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THE POTENTIAL HIERARCHY OF SETS

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Abstract. Some reasons to regard the cumulative hierarchy of sets as potential rather than actual are discussed. Motivated by this, a modal set theory is developed which encapsulates this potentialist conception. The resulting theory is equi-interpretable with Zermelo Fraenkel set theory but sheds new light on the set-theoretic paradoxes and the foundations of set theory.

§1. Introduction. The familiar iterative conception tells us that the sets form a hierarchy. At the bottom level are all *urelemente*, if any. New sets are then formed as we ascend in the hierarchy. At successor levels we form all sets of objects available at the preceding stages. At limit levels we take the union of all the objects available at the preceding stages. But how far does this hierarchy extend?

Various partial answers are available in the form of lower bounds on its height. For instance, the hierarchy extends at least ω many steps. Can a more explicit characterization of its height be provided? We are often told that the hierarchy extends *as far as possible*. Vague though this may be, it is hard to see how a more definite answer could be provided. For given any attempt to pinpoint the extent of the hierarchy in a more definite way, it seems possible for the hierarchy to extend even further by allowing any objects from the proposed characterization to form a set. And since the hierarchy is supposed to extend as far as possible, this means that the proposed characterization cannot have been correct after all. So there seems to be something inherently potential about the set theoretic hierarchy. Given any attempt at a definite characterization, it turns out to be possible for the hierarchy to extend even further. The two set theorists who are chiefly responsible for articulating the iterative conception of sets—Ernst Zermelo and Kurt Gödel—took this potential aspect of the hierarchy very seriously.[1] The same goes for prominent philosophers of mathematics such as Hilary Putnam, Charles Parsons, and Geoffrey Hellman.[2]

The apparent potential character of the hierarchy of sets contrasts starkly with the axiomatic set theory that the iterative conception is ordinarily used to motivate, namely Zermelo–Fraenkel (ZF) set theory. This theory is formulated in a language with no modal vocabulary. And the theory quantifies freely over 'all sets', thus apparently assuming that 'all sets' are simultaneously available as a legitimate range of quantification. So this theory appears to treat the hierarchy of sets as if it was an actual or completed hierarchy.

In this article, I propose to take seriously the apparent potential character of the hierarchy of sets and the need for modal notions in the explication of set theory. I first defend the interest of this project (Section 2) and then proceed to develop a modal set theory based on four groups of axioms: first, a propositional modal logic (Section 3); then, a plural

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[1] See, for instance, the final pages of Zermelo (1930) as well as Gödel (1933).

[2] See Putnam (1967), Parsons (1977, 1983b), and Hellman (1989).

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logic, which allows us to quantify over many objects simultaneously (Section 4); next, an analysis of the nature of sets, if any (Section 6); and finally, a single, very intuitive principle concerning the existence of sets, namely that any objects whatsoever may form a set (Section 7). (Note that this last principle uses both plural and modal resources.) The resulting theory enables us to retrieve all the axioms of Zermelo set theory except the axiom of Infinity; plausible additional assumptions yield Infinity and Replacement and thus all ZF (Section 8). My hope is to improve—both technically and philosophically—on what I regard as the best study of modal set theory to date, namely Parsons (1983b).

The picture that emerges from this investigation is that modal set theory is a valuable supplement to standard nonmodal ZF set theory rather than a rival. The modal theories provide powerful instruments for studying the same subject matter under a finer resolution. For the purposes of almost all of ordinary mathematics and much of set theory, this finer resolution is not needed. In these cases no insight is lost, but great expressive and inferential ease is gained, by relying on standard nonmodal set theory. However, when we confront difficult foundational and conceptual questions concerning set theory, the finer resolution provided by the modal approach can be very valuable. In particular, we will see that the modal approach makes available a very natural motivation for the axioms of ZF set theory.

§2. Why the hierarchy should be regarded as potential. One attraction of the potentialist conception of the hierarchy of sets is that the competing actualist conception threatens to be unstable. According to the actualist conception, the set-theoretic quantifiers range over a definite totality of all sets. Why should the objects that make up this totality not themselves form a set? Since a set is completely specified by its elements, we can give a precise and complete specification of the set that these objects would form if they did form a set. What more could be needed for such a set to exist? To disallow such a set would be to truncate the iterative hierarchy at an arbitrary level.[3] The potentialist conception of the hierarchy appears to avoid this problem. For on this conception there is never any completed totality of absolutely all sets. Whenever a completed totality of objects is available, it is possible to extend the hierarchy to include a set with precisely these objects as elements. Of course, one may wonder whether potentialists too are under pressure to concede that the set theoretic hierarchy corresponds to a set, which would show their view to be unstable. But potentialists have a response. On their conception, the hierarchy is potential in character and thus intrinsically different from sets, each of which is completed and thus actual rather than potential. This intrinsic difference affords potentialists—unlike their absolutists rivals—a reason to disallow the disputed set formation.

A different but related attraction of the potentialist conception emerges in connection with the hard question of the conditions under which some objects are eligible to form a set. For instance, why are the natural numbers eligible to form a set, but not the ordinals? This question was discussed already by the father of transfinite set theory, Georg Cantor, when he asked which 'multiplicities' of objects form a set. Although Cantor did not hold anything like the iterative conception of sets, his analysis remains instructive and inspiring. Here is what he wrote in a famous 1899 letter to Dedekind.

> [I]t is necessary [. . .] to distinguish two kinds of multiplicities (by this I always mean *definite* multiplicities).

[3] This challenge is developed in more detail in my work (Linnebo, 2010), which can be regarded as a philosophical companion piece to the present, more technical paper.

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> For a multiplicity can be such that the assumption that *all* of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities*. [. . .]
>
> If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into '*one* thing', I call it a *consistent multiplicity* or a 'set'. (Ewald, 1996, p. 931-932)

An 1897 letter by Cantor to Hilbert is also suggestive.

> I say of a set that it can be thought of as *finished* [. . .] if it is possible without contradiction (as can be done with finite sets) to think of *all its elements as existing together*, and so to think of the set itself as *a compounded thing for itself*; or (in other words) if it is *possible* to imagine the set as *actually existing* with the totality of its elements. (Ewald, 1996, p. 927)

Cantor's thought is that there is an intrinsic difference between multiplicities that form sets and multiplicities that do not, and that this intrinsic difference *explains* why some but not all multiplicities are eligible for set formation. A set is characterized as a 'finished' collection, all of whose elements can 'exist together' or be imagined as 'actually existing'. So for a multiplicity to be eligible for set formation, it must be capable of being regarded as 'finished', and its elements must be capable of 'existing together'. Multiplicities that are capable of the sort of 'completion' that is characteristic of sets are thus intrinsically suited for set formation, whereas multiplicities that resist this sort of 'completion' are intrinsically unsuited for set formation. For instance, since the multiplicities of everything thinkable and of all ordinals resist 'completion', there can be no universal set or set of all ordinals.

Not surprisingly, Cantor's analysis of set formation has often met with incomprehension. What does it mean to say that a set is 'finished' or that its elements are capable of 'existing together'? Dedekind and Hilbert's responses to Cantor's letters are examples of this reaction.[4] Both mathematicians disapproved of Cantor's use of modal and temporal terminology, which had no more of a place in the standard mathematical language of their day than it does now.[5]

In what follows, I propose to take seriously the need for modal notions in the foundations of set theory. I show that doing so enables us to explicate both the potentialist conception of the hierarchy of sets and Cantor's analysis of set formation. Of course, this project prompts just the kind of challenge that Cantor encountered from Dedekind and Hilbert. Is the introduction of modal notions into discussions of set theory really justified? My hope is that sufficient justification will be provided by the explanatory value of the modal approach. A full-fledged explanation of the modal notions will have to await another occasion.

I here confine myself to one brief but important remark about the modality that I invoke. *This is not metaphysical modality in the usual post-Kripkean sense.* Rather, the modality employed in this article is related to that involved in the ancient distinction between a potential and an actual infinity. This modality is tied to a process of building up larger and

[4] See Ferreirós (2007, chapter XIII, section 8). See also Jané (2010) for a useful discussion of competing potentialist and actualist tenets of Cantor's thought.

