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ARTICLE

Proper minor-closed classes of graphs have Assouad–Nagata dimension 2

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Abstract

Asymptotic dimension and Assouad–Nagata dimension are measures of the large-scale shape of a class of graphs. Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [*J. Eur. Math. Society*] showed that any proper minor-closed class has asymptotic dimension 2, dropping to 1 only if the treewidth is bounded. We improve this result by showing it also holds for the stricter Assouad–Nagata dimension. We also characterise when subdivision-closed classes of graphs have bounded Assouad–Nagata dimension.

Keywords: Assouad-Nagata dimension; minor-closed classes; treewidth; asymptotic dimension

2020 MSC Codes: Primary: 05C83; Secondary: 05C12, 51F99

1. Introduction

Asymptotic dimension, denoted by asdim, is a measure of the large-scale shape of a metric space. First introduced by Gromov [12] for the geometric study of groups, it has since been studied for metrics induced by graphs. While the general definition is somewhat technical, it can reduced to a colouring problem on graphs (see Proposition 5, along with Proposition 1.17 of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2]). Specifically, the asymptotic dimension of a class of graphs $\mathcal{G}$ is the smallest integer $n \geqslant 0$ for which there exists an $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that, for each $G \in \mathcal{G}$ and every $r \in \mathbb{R}^+$, G admits an $n+1$-colouring such that every monochromatic component of the rth-power of G has weak diameter at most $f(r)$. We direct the reader to Section 1.1 for the definition of graph powers, weak diameter, and monochromatic components, and Section 1.2 for a formal definition of asymptotic dimension.

Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2] established the following break through result, improving or generalising previous work in [1, 9, 18].

Theorem 1 [2]. *For every proper minor-closed class of graphs $\mathcal{G}$,*

(a) $asdim(\mathcal{G}) \leqslant 2$; and

(b) $asdim(\mathcal{G}) \leqslant 1$ if and only if $\mathcal{G}$ has bounded treewidth.

Asymptotic dimension has also been studied for various non-minor-closed classes in [2, 5, 19].

Another parameter strongly related to asymptotic dimension is Assouad–Nagata dimension, denoted by ANdim. Originally introduced by Nagata [17] prior to the work of Gromov [12] (which introduced asymptotic dimension) and defined analogously to Lebesgue dimension, it is due to an equivalent definition introduced by Assouad [14] that we can view Assouad–Nagata

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dimension as analogous to asymptotic dimension instead. Specifically, Assouad–Nagata dimension is defined identically to asymptotic dimension, except with the extra requirement that the function f is of the form $f(r) = cr$ for some $c \in \mathbb{R}^+$. See Section 1.2 for a formal definition.

Asymptotic and Assouad–Nagata dimension have applications in geometry group theory [6, 12, 13] and for embeddings into Banach spaces [10, 15]. Furthermore, Assouad–Nagata dimension has recently been tied to finding good approximations for the travelling salesman problem [8]. It is immediate from the definition of Assouad–Nagata dimension that it is at least the asymptotic dimension, however no upper bound on the Assouad–Nagata dimension in terms of the asymptotic dimension is possible; indeed, the class of graphs that can be embedded in the plane with at most one crossing per edge has asymptotic dimension 2 but infinite Assouad–Nagata dimension [2].

Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2] also showed the following.

Theorem 2. *For every integer $g \geqslant 0$, the class of weighted graphs with Euler genus at most g has Assouad–Nagata dimension 2.*

The following question arises naturally from the work of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2]: can Theorem 1 be strengthened by replacing asymptotic dimension with Assouad–Nagata dimension. While it was already known that the class of K_t-minor-free graphs has Assouad–Nagata dimension at most $4^t - 1$ [18], we show that the bound in Theorem 1 can be matched exactly in the stronger setting of Assouad–Nagata dimension.

Theorem 3. *For every proper minor-closed class of graphs $\mathcal{G}$,*

(a) $\mathrm{ANdim}(\mathcal{G}) \leqslant 2$; and

(b) $\mathrm{ANdim}(\mathcal{G}) \leqslant 1$ if and only if $\mathcal{G}$ has bounded treewidth.

We conclude this paper with an exact characterisation of when subdivision-closed classes of graphs have bounded Assouad–Nagata dimension.

Theorem 4. *Let $\mathcal{G}$ be a subdivision-closed graph class. Then $\mathcal{G}$ has bounded Assouad–Nagata dimension if and only if there is a graph H such that $\mathcal{G}$ is H-minor-free.*

Theorem 4 immediately implies the previously mentioned result of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2] that the class of graphs that can be embedded in the plane with at most one crossing per edge has infinite Assouad–Nagata dimension, since this class is subdivision-closed and admits every graph H as a minor.

This paper uses many similar techniques to Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2]; in particular Lemma 15 loosely follows the method of Lemma 3.2 in Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2]. Our novel contribution to this method is to use the "completion" of the graph, where edges in the torso are weighted in order to preserve the metric exactly, as opposed to the gadgets used in Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2] which only approximated the metric. Our other major novel contribution in this paper regards how almost-embeddedable graphs are handled; the technique used in Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2] cannot be applied, and while the vortices and the graph embedded on the surface can be handled individually with relevant ease, handling their union is challenging as the union would normally destroy any control functions. We solve this issue by showing that neighbourhood around where the vortices meet the embedded graph is "well-behaved", and show that this is sufficient to allow us to extend our control functions for the vortices and the embedded graph to a control function for the union.

1.1 Basic definitions

Let $\mathbb{R}^+$ denote the set of all strictly positive real numbers, and let $\mathbb{N} := \{1, 2, \ldots\}$.

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We consider only simple graphs, and unless otherwise stated, all graphs are finite. Thus, all further definitions are given assuming graphs are finite, although we remark that most of the definitions extend to infinite graphs. All weighted graphs will have edge weights in $\mathbb{R}^+ \cup \{\infty\}$, and we consider lengths of paths and distances to be with respect to the edge-weights. We require that a subgraph H of a weighted graph G inherits the weighting. We use dist_G to denote distances in G, which can be between vertices or sets of vertices, and may be infinite. We use standard graph-theoretic definitions; see Diestel [3].

Given an graph G and a function $w : E(G) \to \mathbb{R}^+ \cup \{\infty\}$, to *imbue* G with w is to create a new weighted graph with the same vertex and edge sets as G but with weighting w. By contrast, we can remove the weighting of a weighted graph G to recover the underlying unweighted graph, which we denote unweighted(G).

A *class* of graphs is a collection of graphs that is closed under isomorphism. A graph class $\mathcal{G}$ is *hereditary* if every induced subgraph of a graph in $\mathcal{G}$ is also in $\mathcal{G}$. $\mathcal{G}$ *admits* an H-minor if H is a minor of some graph in $\mathcal{G}$; otherwise $\mathcal{G}$ is *H-minor-free*. $\mathcal{G}$ is *subdivision-closed* if for every $G \in \mathcal{G}$, every subdivision of G is in $\mathcal{G}$. $\mathcal{G}$ is *minor-closed* if for every $G \in \mathcal{G}$, every minor of G is in $\mathcal{G}$. A minor-closed class $\mathcal{G}$ is *proper* if it is not the class of all graphs; in this case, $\mathcal{G}$ must be H-minor-free for some H.

For a weighted graph G, a weighted graph H with $V(H) \subseteq V(G)$ is *isometric* in G if $\text{dist}_H(u, v) = \text{dist}_G(u, v)$ for all $u, v \in V(H)$. The *weak diameter* of $S \subseteq V(G)$ in G, denoted $\text{wdiam}_G(S)$, is the maximum distance between vertices of S in G. We remark that "weak diameter" is often referred to as "diameter" in the metric space literature; we use "weak diameter" to avoid confusion with the graph-theoretic notion of "diameter". We also note that the weak diameter of S can be arbitrarily smaller than the diameter of $G[S]$. For example, a cycle can have arbitrarily large diameter, but its vertices have weak diameter 2 within a wheel.

For a real number $r \geqslant 0$, the *r-neighbourhood* of $S \subseteq V(G)$ in G, denoted $N^r_G(S)$, is the set of vertices $v \in V(G)$ such that $\text{dist}_G(v, S) \leqslant r$; note that $N^0_G(S) = S$. In particular, if S is a singleton $\{s\}$, we instead write $N^r_G(s)$. The *r-th power* of a weighted graph G, denoted G^r, is the graph with vertex set $V(G)$ where two vertices u, v are adjacent if $\text{dist}_G(u, v) \leqslant r$. An *$r$-walk* [resp. *$r$-path*] P from $u \in V(G)$ to $v \in V(G)$ in G is a sequence of vertices of G that form a walk [resp. path] from u to v in G^r. This is equivalent to requiring that consecutive vertices are at distance at most r. For technical reasons, we permit walks, and thus r-walks, to have duplicate consecutive vertices. For $Z \subseteq V(G)$, an *r-subwalk* [resp. *r-subpath*] of $P - Z$ is a subsequence of P consisting of a string of consecutive vertices of P which are all not in Z. In particular, an *r-subwalk* [resp. *r-subpath*] of P is any string of consecutive vertices in P. The *interior* of P, denoted $\text{Int}(P)$, is the (possibly empty) r-subwalk [resp. r-subpath] of P consisting of the entire sequence except the start u and the end v. The vertices of $\text{Int}(P)$ are called the *interior vertices* of P. Note that if $\text{Int}(P)$ has weak diameter in G at most d, then P has weak diameter in G at most $d + 2r$.

We consider colourings to be colourings of some set of vertices, and colourings of a graph to be a colouring of its vertex set.

Let G be a graph, and let $S, T \subseteq V(G)$ with $S \subseteq T$. We say that a colouring c_T of T *extends* a colouring c_S of S if $c_T\big|_S = c_S$. Notice that if $S_1, \ldots, S_n \subseteq V(G)$, and c_i is a colouring of S_i for each $i \in \{1, \ldots, n\}$ such that $c_i = c_j$ on $S_i \cap S_j$ for any $i, j \in \{1, \ldots, n\}$, then there exists a colouring c_G of G that extends $c_1, \ldots, c_n$ simultaneously. Furthermore, if $\bigcup_{i=1}^n S_i = V(G)$, then this colouring c_G is unique, and denoted by $\bigcup_{i=1}^n c_i$.

Under a colouring c of a graph G, for any colour i, a set of vertices $S \subseteq V(G)$ is said to be *i-monochromatic* under c if every vertex of S received the colour i under c. S is also said to be *monochromatic* under c if it is i-monochromatic under c for some colour i.

For a weighted graph G and a real number $r > 0$, we say that an *i-monochromatic r-component* of G under c is a maximal nonempty connected subgraph M of G^r such that $V(M)$ is i-monochromatic under c. We then say that a subgraph M of G^r is a *monochromatic r-component* of G under c if M is an i-monochromatic r-component of G for some colour i. Observe that for any

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monochromatic r-component M of G and any $u, v \in V(M)$, there exists a monochromatic r-path from u to v in G of the same colour, and that for any monochromatic r-path P in G, there exists a monochromatic r-component M of G of the same colour such that $P \subseteq V(M)$.

Given a weighted graph G, an integer $m \geqslant 1$, and real numbers $r > 0$, $d \geqslant 0$, we say that a m-colouring c of G is an (m, r, d)-*colouring* if for each monochromatic r-component M of G under c, $V(M)$ has weak diameter in G at most d. Note that this is equivalent to requiring that every monochromatic r-path in G under c has weak diameter in G at most d. This definition is very similar to the concept of weak diameter colourings of G^r used by Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2], with the key difference that the weak diameter is measured in G, rather than G^r. This is done to avoid having to constantly convert between distances in G and distances in G^r. Observe that for any weighted graph H that is isometric in G and any (m, r, d)-colouring c of G, $c\big|_{V(H)}$ is an (m, r, d)-colouring of H.

1.2 Control functions and dimension

Many of the following definitions are given in terms of weighted graphs, however we remark that they can be defined for (and sometimes originate from the study of) arbitrary metric spaces, with the noteworthy difference of "diameter" (in the metric geometry sense) being used in place of "weak diameter" whenever applicable.

For a weighted graph G and real number $r > 0$, $S, T \subseteq V(G)$ are said to be r-*disjoint* if for all $u \in S$, $v \in T$, we have $\mathrm{dist}_G(u, v) > r$. A collection $\mathcal{C}$ of subsets of $V(G)$ is said to be r-*disjoint* if the sets in $\mathcal{C}$ are pairwise r-disjoint.

Given a weighted graph G and an integer $n \geqslant 0$, a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an n-*dimensional control function* for G if, for every real number $r > 0$, there exist $n + 1$ collections $\mathcal{C}_1, \ldots, \mathcal{C}_{n+1}$ of subsets of $V(G)$ such that:

(a) $\bigcup_{i=1}^{n+1} \bigcup_{S \in \mathcal{C}_i} S = V(G)$;
(b) $\mathcal{C}_i$ is r-disjoint for each $i \in \{1, \ldots, n + 1\}$; and
(c) $\mathrm{wdiam}_G(S) \leqslant f(r)$ for each $i \in \{1, \ldots, n + 1\}$ and each $S \in \mathcal{C}_i$.

Note that if f is an n-dimensional control function for G, it is also an m-dimensional control function for G for each integer $m \geqslant n$.

This definition is the standard definition used for general metric spaces, however it is rather cumbersome. Thankfully, given that we are only interested in weighted graphs, we obtain an alternative definition as a consequence of the following result, whose proof is in the appendix.

Proposition 5. *Let $n \geqslant 0$ be an integer, and let G be a weighted graph. Then $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an n-dimensional control function for G if and only if, for every real number $r > 0$, G admits an $(n + 1, r, f(r))$-colouring.*

We use this as the definition of a control function from now on. We remark that Proposition 5 is very similar to Proposition 1.17 of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2], with the key difference that Proposition 1.17 of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2] is stated in terms of asymptotic dimension (see below) instead of control functions. Furthermore, following the proof, one does not actually get an equivalence between control functions and the bound on the weak diameter of vertex sets of monochromatic r-components, instead picking up an extra factor of r.

The *asymptotic dimension* of a weighted graph G, denoted $\mathrm{asdim}(G)$, is the smallest integer $n \geqslant 0$ such that G admits an n-dimensional control function, or ∞ otherwise. While this is consistent with how asymptotic dimension is defined on arbitrary metric spaces, we remark that this definition is not very interesting, since every finite weighted graph G has asymptotic dimension 0. Thus, we extend the definitions of control function and asymptotic dimension to classes of graphs.

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Given an integer $n \geqslant 0$ and a class of weighted graphs $\mathcal{G}$, we say that a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an *n-dimensional control function for $\mathcal{G}$* if f is an n-dimensional control function for each $G \in \mathcal{G}$. The *asymptotic dimension* of $\mathcal{G}$, denoted $\operatorname{asdim}(\mathcal{G})$, is then the smallest integer $n \geqslant 0$ such that $\mathcal{G}$ admits an n-dimensional control function, or ∞ otherwise.

Finding the asymptotic dimension of an infinite weighted graph is non-trivial. Nevertheless, the following result shows a natural way to reduce this problem to a question about finite graphs.

Proposition 6. *Let $n \geqslant 0$ be an integer and let G be an infinite weighted graph. Let $\mathcal{A}$ be the class of all finite induced subgraphs of G, and let f be an n-dimensional control function for $\mathcal{A}$. Then for any real number $\epsilon > 0$, $r \mapsto f((1+\epsilon)r)$ is an n-dimensional control function for G.*

The proof of Proposition 6 is also in the appendix.

It is because of Proposition 6 that we need only consider finite graphs in this paper. We also remark that Proposition 6 is nearly identical to Proposition A.2 of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2], albeit with a slight tweaking to the control function for G; this will be important once we introduce Assouad–Nagata dimension.

