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Subshifts on an infinite alphabet

XAVIER BRESSAUD†

Instituto de Matematica e Estatistica, Universidade de São Paulo, Rua do Matão, 1010, 05508-900 São Paulo, SP, Brasil
(e-mail: bressaud@ime.usp.br)

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Abstract. We study transfer operators over general subshifts of sequences of an infinite alphabet. We introduce a family of Banach spaces of functions satisfying a regularity condition and a decreasing condition. Under some assumptions on the transfer operator, we prove its continuity and quasi-compactness on these spaces. Under additional assumptions—existence of a conformal measure and topological mixing—we prove that its peripheral spectrum is reduced to one and that this eigenvalue is simple. We describe the consequences of these results in terms of existence and properties of invariant measures absolutely continuous with respect to the conformal measure. We also give some examples of contexts in which this setting can be used—expansive maps of the interval, statistical mechanics.

1. *Introduction*

Given a measure space (X, m) and a measurable map T from X into itself, it is a classical question to ask if it admits an invariant measure absolutely continuous with respect to m [**7, 15, 18**]. Then, it is natural to study the ergodic and statistical properties of the system with respect to this invariant measure.

When T is a non-invertible, piecewise differentiable and expansive map from $\mathbb{R}^n$ into itself, the Ruelle–Perron–Frobenius (RPF) operator has proved to be a rather efficient tool for these aims:

$$P(f)(x) = \sum_{y|T(y)=x} \frac{1}{T'(y)} f(y).$$

Whereas it is a slightly more general setting, the study of spectral properties of the transfer operators [**16, 25, 27**] involved in the thermodynamic formalism [**19**] has been fruitfully applied to answer this kind of question (for the case of differentiable hyperbolic dynamical systems, see [**3**]). A classical and important result in this area [**16, 19**] states that if A is a *finite* set, T is the *full shift* on the set X of sequences of elements of A, and ϕ

† Current address: IML, 163, avenue de Luminy, Case 907, 13288 Marseille Cedex 9, France.

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is a Hölder function on X, then, the so-called *transfer operator*, L_ϕ, defined on the space of Hölder functions by

$$L_\phi(f)(x) = \sum_{y|T(y)=x} e^{\phi(y)} f(y),$$

has a single, isolated, and simple, eigenvalue, λ, of maximal modulus. This result yields the existence and uniqueness of a measure m (conformal measure) for which $\lambda^{-1}L_\phi$ is the RPF operator. The eigenfunction ρ of the transfer operator is the density of the T-invariant probability measure $d\mu = \rho\, dm$. With respect to this measure, Hölder observables have exponential decay of correlations, and the sums $\sum_n f \circ T^n$, for f Hölder, satisfy limit theorems [**10, 17, 24**]. These results easily generalize to the case of subshifts of finite type.

Many systems can be understood through the same formalism, whereas they do not fit the hypothesis required for the quasi-compactness result (infinite state space, non-uniform hyperbolicity, absence of Markovian partition, ...). Different authors have proposed extensions of these results in some particular cases. For interval maps see, for example, [**1, 6, 13, 14, 20, 26**], for symbolic dynamics see [**9, 22, 23, 25**]. Here, we address two important difficulties.

- If the alphabet A is infinite, then the transfer operator may not be defined on the space of bounded functions, even for the simplest potential, $\phi = 0$. The loss of compactness of the space $A^{\mathbb{N}}$ makes the existence of a conformal measure non-trivial. Simple examples (with potentials depending only on two coordinates) show that an invariant probability measure absolutely continuous with respect to a conformal measure, whenever they both exist, may not have exponential decay of correlation.
- If we consider the restriction of the shift to general closed invariant subsets of $A^{\mathbb{N}}$, then the transfer operator may not be defined on the space of continuous functions.

In this paper, we prove a quasi-compactness result for some wide classes of transfer operators defined on a general closed subshift of a shift on an infinite alphabet.

Let us make the content of our result precise.

On one hand, the usual pressure $P(\phi)$ is defined to be the logarithm of the spectral radius of the transfer operator. To overcome the lack of compactness of $A^{\mathbb{N}}$, we introduce another quantity, called *pressure at infinity*, $P_\infty(\phi)$, which plays the same role as the pressure, but taking into account only what happens at infinity. The system has a nice behavior if $P_\infty(\phi) < P(\phi)$.

On the other hand, we measure the regularity of a function, using a weighted sum, $V_\theta(\cdot)$, of its oscillations on each cylinder. The transfer operator will shrink the oscillations of observables if $V_\theta(e^\phi) < +\infty$ and $\|\phi\|_\infty < P(\phi)$.

Such assumptions on the potential guarantee the stability of large Banach spaces. The functions in these spaces satisfy a decreasing condition and a regularity condition. Roughly speaking, they go to zero exponentially fast with the first coordinate and the weighted sum, $V_\theta(\cdot)$, of the oscillations on each cylinder is required to be finite.

More precisely, we prove that the transfer operator acting on these Banach spaces is quasi-compact (Theorem 1).

This first result does not involve any measure. It is possible to prove the existence of a conformal measure under the same kind of assumptions (see [**5**]), but here we assume that

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one exists. In this case and under a topological mixing condition, we prove that the only peripheral eigenvalue is one and that it is simple (Theorem 2).

We deduce that there exists a unique invariant measure absolutely continuous with respect to the conformal measure; it is mixing and has exponential decay of correlations (Corollary 1). Then we apply these results to the case of a statistical mechanics setting (Corollary 2) and to the case of an interval map with an infinite coding partition (Corollaries 3 and 4).

2. *Definitions, notation and statement of the main results*

2.1. *Shift and subshift.* Let A be a countable set (say $A = \mathbb{N}^*$, where $\mathbb{N}^*$ denotes the set of non-zero natural numbers) and let $\mathcal{S} = A^{\mathbb{N}^*}$ be the set of all the sequences of elements of A. The elements of $\mathcal{S}$ are denoted by $x = (x_n)_{n\in\mathbb{N}^*} = (x_1, \dots, x_n, \dots)$. For any integer n and any elements $i_1, \dots, i_n$ of A, the subset $\{x \in \mathcal{S} \mid x_1 = i_1, \dots, x_n = i_n\}$ of $\mathcal{S}$ is called an n-cylinder and is simply denoted by $(i_1, \dots, i_n)$. The set $\mathcal{S}$ is endowed with the product topology and the corresponding Borel σ-algebra $\mathcal{B}$ generated by the finite cylinders.

We consider the map $T : \mathcal{S} \to \mathcal{S}$, $x \to Tx$, where, for all $n \in \mathbb{N}^*$, $(Tx)_n = x_{n+1}$. T is not invertible. Let ix denote the preimage of x in (i), $ix = (i, x_1, \dots, x_n, \dots)$, and, more generally, $i_1 \cdots i_n x = (i_1, \dots, i_n, x_1, \dots, x_n, \dots)$. T is called the *one-sided full shift* (or simply *shift*). We denote by U the linear operator (acting on various spaces) defined by $Uf = f \circ T$.

Let $\mathcal{F}$ be a T-invariant closed subset of $\mathcal{S}$. The induced σ-algebra on $\mathcal{F}$ is generated by the restrictions $(i_1, \dots, i_n) \cap \mathcal{F}$ of the cylinders. The restriction $T_{\mathcal{F}}$ of the map T to the subset $\mathcal{F}$ is called a one-sided *subshift*.

Definition 1. (Topological mixing) We shall say that the subshift $T_{\mathcal{F}}$ is *topologically mixing* if and only if, for all cylinder $(i_1, \dots, i_n)$ intersecting $\mathcal{F}$, there exists an integer N such that,

$$T^N((i_1, \dots, i_n) \cap \mathcal{F}) = \mathcal{F}.$$

The subshift is said to be of *finite type* if there is an $N > 0$ and a subset F of A^N such that $\mathcal{F} = \{x \in \mathcal{S} : \forall n \in \mathbb{N}^*, (x_n, \dots, x_{n+N}) \notin F\}$. Since we want to state results for more general subshifts, we will need the following definition.

Definition 2. Let $N > 1$. We shall say that the N-cylinder $(j_1, \dots, j_N)$ in $\mathcal{S}$ is *non-regular* for $\mathcal{F}$ if and only if there are x and y in $(j_2, \dots, j_N) \cap \mathcal{F}$ such that

$$j_1 x \in (j_1, \dots, j_N) \cap \mathcal{F} \quad \text{and} \quad j_1 y \notin (j_1, \dots, j_N) \cap \mathcal{F}.$$

We shall distinguish the set $\mathcal{M}^o$ of the non-regular cylinders, the set $\mathcal{M}^+$ of the regular cylinders that intersect $\mathcal{F}$ and the set $\mathcal{M}^-$ of the regular cylinders that do not intersect $\mathcal{F}$.

2.2. *Banach spaces.* Let α and θ be two real numbers, $\alpha > 1$ and $0 < \theta < 1$. We denote by $\tilde{\alpha}$ the map $\tilde{\alpha} : \mathcal{S} \to \mathbb{R}$ depending only on the first coordinate, defined for all i and all x in the cylinder (i) by $\tilde{\alpha}(x) = \alpha^{-i}$.

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We define the total variation of a function $f : \mathcal{F} \to \mathbb{C}$ by

$$V_\theta(f) := \sum_{n=1}^{+\infty} \theta^n \Big(\sum_{i_1,\dots,i_n} \mathrm{var}_{i_1,\dots,i_n}(f) \Big),$$

where $\mathrm{var}_{i_1,\dots,i_n}(f)$ denotes the oscillation of f on the cylinder $(i_1, \dots, i_n)$,

$$\mathrm{var}_{i_1,\dots,i_n}(f) := \sup_{x,y \in (i_1,\dots,i_n) \cap \mathcal{F}} (|f(x) - f(y)|).$$

We set,

$$[f]_\alpha := \sup_{j \in \mathbb{N}} \left(\alpha^j \sup_{x \in (j) \cap \mathcal{F}} |f(x)| \right) = \left\| \frac{f}{\tilde{\alpha}} \right\|_\infty .$$

We consider the family of sets $\mathcal{X}_{\alpha,\theta} := \{f : \mathcal{F} \to \mathbb{C} \mid V_\theta(f) < +\infty, [f]_\alpha < +\infty\}$ endowed with the norm $\|\cdot\| := V_\theta(\cdot) + [\cdot]_\alpha$. We will prove that these spaces are Banach spaces (Proposition 1). Let $\mathcal{X}_\theta$ denote the space $\mathcal{X}_{1,\theta}$ and let $\mathcal{X}_\alpha$ denote the space $\mathcal{X}_{\alpha,0}$.

For all cylinders $(j_1, \dots, j_N)$, we let $V_\theta^{(j_1,\dots,j_N)}(f)$ denote the variation of f, restricted to the cylinder $(j_1, \dots, j_N)$,

$$\begin{aligned} V_\theta^{(j_1,\dots,j_N)}(f) &:= \sum_{n=1}^{+\infty} \theta^n \Big(\sum_{i_1,\dots,i_n \mid (i_1,\dots,i_n) \subset (j_1,\dots,j_N)} \mathrm{var}_{i_1,\dots,i_n}(f) \Big) \\ &= \sum_{n>N} \theta^n \Big(\sum_{i_{N+1},\dots,i_n} \mathrm{var}_{j_1,\dots,j_N,i_{N+1},\dots,i_n}(f)) \Big). \end{aligned}$$

Finally, we set,

$$\overline{\mathrm{var}}_C(f) := \begin{cases} \mathrm{var}_C(f) & \text{if } C \in \mathcal{M}^+, \\ \sup_{x \in C} |f(x)| & \text{if } C \in \mathcal{M}^o, \\ 0 & \text{if } C \in \mathcal{M}^-, \end{cases}$$

and we denote by $\overline{V}_\theta(f)$ the corresponding total variation

$$\overline{V}_\theta(f) := \sum_{n=1}^{+\infty} \theta^n \Big(\sum_{i_1,\dots,i_n} \overline{\mathrm{var}}_{i_1,\dots,i_n}(f) \Big).$$

Remark 1. It is equivalent to count the variations of f as a function defined on $T^{-1}\mathcal{F}$ and equal to zero on $T^{-1}\mathcal{F} \setminus \mathcal{F}$.