[5] A similar attitude lives on today in Boolos (1989)'s admonition against taking the iterative conception's talk of 'set formation' too literally.

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larger domains of mathematical objects. A claim is possible, in this sense, if it can be made to hold by a permissible extension of the mathematical ontology; and it is necessary if it holds under any permissible such extension. Metaphysical modality would be unsuitable for our present purposes because pure sets are taken to exist of metaphysical necessity if at all. By contrast, the process we investigate is envisaged as taking place at the unique world that is *metaphysically* actual. The novelty is that the hierarchy of sets that exist at this world is regarded as potential in character: however many sets have been formed, it is *mathematically* possible to form even more.

My investigation consists of an informal phase and a formal one—although the two are often intertwined. In the informal phase, I introduce and explain my modal set theory. I gloss its linguistic primitives and seek to motivate its basic principles. The formal phase consists of a formal investigation of the modal set theory and a proof that this theory can interpret ordinary nonmodal ZF. It is important not to conflate the aims of the two distinct phases. In particular, the fact that I attempt some informal explanations of certain notions and principles does not conflict with regarding these notions and principles as primitives for the purposes of the formal phase.

§3. Identifying an appropriate modal logic. The aim of this section is to motivate the choice of a modal logic intended to capture relevant aspects of the process of set formation.[6] Most of the section thus belongs belongs to the informal phase. For present informal purposes, it will be useful to think of the possible stages of the process of forming sets as possible worlds, where the domain of each world consists of the sets that have been formed thus far.

Recall that a *Kripke model* is standardly defined as a quadruple $\langle W, R, \mathbf{D}, v\rangle$, where W is a set of worlds, R an accessibility relation on W, $\mathbf{D}$ is a function that assigns to each world in W a domain, and v is a valuation that specifies what sequences of objects satisfy the atomic predicates at each world. A Kripke model is thus based on sets. However, on pain of paradox, there cannot be a *set* of all possible stages of the process of set formation. So the system of possible worlds with which we are concerned is unsuited to form a Kripke model in the standard sense. It will nevertheless be useful to motivate our choice of modal logic by appealing to properties of our system of possible worlds. For the purposes of the formal phase developed in later sections, however, the relevant modal logic will be taken as basic, and the modal operators will be regarded as primitives rather than analyzed in terms of quantification over possible worlds.

One world w' is regarded as accessible from another w just in case one can get from w to w' by a legitimate expansion of the mathematical ontology. This connects the accessibility relation R with the domain assignment function $\mathbf{D}$ as follows:

$$w R w' \rightarrow \mathbf{D}(w) \subseteq \mathbf{D}(w')$$

The role of the accessibility relation also makes it natural to require that it be a partial order (i.e., reflexive, anti-symmetric, and transitive). Henceforth I will therefore write the accessibility relation as $\leq$ rather than R. This yields the following principle.

[6] In fact, the motivation I provide is not specific to the process of set formation but applies equally to more general processes for introducing or individuating mathematical objects. See (Linnebo, 2009).

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Partial ordering: The accessibility relation $\leq$ is a partial order.

We also require that the process of set formation be well-founded, which yields the following principle.

Well-Foundedness: The accessibility relation $\leq$ is well-founded.

What if we have a choice which sets to go on and form? Assume we are at a world w_0 where we can go on to form further sets so as to arrive at either w_1 or w_2. It makes sense to require that the licence to form sets never goes away as we build up the hierarchy of sets but can always be exercised at a later stage. This corresponds to the requirement that the two worlds w_1 and w_2 can be extended to a common world w_3. This property of a partial order is called *directedness* and formalized as follows:

$$\forall w_1 \forall w_2 \exists w_3 (w_1 \leq w_3 \wedge w_2 \leq w_3).$$

We therefore adopt the following principle.

Directedness: The accessibility relation $\leq$ is directed.

This principle ensures that whenever we have a choice which sets to form, the order in which we choose to proceed is irrelevant. Whichever sets we choose to form first, the ones we chose not to form can always be formed later. Unless $\leq$ is directed, our choice whether to extend the ontology of w_0 to that of w_1 or that of w_2 might have a lasting effect.

The question of what to do when there are several ways of going on to form sets admits of a stronger answer as well, namely that the licence to form a set must always be exercised right away. This corresponds to the following principle.

> **Maximality**: At every stage all the sets that can be formed are in fact formed.[7]

Although this principle too is fairly natural, it goes beyond the minimal conception of the process of set formation. Any arguments that depends on this principle will therefore be flagged as such.

The mentioned properties of the accessibility relation $\leq$ allow us to identify the modal logic appropriate for studying the process of set formation. Since $\leq$ is reflexive and transitive, the modal logic S4 will be sound with respect our intended system of possible worlds. As is well known, the directedness of $\leq$ ensures the soundness of the following principle as well:

$$\Diamond \Box p \rightarrow \Box \Diamond p. \tag{G}$$

The modal propositional logic that results from adding (G) to a complete axiomatization of S4 is known as S4.2. Since the domains always increase along the accessibility relation $\leq$, there is no need for a free logic or the use of an existence predicate. This motivates the choice of S4.2 as our background modal propositional logic.

Let *the closure* of a formula be the result of prefixing its universal closure with a '$\Box$'. Unless context indicates otherwise, every open formula will be understood as short for its closure. With this convention in place, we adopt the following modal first-order logic.

[7] Maximality implies (but is not exhausted by) the requirement that $\leq$ be not just directed but a linear order. Parsons (1983b, p. 319) explains how the requirement of linearity can be formalized.

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DEFINITION 3.1. *Let* MFO *(modal first-order logic) consist of classical S4.2, the usual introduction and elimination rules for identity and the first-order quantifiers, and the axiom* $x \neq y \rightarrow \Box(x \neq y)$.[8]

It is well known that systems such as MFO prove the Converse Barcan Formula:

$$\Box\forall x\phi(x) \rightarrow \forall x\Box\phi(x) \qquad \text{(CBF)}$$

It is also well known that (CBF) requires the domains to increase along the accessibility relation. Both facts will be used below.

§4. Plural logic and modality. Our next task is to formulate a system of plural logic and motivate some principles concerning its interaction with modal operators. The resulting system—which can be regarded as a version of second-order logic with a special, extensional interpretation of the second-order variables—will be our background logic throughout the remainder of the article.

In ordinary first-order logic we have singular variables such as x and y, which can be bound by the existential and universal quantifiers. In plural logic we have in addition plural variables such as xx and yy, which can also be bound by the existential and universal quantifiers.[9] '$\exists xx$' is read as 'there are some things xx such that …', and '$\forall xx$', as 'given any things xx, …'.[10] In plural logic there is also a two-place logical predicate $\prec$, where '$u \prec xx$' is to be read as 'u is one of xx'.

What about the logic of plurals? We begin by extending the usual introduction and elimination rules to the plural quantifiers. Then we adopt the following plural comprehension scheme:

$$\exists xx\forall u[u \prec xx \leftrightarrow \phi(u)], \qquad \text{(P-Comp)}$$

where $\phi(u)$ does not contain 'xx' free. It is important to note that this comprehension scheme allows an empty plurality. Here we deviate slightly from plural locutions in natural language, where a plurality must consist at least one and possibly two objects.[11] However, this minor deviation can be justified pragmatically. Allowing an empty plurality will be very convenient for the purposes of formalizing set theory. And even if plural locutions in natural language are in fact not used in this way, there might have been natural languages where they were.

It should also be noted that, since $\phi(u)$ may contain bound plural variables, our plural comprehension scheme is impredicative. Impredicative plural comprehension can be motivated and justified by what, following Bernays (1935), we may call a 'quasi-combinatorial conception of pluralities'. The idea is to extrapolate from the finite to the infinite. Just as we can run through a finite plurality, making arbitrary choices as to which elements are

[8] Although S4.2 proves the necessity of identity, it does not prove the necessity of distinctness. For the usual proof of the latter depends on the Brouwerian principle $\Box\Diamond\phi \rightarrow \phi$, which does not hold in S4.2. (This obstacle can be overcome by adding further modal operators, as described in note 16.)