A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a *dilation* if there exists a real number $c > 0$ such that $f(r) \leqslant cr$ for every real number $r > 0$. The *Assouad–Nagata dimension* of a graph G is the smallest integer $n \geqslant 0$ such that G admits an n-dimensional control function f that is also a dilation, or ∞ otherwise. Likewise, the *Assouad–Nagata dimension* of a class of graphs $\mathcal{G}$ is the smallest integer $n \geqslant 0$ such that $\mathcal{G}$ admits an n-dimensional control function f that is also a dilation, or ∞ otherwise. Observe that the asymptotic dimension of a class is at most the Assouad–Nagata dimension.

Notice that, in Proposition 6, if f is a dilation, then so is $r \mapsto f((1+\epsilon)r)$. Thus, we can also use Proposition 6 to bound the Assouad–Nagata dimension of an infinite graph in terms of the Assouad–Nagata dimension of the class of its finite induced subgraphs. Thus, it still makes sense to only consider finite graphs even when discussing Assouad–Nagata dimension rather than asymptotic dimension.

Finally, given a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, a weighted graph G, and an integer $n \geqslant 0$, sometimes we can only find $(n+1, r, f(r))$-colourings of G when r is sufficiently large. As such, for a real number $\ell \geqslant 0$, we say that f is an *ℓ-almost n-dimensional control function* for a G if G admits an $(n+1, r, f(r))$-colouring for every real number $r > 0$ with $r \geqslant \ell$. Note that a 0-almost n-dimensional control function is just a n-dimensional control function.

2. Path rerouting

One of the key ideas of this paper is the ability to "adjust" or "reroute" particular r-paths to obtain an r'-path with the same endpoints but whose interior is contained within a "better" part of the graph G. This is formalised in Proposition 7. Note that this concept of rerouting is not completely new, and exists in some capacity within Lemma 3.2 of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2] without being formally identified.

Proposition 7. *Let $r > 0$, $\ell, d \geqslant 0$ be real numbers, let G be a weighted graph, let $x, y \in V(G)$, and let P be an r-path in G from x to y. Assume there exist sets $Z, S \subseteq V(G)$ and a map $\iota : Z \to S$ such that:*

(a) $\operatorname{dist}_G(v, \iota(v)) \leqslant \ell$ *for each $v \in Z$; and*

(b) *every r-subpath of $P - Z$ has weak diameter in G at most d.*

Let $v_1, \ldots, v_n$ be the (possibly empty) subsequence of P that is in Z, including x, y if applicable. Define P' to be the sequence $x, \iota(v_1), \iota(v_2), \ldots, \iota(v_n), y$. Then P' is a $(d + 2r + 2\ell)$-walk from x to y in G whose interior is contained in S.

Proof. It suffices to show that consecutive vertices of P' are at distance at most $d + 2r + 2\ell$. If P does not intersect Z, P is an r-subpath of $P - Z$, and hence $\operatorname{dist}_G(x, y) \leqslant d \leqslant d + 2r + 2\ell$ as desired. So we may assume that P intersects Z, and that $v_1, \ldots, v_n$ are defined with $n \geqslant 1$.

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Consider consecutive interior vertices $\iota(v_i), \iota(v_{i+1})$ of P'; we must show that $\mathrm{dist}_G(\iota(v_i), \iota(v_{i+1})) \leqslant d + 2r + 2\ell$; it suffices to show $\mathrm{dist}_G(v_i, v_{i+1}) \leqslant d + 2r$. If v_i, v_{i+1} are consecutive in P, then $\mathrm{dist}_G(v_i, v_{i+1}) \leqslant r$, as desired. Otherwise, let u_i be the vertex directly after v_i in P, and let w_{i+1} be the vertex directly before v_{i+1} in P; it suffices to show that $\mathrm{dist}_G(u_i, w_{i+1}) \leqslant d$. Consider the r-subpath P_i of P going from u_i to w_{i+1}. $P_i \cap Z = \emptyset$ by definition of the subsequence $v_1, \ldots, v_n$, thus P_i is an r-subpath of $P - Z$ and has weak diameter in G at most d. Therefore, $\mathrm{dist}_G(u_i, w_{i+1}) \leqslant d$, as desired.

It remains to show that $\mathrm{dist}_G(x, \iota(v_1))$ and $\mathrm{dist}_G(y, \iota(v_n))$ are at most $d + 2r + 2\ell$. We show that $\mathrm{dist}_G(x, \iota(v_1)) \leqslant d + 2r + 2\ell$; the argument for $\mathrm{dist}_G(y, \iota(v_n))$ is symmetric. If $x \in Z$, then $v_1 = x$ and thus $\mathrm{dist}_G(x, \iota(v_1)) = \mathrm{dist}_G(x, \iota(x)) \leqslant \ell \leqslant d + 2r + 2\ell$. Otherwise, let w_1 be the vertex directly before v_1 in P; it suffices to show that $\mathrm{dist}_G(x, w_1) \leqslant d + r + \ell$. Let P' be the r-subpath of P going from x to w_1; we must have $P' \cap Z = \emptyset$ by definition of v_1, hence P' is an r-subpath of $P - Z$. Thus, $\mathrm{wdiam}_G(P') \leqslant d$; in particular $\mathrm{dist}_G(x, w_1) \leqslant d \leqslant d + 2r + 2\ell$, as desired. □

An important application for Proposition 7 is when the set S is small. To help express this, we borrow an idea from Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [3]. Given a weighted graph G, $Z \subseteq V(G)$ is said to be a (k, ℓ)-centred set in G if there exists a set $S \subseteq V(G)$ with $|S| \leqslant k$ such that $Z \subseteq N_G^\ell(S)$. We call S a *centre* of Z. Note that for any weighted graph G of radius at most ℓ, $V(G)$ is a $(1, \ell)$-centred set in G.

Corollary 8. *Let $k \geqslant 1$ be an integer, let $r > 0$, $\ell \geqslant 0$ be real numbers, let G be a weighted graph, let Z be a (k, ℓ)-centred set in G, and let $x, y \in Z$ be such that there exists an r-path P from x to y in G that is contained in Z. Then $\mathrm{dist}_G(x, y) \leqslant (k + 1)(2r + 2\ell)$.*

Proof. Let S denote a centre of Z, and define $\iota : Z \to S$ such that $\iota(v) \in N_G^\ell(v) \cap S$ for each $v \in Z$. Since $P \subseteq Z$, it is immediate that every r-subpath of $P - Z$ has weak diameter in G at most 0. We also know that for any $v \in Z$, $\mathrm{dist}_G(v, \iota(v)) \leqslant \ell$. Thus, we can apply Proposition 7 to obtain a $(2r + 2\ell)$-walk from x to y whose interior is in S. Since $|S| \leqslant k$, P' can involve at most $k + 2$ distinct vertices, and thus $\mathrm{dist}_G(x, y) \leqslant (k + 1)(2r + 2\ell)$, as desired. □

The next result is very similar to Lemma 2.1 of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2], albeit the original's bound as stated is not tight enough for our purposes. We instead use Proposition 7 for a different, quicker, proof that avoids this problem.

Corollary 9. *Let G be a weighted graph, let $r > 0$, $d, \ell \geqslant 0$ be real numbers, let $m \geqslant 1$ be an integer, let C be a set of colours of size m, and let $Z \subseteq V(G)$ be a (k, ℓ)-centred set in G such that $G - Z$ admits an (m, r, d)-colouring c with colours C. Then for any colouring c' of Z with colours C, the colouring $c \cup c'$ of G is a $(m, r, (k + 1)(d + 4r + 2\ell))$-colouring with colours C.*

Proof. Let S denote a centre of Z, and let $Z' := N_G^{r+\ell}(S)$. For any $u, v \in V(G) \setminus Z'$ at distance at most r in G and any shortest path Q from u to v in G, observe that Q is disjoint from Z, as otherwise we would have $u, v \in N_G^r(Z) \subseteq Z'$. Thus, Q is also a path of length at most r in $G - Z$, and $\mathrm{dist}_{G-Z}(u, v) \leqslant r$. Therefore, for any monochromatic r-path P in G under $c \cup c'$, any r-subpath P' of $P - Z'$ is a monochromatic r-path in $G - Z$ under c. Thus, P' has weak diameter in $G - Z$ (and G) at most d. Define a map $\iota : Z' \to S$ by selecting $\iota(z) \in N_G^{r+\ell}(z) \cap S$ for each $z \in Z'$; by Proposition 7, we obtain a $(d + 2r + 2(r + \ell)) = (d + 4r + 2\ell)$-walk which has the same endpoints as P but whose interior is contained in S. Since $|S| \leqslant k$, this $(d + 4r + 2\ell)$-walk involves at most $k + 2$ distinct vertices, and thus has weak diameter at most $(k + 1)(d + 4r + 2\ell)$ in G, as desired. □

The main purpose of Proposition 7 and Corollaries 8 and 9 is to give sufficient conditions under which we can show that a given colouring is a (m, r, D)-colouring. With this in mind, we introduce one more tool to do this in a non-trivial setting, however we need to introduce a bit more terminology first.

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A *separation* of a graph G is a pair of subgraphs (A, B) of G such that $G = A \cup B$. The set $S := V(A \cap B)$ is called the *separator* of the separation.

Next, given a real number $r > 0$, a set of colours C of size at least 2, a weighted graph G, $S \subseteq V(G)$, and $Z \supseteq N_G^{3r}(S)$, we say that a colouring c of Z with colours C has an (S, r)-*barrier* in G if there exist distinct colours $\alpha, \beta \in C$ such that:

(a) for each $v \in N_G^r(S) \setminus S$, there exists $s \in N_G^r(v) \cap S$ with $c(s) = c(v)$

(b) $N_G^{2r}(S) \setminus N_G^r(S)$ and $N_G^{3r}(S) \setminus N_G^{2r}(S)$ are α-monochromatic and β-monochromatic respectively under c.

Note that this definition is not completely new; something similar was used in the proof of Lemma 3.2 in Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2] without being formally identified. Note that for any colouring c_S of S with colours C, there exists a colouring of Z with colours C that extends c_S and admits an (S, r)-barrier. Further, note that the presence of an (S, r)-barrier in G is preserved upon extending a colouring.

The choice of name "barrier" comes from the following observation. For any r-path P from $u \in N_G^{2r}(S)$ to $v \in V(G) \setminus N_G^{2r}(S)$, P must contain a vertex in both $N_G^{2r}(S) \setminus N_G^r(S)$ and $N_G^{3r}(S) \setminus N_G^{2r}(S)$. Since these two sets are monochromatic of different colours, this means that P cannot be monochromatic. Hence, this colouring contains a "barrier" near S that blocks monochromatic r-paths. The requirement on how $N_G^r(S) - S$ is coloured is to prevent a "sudden" change in colouring, in the sense that vertices close to S still use similar colours to S, and the monochromatic sets only appear further away from S. This is not important when G is the full graph, however we use it when G is one half of a separation of a larger graph, and S is the separator. Property (a) then prevents a sudden transition of colours upon crossing the separator, and property (b) prevents monochromatic r-components from leaking far across the separator.

We are now in position to state the main tool of this section, Lemma 10. While the statement of Lemma 10 is new, the proof uses similar techniques to the later parts of the proof of Lemma 3.2 in Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [3].

Lemma 10. *Let $r > 0$, $\ell \geqslant 0$ be real numbers, let $k \geqslant 1$ be an integer, and set $\ell' := (k+1)(6r + 2\ell)$ and $r' := 2r + 2\ell'$. Let $m \geqslant 2$ be an integer, let C be a set of colours of size m, and let $d, D > 0$ be real numbers such that $D \geqslant d + 2r'$. Let G be a weighted graph, and for some integer $a \geqslant 0$, let $G_0, G_1, \ldots, G_a$ be isometric subgraphs of G such that:*

(a) $G = \bigcup_{i=0}^a G_i$;

(b) for each $i \in \{1, \ldots, a\}$, if $\widetilde{G}_i := \bigcup_{j=0, j\neq i}^a G_j$ and $S_i := V(G_i \cap \widetilde{G}_i)$, then:

(i) $S_i \subseteq V(G_0)$; and

(ii) S_i is a (k, ℓ)-centred set in G;

(c) G_0 admits a (m, r', d)-colouring c_0 with colours C;

(d) for each $i \in \{1, \ldots, a\}$, G_i admits a colouring c_i with colours C such that:

(i) c_i is an (m, r, D)-colouring of G_i;

(ii) $c_i = c_0$ on S_i; and

(iii) c_i has an (S_i, r)-barrier.

Then $c := \bigcup_{i=0}^a c_i$ is a well-defined (m, r, D)-colouring of G with colours C.

Proof. If $a = 0$ the claim is trivially true, as $G = G_0$ by (a) and $c = c_0$ is an (m, r, D)-colouring of $G_0 = G$ by (c), since $r' \geqslant r$ and $d \leqslant D$. So assume $a \geqslant 1$.

Observe that c is a well-defined colouring of G by (a), (b)(i) and (d)(ii). So it suffices to show that for every real number $r > 0$ and every monochromatic r-component M of G under c, $\mathrm{wdiam}_G(V(M)) \leqslant D$.

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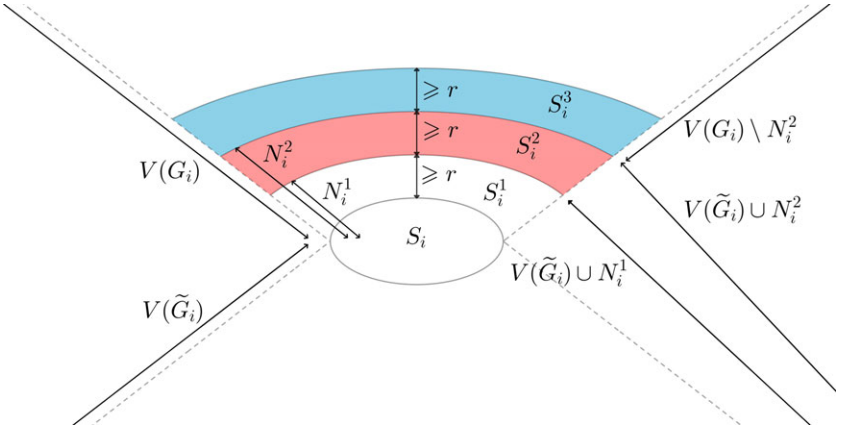


Figure 1. A diagram of the separation $(G_i, \widetilde{G}_i)$ of G for some $i \in \{1, \dots, a\}$, with several relevant sets of vertices near the separator S_i labelled. Vertices and edges of G are not depicted, instead the dotted line denotes a region in which all vertices and edges are contained. Observe that any r-path P from a vertex in $V(G_i) \setminus N_i^2$ to a vertex in $V(\widetilde{G}_i) \cup N_i^2$ must contain a vertex in both S_i^2 and S_i^3, which are monochromatic of different colours, depicted here as red and blue. Thus, P cannot be monochromatic. Additionally, observe that any r-path from a vertex in S_i^2 to a vertex in $\widetilde{G}_i \cup N_i^1$ must contain a vertex in S_i^1.

By (a), for each $i \in \{1, \dots, a\}$, $(G_i, \widetilde{G}_i)$ is a separation of G with separator S_i, which we use implicitly throughout this proof. Also, note that $\bigcap_{i=1}^{a} V(\widetilde{G}_i) = V(G_0)$ by (b)(i).