Remark 2. If all the cylinders are regular, or, equivalently, if the sets $\{j \in \mathbb{N}^* \mid jx \in \mathcal{F}\}$ depend only on $i_1(x)$, then $\overline{V}_\theta(f) = V_\theta(f)$ for all f. In this case, the subshift is called *Markovian*.

2.3. *Transfer operators.* To all functions $\phi : \mathcal{F} \longrightarrow \mathbb{R}$ we associate an operator $\mathcal{L}_\phi$, called the *transfer operator*, by

$$\mathcal{L}_\phi(f)(x) := \sum_{y \mid T(y) = x} e^{\phi(y)} f(y),$$

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or, equivalently,

$$\mathcal{L}_\phi(f)(x) = \sum_{j\in\mathbb{N}^*\,|\,jx\in\mathcal{F}} e^{\phi(jx)} f(jx).$$

Notice that these series may not converge. The domain of the operator depends on the behavior of $\phi(ix)$ for large i. For example, if ϕ satisfies $\|\mathcal{L}_\phi(1)\|_\infty < +\infty$, then $\mathcal{L}_\phi$ is defined on the space of bounded functions.

Let $\phi_N = \sum_{k=0}^{N-1} \phi \circ T^k$. We have

$$\mathcal{L}_\phi^N(f)(x) = \sum_{i_1,\dots,i_N\,|\,j_1\cdots j_N x\in\mathcal{F}} e^{\phi_N(j_1\cdots j_N x)} f(j_1\cdots j_N x).$$

Remark 3. We can think of ϕ as defined over $\mathcal{S}$ and set $\phi(x) = -\infty$ if $x \notin \mathcal{F}$. From this point of view, it makes sense to write $\mathcal{L}_\phi^N(f)(x) = \sum_{i_1,\dots,i_N} e^{\phi_N(j_1\cdots j_N x)} f(j_1\cdots j_N x)$ without defining f out of $\mathcal{F}$.

When it is not ambiguous, we shall write simply $\mathcal{L}$ instead of $\mathcal{L}_\phi$.

We denote by $\Sigma(\mathcal{P})$ the spectrum of a bounded operator $\mathcal{P}$ and by $\varrho(\mathcal{P})$ its spectral radius. The essential spectral radius of $\mathcal{P}$ is the infimum over all the compact operators $\mathcal{Q}$ of the spectral radius of the operators $\mathcal{P} - \mathcal{Q}$. An operator is said to be quasi-compact, or, equivalently, to have a spectral gap, if its essential spectral radius is strictly smaller than its spectral radius. In this case, its dominating behavior is compact-like.

2.3.1. *Pressure.* In the thermodynamic formalism, the 'pressure' is interpreted as the logarithm of the spectral radius of a transfer operator [**16, 19**]. Here, we adopt this interpretation as a definition.

Definition 3. We denote by $P(\phi)$ the real number

$$P(\phi) := \lim_{n\to+\infty} \frac{1}{n} \log(\|\mathcal{L}_\phi^n(1)\|_\infty)$$

and call it the *pressure* of ϕ on L^∞.

Let us introduce the notation

$$\Delta_p(\phi) := \sup_{j>p}\left(\sum_{i=1}^{+\infty} e^{\phi(ij)}\right) \quad \text{and} \quad \Delta_{N,p}(\phi) := \sup_{j>p}\left(\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N j)}\right),$$

where $\phi(i_1\cdots i_N)$ stands for $\sup_{x\in(i_1,\dots,i_N)} \phi(x)$.

If $\Delta_0(\phi) < +\infty$, then the sequence $((1/n)\log(\|\mathcal{L}^n(1)\|_\infty))_{n\in\mathbb{N}^*}$ converges to a limit and $P(\phi) < +\infty$. In effect, it is clear that $\|\mathcal{L}(1)\|_\infty \le \Delta_0(\phi) < +\infty$. Hence, the convergence of the sequence follows from its sub-additivity. We notice also that the identity

$$P(\phi + h\circ T - h + c) = P(\phi) + c,$$

holds for all real numbers c and all bounded functions h, but may not hold if h is unbounded. Since $P(\phi - P(\phi)) = 0$, there is no loss of generality in considering only potentials with pressure zero.

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2.3.2. *Pressure at infinity.* The set $\mathcal{F}$ is not compact. To overcome this difficulty, and be able to state results for wide classes of potentials, we introduce a quantity that measures the importance of what 'happens' outside of the compact subsets of $\mathcal{F}$. Let us introduce,

$$\Delta_\infty(\phi) := \inf_{p\in\mathbb{N}}(\Delta_p(\phi)) = \inf_{p\in\mathbb{N}}\left(\sup_{j>p}\left(\sum_{i=1}^{+\infty} e^{\phi(ij)}\right)\right) = \limsup_{j\to+\infty}\sum_{i=1}^{+\infty} e^{\phi(ij)},$$

and the corresponding quantities for iterations of the operator

$$\begin{aligned}\Delta_{N,\infty}(\phi) := \inf_{p\in\mathbb{N}}(\Delta_{N,p}(\phi)) &= \inf_{p\in\mathbb{N}}\left(\sup_{j>p}\left(\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N j)}\right)\right)\\ &= \limsup_{j\to+\infty}\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N j)}.\end{aligned}$$

We define the pressure at infinity as follows.

Definition 4. If $\Delta_0(\phi) < +\infty$, we define the *pressure at infinity* of ϕ by

$$P_\infty(\phi) := \limsup_{N\to\infty}\frac{1}{N}\log\Delta_{N,\infty}(\phi).$$

We notice that the existence of $\delta > 0$ such that $P_\infty(\phi) < \log\delta$ implies the existence of two integers N and p_N such that $\Delta_{N,p_N} < \delta^N$.

Remark 4. It seems possible to work in a more general—but more complicated—framework, using, in place of the Δ's,

$$\overline{\Delta}_0(\phi) = \|\mathcal{L}(1)\|_\infty \quad\text{and}\quad \overline{\Delta}_\infty(\phi) = \inf_{K \text{ compact}}\left(\sup_{x\in K^c}\mathcal{L}_\phi 1(x)\right).$$

2.4. *Suitable classes of potential.* Let $\alpha > 1$ and $0 < \theta < 1$ be two real numbers and consider the following set of conditions:

($C_0 0$) $P(\phi) = 0$;
($C_0 1$) $\lim_{N\to\infty}\|e^{\phi_N}\|_\infty^{1/N} < \theta^2$;
($C_0 2$) $P_\infty(\phi) < P(\phi)$;
($C_0 3$) $\forall N \in \mathbb{N}^*$, $V_\theta(e^{\phi_N}) < +\infty$.

We want to work with some potentials for which the condition ($C_0 3$) is not satisfied, but only the weaker condition $\sup_i(V_\theta^{(i)}(e^\phi)) < +\infty$. This means that the variation of e^ϕ on each cylinder is uniformly bounded but that the sum of these variations can be infinite (this is, for example, the case in [**26**]).

We have to enforce the other assumptions to compensate for this effect. Mainly, we will ask ($C_0 0$) and ($C_0 2$) to remain satisfied for a potential which is 'less expansive at infinity'. We introduce,

$$\varphi := \phi - \log(\tilde{\alpha}\circ T) + \log\tilde{\alpha}.$$

Roughly speaking, $e^{\varphi(ij)}$ is much greater than $e^{\phi(ij)}$ when i is fixed and j goes to infinity. If x is in the cylinder (ij), we have $e^{\varphi(x)} = \alpha^{j-i}e^{\phi(x)}$.

Remark 5. We set $\tilde{\alpha}(x) = \alpha^{-i}$ for $x \in (i)$ for simplicity, but the results and the proofs remain the same if we replace α^{-i} by any sequence (α_i) with $\sum_i \alpha_i < +\infty$.

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We set $P^\alpha(\phi) := P(\varphi)$ and $P^\alpha_\infty(\phi) := P_\infty(\varphi)$. In general, the exponent α will mean that the quantity is calculated for φ instead of ϕ.

Definition 5. We will say that ϕ is in the class $\mathcal{C}_{\alpha,\theta}$ if it satisfies the following conditions:
(C0) $P^\alpha(\phi) = 0$;
(C1) $\lim_{N\to\infty} \|e^{\phi_N}\|_\infty^{1/N} < \theta^2$;
(C2) $P^\alpha_\infty(\phi) < P^\alpha(\phi)$;
(C3) $\sup_i(\overline{V}_\theta^{(i)}(e^\phi)) < +\infty$.

Condition (C2) is difficult to check. Proposition 5 shows how to obtain an upper bound for $P^\alpha_\infty(\phi)$.

Later, we will introduce another class of potentials more directly adapted to the proofs.

2.5. *Statement of the main results.* Now we are able to state our main result.

THEOREM 1. *Let ϕ be a potential of $\mathcal{C}_{\alpha,\theta}$. The operator $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ is quasi-compact. More precisely, there exists $\tau' < 1$ such that $\mathcal{L}_\phi$ has the following spectral decomposition:*

$$\mathcal{L}_\phi = \mathcal{Q} + \mathcal{R}, \quad \mathcal{Q} \text{ compact}, \quad \{1\} \subset \Sigma(\mathcal{Q}) \subset \{0\} \cup \{z \mid |z| = 1\},$$
$$\varrho(\mathcal{R}) \le \tau' < 1, \quad \mathcal{Q}\mathcal{R} = \mathcal{R}\mathcal{Q} = 0.$$

When the spectral radius of $\mathcal{L}_\phi$ is equal to one, we say that a measure m is *conformal* if it satisfies

$$\forall f \in \mathcal{X}_{\alpha,\theta}, \quad \int_{\mathcal{F}} \mathcal{L}_\phi(f)\, dm = \int_{\mathcal{F}} f\, dm.$$

The question of the existence of a conformal measure when the alphabet is infinite is delicate (see [**5**]). However, in different contexts, such a measure appears naturally. In this case, the topological mixing condition yields a simple spectral decomposition of the transfer operator.

THEOREM 2. *Let ϕ be a potential of $\mathcal{C}_{\alpha,\theta}$. Assume that there exists a conformal measure and that the system is topologically mixing. Then, the operator $\mathcal{L}_\phi$ has only one eigenvalue of modulus one. This eigenvalue is simple. It is equal to one. Let ρ denote the corresponding eigenfunction. It is strictly positive. The conformal measure is unique. It charges all the cylinders in $\mathcal{F}$. The spectral projector on the eigenfunction ρ is*

$$\mathcal{Q}(f) = \left(\int_{\mathcal{F}} f\, dm\right)\rho.$$

Let us make a few remarks about these two theorems.

Remark 6. $\tau_\infty = \max(\theta^{-1}\lim_{N\to\infty}\|e^{\phi_N}\|_\infty^{1/N}, \exp(P^\alpha_\infty(\phi)))$ is an upper bound of the essential spectral radius of $\mathcal{L}_\phi$ over $\mathcal{X}_{\alpha,\theta}$ (see Remark 9). Notice that we do not give any sharper bound for the modulus τ' of the second eigenvalue of the operator than $\tau' < 1$.

Remark 7. As we have already mentioned, there are a lot of such results in the literature. Most of them, even in the case of an infinite alphabet, imply the existence of a lower bound (away from zero) for the density ρ. Our assumptions are in some sense weaker since the density is strictly positive but can tend to zero.