[9] Plural logic was made popular by George Boolos's classic (Boolos, 1984). For an overview, see Linnebo (2012).

[10] I will sometimes use the singular locution 'a plurality' when talking about some things. I do this only for reasons of style and simplicity; the reader should feel free to translate everything back into the official but more clumsy plural locutions.

[11] In doing so I follow Burgess & Rosen (1997) and Burgess (2004).

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to be included in a subplurality and which are not, we can idealize and assume this to be possible for any plurality. This is of course an enormous extrapolation. But it is one that has become commonplace in mathematics and set theory—and very successfully so. We therefore accept the extrapolation and the comprehension scheme that it motivates.

Our next question is how plural logic interacts with the modal operators. My guiding thought here will be that a plurality comprises exactly the same objects at every world at which the plurality exists. Or, in more semantic terms, a plural variable is assigned exactly the same objects as its values at every world at which the variable has any value at all. I submit that this thought coheres well with the use of plural locutions in natural language. For instance, Harry is necessarily one of Tom, Dick, and Harry; and if John is not one of them, then necessarily so.[12]

We therefore adopt two axioms to the effect that being one of some objects, and not being so, are stable from world to world:

$$u \prec xx \rightarrow \Box(u \prec xx) \qquad (\text{STB}^{+} \prec)$$

$$u \not\prec xx \rightarrow \Box(u \not\prec xx) \qquad (\text{STB}^{-} \prec)$$

More generally, we say that a formula $\phi(\mathbf{u})$ is *stable* just in case the following two conditionals hold (where we let variables in boldface abbreviate strings of variables):[13]

$$\phi(\mathbf{u}) \rightarrow \Box\phi(\mathbf{u}) \qquad (\text{STB}^{+}\text{-}\phi)$$

$$\neg\phi(\mathbf{u}) \rightarrow \Box\neg\phi(\mathbf{u}) \qquad (\text{STB}^{-}\text{-}\phi)$$

However, the two stability axioms for '$\prec$' do not fully capture our guiding thought about the modal profile of plurals. Consider the condition that u is one of some given objects xx. Although this condition cannot 'change its mind' about objects existing at one world as we go to larger worlds, the above axioms nevertheless allow its extension to grow as we go to larger worlds—provided that this growth involves only objects that are new at these larger worlds. An example of this phenomenon is provided by the identity predicate. Although our background modal logic MFO proves that this predicate is stable, its extension obviously grows as we go to larger worlds.

We would like to express that pluralities are 'inextensible' and in this way fully to capture our guiding thought. This can be done by adopting the following axiom schema:

$$\forall u(u \prec xx \rightarrow \Box\theta) \rightarrow \Box\forall u(u \prec xx \rightarrow \theta) \qquad (\text{INEXT-}\prec)$$

This formalization requires some explanation. Recall that the Barcan formula

$$\forall u\Box\theta \rightarrow \Box\forall u\theta \qquad (\text{BF})$$

corresponds to the semantic requirement that whenever a world w accesses another world w', then the domain of w' be included in that of w. The axiom schema works by relativizing the Barcan formula to the condition that u is one of some given objects xx. The axiom

[12] See Rumfitt (2005) and Uzquiano (2011) for discussion, and Linnebo (2012, section 2.3) for an overview. Should there turn out to be natural language examples where plural locutions do not function in this way (as argued, e.g., by Hewitt, 2012), my response would be a pragmatic one—as in the case of the empty plurality. Using plural locutions in accordance with my principles would at most be a minor deviation from our ordinary practice. And this alternative usage would be perfectly coherent and theoretically very useful.

[13] This corresponds to the notion of 'absoluteness' in (Parsons, 1983b). I find my term more suggestive.

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schema thus corresponds to the semantic requirement that whenever a world w accesses another world w', then the objects that are among xx at w' be included in the objects that among xx at w. This is exactly the desired property of inextensibility.

Our discussion motivates the adoption of the following modal plural logic.

DEFINITION 4.1. *Let* PFO *be the system that adds to standard first-order logic the standard introduction and elimination rules for the plural quantifiers and the comprehension scheme (P-Comp). Let* MPFO *be the system that adds to PFO the modal logic S4.2, the stability axioms (*STB$^+$ $\prec$*), (*STB$^-$ $\prec$*), and all the instances of the inextensibility schema (*INEXT-$\prec$*).*

I will conclude this section by explaining how our modal plural logic can be used to formalize a notion of *extensional definiteness* that will be important in later sections.[14] It is easy to characterize the notion if we help ourselves to the idiom of possible worlds: a formula ϕ is is extensionally definite at a world w just in case its extension remains the same at any later world $w' \geq w$. For instance, being one of a particular plurality is extensionally definite (or so I have just argued). But how should the notion of extensional definiteness be formalized in our official idiom, which eschews quantification over possible worlds? For formulas with one free singular variable, we can exploit the extensional definiteness that MPFO ascribes to pluralities to provide the following simple formalization of the extensional definiteness of ϕ[15]:

$$\exists xx \Box \forall u[u \prec xx \leftrightarrow \phi(u)] \qquad \text{(ED-}\phi\text{)}$$

The adequacy of this formalization is demonstrated by an easy lemma, which I state without proof.

LEMMA 4.2. *(*ED-ϕ*) entails the stability of ϕ and all instances (in the relevant language) of the inextensibility schema for ϕ:*

$$\phi(x) \rightarrow \Box\phi(x) \qquad \text{(STB}^+\text{-}\phi\text{)}$$

$$\neg\phi(x) \rightarrow \Box\neg\phi(x) \qquad \text{(STB}^-\text{-}\phi\text{)}$$

$$\forall x(\phi(x) \rightarrow \Box\theta) \rightarrow \Box\forall x(\phi(x) \rightarrow \theta) \qquad \text{(INEXT-}\phi\text{)}$$

Moreover, if two extensionally definite conditions are coextensive, then they are necessarily coextensive.

Summing up, our plural logic makes two different kinds of technical contribution, both of which will be important in what follows. It makes available arbitrary pluralities from the relevant domain, as enshrined in its unrestricted comprehension scheme. Additionally, it provides a way of tracking collections of objects from world to world and thus of formalizing the important notion of extensional definiteness.[16] The two contributions are

[14] This corresponds to the notion of 'rigidity' in Parsons (1983b). My alternative term is meant to forestall confusion with the more familiar semantic notion of rigidity famous from Kripke.

[15] Of course, this formalization can be extended to formulas with several free variables in cases (such as set theory) where we can code for ordered pairs.

[16] Alternative formalizations become available if further modal operators are added which enable us to look 'back' at smaller worlds, not just 'forward' at larger ones. For instance, we may add a backtracking operator $\downarrow$ that exempts the formula to which it is applied from the innermost modal operator containing this occurrence in its scope (Hodes, 1984). The extensional definiteness of ϕ can then be expressed as follows: $\Box\forall\mathbf{u}[\phi(\mathbf{u}) \leftrightarrow \downarrow(\phi(\mathbf{u}) \wedge \exists\mathbf{x}(\mathbf{x} = \mathbf{u}))]$. Another option, developed

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connected. The only tracking from world to world that is available for an arbitrary plurality, which need not have any form of intensional definition, is in terms of the objects that the plurality comprises. So the first contribution motivates the second.

§5. Modal and nonmodal theories compared. My next goal is to define a translation from the nonmodal language to the modal one and to show that, under some plausible assumptions, this translation preserves relations of proof-theoretic (and thus also semantic) entailment. The key is the observation that the complex strings $\Box\forall x$ and $\Diamond\exists x$ behave proof-theoretically very much like ordinary quantifiers.