For all $i \in \{1, \dots, a\}$ and $j \in \{1, 2, 3\}$, let $N_i^j := N_{G_i}^{jr}(S_i)$ and $S_i^j := N_{G_i}^{jr}(S_i) \setminus N_{G_i}^{(j-1)r}$. Note that for each $i \in \{1, \dots, a\}$, since c_i has an (S_i, r)-barrier, S_i^2, S_i^3 are both monochromatic of different colours, and for each $v \in S_i^1$ there is an $s \in S_i \cap N_{G_i}^r(v) \subseteq S_i \cap N_G^r(v)$ with the same colour as v. Let $V' := V(G_0) \cup \bigcup_{i=1}^{a} N_i^2$.

Claim: *For each monochromatic r-component M of G under c, $V(M)$ is contained either in $V(G_i)$ for some $i \in \{1, \dots, a\}$, or in V'.*

Proof. For any $i \in \{1, \dots, a\}$, any r-path P from $u \in V(\widetilde{G}_i) \cup N_i^2$ to $v \in V(G_i) \setminus N_i^2$ must contain a vertex in S_i^2 and S_i^3; see Fig. 1. As S_i^2 and S_i^3 are monochromatic of different colours, P cannot be monochromatic. It follows that for any monochromatic r-component M of G, $V(M)$ must be contained in either $V(\widetilde{G}_i) \cup N_i^2$ or $V(G_i) \setminus N_i^2$. Accounting for the fact that this holds for each $i \in \{1, \dots, a\}$, we can conclude that $V(M)$ is either contained in $V(G_i) \setminus N_i^2 \subseteq V(G_i)$ for some $i \in \{1, \dots, a\}$, or in $\bigcap_{i=1}^{a} (V(\widetilde{G}_i) \cup N_i^2) = V'$. □

By assumption, $G_0, \dots, G_a$ are all isometric subgraphs of G. Thus, any monochromatic r-component M of G under c whose vertices are contained in $V(G_i)$ for some $i \in \{1, \dots, a\}$ is also a monochromatic r-component of G_i under c_i; in this case $V(M)$ has weak diameter at most D in G_i and G. Similarly, any monochromatic r-component M of G under c whose vertices are contained in $V(G_0)$ is a subgraph of a monochromatic r'-component of G_0 under c_0 as $r \leqslant r'$, and thus $V(M)$ has weak diameter at most $d \leqslant D$ in G_0 and G. So henceforth, we only need to consider monochromatic r-components whose vertices are contained in V' but are not contained in any $V(G_i)$, $i \in \{0, \dots, a\}$. Consider such a monochromatic r-component M.

Claim: *For each $v \in V(M)$, there exists $s \in V(G_0) \cap N_G^{\ell'}(v)$ with the same colour as v under c.*

Proof. This is trivially true for any $v \in V(G_0)$, and for any $v \in S_i^1$ for some $i \in \{1, \dots, a\}$, the claim follows since c_i has an (S_i, r)-barrier and $r \leqslant \ell'$. Since $V(M) \subseteq V'$, we now only need to consider $v \in S_i^2$ for some $i \in \{1, \dots, a\}$. Observe that $V(M)$ is not contained in S_i^2, as it is not contained in $V(G_i)$, thus there must exist some $u \in V(M) \setminus S_i^2$; since $V(M) \subseteq V'$, this means that

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$u \in V(\widetilde{G}_i) \cup N_i^1$. Because u, v are both in M, there exists a monochromatic r-path P from $v \in S_i^2$ to $u \in V(\widetilde{G}_i) \cup N_i^1$ in G of the same colour. Observe that P must contain a vertex in S_i^1; see Fig. 1. Let x be the first (in the order of the sequence of P) such vertex in $S_i^1 \cap P$; observe that every vertex before x must be in S_i^2. Thus, the r-subpath P' of P from v to x is an r-path contained in $S_i^1 \cup S_i^2 \subseteq N_i^2$. Since c_i has an (S_i, r)-barrier, there must be an $s \in S_i \cap N_G^r(x) \subseteq V(G_0)$ with the same colour as x. Hence, v and s have the same colour, and there is an r-path contained in N_i^2 from v to s obtained by appending s to the end of P'. Since S_i is a (k, ℓ)-centred set in G, N_i^2 is a $(k, \ell + 2r)$-centred set in G. Thus, we can apply Corollary 8 to find that $\mathrm{dist}_G(v, s) \leqslant (k+1)(2r + 2(2r + \ell)) = \ell'$, as desired. □

Therefore, we may define a map $\iota : V(M) \to S$ that sends $v \in V(M)$ to some $s \in V(G_0) \cap N_G^{\ell'}(v)$ with the same colour as v. Now, for any two vertices x, y in $V(M)$, we know that there exists a monochromatic r-path P from x to y in G whose vertices are contained in $V(M)$. Thus, every r-subpath of $P - V(M)$ has weak diameter in G at most 0. By applying Proposition 7, we obtain a monochromatic $(0 + 2r + 2\ell') = r'$-path P' in G of the same colour as P but whose interior is contained in $V(G_0)$. As G_0 is an isometric subgraph, $\mathrm{Int}(P')$ is a monochromatic r'-path in G_0 under c_0, and thus has weak diameter in G_0 and G at most d. Hence, P has weak diameter in G at most $d + 2r'$; in particular $\mathrm{dist}_G(x, y) \leqslant d + 2r' \leqslant D$, as desired. □

3. Colourings on tree-decompositions

Given graphs G, H, an *H-decomposition* of G is a collection $(B_x : x \in V(H))$ of subsets of $V(G)$ such that (a) for each $v \in V(G)$, the subgraph of H induced by the vertices $x \in V(H)$ with $v \in B_x$ is nonempty and connected; and (b) for each $uv \in E(G)$, there exists $x \in V(H)$ such that $u, v \in B_x$. The subsets B_x, $x \in V(H)$, are called *bags*. The *width* of an H-decomposition is $\max_{x \in V(H)} (|B_x| - 1)$. The *adhesion* of an H-decomposition is $\max_{xy \in E(H)} (|B_x \cap B_y|)$. If $V(H) \subseteq V(G)$, $(B_x : x \in V(H))$ is *rooted* if for each $x \in V(H)$, $x \in B_x$. An H-decomposition of G is a *tree-decomposition* of G if H is a tree T; the *treewidth* of G is then the minimum width of a tree-decomposition of G. Note that every graph of treewidth at most k has a tree-decomposition of width and adhesion at most k.

Given a tree-decomposition $(B_t : t \in V(T))$ of a graph G and a vertex $t \in V(T)$, the *torso* of G at t with respect to $(B_t : t \in V(T))$, denoted $G\langle B_t \rangle$, is the graph obtained from $G[B_t]$ by adding an edge uv whenever there exists $t' \in V(T)$ adjacent to t such that $u, v \in B_t \cap B_{t'}$, provided uv does not already exist in $G[B_t]$. This is a standard definition, which we now extend. If G is a weighted graph, we define the *weighted torso* of G at t with respect to $(B_t : t \in V(T))$, denoted $G\langle \widehat{B}_t \rangle$, to be the weighted graph obtained by imbuing $G\langle B_t \rangle$ with the weighting w defined by $w(uv) := \mathrm{dist}_G(u, v)$ for each $uv \in E(G\langle B_t \rangle)$. We emphasise that the weight of uv is determined by the distance between u and v in the whole graph G, not merely in $G[B_t]$, and that even when G is unweighted, the weighted torso is usually distinct from the torso, with edge weights other than 1.

For any subtree T' of T, consider $\widehat{G}_{T'} := \bigcup_{t \in V(T')} G\langle \widehat{B}_t \rangle$; note that this is a well-defined weighted graph, as the weightings on different weighted torsos agree. Observe that shortest paths in G can only exit $V(\widehat{G}_{T'})$ via a vertex in $B_t \cap B_{t'}$ for some $tt' \in E(T)$ with $t \in V(T')$, $t' \in V(T) \setminus V(T')$, and must also reenter $V(\widehat{G}_{T'})$ via another vertex in $B_t \cap B_{t'}$; at this point, the extra edges added to $G[B_t]$ to form $G\langle \widehat{B}_t \rangle$ provide a shortcut of the same length in $\widehat{G}_{T'}$. It follows that $\widehat{G}_{T'}$ is isometric in G. In particular, taking $V(T') := \{t\}$ shows that the weighted torso $G\langle \widehat{B}_t \rangle$ is isometric in G, this is our main motivation for considering weighted torsos. Furthermore, we call $\widehat{G}_T$ the *completion* of G with respect to $(B_t : t \in V(T))$, and denote it simply as $\widehat{G}$. By the previous observation, $\widehat{G}$ is isometric in G; in particular, $\mathrm{dist}_G = \mathrm{dist}_{\widehat{G}}$. As all edges of $\widehat{G}$ had weight dist_G, this means that the weighting on $\widehat{G}$ is $\mathrm{dist}_{\widehat{G}}$. Also notice that $(B_t : t \in V(T))$ is still a tree-decomposition for $\widehat{G}$, and that for each $t \in V(T)$, $\widehat{G}\langle B_t \rangle = \mathrm{unweighted}(\widehat{G}[B_t]) = G\langle B_t \rangle$; because the weighting on $\widehat{G}$ is $\mathrm{dist}_{\widehat{G}}$, this gives $\widehat{G}\langle \widehat{B}_t \rangle = \widehat{G}[B_t] = G\langle \widehat{B}_t \rangle$. It follows that $\widehat{(\widehat{G})} = \widehat{G}$, hence the name "completion".

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We also know that for each subtree T' of T, $(\widehat{\widehat{G}})_{T'} = \bigcup_{t \in V(T')} \widehat{G}[B_t] = \widehat{G}_{T'}$ is isometric in G and hence also $\widehat{G}$; this is critical to later proofs.

For an integer $k \geqslant 0$ and a class of weighted graphs $\mathcal{H}$, we say that a tree-decomposition $(B_t : t \in V(T))$ of a weighted graph G is a $(k, \mathcal{H})$-*construction* of G if it has adhesion at most k and, for each $t \in V(T)$, $G\langle \widehat{B}_t \rangle \in \mathcal{H}$. If G admits a $(k, \mathcal{H})$-construction, we say that G is $(k, \mathcal{H})$-*constructable*.

A *partition* $\mathcal{P}$ of a weighted graph G is a collection of nonempty pairwise disjoint subsets of $V(G)$ such that $\bigcup_{P \in \mathcal{P}} P = V(G)$ and each $P \in \mathcal{P}$ induces a connected subgraph of G. The subsets $P \in \mathcal{P}$ are called the *parts* of $\mathcal{P}$; for a vertex $v \in V(G)$, we use $\mathrm{Part}_{\mathcal{P}}(v)$ to denote the unique part $P \in \mathcal{P}$ for which $v \in P$, and for a set $S \subseteq V(G)$, we use $\mathrm{Parts}_{\mathcal{P}}(S)$ to denote the set of all parts $P \in \mathcal{P}$ for which $S \cap P \neq \emptyset$. The *quotient* of $\mathcal{P}$ in G, denoted $G/\mathcal{P}$, is the graph with vertex set $\mathcal{P}$ and an edge between two parts if the subgraphs of G they induce are adjacent in G; note that $G/\mathcal{P}$ is a minor of G. For a real number $\ell \geqslant 0$, $\mathcal{P}$ is ℓ-*shallow* if, for each part $P \in \mathcal{P}$, $G[P]$ has radius at most ℓ. If, for an integer $k \geqslant 0$, $G/\mathcal{P}$ also has treewidth at most k, then we say that $\mathcal{P}$ is a (k, ℓ)-*partition* of G.

The main results of this section are as follows.

Theorem 11. *Let $k \geqslant 0$ be an integer, let $\mathcal{H}$ be a class of weighted graphs, and let $\mathcal{G}$ be a hereditary class of weighted graphs that are all $(k, \mathcal{H})$-constructable. Then $\mathrm{ANdim}(\mathcal{G}) \leqslant \max(\mathrm{ANdim}(\mathcal{H}), 1)$*

Theorem 12. *For every integer $k \geqslant 0$, there exists a dilation f_k such that for every real number $\ell \geqslant 0$, f_k is an ℓ-almost 1-dimensional control function for every weighted graph that admits a (k, ℓ)-partition.*

For any graph G of treewidth at most k, the partition into singletons is a $(k, 0)$-partition of G; this gives the following as an easy application of Theorem 12.

Theorem 13. *For any integer $k \geqslant 0$, any class of weighted graphs of treewidth at most k has Assouad–Nagata dimension at most 1.*

Theorems 11 and 12 are both special cases of a more general result, Theorem 14, which we work towards stating now.

For any weighted graph G, partition $\mathcal{P}$ of G, and tree-decomposition $(B_t : t \in V(T))$ of $G/\mathcal{P}$, observe that $(\bigcup_{P \in B_t} P : t \in V(T))$ is a tree-decomposition of G. For an integer $k \geqslant 0$, a real number $\ell \geqslant 0$, and a class of weighted graphs $\mathcal{H}$, we say that the pair $(\mathcal{P}, (B_t : t \in V(T)))$ is a $(k, \ell, \mathcal{H})$-*strong-construction* for G if:

(a) For each $t \in V(T)$ and $\mathcal{P}_t \subseteq B_t$, $G\langle \widehat{\bigcup_{P \in B_t} P} \rangle [\bigcup_{P \in \mathcal{P}_t} P] \in \mathcal{H}$;

(b) $\mathcal{P}$ is ℓ-shallow; and

(b) $(B_t : t \in V(T))$ has adhesion at most k.

We say that G is $(k, \ell, \mathcal{H})$-*strongly constructable* if it admits a $(k, \ell, \mathcal{H})$-strong-construction.

Theorem 14. *For every integer $k \geqslant 0$, and function $f : \mathbb{R}^+ \to \mathbb{R}^+$, there exists a function $f_k : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following holds. Let $n \geqslant 0$ be an integer, let $\ell \geqslant 0$ be a real number, and let $\mathcal{H}$ be a class of graphs that admits f as an ℓ-almost n-dimensional control function. Set $n' := \max(n, 1)$, then for every weighted graph G that admits a $(k, \ell, \mathcal{H})$-strong-construction, f_k is an ℓ-almost n'-dimensional control function for G. Further, if f is a dilation, then so is f_k.*

Note that if G is $(k, \mathcal{H})$-constructable and $\mathcal{H}$ is hereditary, then G is $(k, 0, \mathcal{H})$-strongly constructable using the partition into singletons; Theorem 11 then follows. Separately, if $\mathcal{H}$ is the class of weighted graphs H such that $V(H)$ is a $(k+1, \ell)$-centred set in H, then any weighted graph G that admits a (k, ℓ)-partition is $(k, \ell, \mathcal{H})$-strongly constructable. Using Corollary 8, we find that, for any $r \geqslant \ell$ and any $H \in \mathcal{H}$, any r-path in H has weak diameter in H at most $4(k+2)r$; it follows any 1-colouring of H is a $(1, r, 4(k+2)r)$-colouring. Thus, $r \mapsto 4(k+2)r$ is a dilation

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that is an ℓ-almost 0-dimensional control function for $\mathcal{H}$; Theorem 12 follows. The remainder of this section is therefore dedicated to the proof of Theorem 14.

Theorem 14 itself follows from the following, more technical, lemma.

