0(t). \tag{3.48}$$

From $\delta_i\delta_j = \delta_j\delta_i$, $1 \leq i, j \leq n$ we get the equations

$$G^i(x) - G^i(x\cdot\delta_j) = G^j(x) - G^j(x\cdot\delta_i)$$

and on Fourier coefficients

$$(1 - e^{2\pi i\langle k,b_je_j\rangle})\hat{G}^i_0(k) = (1 - e^{2\pi i\langle k,b_ie_i\rangle})\hat{G}^j_0(k) \tag{3.49}$$

for $1 \leq i, j \leq n$ and $k \in \mathbb{Z}^n$.

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Equation (3.48) corresponds to the following equations on Fourier coefficients:

$$(1 - e^{2\pi\langle k, b_j e_j\rangle})\hat{h}_0(k) = \hat{G}_0^j(k). \tag{3.50}$$

Thus, for each $k \in \mathbb{Z}^n$, $\langle k, b_j e_j\rangle \neq 0$ we need

$$\hat{h}_0(k) = \frac{\hat{G}_0^j(k)}{1 - e^{2\pi i\langle k, b_j e_j\rangle}} \tag{3.51}$$

and by (3.49) the value of $\hat{h}_0(k)$ is independent of the choice of j, $\langle k, b_j e_j\rangle \neq 0$.

Now since the numbers b_j, $1 \leq j \leq n$, are Diophantine, then the Fourier series

$$h_0(t) = \sum_{k\in\mathbb{Z}^n} \hat{h}_0(k) e^{2\pi i\langle k,t\rangle} \tag{3.52}$$

converges in the C^∞ topology and defines a smooth function and the corresponding function $h\colon \mathbb{T}^n \to \mathbb{R}$ in (3.40) satisfies the cohomological equations (3.20). One could emphasize that $\hat{G}_0^j(0) = 0$, $j = 1, \ldots, n$, is a necessary condition. It is indeed satisfied, because

$$\begin{aligned}\hat{G}_0^j(0) &= \int_{T^n} G_0^j(t)\, dt \\ &= \int_{T^n}\int_{T^{n+1}} G^j(u, t)\, du\, dt,\end{aligned}$$

and relation (3.17), finishing the proof of Theorem 3.2. □

Remark 3.3. The actions given by Theorem 3.2 are *ergodic*.

In fact, let $f \in L^2(M, d\mu)$ be an A-invariant function, i.e.

$$f(x \cdot \gamma) = f(x) \quad \text{for all } x \in M \text{ and all } \gamma \in \Gamma. \tag{3.53}$$

Taking $x = (u, t)$ and $\gamma = (0, b)$ we get

$$f(u, t + b) = f(u, t) \quad \text{for all } (u, t) \in H_n$$

and as the translation $t \mapsto t + b$ is ergodic, then $f(u, t) = f(u, 0)$ for all $u \in \mathbb{R}^{n+1}$. Now taking $x = (u, 0)$ and $\gamma = (a, 0)$ where $a = (a_0, a_1, \ldots, a_n)$, then the translation $u \mapsto u + a$ of $\mathbb{T}^{n+1}$ is ergodic and from (3.53) $f((u, 0) \cdot (a, 0)) = f(u + a, 0) = f(u, 0)$ for all $u \in \mathbb{T}^{n+1}$ proving that f is constant and A is ergodic.

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