DEFINITION 5.1. *We refer to the complex strings* $\Box\forall$ *and* $\Diamond\exists$ *as* modalized quantifiers. *A formula all of whose quantifiers are modalized is said to be* fully modalized. *When a* $\mathcal{L}$ *is a nonmodal language of first or second order, let* $\mathcal{L}^{\Diamond}$ *be the modal language that results from adding the modal operators* $\Box$ *and* $\Diamond$ *to* $\mathcal{L}$. *Given a nonmodal formula* ϕ *of* $\mathcal{L}$, *its* potentialist translation $\phi^{\Diamond}$ *is the fully modalized formula of* $\mathcal{L}^{\Diamond}$ *that results from replacing each ordinary quantifier in* ϕ *with the corresponding modalized quantifier.*

DEFINITION 5.2. *Let the* stability axioms *for a modal language* $\mathcal{L}^{\Diamond}$ *(which may be a plural one) be the axioms stating that every atomic predication of* $\mathcal{L}^{\Diamond}$ *is stable.*

Recall that the stability of identity is a theorem of our modal logic MFO, and that the stability of the notion of being one of a given plurality is built into MPFO. The stability of the membership predicate $\in$ will be defended in Section 6.

LEMMA 5.3. *Let* ϕ *be a fully modalized formula of a modal language* $\mathcal{L}^{\Diamond}$. *Then S4.2 and the stability axioms for* $\mathcal{L}^{\Diamond}$ *prove that* $\Diamond\phi$, ϕ, *and* $\Box\phi$ *are equivalent and thus in particular that* ϕ *is stable.*

Proof sketch. Assume ϕ is fully modalized. Since we are working in an extension of the modal logic T, it suffices to prove $\Diamond\phi \rightarrow \Box\phi$, which we do by induction on the complexity of ϕ. If ϕ is atomic, the stability axioms allow us to prove our target $\Diamond\phi \rightarrow \Box\phi$. If ϕ is $\neg\psi$, then our target follows straightforwardly from the induction hypothesis applied to ψ. If ϕ is $\psi_1 \wedge \psi_2$, then $\Diamond\phi$ implies $\Diamond\psi_1 \wedge \Diamond\psi_2$, which by the induction hypothesis implies $\Box\psi_1 \wedge \Box\psi_2$, which in turn implies $\Box\phi$. Assume finally that ϕ is of the form $\Diamond\exists x\ \psi$; the case of the plural quantifier is analogous. Since we work in an extension of S4, we have $\Diamond\phi \rightarrow \phi$ for ϕ of this form. So it suffices to prove $\phi \rightarrow \Box\phi$. By our induction hypothesis, we know that ϕ is equivalent to $\Diamond\exists x\,\Box\psi$. Observe next that the Converse Barcan Formula yields $\forall x\Box\exists y(x = y)$ and thus also $\exists x\Box\psi \rightarrow \Box\exists x\ \psi$. The formula $\Diamond\exists x\,\Box\psi$ thus implies $\Diamond\Box\exists x\,\psi$. By (G), the latter formula implies $\Box\Diamond\exists x\ \psi$, which is just $\Box\phi$, as desired. $\dashv$

I claimed above that plausible assumptions ensure that the 'modalized quantifiers' $\Box\forall$ and $\Diamond\exists$ behave logically just like ordinary quantifiers. I now give a precise statement of this claim.

in Studd (forthcoming), is to adopt a second set of modal operators whose accessibility relation is the converse of that of the set of 'forward-looking' operators. Without adding 'backward-looking' modal operators, however, I know of no first-order and singular formalization of extensional definiteness. We thus face a choice between adopting a more complex modal logic or adopting some form of plural or second-order logic. Without denying the value of the former, I here choose the latter, not least because the relation between pluralities and sets is of independent interest.

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THEOREM 5.4. *Let $\vdash$ be the relation of classical deducibility in a language $\mathcal{L}$, although if $\mathcal{L}$ is a plural language, we mean classical deducibility without the use of any plural comprehension axioms. Let $\vdash^{\Diamond}$ be deducibility in $\mathcal{L}^{\Diamond}$ by $\vdash$, S4.2, and the stability axioms for $\mathcal{L}^{\Diamond}$. Let $\phi_1, \ldots, \phi_n$ and ψ be $\mathcal{L}$-formulas. Then we have:*

$$\phi_1, \ldots, \phi_n \vdash \psi \quad \textit{iff} \quad \phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

Proof. The proof goes by induction on the proofs. We start with the left-to-right direction. The only hard cases are the introduction and elimination rules for the quantifiers. I will outline the case of the first-order universal quantifier; the other cases are analogous. We begin with the rule UI. Assume we have $\phi_1, \ldots, \phi_n \vdash \forall x\, \psi$ and conclude by UI that $\phi_1, \ldots, \phi_n \vdash \psi(t)$ for some suitable term t. By the induction hypothesis we have $\phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \Box\forall x\, \psi^{\Diamond}$, from which we can obviously get $\phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}(t)$. Next we consider the rule UG. Assume that we have $\phi_1, \ldots, \phi_n \vdash \psi(t)$, where t does not occur free in any of the ϕ_i and conclude by UG that $\phi_1, \ldots, \phi_n \vdash \forall x\, \psi$. By the induction hypothesis we have $\phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}(t)$, from which UG gives us $\phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \forall x\, \psi^{\Diamond}$. A standard trick available in S4 then gives us $\Box\phi_1^{\Diamond}, \ldots, \Box\phi_n^{\Diamond} \vdash^{\Diamond} \Box\forall x\, \psi^{\Diamond}$. Lemma 5.3 then gives us $\phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \Box\forall x\, \psi^{\Diamond}$, as desired.

The right-to-left direction is easily established by adopting a Hilbert-style axiomatic approach to the relevant modal logic. Consider the operation $\phi \mapsto \phi^{-}$ of deleting all modal operators. This operation maps every axiom of our modal logic to a theorem of the corresponding nonmodal logic and correlates every inference rule of the former with a legitimate inference of the latter. The right-to-left direction then follows from the observation that $(\phi^{\Diamond})^{-} = \phi$. □

The equivalence in Theorem 5.4 is fairly robust, in the sense that it would obtain even if the definition of $\vdash^{\Diamond}$ was altered so as to incorporate a modal logic stronger than S4.2. The left-to-right direction requires only that $\vdash^{\Diamond}$ incorporate at least S4.2. And the right-to-left direction requires only that $\vdash^{\Diamond}$ incorporate not more than the modal logic Triv based on the axiom $p \leftrightarrow \Box p$ of modal collapse.

Theorem 5.4 tells us that, if we are interested in logical relations between fully modalized formulas against the background of a modal theory that includes at least S4.2 and the stability axioms, then we may delete all the modal operators and proceed by the corresponding nonmodal theory. This theorem will be crucial in later comparisons of modal and nonmodal set theories. The theorem is also underlying the observation made in the introduction that modal set theory is compatible with ordinary nonmodal ZF set theory but looks at the same subject matter under a 'finer resolution'.

By contrast, formulas that are not fully modalized are not subject to Theorem 5.4 or any similar result. Such formulas can thus exploit the 'finer resolution' that is made available in our modal set theory to enable arguments and explanations that are not available in ordinary nonmodal set theory. Several examples of this phenomenon are provided below, but the nicest and most important is that of Section 7.

§6. The nature of sets. I now develop an account of the nature of sets based on the simple core idea that sets are collections that are 'constituted' by their elements. First I provide a loose and informal characterization. Then I formulate the official and technically precise version.

One aspect of the core idea is the extensionality of sets. If a set is a collection that is 'constituted' by its elements, then the nature of a set is exhausted by which elements it

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has. Once you specify the elements of a set, you have specified everything that is essential to it. In particular, if two sets have the same elements, then they are identical. Another aspect of the core idea is the priority of the elements of a set to the set itself. If a set is a collection 'constituted' by its elements, then these elements will have to be available 'before' the set itself can be formed. For instance, if at some stage an object a is available, we can immediately form the new set $\{a\}$ but not the new set $\{\{a\}\}$. For $\{\{a\}\}$ is partially constituted by $\{a\}$, which means that the latter must be available 'before' the former can be formed. Note that our account of the nature of sets says nothing about what sets there are but is concerned with what sets have to be like if there are any.

Let us now make the ideas just adumbrated technically precise. I begin by defining the relevant languages.