Lemma 15. *For every integer $k \geqslant 0$ and function $f{:}\mathbb{R}^+ \to \mathbb{R}^+$, there exists a function $f_k{:}\mathbb{R}^+ \to \mathbb{R}^+$ such that the following holds. Let $n \geqslant 0$ be an integer, let $\ell \geqslant 0$ be a real number, let $\mathcal{H}$ be a class of graphs for which f is an ℓ-almost n-dimensional control function, and let G be a weighted graph with a $(k, \ell, \mathcal{H})$-strong-construction $(\mathcal{P}, (B_t : t \in V(T)))$. Set $n' := \max(n, 1)$, let $r > 0$ be a real number with $r \geqslant \ell$, let $q \in V(T)$, and let $S^{\mathcal{P}} \subseteq B_q$ be of size at most k. Set $S := \bigcup_{P \in S^{\mathcal{P}}} P$, $Z^{\mathcal{P}} := \mathit{Parts}_{\mathcal{P}}(N_G^{3r}(S))$, and let c_Z be an $(n'+1)$-colouring of $Z := \bigcup_{P \in Z^{\mathcal{P}}} P$ with colours $\{1, \ldots, n'+1\}$. Then c_Z can be extended to an $(n'+1, r, f_k(r))$-colouring c of G with colours $\{1, \ldots, n'+1\}$. Further, if f is a dilation, then so is f_k.*

Proof. Let $f_0 := f$, and for every integer $k \geqslant 1$ and real number $r > 0$, let:

$$\begin{aligned} g_k'(r) &:= 8(k+1)r, \\ g_k^*(r) &:= 2g_k'(r) + 2r, \\ f_k^*(r) &:= f_{k-1}(g_k^*(r)), \\ f_k^\#(r) &:= (k+1)(f_k^*(r) + 4g_k^*(r) + 12r), \text{ and} \\ f_k(r) &:= f_k^\#(r) + 2g_k^*(r). \end{aligned}$$

Observe that if f is a dilation, then for every integer $k \geqslant 0$, g_k', g_k^*, f_k^*, $f_k^\#$, and f_i are all dilations. Also, note that for every integer $k \geqslant 0$ and real number $r > 0$ with $r \geqslant \ell$, $g_k^*(r) \geqslant r \geqslant \ell$.

Now, let r be defined as in the statement of Lemma 15; we need to show that G admits an $(n'+1, r, f_k(r))$-colouring with colours $\{1, \ldots, n'+1\}$ that extends c_Z. We do this via induction, primarily on k and secondarily on $|V(G)|$.

The base case occurs when $k = 0$; in this case, for each connected component C of G, $V(C) \subseteq \bigcup_{P \in B_t} P$ for some $t \in V(T)$. Furthermore, for this t, C is isometric in $G\langle\widehat{\bigcup_{P \in B_t} P}\rangle$. Since $G\langle\widehat{\bigcup_{P \in B_t} P}\rangle \in \mathcal{H}$ and $r \geqslant \ell$, $G\langle\widehat{\bigcup_{P \in B_t} P}\rangle$ admits an $(n'+1, r, f(r))$-colouring, which we may assume is with colours $\{1, \ldots, n'+1\}$; the restriction of this colouring to $V(C)$ then gives an $(n'+1, r, f(r))$-colouring of C with colours $\{1, \ldots, n'+1\}$. As this holds for every connected component of G, taking the union of these colourings gives an $(n'+1, r, f(r))$-colouring of G with colours $\{1, \ldots, n'+1\}$. Since $f(r) = f_0(r)$ and $Z = \emptyset$ as $|S| = 0$, this is the desired colouring. So we may now assume that $k \geqslant 1$, and that the lemma holds for all smaller values of k and when k is the same but $|V(G)|$ is smaller.

We begin by making a few assumptions. Notice that $(\bigcup_{P \in B_t} P{:}t \in V(T))$ is a tree-decomposition of G; let $\widehat{G}$ be the corresponding completion of G. We may assume that $G = \widehat{G}$. Otherwise, observe that $\mathcal{P}$ is still a partition of $\widehat{G}$, and that $(B_t : t \in V(T))$ is still a tree-decomposition for $\widehat{G}/\mathcal{P}$ of adhesion at most k, since the extra edges added to $G/\mathcal{P}$ to make $\widehat{G}/\mathcal{P}$ only go between parts that share a bag. Furthermore, note that edges of G have weight in $\widehat{G}$ no larger than their weight in G; since no vertices or edges are deleted going from G to $\widehat{G}$, this means for any $P \in \mathcal{P}$, $\mathrm{rad}(\widehat{G}[P]) \leqslant \mathrm{rad}(G[P]) \leqslant \ell$. Thus, $\mathcal{P}$ is also an ℓ-shallow partition for $\widehat{G}$. Additionally, observe that $(\bigcup_{P \in B_t} P{:}t \in V(T))$ is still a tree-decomposition of $\widehat{G}$, and that the completion of $\widehat{G}$ with respect to $(\bigcup_{P \in B_t} P{:}t \in V(T))$ is still $\widehat{G}$. Also, for each $t \in V(T)$, observe that $\widehat{G}\langle\widehat{\bigcup_{P \in B_t} P}\rangle = \widehat{G}[\bigcup_{P \in B_t} P] = G\langle\widehat{\bigcup_{P \in B_t} P}\rangle \in \mathcal{H}$; thus for each $\mathcal{P}_t \subseteq B_t$, $\widehat{G}\langle\widehat{\bigcup_{P \in B_t} P}\rangle[\bigcup_{P \in \mathcal{P}_t} P] = \widehat{G}[\bigcup_{P \in \mathcal{P}_t} P] = G\langle\widehat{\bigcup_{P \in B_t} P}\rangle[\bigcup_{P \in \mathcal{P}_t} P] \in \mathcal{H}$. So $(\mathcal{P}, (B_t : t \in V(T)))$ is still a $(k, \ell, \mathcal{H})$-strong-construction for $\widehat{G}$. Finally, as observed when we first defined the completion, we have $\mathrm{dist}_G = \mathrm{dist}_{\widehat{G}}$, hence $N_{\widehat{G}}^{3r}(S) = N_G^{3r}(S)$, and any $(n'+1, r, f_k(r))$-colouring of $\widehat{G}$ is an $(n'+1, r, f_k(r))$-colouring of G. Thus, we can proceed by setting $G := \widehat{G}$.

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We may also assume that $S^{\mathcal{P}}$ is nonempty. Otherwise, if G is empty, we are clearly done, and if G is nonempty, pick any part P' of $\mathcal{P}$ and pick some vertex $q' \in V(T)$ with $P' \in B_{q'}$. Let $S'^{\mathcal{P}} := \{P\}$, let $S' := \bigcup_{P \in S'^{\mathcal{P}}} P = P'$, let $Z^{\mathcal{P}'}$ be the set of parts that intersect $N_G^{3r}(S')$, and let $c_{Z'}$ be an arbitrary $(n'+1)$-colouring of $Z' := \bigcup_{P \in Z^{\mathcal{P}'}} V(P)$ with colours $\{1, \ldots, n'+1\}$. Observe that Z must be empty, thus any colouring that extends $c_{Z'}$ also extends c_Z. Therefore, since $|S'^{\mathcal{P}}| = 1 \leqslant k$, we may proceed by setting $S^{\mathcal{P}} := S'^{\mathcal{P}}$, $S := S'$, $Z^{\mathcal{P}} := Z'^{\mathcal{P}}$, $Z := Z'$ and $c_Z := c_{Z'}$. Observe also that since parts are nonempty, $S^{\mathcal{P}}$ being nonempty implies that S and hence Z are nonempty.

Henceforth, we can now assume that $S^{\mathcal{P}}$, and consequently Z, are nonempty, and that $G = \widehat{G}$. We use the latter assumption implicitly throughout the remainder of the proof. Note that this gives, as mentioned when we argued that we could take $G = \widehat{G}$, that for each $t \in V(T)$ and each $\mathcal{P}_t \subseteq B_t$, $G[\bigcup_{P \in \mathcal{P}_t} P] \in \mathcal{H}$.

For each $e = tt' \in E(T)$, let $S_e^{\mathcal{P}} := B_t \cap B_{t'}$ and let $S_e := \bigcup_{P \in S_e^{\mathcal{P}}} P$. Note that $|S_e^{\mathcal{P}}| \leqslant k$; consequently, S_e is a (k, r)-centred set in G, as $\ell \leqslant r$.

For each $P \in S^{\mathcal{P}}$, let T_P be the subgraph of T induced by the vertices $t \in V(T)$ such that $B_t \cap \mathrm{Parts}_{\mathcal{P}}(N_G^{3r}(P)) \neq \emptyset$. Since P induces a nonempty connected subgraph of G, $N_G^{3r}(P)$ also induces a nonempty connected subgraph of G, and thus $\mathrm{Parts}_{\mathcal{P}}(N_G^{3r}(P))$ induces a nonempty connected subgraph of $G/\mathcal{P}$. Hence, T_P is nonempty and connected, and for each $tt' \in E(T_P)$, $B_t \cap B_{t'} \cap \mathrm{Parts}_{\mathcal{P}}(N_G^{3r}(P)) \neq \emptyset$. Additionally, T_P contains q as a vertex, as $P \in S^{\mathcal{P}} \subseteq B_q$.

Let $T' := \bigcup_{P \in S^{\mathcal{P}}} T_P$. Notice that T' is connected, as each T_P, $P \in S^{\mathcal{P}}$, is connected and contains q, and nonempty, as $|S^{\mathcal{P}}| > 0$. Further, notice that $t \in V(T)$ is in $V(T')$ if and only if $B_t \cap Z^{\mathcal{P}} \neq \emptyset$, and that $B_t \cap B_{t'} \cap Z^{\mathcal{P}} \neq \emptyset$ for any $tt' \in E(T')$. Let $G' := \bigcup_{t \in V(T')} G[\bigcup_{P \in B_t} P]$. Since T' is a subtree of T, by a prior observation we made when we first defined the completion, G' is isometric in G. Also, notice that for any part $P \in \mathcal{P}$, $G[P]$ is either a subgraph of G', or P is disjoint from $V(G')$; let $\mathcal{P}' := \mathrm{Parts}_{\mathcal{P}}(V(G'))$, we thus have that $\mathcal{P}'$ is an ℓ-shallow partition for G'. Finally, notice that $Z \subseteq V(G')$.

Next, let E' denote the set of edges between T' and $T - V(T')$. For each $e \in E'$, let T_e be the connected component of $T - V(T')$ incident to e; note that e is the only edge between T_e and $T - V(T_e)$. Thus, T_e is disjoint from $T_{e'}$, $e' \in E' \setminus \{e\}$, and there is a unique vertex $q_e \in V(T_e)$ adjacent to $T - V(T_e)$; in particular q_e is adjacent to T' and e is incident with q_e. Additionally, notice that $V(T') \cup \bigcup_{e \in E'} V(T_e) = V(T)$; in particular, for each $e \in E'$, $V(T) \setminus V(T_e) = V(T') \cup \bigcup_{e' \in E' \setminus \{e\}} V(T_{e'})$. Finally, notice that because $V(T_e) \subseteq V(T) \setminus V(T')$, for each $t \in V(T_e)$, $B_t \cap Z^{\mathcal{P}} = \emptyset$.

Now, for each $e \in E'$, let $G_e := \bigcup_{t \in V(T_e)} G[\bigcup_{P \in B_t} P]$, and let $\widetilde{G}_e := \bigcup_{t \in V(T) \setminus V(T_e)} G[\bigcup_{P \in B_t} P]$. Since the only edge between T_e and $T - V(T_e)$ was e, whose endpoints are q_e and some vertex of T', note that $V(G_e \cap \widetilde{G}_e) = S_e \subseteq V(G')$ and that $S_e^{\mathcal{P}} \subseteq B_{q_e}$. Additionally, using the same reasoning as with G', notice that G_e is an isometric subgraph of G, and that $\mathcal{P}_e := \mathrm{Parts}_{\mathcal{P}}(V(G_e))$ is an ℓ-shallow partition of G_e. Finally, notice that $Z \cap V(G_e) = \emptyset$, as $Z^{\mathcal{P}} \cap B_t = \emptyset$ for each $t \in V(T_e) \subseteq V(T) \setminus V(T')$.

We now seek to apply Lemma 10 on G, using G' as G_0 and $(G_e : e \in E')$ as $G_1, \ldots, G_a$. We use r as both r and ℓ, $n'+1$ for m, $\{1, \ldots, n'+1\}$ for C, k as itself, $f_k^{\#}(r)$ for d, and $f_k(r)$ for D. Notice that ℓ' is $g_k'(r)$ and r' is $g_k^*(r)$, hence $D = f_k(r) \geqslant d + 2r' = f_k^{\#}(r) + 2g_k^*(r)$, as required.

To begin, notice that each vertex or edge of G is contained in $G[\bigcup_{P \in B_t} P]$ for some $t \in V(T)$. Since $V(T') \cup \bigcup_{e \in E'} V(T_e) = V(T)$, for each $t \in V(T)$, $G[\bigcup_{P \in B_t} P]$ is a subgraph of either G', or G_e for some $e \in E'$. Thus, $G = G' \cup \bigcup_{e \in E'} G_e$, and we have satisfied Lemma 10 (a).

Next, for each $e \in E'$, recall that $S_e = V(G_e \cap \widetilde{G}_e) \subseteq V(G')$ and that S_e is a (k, r)-centred set in G. Since $V(T) \setminus V(T_e) = V(T') \cup \bigcup_{e' \in E' \setminus \{e\}} V(T_{e'})$, we have $\widetilde{G}_e = G' \cup \bigcup_{e' \in E' \setminus \{e\}} G_{e'}$. Thus, the $\widetilde{G}_e$ [resp. the S_e], $e \in E'$, are the $\widetilde{G}_i$ [resp. the S_i], $i \in \{1, \ldots, a\}$, in the statement of Lemma 10, so we have satisfied Lemma 10 (b).

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Now, consider Lemma 10 (c). Notice that $\mathcal{P}' \setminus Z^{\mathcal{P}}$ is an ℓ-shallow partition of $G' - Z$, and that $(B_t \setminus Z^{\mathcal{P}} : t \in V(T'))$ is tree-decomposition for $(G' - Z)/\mathcal{P}'$; since $B_t \cap B_{t'} \cap Z^{\mathcal{P}} \neq \emptyset$ for each $tt' \in E(T')$, we find that $(B_t \setminus Z^{\mathcal{P}} : t \in V(T'))$ has adhesion at most $k-1$. Also, notice that $(\bigcup_{P \in B_t \setminus Z^{\mathcal{P}}} P : t \in V(T'))$ is a tree-decomposition of $G' - Z$; fix $t \in V(T')$ and consider $(G' - Z)\langle \widehat{\bigcup_{P \in B_t \setminus Z^{\mathcal{P}}} P} \rangle$. For any $t' \in V(T')$ adjacent to t and any $u, v \in \bigcup_{P \in (B_t \cap B_{t'}) \setminus Z^{\mathcal{P}}} P$, observe that $uv \in E(G)$ and hence $uv \in E(G' - Z)$. Additionally, notice that for each $uv \in E(G' - Z)$, uv has weight $\mathrm{dist}_G(u, v)$ in both G and $G' - Z$; this forces $\mathrm{dist}_G(u, v) = \mathrm{dist}_{G'-Z}(u, v)$. It follows that $(G' - Z)\langle \widehat{\bigcup_{P \in B_t \setminus Z^{\mathcal{P}}} P} \rangle = (G' - Z)[\bigcup_{P \in B_t \setminus Z^{\mathcal{P}}} P] = G[\bigcup_{P \in B_t \setminus Z^{\mathcal{P}}} P]$. Thus, for any $\mathcal{P}_t \subseteq B_t \setminus Z^{\mathcal{P}} \subseteq B_t$, $(G' - Z)\langle \widehat{\bigcup_{P \in B_t \setminus Z^{\mathcal{P}}} P} \rangle [\bigcup_{P \in \mathcal{P}_t} P] = G[\bigcup_{P \in \mathcal{P}_t} P] \in \mathcal{H}$. Therefore, $(\mathcal{P}' \setminus Z^{\mathcal{P}}, (B_t \setminus Z^{\mathcal{P}} : t \in V(T'))$ is a $(k-1, \ell, \mathcal{H})$-strong-construction for $G' - Z$.