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Remark 8. The property of the conformal measure charging all the cylinders is a consequence of the topological mixing condition and plays an important role at the end of the proof of Theorem 2. If this property is added to the hypothesis, then Theorem 2 holds under the weaker topological mixing condition

$$\forall n, i_1, \ldots, i_n, \ \forall x \in \mathcal{F}, \ \exists N, j_1, \ldots, j_N, \mid i_1 \cdots i_n j_1 \cdots j_N x \in \mathcal{F}.$$

2.6. *Consequences.*

2.6.1. *Ergodic results.* The following corollary is a direct consequence of the spectral decomposition of the transfer operator.

COROLLARY 1. *Let T be a topologically mixing subshift on a closed invariant subset $\mathcal{F}$ of $\mathcal{S}$. Let $\alpha > 1, 0 < \theta < 1$ be two real numbers and let ϕ be a potential in $(\mathcal{C}_{\alpha,\theta})$. Assume that there is a conformal measure of the system denoted by m.*

T has a unique invariant probability measure μ, absolutely continuous with respect to m. Its density ρ with respect to m is in $\mathcal{X}_{\alpha,\theta}$. The measure $d\mu = \rho\, dm$ is strongly mixing. Functions of $\mathcal{X}_\theta$ have exponential decay of correlations with respect to μ.

Under the same assumptions, and using quite classical methods, it is possible to prove limit theorems for the system. The central limit theorem follows directly from the summability of the iterates $\mathcal{L}^n 1$ (see [**12**]). Large deviations results can be obtained using perturbations of quasi-compact operator theory (see [**5, 6**]). Convergence of the laws of entrance times in families of rare events can be obtained following [**8**].

2.6.2. *Statistical mechanics.* The potential ϕ gives raise to a compatible system of conditional probabilities through

$$\Pi_\phi(X_1 = i_1, \ldots, X_n = i_n \mid \sigma((X_p)_{p>n}))(x) = \frac{e^{\phi_n(i_1 \cdots i_n T^n x)}}{\sum_{j_1,\ldots,j_n} e^{\phi_n(j_1 \cdots j_n T^n x)}}.$$

A Gibbs state for this system of conditional probabilities is a probability measure whose conditional probabilities have precisely these values. Namely, m is a Gibbs state for the system of conditional probabilities associated to ϕ if and only if it satisfies

$$m(f \mid T^n \mathcal{B})(x) = \frac{\mathcal{L}_\phi^n f(T^n x)}{\mathcal{L}_\phi^n 1(T^n x)} \quad \text{for } m\text{-a.e. } x.$$

Adopting this point of view, it is not very difficult to deduce the following corollary from Theorems 1 and 2.

COROLLARY 2. *Let T be a topologically mixing subshift on a closed invariant subset $\mathcal{F}$ of $\mathcal{S}$. Let $\alpha > 1, 0 < \theta < 1$ be two real numbers and let ϕ be a potential in $(\mathcal{C}_{\alpha,\theta})$. Assume that there is a conformal measure of the system denoted by m.*

m is a Gibbs state for the potential ϕ. There exists a unique function ρ and a unique invariant measure μ satisfying

$$\mu(f \mid T^n \mathcal{B})(x) = \mathcal{L}^n_{\phi - \log(\rho \circ T) + \log(\rho)} f(T^n x) \quad \textit{for } \mu\textit{-a.e. } x.$$

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This measure is a Gibbs state for the potential $\phi - \log(\rho \circ T) + \log(\rho)$, *absolutely continuous with respect to* m. *It is translation invariant, ergodic and mixing. The decay of correlations for functions in* $\mathcal{X}_\theta$ *is exponential.*

2.6.3. *Interval maps.* Let (I, F) be a topologically mixing, piecewise C^1 interval map with a countable number of subintervals of monotonicity. If it is expanding, it is possible to encode it using its monotonicity partition. Hence, we shall identify it with a subshift $\mathcal{F}$ on an infinite alphabet.

Let m be the image of the Lebesgue measure on $\mathcal{F}$. It is conformal for the potential

$$\phi = -\log F'.$$

The corresponding transfer operator is then called a Ruelle–Perron–Frobenius operator.

From Theorems 1 and 2 we deduce the following.

COROLLARY 3. *Let* (I, F) *be a topologically mixing interval map, with an infinite coding partition.*

If $-\log F'$ *is in* $\mathcal{C}_{\alpha,\theta}$, *then there exists a unique probability measure* μ, *invariant and absolutely continuous with respect to the Lebesgue measure. Moreover,* μ *is ergodic, mixing, and functions with bounded variations have exponential decay of correlations.*

Now we consider, as an example, a set of assumptions—close to the assumptions of the folklore theorem—under which it is possible to use our result.

COROLLARY 4. *Let* $I = [0, 1]$ *be the unit interval, and suppose* $\{I_1, I_2, \dots\}$ *is a countable collection of disjoint open subintervals of* I *such that* $\cup_i I_i$ *has the full Lebesgue measure in* I. *Suppose there are constants* $K_0 > 1$ *and* $K_1 > 0$ *and mappings* $f_i : I_i \to I$ *satisfying the following conditions:*

(1) f_i *extends to a* C^2 *diffeomorphism from* $\bar{I}_i$ *onto* I *and* $\inf_{x \in I_i} |f_i'(x)| > K_0$ *for all* i;

(2) $\sup_{x \in I_i} (f_i''(x)/f_i'(x))|I_i| < K_1$ *for all* i;

(3) $M = -\sum_i |I_i| \log |I_i| < +\infty$.

Then the mapping F *defined by* $F(x) = f_i(x)$ *for* $x \in I_i$ *has unique invariant ergodic probability measure* μ *absolutely continuous with respect to the Lebesgue measure on* I. *This measure is mixing and functions with bounded variations have exponential decays of correlations.*

In this context, Theorems 1 and 2 do not apply directly because the natural potential does not satisfy $P_\infty^\alpha < 0$ for any suitable sequence α. We introduce a conjugate potential which fits the assumptions of our result and leads to the conclusion for the original system. The proof is in §4.3.2.

3. *Proof of the theorems*

3.1. *Main steps of the proof.* Preliminary results are contained in §3.2. Firstly, we prove that the spaces $\mathcal{X}_{\alpha,\theta}$ are complete (Proposition 1). Then we prove an inequality bounding the variation of the image of a function in $\mathcal{X}_{\alpha,\theta}$ (Lemma 1). This inequality plays the same kind of role as the 'basic inequality' in [**16**].

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In §3.3 we give a criterion for the continuity of a transfer operator over both $\mathcal{X}_\alpha$ and $\mathcal{X}_{\alpha,\theta}$ (Lemma 2).

In §3.4, under the same assumptions, we compute an upperbound for the essential spectral radius of $\mathcal{L}_\phi$ as an operator on $\mathcal{X}_{\alpha,\theta}$ (Proposition 2). We complete the proof of this result in three steps. Firstly, we write an explicit decomposition $\mathcal{L}^N = Q + R$, where Q is compact. Then we introduce a balanced norm for which the difference R between the operator and the compact part can be explicitly bounded (Lemma 3). Finally, we conclude by noticing that the spectral radius of R is an upper bound for the essential spectral radius of $\mathcal{L}^N$.

In §3.5 we introduce a sequence, $\mathcal{C}^{(N)}_{\alpha,\theta}$, of classes of potentials (Definition 6). In view of Proposition 2, it is clear that the corresponding transfer operators are continuous and have essential spectral radius strictly smaller than one, whereas their spectral radius is exactly one. A technical result (Lemma 4) yields for these operators a spectral decomposition with the properties required in the conclusion of Theorem 1.

§3.6 contains the proof of technical results (Lemmas 5 and 6) relating the regularity properties of ϕ_N to those of ϕ. We apply these results to achieve the proof of Theorem 1, showing that all ϕ in $\mathcal{C}_{\alpha,\theta}$ is in $\mathcal{C}^{(N)}_{\alpha,\theta}$ for some N (Proposition 4). Then we state and prove an auxiliary result about pressure at infinity (Proposition 5).

In §3.7 we prove Theorem 2. Using the existence of a conformal measure, we exhibit a positive invariant function (Lemma 7). Using the topological mixing property, we prove that there cannot exist any other invariant direction (Lemma 8). We deduce that the conformal measure is the only one. Finally, we prove that one is the only eigenvalue of modulus one (Lemma 9).

3.2. *Preliminary results.*

3.2.1. *Completeness of the spaces.*

PROPOSITION 1. *The spaces $\mathcal{X}_{\alpha,\theta}$ are Banach spaces.*

Proof. We must prove these spaces are complete. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of $\mathcal{X}_{\alpha,\theta}$. The sequence $(g_n)_{n\in\mathbb{N}}$ defined by $g_n = f_n/\tilde{\alpha}$ is a Cauchy sequence in L^∞, which is a Banach space. Hence, it converges in L^∞ toward a limit g. We deduce that $(f_n)_{n\in\mathbb{N}}$ converges in $\mathcal{X}_\alpha$ toward $f = \tilde{\alpha}g$.

Now, we choose a real number $\epsilon > 0$ ($\epsilon < 1/(2(1+\theta))$), and we exhibit an integer N_1 such that, for all $n > N_1$,

$$V_\theta(f_n - f) \leq 4\epsilon.$$

To do so, we first choose a way of counting the cylinders. Since the set of all the finite cylinders is countable there is a one-to-one map $k \to C_k$ such that $\{C_k, k \in \mathbb{N}\}$ is exactly the set of finite cylinders. Let $l(k)$ denote the length of the cylinder C_k. Then we associate to the kth cylinder $C_k = (i_1, \ldots, i_{l(k)})$ an integer N_k with the following properties:

- $\forall n \geq N_k$, $\|f_n - f\|_\infty \leq \epsilon^{i_1+\cdots+i_{l(k)}}$ ($f_n \to f$ in $\mathcal{X}_\alpha$ and also in L^∞);
- $\forall n, m \geq N_k$, $V_\theta(f_n - f_m) \leq \epsilon^k$ ((f_n) is a Cauchy sequence in $\mathcal{X}_{\alpha,\theta}$).

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Let $n > N_1$. For all integer k (and hence for all cylinder C_k), we have,

$$\operatorname{var}_{C_k}(f_n - f) \le \operatorname{var}_{C_k}(f_n - f_{N_2}) + \sum_{j=3}^{k} \operatorname{var}_{C_k}(f_{N_{j-1}} - f_{N_j}) + \operatorname{var}_{C_k}(f_{N_k} - f).$$

Hence, summing,

$$\begin{aligned} V_\theta(f_n - f) &= \sum_{k=1}^{+\infty} \theta^{l(k)} \operatorname{var}_{C_k}(f_n - f) \\ &\le V_\theta(f_n - f_{N_2}) + \sum_{j=3}^{k} V_\theta(f_{N_{j-1}} - f_{N_j}) + \sum_{k=1}^{+\infty} \theta^{l(k)} \| f_{N_k} - f \|_\infty \\ &\le \sum_{j=2}^{k} \epsilon^{j-1} + \sum_{l=1}^{+\infty} \theta^l \sum_{i_1,\dots,i_l} \epsilon^{i_1+\cdots+i_l} \\ &\le \frac{\epsilon}{1-\epsilon} + \frac{\theta\epsilon}{1-\epsilon-\theta\epsilon} \\ &\le 4\epsilon. \end{aligned}$$

□

3.2.2. *Regularity of the images.* Let

$$\gamma_N = \theta^{-N} \| e^{\phi_N} \|_\infty \quad \text{and} \quad v_N = \theta^{-2N} \sup_{j_1,\dots,j_N \in \mathbb{N}^N} \overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N}).$$

We prove the following.