DEFINITION 6.1. *Let $\mathcal{L}_\in$ be the language of ordinary nonmodal set theory, and let $\mathcal{L}_{P\in}$ be the corresponding plural language. Let $\mathcal{L}_\in^\Diamond$ and $\mathcal{L}_{P\in}^\Diamond$ be the modal languages that results by adding the modal operators to the mentioned nonmodal languages respectively.*

6.1. The extensionality of sets. The extensionality of sets clearly motivates the familiar axiom of extensionality:

$$x = y \leftrightarrow \forall u(u \in x \leftrightarrow u \in y) \tag{Ext}$$

However, there is more to the extensionality of sets than this. Even if (Ext) is necessary, this will only allow us to identify and distinguish sets within each world. But the idea that sets are constituted by their elements supports a 'transworld' version of principle of extensionality as well. Intuitively speaking, a set x from one world w_1 is identical with a set y from another world w_2 just in case x's elements at w_1 are precisely the same as y's elements at w_2.

This 'transworld' principle of extensionality is captured by the 'intraworld' principle (Ext) together with the claim that being an element in a particular set is extensionally definite; that is, that a set has precisely the same elements at every world at which it exists. As we know from Section 4, this claim about extensional definiteness can be formalized as follows:

$$\exists yy\Box\forall u(u \prec yy \leftrightarrow u \in x) \tag{ED-$\in$}$$

By Lemma 4.2, we know that (ED-$\in$) entails the stability axioms for $\in$, as well as all instances of the inextensibility schema for $\in$:

$$\forall x(x \in y \rightarrow \Box\theta) \rightarrow \Box\forall x(x \in y \rightarrow \theta) \tag{INEXT-$\in$}$$

The extensional definiteness of elementhood enables us to prove a counterpart of the familiar result from ordinary nonmodal set theory that bounded first-order formulas are absolute in transitive structures. To state the result, we first need a definition.

DEFINITION 6.2. *An occurrence $\forall x\,\phi$ of the universal singular quantifier is said to be* bounded *iff ϕ is of the form $x \in y \rightarrow \psi$. An occurrence $\exists x\,\phi$ of an existential quantifier is said to be* bounded *iff ϕ is of the form $x \in y \wedge \psi$. We write these formulas as $(\forall x \in y)\psi$ and $(\exists x \in y)\psi$, respectively. A first-order formula is said to be* bounded *(or Δ_0) iff all its quantifiers are bounded.*

LEMMA 6.3. *Let ϕ be a bounded $\mathcal{L}_\in$-formula, and let $\phi^\Diamond$ be its potentialist translation. Then MPFO and the two axioms associated with the transworld principle of extensionality prove $\phi \leftrightarrow \phi^\Diamond$. This theory also proves that ϕ is stable.*

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Proof. The proof of the first claim goes by induction on the number of quantifiers in ϕ. If ϕ is quantifier-free, the claim is trivial. So assume ϕ is $(\forall x \in a)\psi$; the case of a bounded existential quantifier is analogous. Clearly we have $\Box(\forall x \in a)\psi^{\Diamond} \rightarrow (\forall x \in a)\psi^{\Diamond}$. Since the induction hypothesis ensures that ψ and $\psi^{\Diamond}$ are provably equivalent, this can be strengthened to $\Box(\forall x \in a)\psi^{\Diamond} \rightarrow (\forall x \in a)\psi$, that is $\phi^{\Diamond} \rightarrow \phi$.

For the other direction, assume $(\forall x \in a)\psi$, which abbreviates $\forall x(x \in a \rightarrow \psi)$. Observe first that the induction hypothesis yields that ψ and $\psi^{\Diamond}$ are provably equivalent. Observe next that since $\psi^{\Diamond}$ is fully modalized, Lemma 5.3 yields $\psi^{\Diamond} \rightarrow \Box\psi^{\Diamond}$. Using these two observations, our assumption implies $\forall x(x \in a \rightarrow \Box\psi^{\Diamond})$. By Lemma 4.2, we can apply (INEXT-$\in$) to get $\Box\forall x(x \in a \rightarrow \psi^{\Diamond})$, which abbreviates as $\Box(\forall x \in a)\psi^{\Diamond}$. This establishes $\phi \rightarrow \phi^{\Diamond}$, as desired.

The second claim—that ϕ is stable—follows by applying Lemma 5.3 to the first claim. □

6.2. The priority of elements to their set. Recall our core idea that a set is a collection constituted by its elements. I have addressed the first aspect of this idea: the transworld version of the principle of extensionality. I now turn to the second aspect: the principle that the elements of a set are prior to the set itself.

A natural first attempt to express this principle is as follows:

$$x \in y \rightarrow \Diamond(Ex \wedge \neg Ey)$$

Unfortunately, this attempt fails because our accessibility relation only allows a world to access another world if the domain of the former is included in the domain of the latter.[17]

Fortunately, standard ZF set theory suggests an alternative. Recall that one of its axioms is *Foundation*, which can be formalized as follows:

$$\forall x[\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \forall z(z \in x \rightarrow z \notin y))] \qquad \text{(F)}$$

Instead of searching for the ideal formalization of the priority principle, I propose that we simply adopt (F) as an axiom.[18] This is permissible because (F) follows from the priority principle and our assumption that the accessibility relation between possible worlds is well-founded. We see this by the following informal but compelling argument.[19] Consider a set x. Let C be the class of worlds that contain at least one element of x. By the well-foundedness of $\leq$, there is a $\leq$-minimal member of this class, say w_0. Let y be an element of x that exists at w_0. Assume some element z of x is an element of y as well. By the priority principle, z would have to be present at some world $w_1 < w_0$. But this contradicts

[17] Studd (forthcoming) formulates the well-foundedness of the accessibility relation by means of a version of the Löb principle $\Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi$ for his backward-looking necessity operator (see note 16). He also shows how the priority principle can be expressed and how this enables the derivation of (the potentialist translation of) Foundation as a theorem. This is a satisfying feature of his bimodal approach.

[18] Although this approach gives us all we need, it does not (unlike the one described in the previous note) rule out models that violate priority. For instance, my theory has models of the sort suggested by Zermelo (1930) whose worlds are V_κ's for inaccessible κ, with the obvious accessibility relation.

[19] The argument cannot be formalized in any of the theories set out in this article but could easily be formalized in a set theory with classes.

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the minimality of w_0. So no element of x can be an element of y as well, which gives us (F).[20]

6.3. Sets and pluralities. I have explicated the core idea that sets are collections 'constituted' by their elements in a way that makes quite limited use of plural logic, namely, in order to express the notion of extensional definiteness. However, the core idea admits of an interesting alternative explication that makes more substantive use of plural logic. According to this explication, a set is an object that is 'metaphysically derived from' the plurality of its elements and that therefore 'inherits' many of its properties from this plurality; in particular, the modal profile of a set is 'inherited' from that of the associated plurality. Cantor gives voice to a similar idea about the relation between sets and pluralities in some of his famous 'definitions' of set, for instance when we writes that '[b]y a manifold or a set I understand every multiplicity [jedes Viele] which can be thought of as one' (Cantor, 1883, p. 916).[21]

This alternative explication of our core idea suggests a somewhat different technical development. Instead of a single nonlogical primitive $\in$ for elementhood, we adopt a single nonlogical primitive $\equiv$, to be flanked by a singular argument on its left and a plural one on its right, where $x \equiv uu$ is to be read as 'x is the set formed by uu'. Let $xx \approx yy$ abbreviate $\forall w(w \prec uu \leftrightarrow w \prec vv)$. Then the extensionality and extensional definiteness of sets can be given the following alternative formulations:

$$x \equiv uu \wedge y \equiv vv \rightarrow (x = y \leftrightarrow uu \approx vv) \qquad (\text{Ext}')$$

$$x \equiv uu \rightarrow \Box(x \equiv uu) \qquad (\text{STB}^{+}\text{-}\equiv)$$

$$x \not\equiv uu \rightarrow \Box(x \not\equiv uu) \qquad (\text{STB}^{-}\text{-}\equiv)$$

Assume membership is now defined by:

$$x \in y \leftrightarrow \exists vv(y \equiv vv \wedge x \prec vv) \qquad (\text{Def-}\in)$$

One then easily verifies that (Ext′) is equivalent to (Ext), and (STB^{+}-$\equiv$) and (STB^{-}-$\equiv$) to (ED-$\in$).