Thus, using $g_k^*(r) \geqslant \ell$ as r and recalling that $f_{k-1}(g_k^*(r)) = f_k^*(r)$, we may apply the induction hypothesis, keeping q unchanged and letting $S^{\mathcal{P}}$ be empty, to find an $(n'+1, g_k^*(r), f_k^*(r))$-colouring c' of $G' - Z$ with colours $\{1, \ldots, n'+1\}$. Now, recall that $\mathcal{P}'$ is an ℓ-shallow partition for G'; thus, for each $P \in \mathcal{P}'$, $\mathrm{wdiam}_{G'}(P) \leqslant 2\ell \leqslant 2r$. Also, as $N_G^{3r}(S) \subseteq Z \subseteq V(G')$ notice that every $v \in N_G^{3r}(S)$ is still a distance at most $3r$ from S in G'. Let S^* be a set containing, for each $P \in S^{\mathcal{P}}$, exactly one vertex v which is at distance at most $\ell \leqslant r$ from every other vertex in $G'[P]$ (which exists, since $\mathcal{P}$ is ℓ-shallow). Then for any $z \in Z$, $\mathrm{dist}_{G'}(z, N_G^{3r}(S)) \leqslant 2r$, $\mathrm{dist}_{G'}(z, S) \leqslant 5r$, and $\mathrm{dist}_{G'}(z, S^*) \leqslant 6r$. Since $|S^{\mathcal{P}}| \leqslant k$, $|S^*| \leqslant k$, and thus Z is $(k, 6r)$-centred set in G'. Since $(k+1)(f_k^*(r) + 4g_k^*(r) + 12r) = f_k^{\#}(r)$, we thus have that the colouring $c'' := c' \cup c_Z$ of G' is an $(n'+1, g_k^*(r), f_k^{\#}(r))$-colouring with colours $\{1, \ldots, n'+1\}$ via Corollary 9. So Lemma 10 (c) is satisfied with c'' as c_0. Finally, note that c'' also extends c_Z, by definition.

Now, for each $e \in E'$, define $c_{S_e} := c''|_{V(S_e)}$. Let $Z_e^{\mathcal{P}} := \mathrm{Parts}_{\mathcal{P}}(N_{G_e}^{3r}(S_e))$ and $Z_e := \bigcup_{P \in Z_e^{\mathcal{P}}} P$, we can find a colouring c_{Z_e} of Z_e with colours $\{1, \ldots, n'+1\}$ that extends c_{S_e} and has an (S_e, r)-barrier in G_e. Recall that $\mathcal{P}_e$ is an ℓ-shallow partition of G_e, and notice that $(B_t : t \in V(T_e))$ is a tree-decomposition for $G_e/\mathcal{P}_e$ of adhesion at most k, and that $(\bigcup_{P \in B_t} P : t \in V(T_e))$ is a tree-decomposition for G_e. For any $t \in V(T_e)$, a similar argument to the one used for the weighted torsos of $G' - Z$ shows that $G_e\langle \widehat{\bigcup_{P \in B_t} P} \rangle = G_e[\bigcup_{P \in B_t} P] = G[\bigcup_{P \in B_t} P]$; thus, for any $\mathcal{P}_t \subseteq B_t$, $G_e\langle \widehat{\bigcup_{P \in B_t} P} \rangle [\bigcup_{P \in \mathcal{P}_t} P] = G[\bigcup_{P \in \mathcal{P}_t} P] \in \mathcal{H}$. It follows that $(\mathcal{P}_e, (B_t : t \in V(T_e)))$ is a $(k, \ell, \mathcal{H})$-strong-construction for G_e. Finally, note that $|V(G_e)| \leqslant |V(G) - Z| < |V(G)|$, as $Z \cap V(G_e) = \emptyset$ and Z is nonempty. Hence, we can apply induction on G_e, using r as itself, q_e as q, $S_e^{\mathcal{P}} \subseteq B_{q_e}$ as $S^{\mathcal{P}}$, and c_{Z_e} as c_Z. This allows us to extend c_{Z_e} to an $(n'+1, r, f_k(r))$-colouring c_e of G_e with colours $\{1, \ldots, n'+1\}$. Note that since c_e extends c_{Z_e}, c_e has an (S_e, r)-barrier in G_e, and that $c_e = c''$ on S_i, by definition of c_{S_e}. Hence, Lemma 10 (d) is satisfied with the various c_e, $e \in E'$, acting as $c_1, \ldots, c_a$.

Thus, we can now apply Lemma 10 with the parameters specified earlier. This gives us that $c := c'' \cup \bigcup_{e \in E'} c_e$ is an $(n'+1, r, f_k(r))$-colouring with colours $\{1, \ldots, n'+1\}$. Since c extends c'', which extends c_Z, c also extends c_Z. Thus, c is the desired colouring. □

4. Colourings on proper minor-closed classes

A landmark result by Robertson and Seymour [21] known as the "Graph Minor Structure Theorem" gives a structural description of H-minor-free graphs. We now build towards stating this result.

The *Euler genus* of a surface with h handles and c cross-caps is $2h + c$. The *Euler genus* of a graph G is the minimum Euler genus of a surface Σ such that G can be embedded into Σ without crossings.

Given a graph G_0 embedded in some surface Σ without crossings, a closed disc D in Σ is said to be G_0-*clean* if the interior of D does not intersect the embedding of G_0, and the boundary of

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D intersects the embedding of G_0 only at vertices of G_0. Observe that D must sit inside a face of G_0; if the vertices of G_0 that D intersects are precisely the vertices of this face, we say that D is *snug* in G_0. There is a natural cyclic ordering of the vertices of G_0 that D intersects by following the boundary of D; let $v_1, \ldots, v_n$ denote these vertices in this order. Observe that if D is snug, this ordering is the same as the cyclic ordering of the corresponding face. A *D-vortex* of G_0 is then a graph H such that $V(G_0 \cap H) = \{v_1, \ldots, v_n\}$. If P_n denotes the path on n vertices, then a *vortex-decomposition* of a D-vortex H of G_0 is a P_n-decomposition $(B_1, \ldots, B_n)$ of H such that $v_i \in B_i$ for each $i \in \{1, \ldots, n\}$. The *width* of a vortex-decomposition is the width of the P_n-decomposition, and the *width* of a D-vortex H of G_0 is the minimum width of a vortex-decomposition of H. Note that the treewidth of H is at most the width of H.

For integers $g, p, k, a \geqslant 0$, a graph G is said to be (g, p, k, a)*-almost embeddable* if there exists a set $A \subseteq V(G)$ of size at most a and subgraphs $G_0, \ldots, G_s$ of G for some $s \in \{0, \ldots, p\}$ such that:

(a) $G - A = \bigcup_{i=0}^{s} G_i$;

(b) $G_1, \ldots, G_s$ are pairwise disjoint;

(c) G_0 is embedded into a surface Σ of Euler genus at most g;

(d) there exist pairwise disjoint G_0-clean discs $D_1, \ldots, D_s$ in Σ; and

(e) G_i is a D_i-vortex of G_0 of width at most k for each $i \in \{1, \ldots, s\}$.

Additionally, if all the discs $D_1, \ldots, D_s$ are snug in G_0, we say that G is *snugly* (g, p, k, a)*-almost embeddable*.

Refer to the set A as the *apex vertices* of G, the subgraph G_0 as the *embedded subgraph* of G, the G_0-clean closed discs $D_1, \ldots, D_s$ as the *discs* of G, and the subgraphs $G_1, \ldots, G_s$ as the *vortices* of G; for each $i \in \{1, \ldots, s\}$, G_i is the *vortex* at D_i. Also, call $H := \bigcup_{i=1}^{s} G_i$ the *vortex-union subgraph* of G. Next, for $i \in \{1, \ldots, s\}$, refer to the vertices S_i of $G_0 \cap G_i$, ordered by the natural cyclic ordering from the boundary of D_i, as the *boundary vertices* of D_i. The set $S := \bigcup_{i=1}^{s} S_i$ is referred to as the *boundary* of G.

We take the chance to note some properties of snugly (g, p, k, a)-almost embeddable graphs. For each $i \in \{1, \ldots, s\}$, let $(B_{i,1}, \ldots, B_{i,n_i})$ be a vortex-decomposition of G_i of width at most k, and let $S_i = v_{i,1}, \ldots, v_{i,n_i}$ be the cyclic ordering of the boundary vertices of D_i; as D_i is snug $v_{i,1}, \ldots, v_{i,n_i}$ is precisely the vertex set of the face F_i that D_i sits inside, in the cyclic order induced by F_i. Thus, for any interval $a, \ldots, b$ of the path P_{n_i}, $v_{i,a}, \ldots, v_{i,b}$ induces a connected subgraph of $G_0[S_i]$. It follows that $(B_{i,j} : v_{i,j} \in S_i)$ is a rooted $G_0[S_i]$-decomposition of G_i of width at most k. As the vortices (including the boundary vertices) as pairwise disjoint, we find that $(B_{i,j} : v_{i,j} \in S)$ is a rooted $G_0[S]$-decomposition of H of width at most k. Additionally, since for each $i \in \{1, \ldots, s\}$, S_i is the vertex set of F_i, S is the vertex set of the union of the faces $F_1, \ldots, F_s$. Finally, observe that (G_0, H) forms a separation of $G - A$ with separator $V(G_0 \cap H) = S$.

We can now state the Graph Minor Structure Theorem of Robertson and Seymour [21].

Theorem 16. (Graph Minor Structure Theorem [21]). *For every graph H, there exists an integer $k \geqslant 0$ such that every H-minor-free graph G admits a tree-decomposition of adhesion at most k such that every torso is (k, k, k, k)-almost embeddable.*

Observe that if G as above is a weighted graph, every weighted torso is also (k, k, k, k)-almost embeddable. Thus, if $\mathcal{H}$ is the class of all (k, k, k, k)-almost embeddable weighted graphs, G is $(k, \mathcal{H})$-constructable. However, since induced subgraphs of (k, k, k, k)-almost embeddable graphs are not necessarily (k, k, k, k)-almost embeddable, $\mathcal{H}$ is not hereditary, and we consequently cannot make use of Theorem 11. Instead, consider the class $\mathcal{H}'$ of all induced subgraphs of (k, k, k, k)-almost embeddable weighted graphs; note that $\mathcal{H}'$ is hereditary and G is also $(k, \mathcal{H}')$-constructable. Thus, if we could bound the Assouad–Nagata dimension of $\mathcal{H}'$, we could apply Theorem 11 to get a bound on the Assouad–Nagata dimension of any H-minor-free class. Since

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proper minor-closed classes are H-minor-free for some graph H, to prove Theorem 3 (a) it suffices to show that $\mathrm{ANdim}(\mathcal{H}') \leqslant 2$.

We now focus on finding a dilation f that is a 2-dimensional control function for every weighted graph that is an induced subgraph of a (g, p, k, a)-almost embeddable weighted graph. We show that every such weighted graph is isometric in a snugly (g, p, k, a)-almost embeddable weighted graph; see Proposition 22. Thus, it suffices to show that f is a 2-dimensional control function for every snugly (g, p, k, a)-almost embeddable weighted graph G. We notice that if we can find a control function for $G - A = G_0 \cup H$, we can then apply Corollary 9 to find f as A is a $(a, 0)$-centred set. So we just need to focus on the union $G_0 \cup H$. Individually, G_0 and H admit 2-dimensional control functions, as the former has bounded Euler genus, so Theorem 2 applies, and the latter has bounded treewidth, so Theorem 13 applies. However, taking the union creates issues, as it no longer has bounded treewidth nor Euler genus. Indeed, we have no direct way of dealing with this union; trying to colour the two halves of the separation independently fails, as by jumping back and forth between G_0 and H we can find short paths in the union that exist in neither half individually. However, we can recognise that the problem occurs "close" to where the two subgraphs meet, the boundary S of G, and that "far away" from the boundary, naively colouring as either a graph of bounded Euler genus or as a graph of bounded treewidth suffices. So we just need to deal with the part of the graph "close" to the boundary. We formalise this idea in the following proposition.

Proposition 17. *Let $n \geqslant 0$ be an integer, let $f, g{:}\mathbb{R}^+ \to \mathbb{R}^+$, let G be a weighted graph, and let (A, B) be a separation of G with separator S such that:*

(a) *f is an n-dimensional control function for both A and B; and*

(b) *for any real number $\ell \geqslant 0$, g is an ℓ-almost n-dimensional control function for $G[N_G^{\ell}(S)]$.*

Then the function $r \mapsto g(f(r) + 4r) + 2(f(r) + 4r)$ is an n-dimensional control function for G.

Proof. Fix a real number $r > 0$, set $r' := f(r) + 4r$ and $d' := g(r')$. We need to show that G admits an $(n + 1, r, d' + 2r')$-colouring. Let c_A and c_B be $(n + 1, r, f(r))$-colourings of A, B respectively, set $Z := N_G^r(S)$ and $Z' := N_G^{r'}(S)$, and let $c_{Z'}$ be an $(n + 1, r', d')$-colouring of $G[Z']$. Let $\iota{:}Z \to S$ be the identity on S, and for $x \in Z \setminus S$, let $\iota(x)$ be a vertex in $S \cap N_G^r(x)$. Now, define a colouring c of G via, for each $x \in V(G)$:

$$c(x) := \begin{cases} c_A(x) & \text{if } x \in V(A) \setminus Z, \\ c_B(x) & \text{if } x \in V(B) \setminus Z, \\ c_{Z'}(\iota(x)) & \text{otherwise.} \end{cases}$$

Observe that for any $x, y \in V(A) \setminus Z$ at distance at most r in G and any shortest path Q from x to y in G, Q cannot intersect S as otherwise we would have $x, y \in N_G^r(S) = Z$. Since Q induces a connected subgraph of G, this implies that $Q \subseteq V(A) - S$. Noting that every edge of G between vertices in $V(A) - S$ is an edge of A, Q is therefore also a path of length at most r in A, and $\mathrm{dist}_A(x, y) \leqslant r$. Consequently, any r-path in G whose vertices are contained in $V(A) \setminus Z$ is also an r-path in A. A symmetric argument shows that any r-path in G whose vertices are contained in $V(B) \setminus Z$ is also an r-path in B.

Now, notice that for any $x \in V(A)$ and $y \in V(B)$, if $\mathrm{dist}_G(x, y) \leqslant r$, then $x, y \in N_G^r(S) = Z$, as the shortest path between them must intersect S. Consider an r-path P' in G that is disjoint from Z; we claim that P' must be contained in either $V(A) \setminus Z$ or $V(B) \setminus Z$. Otherwise, somewhere on P', we would have consecutive vertices $x \in V(A), y \in V(B)$; the previous observation then tells us that $x, y \in Z$, a contradiction. Therefore, for any monochromatic r-path P from $u \in V(G)$ to $v \in V(G)$ in G under c, any r-subpath P' of $P - Z$ must be contained in either $V(A) \setminus Z$, or in $V(B) \setminus Z$. The observation from the previous paragraph then tells us that P' is an r-path in either A, or in B, respectively. Since c is c_A on $V(A) \setminus Z$ [resp. c_B on $V(B) \setminus Z$], P' is also monochromatic in either

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A under c_A, or in B under c_B, respectively. Thus, P' has weak diameter at most $f(r)$ in either A, or in B, respectively, and consequently in G as well.