LEMMA 1. *If γ_N and v_N are finite, then*

$$V_\theta(\mathcal{L}^N f) \le \gamma_N \sum_{j_1,\dots,j_N} V_\theta^{(j_1,\dots,j_N)}(f) + \theta^{-N} \sum_{j_1,\dots,j_N} \overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N}) \| f.1_{(j_1,\dots,j_N)} \|_\infty. \tag{1}$$

Proof. We complete the proof for $N = 1$. Firstly, we assume that all the cylinders are regular. For all x and y in $\mathcal{F}$, we have

$$\begin{aligned} |\mathcal{L}f(x) - \mathcal{L}f(y)| &= \left| \sum_{j=1}^{+\infty} (e^{\phi(jx)} f(jx) - e^{\phi(jy)} f(jy)) \right| \\ &\le \sum_{j=1}^{+\infty} e^{\phi(jx)} |f(jx) - f(jy)| + \sum_{j=1}^{+\infty} |f(jy)(e^{\phi(jx)} - e^{\phi(jy)})|. \end{aligned}$$

Hence, for all $x, y \in (i_1, \dots, i_n) \cap \mathcal{F}$,

$$|\mathcal{L}f(x) - \mathcal{L}f(y)| \le \| e^\phi \|_\infty \sum_{j=1}^{+\infty} |f(jx) - f(jy)| + \sum_{j=1}^{+\infty} \| f.1_{(ji_1\cdots i_n)} \|_\infty |e^{\phi(jx)} - e^{\phi(jy)}|.$$

Taking the supremum, we obtain

$$\begin{aligned} \operatorname{var}_{i_1,\dots,i_n}(\mathcal{L}f) &= \sup_{x,y \in (i_1,\dots,i_n)\cap\mathcal{F}} |\mathcal{L}f(x) - \mathcal{L}f(y)| \\ &\le \| e^\phi \|_\infty \sum_{j=1}^{+\infty} \operatorname{var}_{j,i_1,\dots,i_n}(f) + \sum_{j=1}^{+\infty} \| f.1_{(ji_1\cdots i_n)} \|_\infty \operatorname{var}_{j,i_1,\dots,i_n}(e^\phi). \end{aligned}$$

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A sum over all the cylinders yields

$$V_\theta(\mathcal{L}f) = \sum_{n=1}^{+\infty} \theta^n \sum_{i_1,\dots,i_n} \mathrm{var}_{i_1,\dots,i_n}(\mathcal{L}f)$$
$$\leq \|e^\phi\|_\infty \theta^{-1} V_\theta(f) + \sum_{n=1}^{+\infty} \theta^n \sum_{j,i_1,\dots,i_n} \mathrm{var}_{j,i_1,\dots,i_n}(e^\phi)\|f.1_{(j)}\|_\infty.$$

If some cylinders are not regular, then the sets $\{j \in \mathbb{N}^* \mid jx \in \mathcal{F}\}$ and $\{j \in \mathbb{N}^* \mid jy \in \mathcal{F}\}$ can be distinct even if x and y are in the same cylinder. In this case, the sums defining $\mathcal{L}_\phi(f)(x)$ and $\mathcal{L}_\phi(f)(y)$ are made over two different subsets of the alphabet. To overcome this difficulty, we use $\overline{V}_\theta(e^\phi)$ instead of $V_\theta(e^\phi)$. For $x, y \in (i_1, \dots, i_n)$,

$$|\mathcal{L}f(x) - \mathcal{L}f(y)| \leq \left| \sum_{j|(j,i_1,\dots,i_n)\in\mathcal{M}^+} (e^{\phi(jx)} f(jx) - e^{\phi(jy)} f(jy)) \right|$$
$$+ \sum_{j|(j,i_1,\dots,i_n)\in\mathcal{M}^o} e^{\phi(ji_1\cdots i_n)} \|f.1_{(j,i_1,\dots,i_n)}\|_\infty.$$

We obtain,

$$V_\theta(\mathcal{L}f) = \sum_{n=1}^{+\infty} \theta^n \sum_{i_1,\dots,i_n} \mathrm{var}_{(i_1,\dots,i_n)}(\mathcal{L}f)$$
$$\leq \|e^\phi\|_\infty \theta^{-1} V_\theta(f)$$
$$+ \sum_{n=1}^{+\infty} \theta^n \sum_{j,i_1,\dots,i_n\in\mathcal{M}^+} \mathrm{var}_{(j,i_1,\dots,i_n)}(e^\phi)\|f.1_{(j)}\|_\infty$$
$$+ \sum_{n=1}^{+\infty} \theta^n \sum_{j,i_1,\dots,i_n\in\mathcal{M}^o} e^{\phi(ji_1\cdots i_n)} \|f.1_{(j,i_1,\dots,i_n)}\|_\infty,$$

and, finally,

$$V_\theta(\mathcal{L}f) \leq \|e^\phi\|_\infty \theta^{-1} V_\theta(f) + \sum_{n=1}^{+\infty} \theta^n \sum_{j,i_1,\dots,i_n} \overline{\mathrm{var}}_{(j,i_1,\dots,i_n)\cap\mathcal{F}}(e^\phi)\|f.1_{(j)}\|_\infty,$$

or,

$$V_\theta(\mathcal{L}f) \leq \gamma_1 V_\theta(f) + \theta^{-1} \sum_{j=1}^{+\infty} \overline{V}_\theta^{(j)}(e^\phi)\|f.1_{(j)}\|_\infty.$$

For $N > 1$, the proof is analogous. The only point to notice is that the non-regular cylinders for $\mathcal{L}^N$ (let $\mathcal{M}_N^o$ denote their set) are also non-regular for $\mathcal{L}$ (meaning, $\mathcal{M}_N^o \subset \mathcal{M}^o$). In effect, if $(i_1, \dots, i_n)$ is such that there exist $x, y \in (i_{N+1}, \dots, i_n)$ with $i_1 \cdots i_N x \in \mathcal{F}$ and $i_1 \cdots i_N y \notin \mathcal{F}$, then $x' = i_2 \cdots i_N x$ and $y' = i_2 \cdots i_N y$ are such that $i_1 x' \in \mathcal{F}$ and $i_1 y' \notin \mathcal{F}$. □

3.3. *Continuity of* $\mathcal{L}$. Let

$$a_N^\alpha = \theta^{-1} \sum_{i=1}^{+\infty} \alpha^{-i} \overline{V}_\theta^{(i)}(e^{\phi_N}) = \sum_{n=1}^{+\infty} \sum_{i_1,\dots,i_n} \theta^n \alpha^{-i_1} \overline{\mathrm{var}}_{i_1,\dots,i_n}(e^{\phi_N}).$$

We prove the following.

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LEMMA 2. *If γ_N, a_N^α, and $\Delta_{N,0}^\alpha$ are finite, then the operators $\mathcal{L}_\phi : \mathcal{X}_\alpha \to \mathcal{X}_\alpha$ and $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ are continuous.*

Proof. On one hand, according to (1), we have

$$V_\theta(\mathcal{L}^N f) \le \gamma_N \sum_{j_1,\dots,j_N} V_\theta^{(j_1,\dots,j_N)}(f) + \theta^{-N} \sum_{j_1,\dots,j_N} \overline{V}_\theta^{(j_1,\dots,j_N)}(e^{\phi_N}) \| f.1_{(j_1,\dots,j_N)} \|_\infty.$$

Using the finiteness of a_N^α, we deduce

$$\begin{aligned} V_\theta(\mathcal{L}^N f) &\le \gamma_N V_\theta(f) + \sum_{n=1}^{+\infty} \theta^n \sum_{j_1,\dots,j_N,i_1,\dots,i_n} \overline{\mathrm{var}}_{j_1,\dots,j_N,i_1,\dots,i_n}(e^{\phi_N}) \| f.1_{(j_1,\dots,j_N)} \|_\infty \\ &\le \gamma_N V_\theta(f) + \sum_{j_1,\dots,j_N} \alpha^{-j_1} \sum_{n=1}^{+\infty} \theta^n \sum_{i_1,\dots,i_n} \overline{\mathrm{var}}_{j_1,\dots,j_N,i_1,\dots,i_n}(e^{\phi_N}) \alpha^{j_1} \| f.1_{(j_1)} \|_\infty \\ &\le \gamma_N V_\theta(f) + a_N^\alpha [f]_\alpha. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\alpha^i \mathcal{L}^N f(ix)| &\le \left(\sum_{i_1,\dots,i_N} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i x)} \alpha^{i_1} \| f 1_{(i_1)} \|_\infty \right) \\ &\le \Delta_{N,0}^\alpha [f]_\alpha. \end{aligned}$$

We conclude that $\mathcal{L}^N$ is bounded on $\mathcal{X}_\alpha$ and $\mathcal{X}_{\alpha,\theta}$. □

3.4. *Essential spectral radius.* Let $\delta_N > \Delta_{N,\infty}^\alpha(\phi)$. In this section, we will prove the following.

PROPOSITION 2. *If γ_N, a_N^α and $\Delta_{N,0}^\alpha$ are finite, then the essential spectral radius of $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ is bounded by*

$$\tau_N = \sqrt[N]{4 \max(\gamma_N; v_N; \delta_N)}.$$

3.4.1. *Explicit decomposition of $\mathcal{L}$.* Let m be a measure and consider, for all functions f,

$$E^{(N)}(f) = \sum_{i_1,\dots,i_n} 1_{(i_1,\dots,i_n)} \frac{1}{m(i_1,\dots,i_n)} \int_{(i_1,\dots,i_n)} f \, dm.$$

For all finite families of N-cylinders, $\mathcal{K}$ (this letter shall denote the family as well as the union of the corresponding cylinders), we define

$$E_\mathcal{K}^{(N)}(f) = E^{(N)}(f).1_\mathcal{K}.$$

The operator $Q_\mathcal{K}^{(N)} = \mathcal{L}^N E_\mathcal{K}^{(N)}$ describes the action of $\mathcal{L}^N$ on approximations of functions measurable with respect to a finite number of N-cylinders. It is compact. We write

$$\mathcal{L}^N = Q_\mathcal{K}^{(N)} + (\mathcal{L}^N - Q_\mathcal{K}^{(N)}).$$

Let us set,

$$\bar{f} = f - E^{(N)}(f) \quad \text{and} \quad f^* = E^{(N)}(f) - E_\mathcal{K}^{(N)}(f).$$

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3.4.2. *Inequalities.* Let p_N be an integer such that $\Delta^{\alpha}_{N,p_N} < \delta_N$ and set $[f] = \alpha^{-p_N}[f]_{\alpha}$. We define the balanced norm $\|f\|_b = V_\theta(f) + [f]$. The next lemma provides an upper bound for the norm $\|\cdot\|_b$ of the operator $\mathcal{L}^N - Q^{(N)}_{\mathcal{K}}$.