The priority thesis too is natural on the alternative explication. If a set is metaphysically derived from the plurality of its elements, then these elements need to be available before the set can be formed.

6.4. The extensional definiteness of subsethood. I argued in Section 6.1 that the notion of being an element of a particular set is extensionally definite. What about the notion of being a *subset* of a particular set? Is this too extensionally definite? The question is whether the following is true:

$$\exists xx\Box\forall u[u \prec xx \leftrightarrow u \subseteq a] \qquad (\text{ED-}\subseteq)$$

The following lemma provides a partial answer.

---

[20] Attentive readers may have noticed that this argument does not distinguish very carefully between (F) and its potentialist translation ($\text{F}^{\Diamond}$). This is permissible because Lemma 6.3 ensures that ($\text{F}^{\Diamond}$) is equivalent to the result of prefixing (F) with a single necessity operator, which we may anyway do to an axiom.

[21] The page reference is to the translation listed in the bibliography. The same goes for all other page references to works where a translation or reprinting is listed.

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LEMMA 6.4. *Consider the $\mathcal{L}^{\Diamond}_{P\in}$-theory consisting of MPFO and the axioms (Ext), (*ED-$\in$*), and (F). This theory proves:*

$$u \subseteq a \to \Box(u \subseteq a), \qquad (\text{STB}^{+}\text{-}\subseteq)$$

$$u \not\subseteq a \to \Box(u \not\subseteq a). \qquad (\text{STB}^{-}\text{-}\subseteq)$$

However, the theory does not prove all instances of

$$\forall u(u \subseteq a \to \Box\theta) \to \Box\forall u(u \subseteq a \to \theta) \qquad (\text{INEXT-}\subseteq)$$

*and hence also not (*ED-$\subseteq$*).*[22]

Proof. The two positive claims follow from Lemma 6.3 and the observation that the formulas $u \subseteq a$ and $u \not\subseteq a$ are bounded. The negative claims can be established by constructing a simple countermodel. Consider a world with two objects a and b but no sets containing either of these objects. Then introduce $\{a, b\}$ first and $\{a\}$ later. Let θ express that u has two elements. This yields a counterexample to (INEXT-$\subseteq$). By Lemma 4.3, this is also a counterexample to (ED-$\subseteq$). □

Lemma 6.4 raises the question whether (ED-$\subseteq$) should be adopted as an axiom. Is (ED-$\subseteq$) supported by our core idea that sets are constituted by their elements? The answer depends on how we understand the process of set formation. Consider a set a and some subset $b \subseteq a$ that is present at some later world. When a was formed, all of its elements must already have been available. So *a fortiori* all the elements of b must have been available. When a was formed, we therefore had the *ability* to form b. But was this ability *exercised*? According to the principle of Maximality—which says that we always form all the sets that we are capable of forming—the answer is *yes* and (ED-$\subseteq$) will thus hold. But as we have seen, without Maximality, it is easy to construct a counterexample to (ED-$\subseteq$).

I will distinguish between theories that assume the Principle of Maximality and those that do not by attaching a minus sign to the name of each theory that does not assume this principle. The following definition sums up our discussion of the nature of sets.

DEFINITION 6.5. *Let* NS *be the $\mathcal{L}^{\Diamond}_{P\in}$-theory that adds to MPFO the following axioms: (Ext), (*ED-$\in$*), (F), and (*ED-$\subseteq$*). Let* NS^{-} *be like NS except without the axiom (*ED-$\subseteq$*).*

In NS we can prove an extension of Lemma 6.3 to formulas with bounded plural quantifiers. This corresponds to the familiar set-theoretic fact that bounded second-order formulas are absolute in supertransitive structures, that is, in structures that are both transitive and contain as an element each subset of each of their elements.

§7. The principle of set existence.

My analysis of the nature of sets has resulted in various *conditional* claims about what sets there are. But quite reasonably, it did not yield any *categorical* claims about the existence of sets. Our next task is to investigate how categorical set existence claims can be established.

7.1. From pluralities to sets.

We take our lead from the iterative conception, which tells us that any 'given' objects can be used to form a set. That is, when some objects are available at some stage of the process of set formation, there is a later stage containing

[22] Nor does (INEXT-$\subseteq$) ensure (INEXT-$\in$). For there is a model of MPFO, (Ext), the stability axioms for $\in$, (INEXT-$\subseteq$) and the assumption that every object has a singleton in which (INEXT-$\in$) fails.

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a set with precisely these objects as its elements. This principle admits of a very natural expression in terms of our plural and modal resources, namely as the claim that necessarily, given any objects, it is possible for them to form a set; or in symbols:

$$\Box\forall xx\, \Diamond\exists y\, \Box\forall u(u \in y \leftrightarrow u \prec xx). \tag{C}$$

This will be our sole categorical claim about the existence of sets.

In fact, (C) provides a good explication of Cantor's principle that any consistent multiplicity forms a set—thus the label. For recall that a multiplicity is said to be consistent just in case it can be 'completed', such that all its members 'exist together'. On the analysis I defended in Section 4, pluralities have exactly this character. So if this analysis is right, our ordinary notion of a plurality corresponds to Cantor's notion of a consistent multiplicity.

7.2. From conditions to sets. Consider the naive set comprehension scheme:

$$\exists x\forall u[u \in x \leftrightarrow \phi(u)]. \tag{N-Comp}$$

Which instances are valid? Not surprisingly, the principle (C) turns out to entail a broadly Cantorian answer.

Transposed to our potentialist setting and slightly generalized, the question just posed corresponds to the question of which instances of the following scheme are valid:

$$\Diamond\exists x\, \Box\forall u[u \in x \leftrightarrow \phi(u)] \tag{N-Comp$^\Diamond$}$$

Our analysis of the nature of sets provides half of the answer. I argued that any set x is extensionally definite, in the sense that the objects that comprise x's members necessarily comprise its members:

$$\exists yy\Box\forall u[u \in x \leftrightarrow u \prec yy].$$

It follows that, if (N-Comp$^\Diamond$) holds for $\phi(u)$, then possibly there are some objects that necessarily are the extension of $\phi(u)$:

$$\Diamond\exists yy\Box\forall u[u \prec yy \leftrightarrow \phi(u)].$$

Our analysis of the nature of sets thus implies that, if a condition $\phi(u)$ defines a set—in the sense that (N-Comp$^\Diamond$) holds—then it is possible for $\phi(u)$ to be extensionally definite.

What about the converse? That is, if it is possible for a condition to be extensionally definite, does the condition define a set—in the sense that (N-Comp$^\Diamond$) holds? It is easily seen that (C) not only entails this converse conditional but in fact is equivalent to it.

Putting the two conditionals together, we obtain the desired answer to our question: a condition defines a set—in the sense that (N-Comp$^\Diamond$) holds of it—just in case it is possible for the condition to be extensionally definite. I find this answer very satisfactory. It is part of the nature of sets to be extensionally definite. So a condition is intrinsically suited for defining a set just in case it is possible for it to be extensionally definite. On the present account, a condition therefore defines a set just in case it is intrinsically suited for doing so. There is nothing mysterious or surprising about the fact that some conditions fail to define sets. Such conditions have an inexhaustible character that renders them intrinsically unsuitable for defining sets.