Therefore, we can apply Proposition 7 to find that there exists a $f(r) + 2r + 2r = r'$-walk P'' from u to v in G whose interior consists of vertices of the form $\iota(x)$ with $x \in P \cap Z$. Observe that for $x \in P \cap Z$, $c(x) = c_{Z'}(\iota(x)) = c_{Z'}(\iota(\iota(x))) = c(\iota(x))$, as $\iota(\iota(x)) = \iota(x)$ since ι is the identity on S. Therefore, P'' is also monochromatic.

Now, for any $x, y \in S$ at distance at most r' in G and any shortest path Q from x to y in G, observe that Q is contained in $N_G^{r'}(S) = Z'$. Thus, Q is also a path of length at most r' in $G[Z']$, and $\mathrm{dist}_{G[Z']}(x, y) \leqslant r'$. Since $\mathrm{Int}(P'')$ is contained in S, $\mathrm{Int}(P'')$ is also a (possibly empty) r'-path in $G[Z']$. Further, since c is $c_{Z'}$ on S, as ι is the identity map on S, we also have that $\mathrm{Int}(P'')$ is monochromatic under $c_{Z'}$. Hence, $\mathrm{Int}(P'')$ has weak diameter at most d' in both $G[Z']$ and G. Thus, P'' has weak diameter at most $d' + 2r'$ in G; in particular, $\mathrm{dist}_G(u, v) \leqslant d' + 2r'$, as desired. □

So we just need to find a dilation that is an ℓ-almost 2-dimensional control function for $(G - A)[N_{G-A}^{\ell}(S)]$. In fact, we find ℓ-almost 1-dimensional control function for $(G - A)[N_{G-A}^{\ell}(S)]$. We do this by showing $(G - A)[N_{G-A}^{\ell}(S)]$ admits a (w, ℓ)-partition, across multiple steps; Theorem 12 then gives the desired dilation. The first step is to find a "natural" ℓ-shallow partition of $G_0[N_{G_0}^{\ell}(S)]$.

Proposition 18. *Let $r \geqslant 0$ be a real number, let G be a weighted graph, and let $S \subseteq V(G)$. Then $G' := G[N_G^r(S)]$ admits an r-shallow partition $\mathcal{P}$ such that each part of $\mathcal{P}$ contains exactly one vertex of S.*

Proof. Let $\preceq$ be an arbitrary ordering of the vertices of S. For $v \in N_G^r(S)$, let $\iota(v)$ be the smallest, with respect to $\preceq$, $s \in S$ such that $\mathrm{dist}_G(v, S) = \mathrm{dist}_G(v, s)$. Then, for each $s \in S$, define $P_s := \{v \in N_G^r(S) : \iota(v) = s\}$, and finally, define $\mathcal{P} := \{P_s : s \in S\}$. We argue that $\mathcal{P}$ is the desired partition of G'.

Each vertex of G' is in exactly one part of $\mathcal{P}$, as $\preceq$ acts as a tiebreaker. Additionally, for each $s \in S$, $s \in P_s$ as $\mathrm{dist}_G(s, s) = 0$ and $\mathrm{dist}_G(s, s') > 0$ for $s' \in S \setminus \{s\}$; it follows that each part is nonempty and contains exactly one vertex of S. So it remains only to show that each part P_s, $s \in S$, induces a connected subgraph of radius at most r; it suffices to show that for each $v \in P_s$, there exists a path from v to s in $G'[P_s]$ of length at most r.

Let Q be a shortest path from v to s in G; by definition of P_s, Q must have length $\mathrm{dist}_G(v, S) \leqslant r$. Thus, $Q \subseteq N_G^r(S) = V(G')$, as each vertex of Q is at least as close to s as v is. Consequently, each vertex of Q lies in some part of $\mathcal{P}$; assume, for a contradiction, that some $u \in Q$ is not in P_s, and instead lies in $P_{s'}$ for some $s' \in S \setminus \{s\}$. Since Q is a shortest path, $\mathrm{dist}_G(s, v) = \mathrm{dist}_G(s, u) + \mathrm{dist}_G(u, v)$, and since $u \in P_{s'}$, $v \in P_s$, $\mathrm{dist}_G(s, u) \geqslant \mathrm{dist}_G(s', u)$ and $\mathrm{dist}_G(s', v) \geqslant \mathrm{dist}_G(s, v)$. If $\mathrm{dist}_G(s, u) > \mathrm{dist}_G(s', u)$, then $\mathrm{dist}_G(s, v) > \mathrm{dist}_G(s', u) + \mathrm{dist}_G(u, v) \geqslant \mathrm{dist}_G(s', v)$, a contradiction. So we must have that $\mathrm{dist}_G(s, u) = \mathrm{dist}_G(s', u)$ and $\mathrm{dist}_G(s, v) = \mathrm{dist}_G(s', u) + \mathrm{dist}_G(u, v) \geqslant \mathrm{dist}_G(s', v)$; this forces $\mathrm{dist}_G(s, v) = \mathrm{dist}_G(s', v)$. If $s' \prec s$, since $\mathrm{dist}_G(s', v) = \mathrm{dist}_G(s, v) = \mathrm{dist}_G(v, S)$, $\preceq$ would put v in $P_{s'}$ over P_s, contradicting the fact that $v \in P_s$. By contrast, if $s' \succ s$, since $\mathrm{dist}_G(s, u) = \mathrm{dist}_G(s', u) = \mathrm{dist}_G(u, S)$, u would have been placed in P_s over $P_{s'}$, another contradiction. Thus, we must conclude that our assumption was false. This gives $Q \subseteq P_s$, so Q is also a path from v to s in $G'[P_s]$, and is of the same length in $G'[P_s]$ as in G. Since Q had length at most r in G, Q is therefore the desired path. □

We need the following lemma from Dujmović, Morin, and Wood [4]; see [7] for an earlier $O(gr)$ bound.

Lemma 19. *For all integers $g, r \geqslant 0$, every unweighted graph of radius at most r with Euler genus at most g has treewidth at most $(2g + 3)r$.*

We now show that the partition of $G_0[N_{G_0}^{\ell}(S)]$ we obtain from Proposition 18 is actually a (w, ℓ)-partition.

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Proposition 20. *Let $g \geqslant 0$, $p \geqslant 0$ be integers, let $r \geqslant 0$ be a real number, and let G be a weighted graph embedded in a surface Σ of Euler genus at most g without crossings. Let $F_1, \ldots, F_s$, $s \leqslant p$, be faces of G, let S be the vertex set of the union of $F_1, \ldots, F_s$, set $G' := G[N^r_G(S)]$, and let $\mathcal{P}$ be an r-shallow partition of G' such that each part contains exactly one vertex in S. Then $\mathcal{P}$ is a $(2g + 4p + 3, r)$-partition of G'.*

Proof. We must show that $G/\mathcal{P}$ has treewidth at most $2g + 4p + 3$. For each $i \in \{2, \ldots, s\}$, add a handle connecting the interior of F_1 to the interior of F_i; this gives an embedding of G' into a surface Σ' of Euler genus at most $g + 2(\max(s - 1, 0)) \leqslant g + 2p$ such that all the vertices in S are in a common face. Thus, by viewing $G'/\mathcal{P}$ as contracting each part down to the unique vertex in S, we can see that $G'/\mathcal{P}$ can be embedded in Σ' so that every vertex lies on a common face. We can then add a new vertex to $G'/\mathcal{P}$ adjacent to every existing vertex within the interior of the common face to create a new unweighted graph H of radius at most 1 embedded in Σ'. Thus, by Lemma 19, H has treewidth at most $2(g + 2p) + 3 = 2g + 4p + 3$. Since $G'/\mathcal{P} \subseteq H$, $G'/\mathcal{P}$ also has treewidth at most $2g + 4p + 3$, as desired. □

Finally, we extend the (w, ℓ)-partition of $G_0[N^\ell_{G_0}(S)]$ to a (w', ℓ)-partition of $(G - A)[N^\ell_{G-A}(S)]$.

Proposition 21. *Let $t, w \geqslant 0$ be integers, let $r \geqslant 0$ be a real number, and let G' be a graph that admits a separation (G, H) with separator S such that:*

(a) *G admits a (t, r)-partition $\mathcal{P}$ such that each part contains exactly one vertex in S; and*

(b) *H admits a rooted $G[S]$-decomposition of width at most w.*

Then $\mathcal{P}' := \mathcal{P} \cup \bigcup_{v \in V(H) \setminus S}\{\{v\}\}$ is a $((t + 1)(w + 1) - 1, r)$-partition of G'.

Proof. It is immediate that $\mathcal{P}'$ is an r-shallow partition of G', so we only need to show that the treewidth of $G'/\mathcal{P}'$ is at most $(t + 1)(w + 1) - 1$.

Let $(B_t : t \in V(T))$ be a tree-decomposition of $G/\mathcal{P}$ of width at most t, and let $(J_s : s \in S)$ be a rooted $G[S]$-decomposition of H of width at most w. For each $t \in V(T)$, let $S_t := \bigcup_{P \in B_t} P \cap S$; observe that $|S_t| \leqslant t + 1$ as $(B_t : t \in V(T))$ has width at most t and each $P \in B_t$ contains only one vertex in S. For each $t \in V(T)$, let $K_t := \bigcup_{s \in S_t} \mathrm{Parts}_{\mathcal{P}'}(J_s)$. Note that for each $P \in B_t$, there exists $s \in P \cap S_t$; since $s \in J_s$, this gives $P = \mathrm{Part}_{\mathcal{P}'}(s) \in K_t$. Thus, $B_t \subseteq K_t$. We now argue that $(K_t : t \in V(T))$ is the desired tree-decomposition of $G'/\mathcal{P}'$.

First, we show that $(K_t : t \in V(T))$ is a tree-decomposition of $G'/\mathcal{P}'$. Consider any $h \in V(H)$, the vertices $s \in S$ such that $h \in J_s$ induce an nonempty connected subgraph C_h of $G[S]$. Let $C^{\mathcal{P}}_h = G/\mathcal{P}[\mathrm{Parts}_{\mathcal{P}}(V(C_h))]$ note that $C^{\mathcal{P}}_h$ must also be nonempty and connected. Thus, the vertices $t \in V(T)$ such that $B_t \cap V(C^{\mathcal{P}}_h) \neq \emptyset$ induce a nonempty subtree T_h of T. Also, observe that for each $s \in S$, $\mathrm{Part}_{\mathcal{P}}(s) \in V(C^{\mathcal{P}}_s)$, as $(J_s : s \in S)$ is rooted. Now, for each $P \in \mathcal{P}'$, let T'_P be the subgraph of T induced by the vertices $t \in V(T)$ such that $P \in K_t$; we need to show that T'_P is nonempty and connected. Note that there is exactly one vertex h_P in $P \cap V(H)$; either the unique s in $P \cap S$ if $P \in \mathcal{P}$, or the h such that $P = \{h_P\}$ for $P \in \mathcal{P}' \setminus \mathcal{P}$. We argue that $T'_P = T_{h_P}$; since T_{h_P} is nonempty and connected, this gives the desired result. Note that it suffices to show that $V(T'_P) = V(T_{h_P})$, as both subgraphs are induced.

For a given $P \in \mathcal{P}'$, if $t \in V(T)$ is in $V(T'_P)$, then $P \in \mathrm{Parts}_{\mathcal{P}'}(J_s)$ for some $s \in S_t$. However, since $P \cap V(H) = \{h_P\}$ and $J_s \subseteq V(H)$, this forces $h_P \in J_s$. This gives $s \in V(C_{h_P})$, and $\mathrm{Part}_{\mathcal{P}}(s) \in V(C^{\mathcal{P}}_{h_P})$. By definition of S_t, $\mathrm{Part}_{\mathcal{P}}(s) \in B_t$, and thus $V(C^{\mathcal{P}}_{h_P}) \cap B_t \neq \emptyset$. Therefore, $t \in V(T_{h_P})$. By contrast, for each $t \in V(T_{h_P})$, there exists $P' \in B_t \cap V(C^{\mathcal{P}}_{h_P})$; this part P' must contain a $s \in V(C_{h_P}) \subseteq S$. By definition of C_{h_P}, we have $h_P \in J_s$, so $P = \mathrm{Part}_{\mathcal{P}'}(h_P) \in \mathrm{Parts}_{\mathcal{P}'}(J_s)$. However, we also know that $s \in P' \cap S \subseteq S_t$, as $P' \in B_t$, so $P \in \bigcup_{s \in S_t} \mathrm{Parts}_{\mathcal{P}'}(J_s) = K_t$. Therefore, $t \in V(T'_P)$; this combined with the other direction above gives $V(T'_P) = V(T_h)$, as desired.

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To complete our proof that $(K_t : t \in V(T))$ is a tree-decomposition of $G'/\mathcal{P}'$, it remains only to show that for any $P_1P_2 \in E(G'/\mathcal{P}')$, $P_1, P_2 \in K_t$ for some $t \in V(T)$. We know that there is some $uv \in E(G')$ such that $u \in P_1$, $v \in P_2$, which must be an edge in either G or H. In the former case, P_1, P_2 are both parts in $\mathcal{P}$ and P_1P_2 is also an edge of $E(G/\mathcal{P})$; thus, there is some $t \in V(T)$ for which $P_1, P_2 \in B_t \subseteq K_t$. Otherwise, there exists $s \in S$ with $u, v \in J_s$, and thus $P_1 = \text{Part}_{\mathcal{P}'}(u)$, $P_2 = \text{Part}_{\mathcal{P}'}(v) \in K_t$ for some $t \in V(T)$ for which $\text{Part}_{\mathcal{P}}(s) \in B_t$.

Lastly, we just need to consider the width of our tree-decomposition. Observe that for each $t \in V(T)$, $|K_t| \leqslant \sum_{s \in S_t} |J_s| \leqslant |S_t|(w+1) \leqslant (t+1)(w+1)$, as desired. □

All that is left is to prove that any induced subgraph of a (g, p, k, a)-almost embeddable weighted graph is isometric in a snugly (g, p, k, a)-almost-embeddable weighted graph.

Proposition 22. *Let $g, p, a, k \geqslant 0$ be integers, and let H be an induced subgraph of a (g, p, k, a)-almost embeddable weighted graph. Then H is an isometric subgraph of a snugly (g, p, k, a)-almost embeddable weighted graph.*

Proof. We show that H is an isometric subgraph of a (g, p, k, a)-almost embeddable weighted graph G, and that G is an isometric subgraph of a snugly (g, p, k, a)-almost embeddable weighted graph G'. The result follows since being isometric is a transitive property. We start with the former statement.

By definition of H, H is an induced subgraph of a (g, p, k, a)-almost embeddable weighted graph $G^{\#}$. Let w_H be the weighting function of H, and let $w_G{:}E(G) \to \mathbb{R}^+ \cup \{\infty\}$ be defined by setting $w_G(e) := w_H(e)$ for each $e \in E(H)$ and setting $w_G(e) := \infty$ otherwise. Let G be the result of imbuing unweighted$(G^{\#})$ with w_G; observe that G is still (g, p, k, a)-almost embeddable, and that $H \subseteq G$ as $w_H = w_G\big|_{E(H)}$. It remains to show that H is isometric in G.

Observe that every path in G is either a path in H, in which case it has the same length in G as in H as all the edges have the same weight in G as H, or has an edge in $E(G) \setminus E(H)$ and hence has length ∞. Since $H \subseteq G$, it follows that H is isometric in G, as desired. So we now tackle the latter statement.