LEMMA 3. *There exists $\mathcal{K}$ such that, for all f in $\mathcal{X}_{\alpha,\theta}$,*

$$V_\theta(\mathcal{L}^N f^*) \le \delta_N [f], \tag{2}$$

$$V_\theta(\mathcal{L}^N \bar{f}) \le (\gamma_N + v_N) V_\theta(f), \tag{3}$$

$$[\mathcal{L}^N f^*] \le \delta_N [f], \tag{4}$$

$$[\mathcal{L}^N \bar{f}] \le \gamma_N V_\theta(f) + 2\delta_N [f]. \tag{5}$$

Proof. Using (1) and $V^{(i_1,\dots,i_N)}_\theta(f^*) = 0$, we obtain

$$\begin{aligned} V_\theta(\mathcal{L}^N f^*) &\le \sum_{j_1,\dots,j_N} \overline{V}^{(j_1,\dots,j_N)}_\theta(e^{\phi_N}) \|f^*.1_{(j_1,\dots,j_N)}\|_\infty \\ &\le \sum_{j_1,\dots,j_N \in \mathcal{K}^c} \alpha^{-j_1} \sum_{n=1}^{+\infty} \theta^n \sum_{i_1,\dots,i_n} \overline{\mathrm{var}}_{j_1,\dots,j_N,i_1,\dots,i_n}(e^{\phi_N}) \alpha^{j_1} \|f.1_{(j_1)}\|_\infty \\ &\le a^{\alpha}_N(\mathcal{K}) \alpha^{p_N} [f], \end{aligned}$$

where $a^{\alpha}_N(\mathcal{K})$ denotes the sum a^{α}_N restricted to the cylinders which are not in $\mathcal{K}$. Since a^{α}_N is finite, one can choose $\mathcal{K}$ big enough for $a^{\alpha}_N(\mathcal{K})$ to be small. In particular, it can be chosen such that $a^{\alpha}_N(\mathcal{K}) \le \delta_N \alpha^{-p_N}$.

Using (1) and $\|\bar{f} 1_C\|_\infty \le \mathrm{var}_C(f)$, we obtain

$$\begin{aligned} V_\theta(\mathcal{L}^N \bar{f}) &\le \gamma_N \sum_{j_1,\dots,j_N} (V^{(j_1,\dots,j_N)}_\theta(\bar{f})) + \theta^{-N} \sum_{j_1,\dots,j_N} \overline{V}^{(j_1,\dots,j_N)}_\theta(e^{\phi_N}) \|\bar{f}.1_{(j_1,\dots,j_N)}\|_\infty \\ &\le \gamma_N \sum_{j_1,\dots,j_N} (V^{(j_1,\dots,j_N)}_\theta(f)) \\ &\quad + \theta^{-N} \sup_{j_1,\dots,j_N} (\overline{V}^{(j_1,\dots,j_N)}_\theta(e^{\phi_N})) \sum_{j_1,\dots,j_N} \mathrm{var}_{j_1,\dots,j_N}(f) \\ &\le (\gamma_N + v_N) V_\theta(f). \end{aligned}$$

The hypothesis says that the sum $\sum_{i_1,\dots,i_N} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)}$ is finite for all i and, hence, converges uniformly for $i \le p_N$. Subsequently, $\mathcal{K}$ can be chosen such that,

$$\max_{i \le p_N} \left(\sum_{i_1,\dots,i_N \in \mathcal{K}^c} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)} \right) < \delta_N.$$

Then, for $i \le p_N$,

$$\begin{aligned} |\alpha^i \mathcal{L}^N f^*(ix)| &\le \max_{i \le p_N} \left(\sum_{i_1,\dots,i_N \in \mathcal{K}^c} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)} \right) [f]_\alpha \\ &\le \delta_N [f]_\alpha. \end{aligned}$$

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For $i > p_N$, we use the definition of δ_N,

$$|\alpha^i \mathcal{L}^N f^*(ix)| \le \sup_{i>p_N} \left(\sum_{i_1,\dots,i_N} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)} \right) [f]_\alpha$$
$$\le \delta_N [f]_\alpha.$$

Finally,

$$[\mathcal{L}^N f^*] \le \delta_N [f].$$

For $i \le p_N$,

$$|\alpha^i \mathcal{L}^N \bar{f}(ix)| \le \alpha^i \sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N i)} \operatorname{var}_{i_1,\dots,i_N}(f)$$
$$\le \alpha^{p_N} \|e^{\phi_N}\|_\infty \sum_{i_1,\dots,i_N} \operatorname{var}_{i_1,\dots,i_N}(f)$$
$$\le \alpha^{p_N} \gamma_N V_\theta(f).$$

The role of the balanced norm appears here. The weights compensate for the term α^{p_N}.

For $i > p_N$, noticing that $[\bar{f}]_\alpha \le 2[f]_\alpha$,

$$|\alpha^i \mathcal{L}^N \bar{f}(ix)| \le \sup_{i>p_N} \left(\sum_{i_1,\dots,i_N} \alpha^{i-i_1} e^{\phi_N(i_1\cdots i_N i)} \right) [\bar{f}]_\alpha$$
$$\le \delta_N 2[f]_\alpha.$$

Hence,

$$[\mathcal{L}^N \bar{f}] \le \gamma_N V_\theta(f) + 2\delta_N [f]. \qquad \square$$

3.4.3. *Conclusion.* Let us choose $\mathcal{K}$ such that the four inequalities of Lemma 3 hold. A combination of these inequalities yields

$$\|\mathcal{L}^N(\bar{f} + f^*)\|_b \le (2\gamma_N + v_N) V_\theta(f) + 4\delta_N [f].$$

We have established, for all N, the existence of a compact operator $Q_{\mathcal{K}}^{(N)}$ such that

$$\|\mathcal{L}^N - Q_{\mathcal{K}}^{(N)}\|_b \le \max(2\gamma_N + v_N; 4\delta_N) \le \tau_N^N.$$

We conclude that the essential spectral radius of $\mathcal{L}$ over $\mathcal{X}_{\alpha,\theta}$ is bounded by τ_N. Since the spectrum of an operator does not change when a norm is replaced by an equivalent norm, this achieves the proof of Proposition 2. $\square$

3.5. *Spectral decomposition of $\mathcal{L}$.* We consider the following classes of potentials.

Definition 6. We shall say that ϕ is in $\mathcal{C}_{\alpha,\theta}^{(N)}$ if:

($C_N 0$) $P^\alpha(\phi) = 0$;

($C_N 1$) $\|e^{\phi_N}\|_\infty < \theta^N/4$;

($C_N 2$) $\Delta_{N,0}^\alpha < +\infty$ and $\Delta_{N,\infty}^\alpha < \frac{1}{4}$;

($C_N 3.1$) $\sup_{i_1,\dots,i_N} (\overline{V}_\theta^{(i_1,\dots,i_N)}(e^{\phi_N})) < \theta^{2N}/4$;

($C_N 3.2$) $a_N^\alpha < +\infty$.

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In this section, we will prove the following.

PROPOSITION 3. *If ϕ is in $\mathcal{C}^{(N)}_{\alpha,\theta}$, for some $N > 0$, then there exists $\tau' < 1$ such that the operator $\mathcal{L}_\phi : \mathcal{X}_{\alpha,\theta} \to \mathcal{X}_{\alpha,\theta}$ has the following spectral decomposition:*

$$\mathcal{L}_\phi = \mathcal{Q} + \mathcal{R}, \quad \mathcal{Q} \text{ compact}, \quad \Sigma(\mathcal{Q}) \subset \{0\} \cup \{z \mid |z| = 1\},$$
$$\varrho(\mathcal{R}) \leq \tau' < 1, \quad \mathcal{Q}\mathcal{R} = \mathcal{R}\mathcal{Q} = 0.$$

This result is slightly more general than Theorem 1. Its proof requires the following lemma.

LEMMA 4. *Let a positive operator $\mathcal{L}$ be continuous on $\mathcal{X}_\alpha$, with a spectral radius ϱ. Assume it is also continuous on $\mathcal{X}_{\alpha,\theta}$, with an essential spectral radius $\varrho_e < \varrho$.*

The spectral radius of $\mathcal{L}$ on $\mathcal{X}_{\alpha,\theta}$ is exactly ϱ. Moreover, the real ϱ is an eigenvalue of $\mathcal{L}$.

Proof. The points of the spectrum of $\mathcal{L}$ on $\mathcal{X}_{\alpha,\theta}$ whose modulus is greater than the essential spectral radius are eigenvalues. However, an eigenfunction of $\mathcal{L}$ in $\mathcal{X}_{\alpha,\theta}$ is also in $\mathcal{X}_\alpha$. The modulus of the corresponding eigenvalue is smaller than the spectral radius of $\mathcal{L}$ on $\mathcal{X}_\alpha$. Hence, the spectral radius of $\mathcal{L}$ on $\mathcal{X}_{\alpha,\theta}$ is smaller than ϱ.

Since $\tilde{\alpha}$ is in $\mathcal{X}_{\alpha,\theta}$, the inequalities

$$\frac{1}{n} \log \|\mathcal{L}^n(\tilde{\alpha})\| \geq \frac{1}{n} \log [\mathcal{L}^n(\tilde{\alpha})]_\alpha \geq \log \varrho,$$

prove that the spectral radius of $\mathcal{L}$ on $\mathcal{X}_{\alpha,\theta}$ is greater than ϱ.

Let λ be one of the eigenvalues of modulus ϱ. There exists a sequence of complex numbers with modulus strictly greater than ϱ, $(\lambda_n)_{n\in\mathbb{N}}$, going to λ as n tend to infinity, and a function h such that the sequence $(\|R(\lambda_n, \mathcal{L})h\| = \|\sum_{k=0}^{+\infty} \lambda_n^k, \mathcal{L}^k(h)\|)_{n\in\mathbb{N}}$ is unbounded.

We deduce that the sequence $(\|R(|\lambda_n|, \mathcal{L})|h|\|)_{n\in\mathbb{N}}$ is not bounded and that, hence, $\lim_n |\lambda_n| = \varrho$ is in the spectrum of $\mathcal{L}$ (for more details, see Proposition V.4.1 in [**21**]). Since the elements of the spectrum with modulus ϱ are isolated, they are eigenvalues. Hence, ϱ is an eigenvalue of $\mathcal{L}$. □

Together with Proposition 2, this lemma yields Proposition 3. In effect, if ϕ is in $\mathcal{C}^{(N)}_{\alpha,\theta}$, then the assumptions of Proposition 2 are satisfied and we have $\tau_N < 1$. Lemma 4 implies the existence of a spectral gap. Hence, there is a curve (for example, a circle) that separates the spectrum into two different subsets:

- one, Σ_0, included in the disk of radius τ_N;
- the other one made of a finite number, k, of isolated points, $\Sigma_1 = (\lambda_i)_{i=1\ldots k}$ out of this circle.

According to Kato [**11**, Theorem 6.17, p. 178], we can define two spectral projectors, P_0 and P_1, associated to Σ_0 and Σ_1, such that, if $\mathcal{R}_0 = \mathcal{L}P_0$ and $\mathcal{Q}_0 = \mathcal{L}P_1$, then

$$\mathcal{L} = \mathcal{Q}_0 + \mathcal{R}_0,$$

with $\mathcal{Q}_0\mathcal{R}_0 = \mathcal{R}_0\mathcal{Q}_0 = 0$, $\Sigma(\mathcal{Q}_0) = (\lambda_i)_{i=1\ldots p}$ and $\Sigma(\mathcal{R}_0) = \Sigma_0$.

The operator $\mathcal{Q}_0$ can be written as the finite sum of the spectral projectors on the (finite-dimensional) eigenspaces associated to each non-zero eigenvalue $(\lambda_i)_{i=1\ldots k}$. Let τ' be

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the biggest modulus of the $(\lambda_i)_{i=1\ldots k}$ whose modulus is strictly smaller than one. To conclude the proof of Proposition 3, we consider the decomposition $\mathcal{L} = \mathcal{Q} + \mathcal{R}$, where the eigenvalues of the compact operator $\mathcal{Q}$ are all of modulus one and the spectrum of $\mathcal{R}$ is included in the disk of radius $\tau' < 1$. □

3.6. *Classes of potential.* Firstly, we will prove intermediate results in order to prove the following proposition.

PROPOSITION 4.

$$\mathcal{C}_{\alpha,\theta} \subset \bigcup_{M\geq 1} \bigcap_{N\geq M} \mathcal{C}^{(N)}_{\alpha,\theta}.$$

Then, we will give, independently, a proof of Proposition 5 providing a way to estimate the pressure at infinity.