Two remarks about this analysis are in order. Firstly, the analysis is essentially just Cantor's, except that talk about Cantorian multiplicities has been replaced by talk about

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their defining conditions.[23] A consistent multiplicity corresponds to a condition that is possibly extensionally definite. An inconsistent multiplicity corresponds to a condition that cannot be extensionally definite. In fact, if a condition $\phi(u)$ is stable (as Lemma 4.2 tells us any fully modalized condition is), then the claim that $\phi(u)$ cannot be extensionally definite is easily seen to be equivalent to the claim that it is *indefinitely extensible*, in the sense that

$$\Box\forall x x \Diamond\exists u[\phi(u) \wedge u \prec\!\!\!\!\neq x x].^{24} \qquad \text{(IE-}\phi\text{)}$$

Secondly, it is important to note that my account of which conditions define sets makes essential use of my modal approach. The original question is easily stated without any modal resources, namely as the question of which conditions admit of naive set comprehension (N-Comp). However, the answer that is provided by our Cantorian analysis is only available under the finer resolution afforded by the modal approach. For the key notion of a condition's possibly being extensionally definite is not the potentialist translation of any nonmodal formula and is thus expressible only in the richer modal framework. (Of course, when we adopt this finer resolution, (N-Comp) must be replaced by its potentialist counterpart (N-Comp$^{\Diamond}$).) In short, the nonmodal question of which conditions define sets admits of a more attractive and natural answer in a modal framework than in a traditional nonmodal one. And Cantor himself was fully aware of this, as the modal vocabulary in his letters to Dedekind and Hilbert reveal (Section 2).

§8. Recovering the axioms of ZF. Cantor's principle (C) plays a key role in our recovery of many of the familiar axioms of Zermelo Fraenkel set theory. Indeed, many of these axioms follow already from (C) and our theory of the nature of sets. We begin by reminding ourselves of these axioms.

DEFINITION 8.1. Zermelo Fraenkel set theory *(or* ZF*) is the $\mathcal{L}_{\in}$-theory whose axioms are the extensionality principle (Ext), Foundation (F), and the following set existence claims:*

$$\exists x\forall u\,(u \notin x) \qquad \text{(Empty Set)}$$

$$\exists x\forall u\,(u \in x \leftrightarrow u = a \vee u = b) \qquad \text{(Pairs)}$$

$$\exists x\forall u\,(u \in x \leftrightarrow \exists v(u \in v \wedge v \in a)) \qquad \text{(Union)}$$

$$\exists x\forall u\,(u \in x \leftrightarrow u \in a \wedge \phi(u)) \qquad \text{(Separation)}$$

$$\exists x\forall u\,(u \in x \leftrightarrow u \subseteq a) \qquad \text{(Power)}$$

$$\exists x(\varnothing \in x \wedge \forall u(u \in x \rightarrow \{u\} \in x) \qquad \text{(Infinity)}$$

$$\forall u\exists!v\,\psi(u,v) \rightarrow \forall x\exists y(\forall u \in x)(\exists v \in y)\,\psi(u,v) \qquad \text{(Replacement)}$$

Zermelo set theory *(or* Z*) is ZF without Replacement. We add a minus sign to the name of a theory to indicate the theory that results from removing the Power set axiom.*

[23] Compare Parsons (1977, section 3), who argues that Cantorian multiplicities should be understood as intensional entities, specified by means of application conditions.

[24] According to Dummett (1993, p. 441), a concept is indefinitely extensible if, for any definite totality of objects falling under the concept, there is another object that falls under the concept but is not a member of the totality. I believe the above characterization of indefinite extensibility is a good explication of Dummett's notion. See Shapiro & Wright (2006) for an attempt at a nonmodal characterization.

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8.1. Basic modal set theory.

THEOREM 8.2.

(a) NS + (C) proves the potentialist translations of all the axioms of Z—Infinity.

(b) NS⁻ + (C) proves the potentialist translations of all the axioms of Z⁻—Infinity.

(c) Each of the nonmodal set theories in (a) and (b) is interpretable in the corresponding modal set theory.

Proof. Observe first that the following conditions are extensionally definite:

(i) $u \neq u$

(ii) $u = a \vee u = b$

(iii) $\Diamond\exists x(u \in x \wedge x \in a)$

(iv) $u \in a \wedge \phi(u)$ where $\phi(u)$ is stable

(v) $u \subseteq a$

As for (i) and (ii), this is trivial. (iii) and (iv) follow easily from the extensional definiteness of membership. (v) follows from (ED-$\subseteq$). Next we observe that when (C) is applied to the pluralities defined by these conditions, we obtain the potentialist translations of the axioms of Empty Set, Pairing, Union, Separation, and Power. (For Separation we appeal to Lemma 5.3 to ensure the stability of the modal translation of the formula in question.[25]) The claim about interpretability is now immediate from (a), (b), and Theorem 5.4. □

In fact, we will see that the power set axiom is the only axiom of ZF whose justification requires (ED-$\subseteq$) and thereby also the Maximality principle (which, we recall from Section 3, says that at every stage, all sets that can be formed are formed). This observation shows how precise content can be attached to the widespread sentiment that the power set axiom is very strong.

8.2. Full modal set theory. In order to arrive at full ZF, we still need the axioms of Infinity and Replacement. We will now see how potentialist translations of these axioms can be derived if some further plausible assumptions are added to our modal set theory. As in Theorem 8.2, this will establish that larger fragments of ordinary ZF are interpretable in the various systems of modal set theory.

One natural further assumption is that extensional definiteness is a matter of size.[26] Consider two one-place conditions ϕ and ψ. If ϕ is extensionally definite and every ϕ is correlated with a unique ψ, then ψ too is extensionally definite. This motivates a replacement principle for the notion of extensional definiteness. Let $\text{FUNC}(\psi^{\Diamond}(u, v))$ abbreviate the following formalization of the claim that $\psi^{\Diamond}(u, v)$ is functional:

$$\Box\forall u\, \Diamond\exists v\, \Box\forall v'\, (\psi^{\Diamond}(u, v') \leftrightarrow v = v').$$

Then we have

$$\text{FUNC}(\psi^{\Diamond}(u, v)) \rightarrow \Box\forall x x \Diamond\exists y y(\forall u \prec x x)(\exists v \prec y y)\, \psi^{\Diamond}(u, v) \qquad \text{(ED-Repl)}$$

---

[25] Without the directedness of $\leq$, Lemma 5.3 would fail and we would only be able to justify Bounded Separation. Compare Theorems 1 and 2 in Parsons (1983b, p. 337). However, as Parsons (1983b, p. 339) observes, full Separation can be justified when the potentialist set theory is supplemented with a reflection principle, as will be discussed in the next section.

[26] Shapiro & Wright (2006) make an analogous assumption in their nonmodal approach to the notion of indefinite extensibility.

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THEOREM 8.3. *NS^{-} + (C) + (ED-Repl) entails the potentialist translation of Replacement.*

Proof. Assume $\text{FUNC}(\psi^{\Diamond}(u, v))$ and consider a set x. By Lemma 6.3, it suffices to show that there may be a set y such that $(\forall u \in x)(\exists v \in y)\psi^{\Diamond}(u, v)$. Let xx be the elements of x and apply (ED-Repl) to derive $\Diamond\exists yy(\forall u \prec xx)(\exists v \prec yy)\,\psi^{\Diamond}(u, v)$. By (C) we know that yy possibly form a set y: $\Diamond\exists y\Box\forall u(u \in y \leftrightarrow u \prec yy)$. It follows that $\Diamond\exists y(\forall u \in x)(\exists v \in y)\,\psi^{\Diamond}(u, v)$, as desired. □

Another natural principle says that truths about the potential hierarchy of sets are 'reflected' in truths about individual possible worlds. This corresponds to the following reflection principle:

$$\phi^{\Diamond} \rightarrow \Diamond\phi \qquad (\Diamond\text{-Refl})$$

(Recall that this is short for the closure of the displayed formula.) This principle is best understood as stating that the truth of a claim in 'the model' provided by the potential hierarchy of sets ensures that the claim is possible. For a claim ϕ to be true in this 'model' is for ϕ to be true when all its quantifiers are understood as ranging over all possible sets, including ones not yet formed. But for ϕ to be true when understood in this way is simply for its potentialist translation $\phi^{\Diamond}$ to be true. The principle therefore says that the truth of $\phi^{\Diamond}$ ensures the possibility of ϕ.