Let G_0 be the embedded subgraph of G, which is embedded in some surface Σ of Euler genus at most g, and let $D_1, \ldots, D_s$ be the discs of G. For each $i \in \{1, \ldots, s\}$, let $V_i := \{v_{i,1}, \ldots, v_{i,n_i}\}$ represent the boundary vertices of D_i, where $v_{i,1}, \ldots, v_{i,n_i}$ is ordered according to the natural cyclic ordering obtained from following the boundary of D_i. Note that we may assume $n_i \geqslant 3$ for each $i \in \{1, \ldots, s\}$; otherwise add isolated vertices $v_{i,n_i+1}, \ldots, v_{i,3}$ to G, G_i and G_0 along the boundary of D_i between v_{i,n_i} and $v_{i,1}$ (if they exist) in that order, and update the vertex-decomposition of G_i by adding the bags $B_j := v_j$ for $j \in \{n_{i+1}, \ldots, 3\}$. G is isometric in this updated graph G^*, so whatever G^* is isometric in G will be isometric in as well; thus we may proceed by setting $G := G^*$.

Create G_0' from G_0 by, for each pair $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, n_i\}$, adding the edge $v_{i,j}v_{i,j+1}$ if it doesn't already exist. If $v_{i,j}v_{i,j+1} \in E(G)$, we give $v_{i,j}v_{i,j+1}$ the same weight in G_0' as its weight in G, otherwise we give it weight ∞. We then create G' from G by adding all of the above edges that didn't already exist in G, which will all have weight ∞.

Define an embedding of G_0' in some surface of Euler genus at most g as follows. Starting from the embedding of G_0, for each pair $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, n_i\}$, we embed $v_{i,j}v_{i,j+1}$ by tracing just outside the boundary of D_i; if $v_{i,j}v_{i,j+1} \in E(G_0)$, this overrides the existing embedding of $v_{i,j}v_{i,j+1}$ used for G_0. Provided that the edges cling sufficiently tightly to the boundaries of the discs, the resulting embedding has no crossings. These edges also form a cycle around D_i as $n_i \geqslant 3$. Thus, for each $i \in \{1, \ldots, s\}$, D_i is G_0'-clean, the vertices of G_0' that D_i intersects is V_i, and the cyclic ordering from following the boundary of D_i is just the cyclic ordering of V_i. Additionally, D_i is nested inside a face F_i whose vertices are V_i and whose induced cyclic ordering is the cyclic ordering of V_i. Thus, D_i is snug, and if G_i is the vortex of G at D_i, then G_i is also a D_i-vortex of G_0', and any vortex-decomposition of G_i as a D_i-vortex of G_0 is still a vortex-decomposition of G_i as a D_i-vortex of G_0'. Thus, the width of G_i as a D_i-vortex of G_0' is still at most k.

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Therefore, if A denotes the set of apex vertices of G, then $G' - A = G'_0 \bigcup_{i=1}^{s} G_i$, where $G_1, \ldots, G_s$ are pairwise disjoint, G'_0 is embedded onto a surface of genus at most g, and each G_i, $i \in \{1, \ldots, s\}$, is a D_i-vortex of G'_0 of width at most $k+1$, where the D_i, $i \in \{1, \ldots, s\}$ are pairwise disjoint G'_0-clean snug discs. Since $|A| \leqslant a$, this shows that G' is snugly (g, p, k, a)-almost embeddable, as desired.

So it remains only to show that G is isometric in G'. However, since $G \subseteq G'$, each $e \in E(G)$ has the same weight in G', and each $e \in E(G') \setminus E(G) = \bigcup_{i=1}^{s} E'_i$ has weight ∞, an argument identical to that we used to show that H was isometric in G can be used to show that G is isometric in G'. □

We are finally in a position to show that the class of all induced subgraphs of (g, p, k, a)-almost embeddable weighted graphs has Assouad–Nagata dimension 2.

Proposition 23. *Let $g, p, a, k \geqslant 0$ be integers. Let $\mathcal{G}$ be the class of all (g, p, k, a)-almost embeddable weighted graphs, and let $\mathcal{H}$ be the class of all induced subgraphs of graphs in $\mathcal{G}$. Then $\mathcal{H}$ is hereditary and* $\mathrm{ANdim}(\mathcal{H}) \leqslant 2$.

Proof. The fact that $\mathcal{H}$ is hereditary is immediate, so we only need to show the existence of a dilation f that is a 2-dimensional control function for $\mathcal{H}$. By Theorem 2, there exists a dilation $f_1^{\#}$ that is a 2-dimensional control function for the class of weighted graphs of Euler genus at most g, and by Theorem 13, there exists a dilation $f_2^{\#}$ that is a 1-dimensional control function for the class of weighted graphs of treewidth at most k. Let $f^{\#} := \max(f_1^{\#}, f_2^{\#})$; note that $f^{\#}$ must also be a dilation. By Theorem 12, there exists a dilation f' such that for every real number $\ell \geqslant 0$, f' is an ℓ-almost 1-dimensional control function for every weighted graph G that admits a $((2g+4p+4)(k+1)-1, \ell)$-partition. Let $f^*(r) := f'(f^{\#}(r) + 4r) + 2(f^{\#}(r) + 4r)$ for any real number $r > 0$; note that f^* is also a dilation as both $f^{\#}, f'$ are. Finally, let $f(r) := (a+1)(f^*(r) + 4r)$ for any real number $r > 0$; note that f is also a dilation. We show that f is the desired 2-dimensional control function for $\mathcal{H}$.

Consider any $H^{\#} \in \mathcal{H}$, using Proposition 22 we know that $H^{\#}$ is an isometric subgraph of some $G \in \mathcal{G}$ that is snugly (g, p, k, a)-almost embeddable. Let A be the apex vertices of G, let G_0 denote the embedded subgraph of G, let H denote the vortex-union subgraph of G, and let S denote the boundary of G. As observed when we discussed some properties of snugly almost embeddable graphs, S is the vertex set of the union of the faces $F_1, \ldots, F_s$ for which the discs $D_1, \ldots, D_s$, $s \leqslant p$, of G are snug in, (G_0, H) is a separation of $G - A$ with separator S, and H admits a $G_0[S]$-decomposition of width at most k. We seek to apply Proposition 17 using $G - A$ as G, (G_0, H) as (A, B), $f^{\#}$ as f and f' as g. Proposition 17 (a) is immediately satisfied by definition of $f^{\#}$, as G_0 has Euler genus at most g and H has treewidth at most k. Therefore, we only need to show Proposition 17 (b); that for any $\ell \geqslant 0$, f' is an ℓ-almost 2-dimensional control function for $(G - A)[N^{\ell}_{G-A}(S)]$. We show that $(G - A)[N^{\ell}_{G-A}(S)]$ admits a $((2g+4p+4)(k+1)-1, \ell)$-partition; the desired result then follows by definition of f'.

Let $\ell \geqslant 0$ be a real number, and set $S^{\ell} := N^{\ell}_{G-A}(S)$, $S^{\ell}_{G_0} := S^{\ell} \cap V(G_0)$ and $S^{\ell}_{H} := S^{\ell} \cap V(H)$. Note that $S^{\ell}_{G_0} = N^{\ell}_{G_0}(S)$, and that $S^{\ell}_{H} = N^{\ell}_{H}(S)$; we use these facts implicitly in the remainder of this proof. By Proposition 18, $G_0[S^{\ell}_{G_0}]$ admits an ℓ-shallow partition $\mathcal{P}$ such that each part contains exactly one vertex of S. Since G_0 is embedded on a surface of Euler genus at most g and S is the vertex set of the union of the faces $F_1, \ldots, F_s$, with $s \leqslant p$, by Proposition 20, $\mathcal{P}$ is $(2g+4p+3, \ell)$-partition.

Now, notice that $(G_0[S^{\ell}_{G_0}], H[S^{\ell}_{H}])$ is a separation of $(G - A)[S^{\ell}]$ with separator S, and that $H[S^{\ell}_{H}]$ also admits a rooted $G_0[S] = (G_0[S^{\ell}_{G_0}])[S]$-decomposition of width at most k. Thus, by Proposition 21, $\mathcal{P}' := \mathcal{P} \cup \bigcup_{v \in S^{\ell}_{H} \setminus S} \{\{v\}\}$ is a $((2g+4p+4)(k+1)-1, \ell)$-partition of $(G - A)[S^{\ell}]$, as desired.

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Thus, we can apply Proposition 17 with the parameters specified above to find that $f'(f^{\#}(r) + 4r) + 2(f^{\#}(r) + 4r) = f^*(r)$ is a 2-dimensional control function for $G - A$. Observing that A is a $(a, 0)$-centred set in G, by Corollary 9, $(a + 1)(f^*(r) + 4r) = f(r)$ is a 2-dimensional control function for G. Since $H^{\#}$ is isometric in G, f is also a 2-dimensional control function for $H^{\#}$, as desired. □

We can now bound the Assouad–Nagata dimension of H-minor-free classes of weighted graphs.

Theorem 24. *For any graph H and any H-minor-free class of weighted graphs $\mathcal{G}$,*

(a) $\mathrm{ANdim}(\mathcal{G}) \leqslant 2$; and

(b) $\mathrm{ANdim}(\mathcal{G}) \leqslant 1$ if $\mathcal{G}$ has bounded treewidth.

Proof. Theorem 24 (b) follows from Theorem 13, so we focus on Theorem 24 (a).

By Theorem 16, there exists an integer $k \geqslant 0$ such that every H-minor-free graph has a tree-decomposition of adhesion at most k such that every torso is (k, k, k, k)-almost embeddable. Let $\mathcal{H}$ be the class of weighted graphs that are an induced subgraph of a (k, k, k, k)-almost embeddable weighted graph; by Proposition 23 $\mathcal{H}$ is hereditary and $\mathrm{ANdim}(\mathcal{H}) \leqslant 2$. Observe that if every torso of a weighted graph G is (k, k, k, k)-almost embeddable, then so is every weighted torso. Thus, every $G \in \mathcal{G}$ is $(k, \mathcal{H})$-constructable. By Theorem 11, $\mathrm{ANdim}(\mathcal{G}) \leqslant \max(\mathrm{ANdim}(\mathcal{H}), 1) \leqslant 2$. □

Theorem 3 then follows as a consequence of Theorem 24 and Theorem 1 (b), as the Assouad–Nagata dimension is at least the asymptotic dimension. We remark that the converse of Theorem 24 (b) is not true; for example, take the class $\mathcal{G}$ of 2-dimensional grids (which is K_5 and $K_{3,3}$-minor-free as all 2-dimensional grids are planar), and for each $G \in \mathcal{G}$, fix $r \in V(G)$, and weight each $uv \in E(G)$ by $2^{\mathrm{dist}_G(r,\{u,v\})}$. It can be seen that any ℓ-path in the weighted graph is either contained in 2ℓ-neighbourhood of r, or consists of only a single vertex. It follows that $\ell \mapsto 4\ell$ is a 0-dimensional control function for the corresponding class of weighted graphs $\mathcal{G}'$, despite $\mathcal{G}'$ also having unbounded treewidth [20].

5. Assouad–Nagata dimension of non-minor-closed classes

We close this paper by looking at classes of weighted graphs that admit every graph as a minor. In particular, we establish some necessary conditions for such a class to have bounded asymptotic or Assouad–Nagata dimension, which leads to a proof of Theorem 4. We achieve this by migrating a control function for a graph G in the class to a control function for any minor of G, achieved by reweighting edges appropriately.

We require the following observation.

Observation 25. *Let $n \geqslant 0$ be an integer, let G, H be weighted graphs, and let f be an n-dimensional control function for G. If there exist real numbers $\alpha, \beta > 0$ and a map $\iota : V(H) \to V(G)$ such that for all $u, v \in V(H)$, $\beta \mathrm{dist}_H(u, v) \leqslant \mathrm{dist}_G(\iota(u), \iota(v)) \leqslant \alpha \mathrm{dist}_H(u, v)$, then $r \mapsto \frac{1}{\beta} f(\alpha r)$ is an n-dimensional control function for H.*

Proof. Fix a real number $r > 0$, let c_G be an $(n + 1, \alpha r, f(\alpha r))$-colouring of G. Define a colouring c_H of H via $c_H(v) := c_G(\iota(v))$ for each $v \in V(H)$. Since $\mathrm{dist}_G(\iota(u), \iota(v)) \leqslant \alpha \mathrm{dist}_H(u, v)$, for any monochromatic r-path $v_0, v_1, \ldots, v_m$ in H under c_H, $\iota(v_0), \iota(v_1), \ldots, \iota(v_m)$ is a monochromatic αr-walk in G under c_G, and thus $\mathrm{dist}_G(\iota(v_0), \iota(v_n)) \leqslant f(\alpha r)$. Since $\beta \mathrm{dist}_H(v_0, v_n) \leqslant \mathrm{dist}_G(\iota(v_0), \iota(v_n))$, we have that $\mathrm{dist}_H(v_0, v_n) \leqslant \frac{1}{\beta} f(\alpha r)$. It follows that c_H is an $(n + 1, r, \frac{1}{\beta} f(\alpha r))$-colouring of H. □

Notice that the fact that being isometric preserves control functions is a special case of Observation 25, using $\alpha = \beta := 1$ and the identity map for ι.

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We also make use of the following characterisation of minors. A *model* of a graph H in a graph G is a partition $\mathcal{P} := (P_x : x \in V(H))$ of some subgraph A of G such that $A/\mathcal{P}$ is isomorphic to H. Such a model exists if and only if $H \leqslant G$.

We first show that for any graphs G, H with H connected and $H \leqslant G$, there is a G' obtained from imbuing G with rational edge weights whose control functions translate to control functions for H.

Proposition 26. *Let $\epsilon > 0$ be a real number, let G be a graph, and let H be a connected unweighted graph that is a minor of G. Then there exists a function $w{:}E(G) \to \mathbb{R}^+$ such that the following holds. For any integer $n \geqslant 0$, if $f{:}\mathbb{R}^+ \to \mathbb{R}^+$ is an n-dimensional control function for the weighted graph G' obtained from imbuing G with w, then the function $r \mapsto f((1+\epsilon)r)$ is an n-dimensional control function for H. Furthermore, if ϵ is rational, then w is a rational-valued function.*

Proof. Let $\mathcal{P} := (P_x : x \in V(H))$ be a model of H in G, which exists as $H \leqslant G$. By definition, $\mathcal{P}$ is a partition of some subgraph H' of G such that $H'/\mathcal{P}$ is isomorphic to H.

Let E be the set of edges $uv \in E(H')$ such that u, v belong to different parts of $\mathcal{P}$, let E' be the set of edges $uv \in E(H')$ such that u, v belong to the same part of $\mathcal{P}$, and let $E'' := E(G) \setminus E(H')$. Define $\iota{:}V(H) \to V(G)$ via setting, for each $x \in V(H)$, $\iota(x) \in P_x$. Set $p := \max(|E'|, 1)$ and $d := \operatorname{diam}(H)$; note that $d < \infty$ since H is connected. Define w via

$$w(e) := \begin{cases} 1 & \text{if } v \in E, \\ \frac{\epsilon}{p} & \text{if } v \in E', \\ (1+\epsilon)d + 1 & \text{otherwise.} \end{cases}$$

Note that if ϵ is rational, then w is rational-valued, as d and p are integers; the former being because H is unweighted. We show that w is the desired weighting via showing that $\operatorname{dist}_H(u,v) \leqslant \operatorname{dist}_{G'}(\iota(u), \iota(v)) \leqslant (1+\epsilon)\operatorname{dist}_H(u,v)$ for any $u, v \in V(H)$; the desired result then follows from Observation 25. This is clearly true when $u = v$, so assume they are distinct, and set $\ell := \operatorname{dist}_H(u,v)$, $\ell' := \operatorname{dist}_{G'}(\iota(u), \iota(v))$; note that $\ell \geqslant 1$ as H is unweighted. Let P be a shortest path from u to v in H, observe that P corresponds to a path P_G in G' from $\iota(u)$ to $\iota(v)$ that uses exactly ℓ edges in E and no edges in E''. Thus, the length of P_G is at most $\ell + |E'|\frac{\epsilon}{p} \leqslant \ell + p\frac{\epsilon}{p} \leqslant (1+\epsilon)\ell$ as $\ell \geqslant 1$. Thus, $\ell' \leqslant (1+\epsilon)\ell$. Next, since $d = \operatorname{diam}(H)$, by the prior observation $\ell' \leqslant (1+\epsilon)d$. Therefore, if P' is a shortest path from $\iota(u)$ to $\iota(v)$ in G', then P' does not use any edges in E'', as they all have weight larger than ℓ'. Thus, P' corresponds to a path in H of length at most ℓ'. This gives $\ell \leqslant \ell'$, as desired. □

We now show that there is a G'' obtained from imbuing G with integer edge weights such that control functions for G'' translate to control functions for G'.