3.6.1. *Oscillations.* A first result refers to the local oscillations of the iterated potentials.

LEMMA 5. *For all integer p, there is a constant C_p such that, for all N, r integers, with $0 \leq r \leq p$,*

$$\begin{cases} V_\theta^{(j_1,\ldots,j_{pN+r})}(e^{\phi_{pN+r}}) \leq C_p \|e^{\phi_p}\|_\infty^N V_\theta^{(j_1,\ldots,j_p)}(e^{\phi_p}), \\ v_{pN+r} \leq C_p \|e^{\phi_p}\|_\infty^N v_p. \end{cases}$$

Proof. We prove the result only for $p = 1$ and $r = 0$. The general proof works in the same way. So, we have to prove the existence of a constant C such that, for all integer N,

$$\sup_{i_1,\ldots,i_N} (V_\theta^{(i_1,\ldots,i_N)}(e^{\phi_N})) \leq C\|e^{\phi}\|_\infty^{N-1} \sup_i (V_\theta^{(i)}(e^{\phi})). \tag{6}$$

Firstly, we prove that, for all integer N, all integer $n > N$ and all cylinder $(i_1, \ldots, i_n)$ of length n,

$$\mathrm{var}_{i_1,\ldots,i_n}(e^{\phi_N}) \leq \|e^{\phi}\|_\infty^{N-1} \sum_{k=1}^{N} (\mathrm{var}_{i_k,\ldots,i_n}(e^{\phi})). \tag{7}$$

It clearly holds for $N = 1$ (equality). Assume it holds at step $N - 1$ and write

$$\begin{aligned} \mathrm{var}_{i_1,\ldots,i_n}(e^{\phi_N}) &= \mathrm{var}_{i_1,\ldots,i_n}(e^{\phi_{N-1}} e^{\phi} \circ T^{N-1}) \\ &\leq \|e^{\phi}\|_\infty \mathrm{var}_{i_1,\ldots,i_n}(e^{\phi_{N-1}}) + \mathrm{var}_{i_N,\ldots,i_n}(e^{\phi}) \|e^{\phi}\|_\infty^{N-1} \\ &\leq \|e^{\phi}\|_\infty \left(\|e^{\phi}\|_\infty^{N-2} \sum_{k=1}^{N-1} (\mathrm{var}_{i_k,\ldots,i_n}(e^{\phi})) \right) + \mathrm{var}_{i_N,\ldots,i_n}(e^{\phi}) \|e^{\phi}\|_\infty^{N-1} \\ &\leq \|e^{\phi}\|_\infty^{N-1} \sum_{k=1}^{N} (\mathrm{var}_{i_k,\ldots,i_n}(e^{\phi})). \end{aligned}$$

Hence, (7) holds for all N, $n > N$, and all cylinders $(i_1, \ldots, i_n)$.

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Inequality (6) follows by summing over all the cylinders:

$$\begin{aligned}
V_\theta^{(j_1,\dots,j_N)}(e^{\phi_N}) &= \sum_{n>N}\theta^n\Bigg(\sum_{i_{N+1},\dots,i_n}\operatorname{var}_{j_1,\dots,j_N,i_{N+1},\dots,i_n}(e^{\phi_N})\Bigg)\\
&\le \sum_{n>N}\theta^n\sum_{i_N,\dots,i_n}\left(\|e^\phi\|_\infty^{N-1}\sum_{k=1}^{N-1}(\operatorname{var}_{j_k,\dots,j_N,i_{N+1},\dots,i_n}(e^\phi))\right)\\
&\le \|e^\phi\|_\infty^{N-1}\sum_{k=1}^{N-1}\theta^k\Bigg(\sum_{n>N}\theta^{n-k}\sum_{i_N,\dots,i_n}\operatorname{var}_{j_k,\dots,j_N,i_{N+1},\dots,i_n}(e^\phi)\Bigg)\\
&\le \|e^\phi\|_\infty^{N-1}\sum_{k=1}^{N-1}\theta^k\sup_{j_k,\dots,j_N}(V_\theta^{(j_k,\dots,j_N)}(e^\phi))\\
&\le C\|e^\phi\|_\infty^{N-1}\sup_i(V_\theta^{(i)}(e^\phi)).
\end{aligned}$$

Taking into account the non-regular cylinders, (6) yields,

$$\begin{aligned}
v_N &\le \sup_{i_1,\dots,i_N}(V_\theta^{(i_1,\dots,i_N)}(e^{\phi_N}))+\sum_{n=0}^{+\infty}\theta^n\sum_{i_1,\dots,i_n\in\mathcal{M}_N^o}e^{\phi_N(i_1\cdots i_n)}\\
&\le C\|e^\phi\|_\infty^N\sup_i(V_\theta^{(i)}(e^\phi))+\|e^{\phi_{N-1}}\|_\infty\sum_{n=0}^{+\infty}\theta^n\sum_{i_1,\dots,i_n\in\mathcal{M}^o}e^{\phi(i_1\cdots i_n)}\\
&\le C'\|e^\phi\|_\infty^N v_1.
\end{aligned}$$

□

3.6.2. *Total weighted variation.* The following lemma refers to the quantities a_N^α, weighted sums of the local oscillations of the potential defined in §3.3. It provides a simple criterion for their finiteness.

LEMMA 6. *If $a_1^\alpha<+\infty$, then, for all $N\ge 1$, $a_N^\alpha<+\infty$.*

Proof. We need an inequality sharper than (7). For all $N>1$, and $n>1$,

$$\begin{aligned}
\operatorname{var}_{i_1,\dots,i_n}(e^{\phi_N}) &= \operatorname{var}_{i_1,\dots,i_n}(e^\phi e^{\phi_{N-1}\circ T})\\
&\le e^{\phi(i_1\cdots i_n)}\operatorname{var}_{i_1,\dots,i_n}(e^{\phi_{N-1}}\circ T)+\operatorname{var}_{i_1,\dots,i_n}(e^\phi)e^{\phi_{N-1}\circ T(i_1\cdots i_n)}\\
&\le e^{\phi(i_1i_2)}\operatorname{var}_{i_2,\dots,i_n}(e^{\phi_{N-1}})+\operatorname{var}_{i_1,\dots,i_n}(e^\phi)\|e^{\phi_{N-1}}\|_\infty.
\end{aligned}$$

For $n=1$,

$$\operatorname{var}_{i_1}(e^{\phi_N})\le e^{\phi(i_1)}\|e^{\phi_{N-1}}\|_\infty+\operatorname{var}_{i_1}(e^\phi)\|e^{\phi_{N-1}}\|_\infty.$$

If all the cylinders are regular, then, by recurrence,

$$\begin{aligned}
a_N^\alpha &= \sum_{n=1}^{+\infty}\theta^n\sum_{i_1,\dots,i_n}\alpha^{-i_1}\operatorname{var}_{i_1,\dots,i_n}(e^{\phi_N})\\
&\le \theta\sum_{i_1}\alpha^{-i_1}(e^{\phi(i_1)}\|e^{\phi_{N-1}}\|_\infty+\operatorname{var}_{i_1}(e^\phi)\|e^{\phi_{N-1}}\|_\infty)\\
&\quad+\sum_{n=2}^{+\infty}\theta^n\sum_{i_1,\dots,i_n}\alpha^{-i_1}((e^{\phi(i_1i_2)})\operatorname{var}_{i_2,\dots,i_n}(e^{\phi_{N-1}})+\operatorname{var}_{i_1,\dots,i_n}(e^\phi)\|e^{\phi_{N-1}}\|_\infty)
\end{aligned}$$

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$$
\begin{aligned}
&\leq \theta \|e^{\phi}\|_{\infty}^{N}\left(\sum_{i_1}\alpha^{-i_1}\right) + \|e^{\phi_{N-1}}\|_{\infty}\theta\sum_{i_1}\alpha^{-i_1}\operatorname{var}_{i_1}(e^{\phi}) \\
&\quad + \sum_{n=2}^{+\infty}\theta^{n}\sum_{i_1,\ldots,i_n}\alpha^{i_2-i_1}e^{\phi(i_1i_2)}\alpha^{-i_2}\operatorname{var}_{i_2,\ldots,i_n}(e^{\phi_{N-1}}) \\
&\quad + \|e^{\phi_{N-1}}\|_{\infty}\sum_{n=2}^{+\infty}\theta^{n}\sum_{i_1,\ldots,i_n}\alpha^{-i_1}\operatorname{var}_{i_1,\ldots,i_n}(e^{\phi}) \\
&\leq \theta \|e^{\phi}\|_{\infty}^{N}\left(\sum_{i}\alpha^{-i}\right) \\
&\quad + \sup_{i_2}\left(\sum_{i_1}\alpha^{i_2-i_1}e^{\phi(i_1i_2)}\right)\theta\sum_{n=2}^{+\infty}\theta^{n-1}\sum_{i_2,\ldots,i_n}\alpha^{-i_2}\operatorname{var}_{i_2,\ldots,i_n}(e^{\phi_{N-1}}) \\
&\quad + \|e^{\phi_{N-1}}\|_{\infty}\sum_{n=1}^{+\infty}\theta^{n}\sum_{i_1,\ldots,i_n}\alpha^{-i_1}\operatorname{var}_{i_1,\ldots,i_n}(e^{\phi}) \\
&\leq \theta \|e^{\phi}\|_{\infty}^{N}\left(\sum_{i}\alpha^{-i}\right) + \Delta_0^{\alpha}\theta a_{N-1}^{\alpha} + \gamma_{N-1}a_1^{\alpha} \\
&< +\infty.
\end{aligned}
$$

If some cylinders are not regular, the same argument as in the proof of Lemma 5 yields the result. □

3.6.3. *Conclusion.* Let ϕ be a potential in $\mathcal{C}_{\alpha,\theta}$. We claim that there is an integer N_3 such that, for $N > N_3$, $\phi \in \mathcal{C}_{\alpha,\theta}^{(N)}$. In effect:

- (C0) is the same as (C_N0).
- There exists an integer N_0 such that $\|e^{\phi_{N_0}}\|_{\infty}$ is bounded by θ^2. For $N > N_0$, $\|e^{\phi_N}\|_{\infty}$ decreases exponentially fast, strictly faster than θ^{2N}. Hence, there exists an integer N_1 such that, for $N > N_1$, $\|e^{\phi_N}\|_{\infty} \leq \theta^{2N}/4$. We deduce that, as soon as $N > N_1$, (C_N1) holds for ϕ.
- According to (C2), there exists an integer N_2 (say $N_2 \geq N_1$) such that, for $N > N_2$, (C_N2) holds.
- (C3) and inequality (6) prove that $v_{N_1} < +\infty$. Lemma 5 proves that for $N > N_1$, v_N decreases exponentially fast at rate $\sqrt[N_1]{\|e^{\phi_{N_1}}\|_{\infty}} < \theta^2$. Hence, there exists $N_3 > N_2$ such that, for $N > N_3$, $v_N < \theta^{2N}/4$. This proves that ($C_N3.1$) holds.
- According to Lemma 6, ($C_N3.2$) holds for all N, because $a_1^{\alpha} \leq \sum_i \alpha^{-i} v_1 < +\infty$.

It appears that $\phi \in \mathcal{C}_{\alpha,\theta}^{(N)}$ for all $N > N_3$. □

Propositions 3 and 4 yield Theorem 1.

Remark 9. The proof also yields a bound for the essential spectral radius τ of the operator. Since it is smaller than τ_N, for all $N > N_3$, it is bounded by $\liminf_{N\to+\infty}\tau_N$. Hence, taking Lemma 5 into account, we obtain

$$\log\tau \leq \max\left(\lim_{N\to\infty}\frac{1}{N}\log\gamma_N,\, P_{\infty}^{\alpha}(\phi)\right).$$

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3.6.4. *Pressure at infinity.* The following proposition is useful for obtaining an upper bound of the pressure at infinity of a potential.

PROPOSITION 5. *If* (a) $\Delta_0(\phi) < +\infty$, (b) $\Delta_\infty(\phi) < \delta$ *and* (c) $\lim_{j\to\infty} e^{\phi(ij)} = 0$, *then,*

$$P_\infty(\phi) < \log\delta.$$

Remark 10. This proposition can be stated in the following way. All potential ϕ satisfying (C0), (C1), (C3) and (a) $\Delta_0(\varphi) < +\infty$, (b) $\Delta_\infty(\varphi) < \delta$, (c) $\lim_{j\to\infty} e^{\varphi(ij)} = 0$ for $\varphi = \phi - \log\tilde{\alpha}\circ T + \log\tilde{\alpha}$ are in $\mathcal{C}_{\alpha,\theta}$.

Proof. Notice that, since the sequence $(\Delta_{N,0})_{N\in\mathbb{N}}$ is sub-additive, $\Delta_0 < +\infty$ implies for all N, $\Delta_{N,0} < +\infty$. For all $N \in \mathbb{N}^*$, consider the statement H_N:

$$\exists p_N \in \mathbb{N},\ \sup_{j>p_N}\left(\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N j)}\right) < \delta^N.$$

The statement H_N holds for $N = 1$ (assumption (b)). Assume it holds for an integer N and let p_N be the corresponding integer (it can be chosen greater than p). Then

$$\begin{aligned}
\sum_{i_1,\dots,i_{N+1}} e^{\phi_{N+1}(i_1\cdots i_{N+1}j)} &= \sum_{i_{N+1}} e^{\phi(i_{N+1}j)} \sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_{N+1}j)} \\
&= \sum_{i_{N+1}\le p_N} e^{\phi(i_{N+1}j)} \sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_{N+1}j)} \\
&\quad + \sum_{i_{N+1}>p_N} e^{\phi(i_{N+1}j)} \sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_{N+1}j)} \\
&< \left(\sum_{i\le p_N} e^{\phi(ij)}\right) \sup_{i\le p_N}\left(\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N i)}\right) \\
&\quad + \left(\sum_{i_{N+1}>p_N} e^{\phi(i_{N+1}j)}\right) \sup_{i>p_N}\left(\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_{N+1}i)}\right) \\
&< \left(\sum_{i\le p_N} e^{\phi(ij)}\right)\Delta_{N,0} + \left(\sum_{i_{N+1}>p_N} e^{\phi(i_{N+1}j)}\right)\delta^N.
\end{aligned}$$

For all $\epsilon > 0$, we can choose p_ϵ big enough so that, as soon as $j > p_\epsilon$, $\sum_i e^{\phi(ij)} < \delta$ (assumption (b)) and $\sum_{i\le p_N} e^{\phi(ij)} \le \epsilon$ (assumption (c)). We obtain

$$\sup_{j>p_\epsilon}\left(\sum_{i_1,\dots,i_N} e^{\phi_N(i_1\cdots i_N j)}\right) \le \epsilon\Delta_{N,0} + \delta^{N+1}.$$

This is sufficient to see that H_{N+1} is satisfied. Hence, H_N holds for all $N \ge 1$. We deduce that $P_\infty(\phi) < \log\delta$. □

3.7. *Mixing.* From now on, we assume that the system has a conformal measure, say m, and that it is topologically mixing.

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3.7.1. *Existence of a positive invariant function.*

LEMMA 7. *There exists a positive invariant function.*

Proof. Firstly, we prove that the conformal measure charges all the cylinders of $\mathcal{F}$. Let $(i_1, \ldots, i_n)$ be a cylinder intersecting $\mathcal{F}$. The topological mixing property says that an integer N can be chosen large enough so that, for all x in $\mathcal{F}$, there is a $(j_1, \ldots, j_N)$ with $i_1 \cdots i_n j_1 \cdots j_N x \in \mathcal{F}$, and, hence, $e^{\phi_{n+N}(i_1 \cdots i_n j_1 \cdots j_N x)} > 0$. Since m is conformal,

$$m(i_1, \ldots, i_n) = m(\mathcal{L}^{n+N} 1_{i_1,\ldots,i_n}) = \sum_{j_1,\ldots,j_N} m(e^{\phi_{n+N}(i_1 \cdots i_n j_1 \cdots j_N x)}) > 0.$$

Hence, we can use the conformal measure m in the definition of the part $E^{(N)}$ of the explicit decomposition of $\mathcal{L}^N$ (§3.4). If w is in $\mathcal{X}_{\alpha,\theta}$ and $\int_{\mathcal{F}} |w|\, dm = 0$, then $E^{(N)} w = 0$. We deduce from Lemma 3 that for such a w, $\|\mathcal{L}^N w\|_b \leq \tau_N^N \|w\|_b$.

According to Lemma 4, the real number one is an eigenvalue. Let ρ be an invariant function. The function $\rho^+ := \max(\rho, 0)$ is invariant m-almost everywhere in the following sense. We have $\mathcal{L}(\rho^+) \geq \rho^+$ and $m(\mathcal{L}(\rho^+)) = m(\rho^+)$.

Set $w = \mathcal{L}(\rho^+) - \rho^+$. Since $w \geq 0$ and $m(w) = 0$, we see that $\|\mathcal{L}^N w\|_b \leq \tau_N^N \|w\|_b$. We deduce that $\mathcal{L}^n(\rho^+)$ converges in $\mathcal{X}_{\alpha,\theta}$ as n goes to infinity. The limit is a positive invariant function. □

3.7.2. *Simplicity of the eigenvalue one.*

LEMMA 8. *Let $\rho \in \mathcal{X}_{\alpha,\theta}$ be a (non-identically null) positive invariant function. It is strictly positive. All the invariant functions of $\mathcal{L}_\phi$ are proportional to ρ. There are no other elements in the characteristic space associated with the eigenvalue one. The conformal measure is unique.*

Proof. If $\inf_{C \cap \mathcal{F}} \rho = 0$ for all cylinders C, then for all $x \in \mathcal{F}$ and all integer n,

$$\rho(x) = \mathcal{L}^n \rho(x) = \sum_{i_1,\ldots,i_n} e^{\phi_n(i_1 \cdots i_n x)} \rho(i_1 \cdots i_n x) \leq \theta^n \sum_{i_1,\ldots,i_n} \mathrm{var}_{i_1,\ldots,i_n}(\rho).$$

Hence, $V_\theta(\rho) < +\infty$ would imply $\rho(x) = 0$, $\forall x \in \mathcal{F}$. Since ρ is non-zero, there must be a cylinder C on which ρ is bounded away from zero.

The topological mixing condition ensures the existence of an integer n such that $T^n(C \cap \mathcal{F}) = \mathcal{F}$. Hence, for all $x \in \mathcal{F}$, there exists $(i_1, \ldots, i_n)$ such that $i_1 \cdots i_n x \in C \cap \mathcal{F}$. Hence, $\rho(x) = \mathcal{L}(\rho)(x) \geq e^{\phi(i_1 \cdots i_n x)} \rho(i_1 \cdots i_n x) \geq e^{\phi(i_1 \cdots i_n x)} \inf_C \rho$. Finally,

$$\forall x \in \mathcal{F},\ \rho(x) > 0.$$

Let ρ' be another invariant function, let λ be a real number and set $\rho_\lambda := \rho' - \lambda\rho$. The function ρ_λ is invariant. For the same reasons as in Lemma 7, there exists a set $X \subset \mathcal{F}$, with $m(X) = 1$, such that for all $x \in X$, $\mathcal{L}(\rho_\lambda^+)(x) = \rho_\lambda^+(x)$. On this set, the same argument as for ρ proves that either ρ_λ^+ is zero, or it is strictly positive. Hence, the set $\{\rho' \leq \lambda\rho\} = \{\rho_\lambda^+ = 0\}$ is either of measure zero, or of measure one.

Now we choose λ as the infimum of the real numbers such that $m\{\rho' \leq \lambda\rho\} = 0$. Then we have $\rho' = \lambda\rho$, m-almost everywhere. The same argument as in the proof of Lemma 7

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ensures that $\|\rho_\lambda\|_b = \|\mathcal{L}^N(\rho_\lambda)\|_b \leq \tau_N^N \|\rho_\lambda\|_b$ and, hence, $\|\rho_\lambda\|_b = 0$. We conclude that $\rho' = \lambda\rho$ in $\mathcal{X}_{\alpha,\theta}$. ρ is the only eigendirection associated to the eigenvalue one.

Let us now assume that ρ' is such that $\mathcal{L}_\phi(\rho') = \rho + \rho'$. This implies $m(\rho + \rho') = m(\mathcal{L}_\phi(\rho')) = m(\rho')$ and, hence, $m(\rho) = 0$. However, ρ is strictly positive. We deduce that such a ρ' cannot exist. Hence, the eigenvalue one is simple.

The uniqueness of the conformal measure follows from [**11**, Theorem 6.22, p. 184]. It ensures that the eigenvalue one of the adjoint operator of $\mathcal{L}$ has also multiplicity one. □

3.7.3. *Renormalization.*

Definition 7. Let ρ be the only strictly positive invariant function. We call the *renormalized operator* the operator $\mathcal{P} = \mathcal{L}_{\phi+\log\rho-\log\rho\circ T}$,

$$\forall f \in \mathcal{X}_{\alpha,\theta},\ \mathcal{P}(f) = \frac{1}{\rho}\mathcal{L}_\phi(f\rho).$$

The real number one is an eigenvalue of $\mathcal{P}$ with eigenfunction one and $\mathcal{P}$ is the adjoint of U in L^2_μ, since

$$\int_{\mathcal{F}} U(f)g\,d\mu = \int_{\mathcal{F}} U(f)g\rho\,dm = \int_{\mathcal{F}} f\mathcal{L}_\phi(g\rho)\,dm = \int_{\mathcal{F}} f\mathcal{P}(g)\,d\mu.$$

3.7.4. *Other eigenvalues.*

LEMMA 9. *The only eigenvalue of modulus one is one.*

Proof. Firstly, we prove that, if λ is an eigenfunction of modulus one of $\mathcal{L}$, then λ is a root of unity. The set of the eigenvalues of U is a subgroup of the complex unit circle. Since $\mathcal{P}$ is the adjoint of U, the eigenvalues of modulus one of $\mathcal{P}$ also form a subgroup of the circle.

Let $h \in \mathcal{X}_{\alpha,\theta}$ be an eigenfunction of $\mathcal{L}$, associated to the eigenvalue λ of modulus one. Then $|h| = \rho$ (because it is an invariant function μ-almost everywhere), and, if we set $h = \rho e^{i\omega}$, $\mathcal{P}(e^{i\omega}) = \lambda e^{i\omega}$. We see that λ^p is also an eigenvalue of $\mathcal{P}$ (with eigenfunction $e^{ip\omega}$). Hence λ^p is an eigenvalue of $\mathcal{L}$.

Since the total number of eigenvalues with modulus greater than τ is finite, the subgroup of eigenvalues of modulus one is finite. Hence, it is made of a finite number of roots of the unity.

The eigenfunctions of $\mathcal{L}$ corresponding to eigenvalues of modulus one are invariant with respect to an iterate of $\mathcal{L}$. We prove that they are strictly positive and, hence, proportional to ρ, using the same argument as in Lemma 8.

We conclude that one is the only eigenvalue of modulus one of $\mathcal{L}$ over $\mathcal{X}_{\alpha,\theta}$. □

3.7.5. *Conclusion.* Lemmas 7–9 show that the spectral decomposition $\mathcal{L} = \mathcal{Q} + \mathcal{R}$ holds with

$$\mathcal{Q}(f) = \rho \int_{\mathcal{F}} f\,dm$$

and $\mathcal{R}$ of spectral radius smaller than $\tau' < 1$. Hence, Theorem 2 is proved.