Proposition 27. *For every dilation f, there exists a dilation f' such that the following holds. Let G be a graph, and let H be the result of imbuing G with some rational-valued weighting function w. Then there exists a function $w'{:}E(G) \to \mathbb{N}$ such that for any integer $n \geqslant 0$, if f is an n-dimensional control function for the weighted graph G' that is the result of imbuing G with w', then f' is an n-dimensional control function for H.*

Proof. Since f is a dilation, there exists a real number $\alpha > 0$ such that $f(r) \leqslant \alpha r$ for every real number $r > 0$; set $f'(r) := \alpha r$ for every real number $r > 0$. We claim that f' is the desired function.

Since w is rational-valued and $|E(G)| < \infty$, there exists some integer $k > 0$ such that kw is integer-valued; set $w' := kw$. Observe that for every $u, v \in V(H) = V(G')$, $k\operatorname{dist}_H(u,v) = \operatorname{dist}_{G'}(u,v)$. Thus, if f is an n-dimensional control function for G, by Observation 25 the function $r \mapsto \frac{1}{k}f(kr)$ is an n-dimensional control function for H. Since $f(r) \leqslant \alpha r$ for every real number $r > 0$, $\frac{1}{k}f(kr) \leqslant \frac{1}{k}\alpha(kr) = \alpha r = f'(r)$. Thus, f' is an n-dimensional control function for H, as desired. □

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The last step is to find a $G''' \in \mathcal{G}$ whose control functions translate to control functions for G'', this is achieved by subdividing G according to the weights of G''. Before we put all this together formally, we introduce one more piece of terminology. Given a set $S \subseteq \mathbb{R}^+$, the *S-weights completion* of a class of graphs $\mathcal{G}$, denoted $\mathcal{G}_S$, is the class of weighted graphs G such that all edge weights of G are in S and unweighted$(G) \in \mathcal{G}$.

Theorem 28. *For every class of graphs $\mathcal{G}$ that admits every graph as a minor,*

(a) $asdim(\mathcal{G}_{\mathbb{R}^+}) = \infty$;

(b) $ANdim(\mathcal{G}_{\mathbb{N}}) = \infty$; *and*

(c) if $\mathcal{G}$ is closed under subdivision, then $ANdim(\mathcal{G}) = \infty$.

Proof. Let $\mathcal{H}$ be any class of connected unweighted graphs with infinite asymptotic (and Assouad–Nagata) dimension, such as the class of all grids of all dimensions [12]. Fix any integer $n \geqslant 0$ and any function $f{:}\mathbb{R}^+ \to \mathbb{R}^+$. We show that if f is an n-dimensional control function for $\mathcal{G}_{\mathbb{R}^+}$, then there is an n-dimensional control function for $\mathcal{H}$, a contradiction by choice of $\mathcal{H}$; Theorem 28 (a) follows. Additionally, we show that if f is a dilation and an n-dimensional control function for $\mathcal{G}_{\mathbb{N}}$, we again have that there is an n-dimensional control function for $\mathcal{H}$, another contradiction; Theorem 28 (b) follows. Finally, we show that if $\mathcal{G}$ is closed under subdivision, and f is a dilation and an n-dimensional control function for $\mathcal{G}$, then f is a dilation that is an n-dimensional control function for $\mathcal{G}_{\mathbb{N}}$, contradicting Theorem 28 (b); Theorem 28 (c) follows.

We start with the first claims. Presume that f is an n-dimensional control function for $\mathcal{G}_{\mathbb{R}^+}$, and fix a rational number $\epsilon > 0$. Now, for every $H \in \mathcal{H}$, there exists $G \in \mathcal{G}$ that contains H as minor. By Proposition 26, there exists a rational-valued function $w{:}E(G) \to \mathbb{R}^+$ such that if f is an n-dimensional control function for the weighted graph G' obtained by imbuing G with w, then the function $r \mapsto f((1+\epsilon)r)$ is an n-dimensional control function for H. Observe that $G' \in \mathcal{G}_{\mathbb{R}^+}$, thus f is an n-dimensional function for G' and $r \mapsto f((1+\epsilon)r)$ is an n-dimensional control function for H. Since this holds for every $H \in \mathcal{H}$, this would make $r \mapsto f((1+\epsilon)r)$ an n-dimensional control function for $\mathcal{H}$, as claimed. This completes the proof of Theorem 28 (a).

Now, we tackle the second claim. Presume that f is a dilation, and that f is an n-dimensional control function for $\mathcal{G}_{\mathbb{N}}$. Since f is a dilation, we can apply Proposition 27, let f' be the function we obtain. For every $H \in \mathcal{H}$, let G and G' be defined as in the previous paragraph; note that if f' is an n-dimensional control function for G', then the function $r \mapsto f'((1+\epsilon)r)$ is an n-dimensional control function for H. So we aim to show that f' is indeed an n-dimensional control function for G'.

Since G' is the result of imbuing G by the rational-valued weighting function w, by Proposition 27 (using G' as H), there exists a function $w'{:}E(G) \to \mathbb{N}$ such that if f is an n-dimensional control function for the weighted graph G'' obtained from imbuing G with w', then f' is an n-dimensional control function for G'. Observe that $G'' \in \mathcal{G}_{\mathbb{N}}$, thus f is an n-dimensional control function for G'', f' is an n-dimensional control function for G', and $r \mapsto f'((1+\epsilon)r)$ is an n-dimensional control function for H. Since this holds for every $H \in \mathcal{H}$, this implies that $r \mapsto f'((1+\epsilon)r)$ would be an n-dimensional control function for $\mathcal{H}$, as claimed. This completes the proof of Theorem 28 (b).

We now proceed to the final caim. Presume that $\mathcal{G}$ is closed under subdivision, that f is a dilation, and that f is an n-dimensional control function for $\mathcal{G}$. For every $G \in \mathcal{G}_{\mathbb{N}}$, let w be the weighting of G. Then, let G' be the graph obtained from unweighted(G) by subdividing each $e \in E(\text{unweighted}(G)) = E(G)$ $w(e) - 1$ times. By definition of $\mathcal{G}_{\mathbb{N}}$, unweighted$(G) \in \mathcal{G}$, and because $\mathcal{G}$ is closed under subdivision, G' is also in $\mathcal{G}$. Thus, f is an n-dimensional control function for G'. Now, notice that G is isometric in G'; this follows from the fact that for any $e \in E(G)$, e has been split into a path of length exactly $w(e)$ in G', and because no new paths have been created. Thus, f is also an n-dimensional control function for G. Since this holds for every $G \in \mathcal{G}$, this implies

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that f would be a n-dimensional control function for $\mathcal{G}$, as claimed. This completes the proof of Theorem 28 (c), and the overall proof of Theorem 28. □

Theorem 4 follows directly from Theorem 24 and Theorem 28 (c).

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A. Proof of Proposition 5

We recall Proposition 5 for convenience.

Proposition 5. *Let $n \geqslant 0$ be an integer, and let G be a weighted graph. Then $f{:}\mathbb{R}^+ \to \mathbb{R}^+$ is an n-dimensional control function for G if and only if, for every real number $r > 0$, G admits an $(n+1, r, f(r))$-colouring.*

This method of this proof is essentially identical to the proof of Proposition 1.17 in Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [2], which shows implicitly that if $f(r)$ is an n-dimensional control function for G, then G admits an $(n+1, r, rf(r))$-colouring for every real number $r > 0$. Our improvement to an $(n+1, r, f(r))$-colouring comes not from any new idea, but rather due to only measuring distances in G rather than both G and G^r, and avoiding a conversion overestimate in the process. This improvement is significant as it gives an equivalent definition for control functions (and thus Assouad–Nagata dimension), as opposed to just an equivalent definition for asymptotic dimension.

We now begin the proof of Proposition 5.

Proof. First, assume that $f(r)$ is an n-dimensional control function for G. Fix a real number $r > 0$, by definition of f there exist collections $\mathcal{C}_1, \dots, \mathcal{C}_{n+1}$ such that:

(a) $\bigcup_{i=1}^{n+1} \bigcup_{S \in \mathcal{C}_i} S = V(G)$;
(b) $\mathcal{C}_i$ is r-disjoint for each $i \in \{1, \dots, n+1\}$; and
(c) $\operatorname{wdiam}_G(S) \leqslant f(r)$ for each $i \in \{1, \dots, n+1\}$ and $S \in C_i$.

Let $c{:}V(G) \to \{1, \dots, n+1\}$ be defined such that $v \in \bigcup_{S \in \mathcal{C}_{c(v)}} S$; note that this is possible by property (a), but not necessarily unique. Observe that for any $i \in \{1, \dots, n+1\}$, if $u, v \in V(G)$ are both coloured i under c and are at distance at most r in G, then u, v both belong to the same set of $\mathcal{C}_i$ by property (b). It follows that any i-monochromatic r-path in G under c is contained in some set of $\mathcal{C}_i$, and thus has weak diameter in G at most $f(r)$, by property (c). Therefore, c is an $(n+1, r, f(r))$-colouring, as desired.

For the reverse direction, fix a real number $r > 0$, and let c be a given $(n+1, r, f(r))$-colouring of G, which we may assume is with colours $\{1, \dots, n+1\}$. For each $i \in \{1, \dots, n+1\}$, let $\mathcal{C}_i$ be the set $\{V(M){:}M \text{ an } i\text{-monochromatic } r\text{-component under } c\}$. Observe that for any colour i, any two distinct i-monochromatic r-components M_1, M_2 of G under c must be r-disjoint. Otherwise, $M_1 \cup M_2$ would be connected in G^r; since $V(M_1 \cup M_2)$ is i-monochromatic and M_1, M_2 are strict subgraphs of $M_1 \cup M_2$, this contradicts the maximality of M_1 and M_2. Thus, $\mathcal{C}_i$ is r-disjoint. By definition of c, for any monochromatic r-component M of G under c, $V(M)$ has weak diameter at most $f(r)$ in G. Additionally, each $v \in V(G)$ must be in some monochromatic r-component of G (of the same colour as v), so $\bigcup_{i=1}^{n+1} \bigcup_{V(M) \in \mathcal{C}_i} V(M) = V(G)$. Therefore, $\mathcal{C}_1, \dots, \mathcal{C}_{n+1}$ are the desired collections. The result follows. □

B. Proof of Proposition 6

We recall Proposition 6 for ease of reference.

Proposition 6. *Let $n \geqslant 0$ be an integer and let G be an infinite weighted graph. Let $\mathcal{A}$ be the class of all finite induced subgraphs of G, and let f be an n-dimensional control function for $\mathcal{A}$. Then for any real number $\epsilon > 0$, $r \mapsto f((1+\epsilon)r)$ is an n-dimensional control function for G.*

This theorem is very similar to that Theorem A.2 of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [3]. The difference is that we obtain the control function $r \mapsto f((1+\epsilon)r)$

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for G, whereas Theorem A.2 of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [3] gives the control function $r \mapsto f(r+1)$. This difference is significant, as if f is a dilation, then so is $r \mapsto f((1+\epsilon)r)$; however $r \mapsto f(r+1)$ need not be a dilation. This allows us to give an upper bound on the Assouad–Nagata dimension instead of just the asymptotic dimension.

Before proceeding to the proof of Proposition 6, we observe a key difference between infinite and finite graphs. In a finite graph, a shortest path between any two vertices always exists, since there are only finitely many paths. However, this is not true in an infinite graph. Consequently, in an infinite graph, the distance between two vertices is instead the infimum across all lengths of paths between the vertices. However, we remark that most other ideas translate to the infinite case without issue; importantly, objects such as r-paths and monochromatic r-components can be defined identically. Additionally, while not explicitly stated, Proposition 5 holds even when G is infinite, using an identical proof.

The proof for Proposition 6 is essentially identical to the proof of Theorem A.2 of Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [3]. The improved control function for G comes from a slight tweak in the error of the length of a path that is very close to the infimum.

We require a technical lemma that is a special case of Gottschalk [11].

Lemma 29. *Let C, I be sets, let $\mathcal{S}$ be the set of all finite subsets of I, and for each $S \in \mathcal{S}$, let $c_S : S \to C$. Then there exists a map $c : I \to C$ such that for every $S \in \mathcal{S}$, there exists $S' \in \mathcal{S}$ with $S \subseteq S'$ such that $c_{S'} = c\big|_{S'}$.*

We now begin the proof of Proposition 6.

Proof. For any real number $r > 0$, let $f'(r) := f((1+\epsilon)r)$; we must find an $(n+1, r, f'(r))$ of G. Set $C := \{1, \dots, n+1\}$, and let $\mathcal{S}$ be the set of all finite subsets of $I := V(G)$. For each $S \in \mathcal{S}$, let c_S be an $(n+1, (1+\epsilon)r, f'(r))$-colouring of $G[S]$, which we may assume is with colours C. We can now apply Lemma 29 to obtain a map $c : V(G) \to C$ such that for each $S \in \mathcal{S}$, there exists $S' \in \mathcal{S}$ with $c_{S'} = c\big|_{S'}$. We show that c is an $(n+1, r, f'(r))$-colouring of G.

Let P be any monochromatic r-path in G under c. Since consecutive vertices in P are at distance at most r in G, we can find a path of length at most $(1+\epsilon)r$ between them in G. Stringing these paths together gives us a walk P' containing P such that consecutive vertices of P are at distance at most $(1+\epsilon)r$ in $G[P']$. Let $S' \in \mathcal{S}$ be such that $P' \subseteq S'$ and $c_{S'} = c\big|_{S'}$. Since $G[P'] \subseteq G[S']$, consecutive vertices of P are also at distance at most $(1+\epsilon)r$ in $G[S']$, and since P is monochromatic in G under c and $c_{S'} = c\big|_{S'}$, P is also monochromatic in $G[S']$ under $c_{S'}$. Hence, P forms a monochromatic $((1+\epsilon)r)$-path in $G[S']$ under $c_{S'}$, and thus it has weak diameter in $G[S']$ and G is at most $f'(r)$, as desired. □

It follows from Proposition 6 that the Assouad–Nagata dimension of an infinite weighted graph G is at most the Assouad–Nagata dimension of the class of all finite induced subgraphs of G. We remark that the converse is not true; for example, take the disjoint union of all graphs in any class of finite unweighted graphs with infinite Assouad–Nagata dimension, such as the class of of all grids of all dimensions [12], and then add a single vertex adjacent to every other vertex. This infinite graph G has Assouad–Nagata dimension 0 as $\mathrm{wdiam}_G(V(G)) \leqslant 2$, but the class of induced subgraphs of G contains a class with infinite Assouad–Nagata dimension, and thus also has infinite Assouad–Nagata dimension. We also remark that an analogous argument can be used to show that the converse is not true even when Assouad–Nagata dimension is replaced by asymptotic dimension.

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