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Remark 11. The topological mixing property is equivalent to the irreducibility of the operator in the sense of Schaefer [**21**, Definition III.8.1, p. 186]. Theorem 2 can be proved using the results describing the peripheral spectrum of a positive irreducible operator [**21**, V.5, pp. 328–333].

4. *Consequences*

We briefly present the links between the main results and the settings in which they might be used. The only aim is to give a taste of the possible applications of these results.

4.1. *Ergodic results.*

The existence of a real eigenfunction associated to the eigenvalue one of $\mathcal{L}$ in $\mathcal{X}_{\alpha,\theta}$ implies the existence of an a.c.i.m. Choose ρ such that $m(\rho) = 1$ and set $d\mu = \rho\, dm$. The measure μ is invariant and absolutely continuous with respect to m.

PROPOSITION 6. (Exponential decay of correlations) *There exists* $\tau' \in]0, 1[$ *such that, for all* f, g *in* $\mathcal{X}_\theta$*, there is a constant* C *such that*

$$\mathrm{Cor}_n(f, g) := \left| \int_{\mathcal{F}} fg \circ T^n \, d\mu - \int_{\mathcal{F}} f \, d\mu \int_{\mathcal{F}} g \, d\mu \right| \leq C\tau'^n.$$

Proof. We have $f\rho \in \mathcal{X}_{\alpha,\theta}$ and $g \in L^\infty$. Hence,

$$\begin{aligned}
\mathrm{Cor}_n(f, g) &= \left| \int_{\mathcal{F}} fg \circ T^n \, d\mu - \int_{\mathcal{F}} f \, d\mu \int_{\mathcal{F}} g \, d\mu \right| \\
&\leq \int_{\mathcal{F}} \left| \mathcal{L}_\phi^n(f\rho) - \int_{\mathcal{F}} f\rho \, dm\rho \right| |g| \, dm \\
&\leq \int_{\mathcal{F}} |\mathcal{R}^n(f\rho)||g| \, dm \leq [\mathcal{R}^n(f\rho)]_\alpha \int_{\mathcal{F}} |g| \tilde{\alpha} \, dm \leq \tau'^n \|f\rho\| \|g\|_\infty \\
&\leq C\tau'^n. \qquad \square
\end{aligned}$$

The set $\{f \mid f\rho \in \mathcal{X}_{\alpha,\theta}\}$ contains $\mathcal{X}_\theta$ and is dense in L^2_μ. The convergence of $\int_{\mathcal{F}} ff \circ T^n \, d\mu$ toward $\left(\int_{\mathcal{F}} f \, d\mu\right)^2$, for all f in a dense subset of L^2_μ, proves the strong mixing. Corollary 1 is proved. $\square$

4.2. *Statistical mechanics.*

The only point we have to notice is the following.

LEMMA 10. *The conformal measure* m_ϕ *of the operator* $\mathcal{L}_\phi$ *is a Gibbs state for the potential* ϕ*.*

Proof. We have to check that, under the measure m, a version of the conditional expectation of f, given that $T^n x = T^n y$, can be written

$$m(f \mid T^n \mathcal{B})(y) = \frac{(U\mathcal{L})^n(f)(y)}{(U\mathcal{L})^n(1)(y)}.$$

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The right-hand side is measurable with respect to $T^n\mathcal{B}$. Let g be a function measurable with respect to $T^n\mathcal{B}$ and write $g = h \circ T^n$ for some h. We have

$$\begin{aligned}\int_{\mathcal{F}} g\frac{(U\mathcal{L})^n(f)}{(U\mathcal{L})^n(1)}\,dm &= \int_{\mathcal{F}}\left(h\frac{\mathcal{L}^n(f)}{\mathcal{L}^n(1)}\right)\circ T^n\,dm\\ &= \int_{\mathcal{F}} h\mathcal{L}^n(f)\,dm\\ &= m(gf).\end{aligned}$$

□

The same computation proves that if ρ is the unique eigenfunction of $\mathcal{L}$, then the measure $d\mu = \rho\,dm$ is a Gibbs state on $A^{\mathbb{N}^*}$, translation invariant, associated to the potential $\phi - \log\rho\circ T + \log\rho$. We prove the other statements of Corollary 2 following the lines of the proof of Corollary 1. □

4.3. *Interval maps.*

4.3.1. *General result.* A simple change of variables proves that the Lebesgue measure is conformal for the RPF operator. Hence, the only point to check to prove Corollary 3 is that the functions of the interval with bounded variations are in $\mathcal{X}_\theta$. The partition of $\mathcal{F}$ into cylinders $(i_1,\dots,i_n)$ corresponds to a partition of the original interval into subintervals $I_{i_1,\dots,i_n}$. Hence, if $\mathrm{Var}_J(f)$ denotes the (usual) variation of f on the interval J,

$$\sum_{i_1,\dots,i_n}\mathrm{var}_{i_1,\dots,i_n}(f) \le \sum_{I_{i_1,\dots,i_n}}\mathrm{Var}_{I_{i_1,\dots,i_n}}(f) \le \mathrm{Var}_I(f)$$

and, hence,

$$V_\theta(f) \le \left(\sum_n \theta^n\right)\mathrm{Var}_I(f).$$

□

4.3.2. *Example.* We give the proof of Corollary 4.

Proof. We denote by dl the Lebesgue measure on I, as well as its image on the shift. We denote by $a_i := |I_i| = l(I_i) = l(i)$. Without loss of generality, we can assume that this sequence is decreasing.

We notice that condition 2 of Corollary 4 implies

$$\forall x, y \in (i),\ \exp\left(-K_1\frac{|x-y|}{a_i}\right) \le \frac{F'(y)}{F'(x)} \le \exp\left(K_1\frac{|x-y|}{a_i}\right)$$

and

$$a_i e^{-K_1} \le \frac{1}{F'(x)} \le a_i e^{K_1}.$$

Let $\psi(x) := -\log F'(x)$. This potential does not satisfy the required properties for the application of Theorem 1. We shall use another potential. Let i_0 be the first integer i such that $a_i \le e^{-K_1}/K_0$. Let $(\underline{a}_n)$ be the sequence defined as follows:

$$\underline{a}_i := \begin{cases} a_{i_0} & \text{if } i < i_0,\\ a_i & \text{if } i \ge i_0.\end{cases}$$

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We set

$$\forall x \in (ij),\ \phi(x) = \psi(x) + \log \underline{a}_j - \log \underline{a}_i.$$

Let us set $1/\sqrt{K_0} < \theta < 1$ and for all integer i, $\alpha_i = -\underline{a}_i \log \underline{a}_i$. According to condition 3, the sequence (α_i) is summable. We shall use our result taking Remark 5 into account. Let us set $\varphi(ijx) = \phi(ijx) - \log \alpha_j + \log \alpha_i$ and notice that there is a constant $\underline{M} > 0$ such that

$$\sum_{i=1}^{+\infty} e^{\varphi(ij)} \le \sum_{i=1}^{+\infty} \frac{\alpha_i}{\alpha_j} a_i e^{K_1} \frac{\underline{a}_j}{\underline{a}_i} \le \frac{\underline{a}_j}{\alpha_j} e^{K_1} \left(\sum_{i=1}^{+\infty} \frac{a_i \alpha_i}{\underline{a}_i} \right) \le \frac{e^{K_1}}{\log \underline{a}_j} \left(\sum_{i=1}^{+\infty} a_i \log \underline{a}_i \right) \le \frac{-\underline{M}}{\log \underline{a}_j}. \tag{8}$$

Now, let us check that the potential ϕ is in $\mathcal{C}_{\alpha,\theta}$.

(C0) The pressure $P^\alpha(\phi) = P(\varphi)$ is finite because $\sum_{i=1}^{+\infty} e^{\varphi(ij)}$ is uniformly bounded. The measure $dm_\varphi = (\alpha_i/\underline{a}_i)\, dl$ is finite and satisfies $m_\varphi(1) = m_\varphi(\mathcal{L}_\varphi^n 1) \le m_\varphi(1) \|\mathcal{L}_\varphi^n 1\|_\infty$. We deduce that $P^\alpha(\phi) \ge 0$. It is enough here because in what follows we can replace ϕ by $\phi - P^\alpha(\phi)$. In fact, it is possible to prove $P^\alpha(\phi) = 0$.

(C1) Let us estimate $e^{\phi(ijx)} = e^{\psi(ijx)}(\underline{a}_j/\underline{a}_i)$. We distinguish three cases:

— if $i \le j$, $e^{\phi(ijx)} \le e^{\psi(ijx)} \le 1/K_0$;

— if $j < i < i_0$, $e^{\phi}(ijx) = e^{\psi(ijx)} \le 1/K_0$;

— if $j < i$ and $i > i_0$, $e^{\phi(ijx)} \le a_i e^{K_1}(\underline{a}_j/a_i) \le \underline{a}_j e^{K_1} \le a_{i_0} e^{K_1} \le 1/K_0$.

(C2) From (8) we deduce that:

(a) $\Delta_0(\varphi) = \sup_j \left(\sum_{i=1}^{+\infty} e^{\varphi(ij)} \right) \le \sup_j (-1/\log \underline{a}_j)\underline{M} < +\infty$;

(b) $\Delta_p(\varphi) \le \sup_{j>p}(-1/\log \underline{a}_j)\underline{M}$ is smaller than one if p is large enough;

(c) $\lim_{j\to\infty} e^{\varphi(ij)} \le \underline{M} \lim_{j\to\infty}(-1/\log \underline{a}_j) = 0$.

Now, we apply Proposition 5 to conclude that $P_\infty^\alpha(\phi) < 0$.

(C3) If $x, y \in (ij)$,

$$\begin{aligned} |e^{\phi(x)} - e^{\phi(y)}| = e^{\phi(y)} |e^{\phi(x)-\phi(y)} - 1| &\le a_i e^{K_1} \frac{\underline{a}_j}{\underline{a}_i} |e^{K_1(|x-y|/a_i)} - 1| \\ &\le a_i e^{2K_1} \frac{\underline{a}_j}{\underline{a}_i} K_1 \frac{|x-y|}{a_i}. \end{aligned}$$

We deduce that

$$\sum_{i_3,\ldots,i_n} \mathrm{var}_{i,j,i_3,\ldots,i_n}(e^\phi) \le \sum_{i_3,\ldots,i_n} a_i K_1 e^{2K_1} \frac{\underline{a}_j}{\underline{a}_i} \frac{l(i_3,\ldots,i_n)}{a_i} \le K_1 e^{2K_1} \frac{\underline{a}_j}{\underline{a}_i} l(i,j).$$

Hence,

$$V_\theta^{(i)}(e^\phi) = \sum_{n>1} \theta^n \sum_{i_2,\ldots,i_n} \mathrm{var}_{i,i_2,\ldots,i_n}(e^\phi) \le \left(\sum_{n>1} \theta^n \right) K_1 e^{2K_1} \frac{a_i}{\underline{a}_i} \le K < +\infty.$$

We have proved that the potential ϕ is in $\mathcal{C}_{\alpha,\theta}$. The measure $\underline{a}_i^{-1}\, dl$ is conformal. The system is topologically mixing. Hence, applying Corollary 1 we obtain the existence of an invariant function ρ for $\mathcal{L}_\phi$ in $\mathcal{X}_{\alpha,\theta}$. The measure $d\mu = \rho \underline{a}_i^{-1}\, dl$ is invariant for the original system and absolutely continuous with respect to the Lebesgue measure on I. The ergodic properties of this invariant measure also follow from Corollary 1. The proof is complete. □

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