Three remarks about the reflection principle ($\Diamond$-Refl) are in order. Firstly, the principle is not specifically about sets but about the process of forming mathematical objects more generally. The principle says that we should recognize any situation that is realized in the potential hierarchy as genuinely possible. ($\Diamond$-Refl) thus complements the modal principles described in Section 3. Where those principles characterize *the structure* of the space of possible worlds, ($\Diamond$-Refl) says something about its *extent*.

Secondly, it is often claimed that reflection principles have a 'top-down' character that sits poorly with the view of the hierarchy of sets as potential. Indeed, the usual form of a reflection principle

$$\phi \rightarrow \exists\alpha\, \phi^{V_\alpha}$$

suggests that the entire hierarchy has to be available in order for ϕ to be evaluated and reflected down to an appropriate initial segment V_α. The formulation ($\Diamond$-Refl) shows that this impression is incorrect and illustrates how a reflection principle can enjoy a more 'bottom-up' motivation.

Finally, there is one serious worry about ($\Diamond$-Refl). When we apply the principle to (C) and do a *modus ponens*, we get the claim that possibly every plurality forms a set:

$$\Diamond\forall xx\exists y\forall u(u \in y \leftrightarrow u \prec xx)$$

But what is here claimed to be possible is in fact inconsistent, as is seen by instantiating the quantifier $\forall xx$ with respect to the plurality of all nonself-membered sets and reproducing the reasoning of Russell's paradox.

Fortunately, the motivation I offered for ($\Diamond$-Refl) allows me to explain what has gone wrong. The motivation was that every possibility that is realized in the potential hierarchy of sets should also be realized at some possible world. But when ϕ contains plural variables, the possibility witnessed by the truth of $\phi^{\Diamond}$ can be seen not quite to correspond to the possibility described by ϕ. In the potential hierarchy of sets every plurality corresponds to a set, as is ensured by (C). So in this hierarchy there is a one-to-one correspondence between pluralities and sets. The possibility witnessed by $\phi^{\Diamond}$ is thus a possibility in which any plural quantifiers range over precisely such pluralities as form sets. By contrast, the

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possibility described by ϕ is a possibility in which any plural quantifiers range over all pluralities regardless of whether they form sets.[27] Thus, when ϕ contains plural variables, the possibility witnessed by $\phi^{\Diamond}$ is importantly different from the possibility described by ϕ.

How can we avoid this mismatch and be faithful to the motivation that was offered for the reflection principle? The simplest option is to restrict the reflection principle ($\Diamond$-Refl) to singular formulas ϕ, for which the mismatch does not arise. One may worry, however, that this restriction is too blunt and fails to capture fully the mentioned motivation. Fortunately, this worry is assuaged by our next lemma, which shows that every possibility $\phi^{\Diamond}$ to which we may wish to apply the reflection principle admits of an equivalent singular characterization. The desired instance of reflection is thus fully captured by ($\Diamond$-Refl) even when this principle is subjected to the mentioned restriction. So henceforth we will do just that.

LEMMA 8.4. *Let ϕ be a fully modalized sentence of $\mathcal{L}^{\Diamond}_{P\in}$. Let ϕ' be the result of replacing every plural variable uu_i of ϕ with a singular variable x_i (assumed not already to occur in ϕ') and replacing every occurrence of '$\prec$' with an occurrence of '$\in$'. Then NS$^-$ + (C) proves $\phi \leftrightarrow \phi'$.*

Proof. We prove a more general claim where ϕ may contain free variables. Assume the free plural variables in ϕ are $uu_1, \ldots, uu_n$. Then I claim that the mentioned theory proves

$$x_1 = \{uu_1\} \wedge \ldots \wedge x_n = \{uu_n\} \rightarrow (\phi \leftrightarrow \phi')$$

where $x = \{uu\}$ abbreviates $\forall v(v \in x \leftrightarrow v \prec uu)$. This claim is easily established by induction on the number of (modalized) quantifiers in ϕ. □

With an adequate formulation of the reflection principle now in place, the road is open to the desired theorem.[28]

THEOREM 8.5. *NS$^-$ + (C) + ($\Diamond$-Refl) proves the potentialist translation of Infinity.*

Proof. By Theorem 8.2, NS$^-$ + (C) proves $\Diamond\exists x(x = \varnothing)$ and $\Box\forall u\,\Diamond\exists v(v = \{u\})$. Applying ($\Diamond$-Refl) to the conjunction of these two formulas yields:

$$\Diamond(\exists x(x = \varnothing) \wedge \forall u \exists v(v = \{u\})). \tag{1}$$

---

[27] In fact, (P-Comp) allows us to prove an analogue of Cantor's theorem which says that there are more pluralities than there are objects, and *a fortiori* more than there are sets.

[28] A stronger reflection principle can be formulated as follows:

$$\Diamond\forall\mathbf{x}(\phi^{\Diamond}(\mathbf{x}) \rightarrow \phi(\mathbf{x})) \qquad (\Diamond\text{-Refl}^{+})$$

where $\mathbf{x}$ is a string of singular variables, and where again ϕ must be a purely singular formula. This principle implies ($\Diamond$-Refl), as is seen by letting $\mathbf{x}$ be the empty string and observing that $\Diamond(\phi^{\Diamond} \rightarrow \phi)$ implies $\phi^{\Diamond} \rightarrow \Diamond\phi$. In fact, ($\Diamond$-Refl^{+}) corresponds to the 'complete' nonmodal reflection principle $\forall\alpha(\exists\beta > \alpha)(\forall\mathbf{x} \in V_\alpha)(\phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{x})^{V_\alpha})$, whereas ($\Diamond$-Refl) corresponds to the 'partial' principle $\forall\mathbf{x}(\phi(\mathbf{x}) \rightarrow \exists\alpha\,\phi(\mathbf{x})^{V_\alpha})$. ($\Diamond$-Refl^{+}) implies the potentialist translation of all instances of Replacement, as can be seen by imitating the standard proof of the nonmodal analogue of this result (Lévy, 1960, theorem 6 and Drake, 1974, p. 102, exercise 2; see also Studd, forthcoming, section 4.6). So ($\Diamond$-Refl^{+}) holds out the promise of a unified route to Infinity and Replacement. Unfortunately, I cannot see how the appealing motivation provided for ($\Diamond$-Refl) can be extended to ($\Diamond$-Refl^{+}). And the promised unification requires the full strength of ($\Diamond$-Refl^{+}): for although the two corresponding nonmodal principles are equivalent in a higher-order setting, Lévy & Vaught (1961) show the complete principle to be strictly stronger than the partial one when only (singular) first-order logic is available.

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A routine trick available in S4 allows us to turn a proof of $p \rightarrow \Diamond q$ into a proof of $\Diamond p \rightarrow \Diamond q$. In order to prove (1) $\rightarrow$ Infinity$^{\Diamond}$ we may therefore ignore the initial modal operator of (1). So assume $\exists x(x = \varnothing) \wedge \forall u \exists v(v = \{u\})$. Apply (P-Comp) to a tautology to get a plurality xx (namely the plurality of all objects at the relevant world) such that

$$\varnothing \prec xx \wedge \forall u(u \prec xx \rightarrow \{u\} \prec xx). \tag{2}$$

In fact, this is easily strengthened to $\Box(2)$. Next, by (C) it is possible for xx to form a set:

$$\Diamond \exists x \Box \forall u(u \in x \leftrightarrow u \prec xx). \tag{3}$$

We now observe that $\Box \forall u(u \in x \leftrightarrow u \prec xx)$ and $\Box(2)$ imply $\varnothing \in x \wedge \Box \forall u(u \in x \rightarrow \{u\} \in x)$. This makes it straightforward to advance from (3) to Infinity$^{\Diamond}$:

$$\Diamond \exists x[\varnothing \in x \wedge \Box \forall u(u \in x \rightarrow \{u\} \in x)]$$

This concludes our proof. □

8.3. Interpreting modal set theory in nonmodal ZF. Our final theorem is concerned with the reverse interpretation and thus establishes a relative consistency result for our modal set theory.

DEFINITION 8.6. *Let* MS *be the modal theory based on NS, (C), (ED-Repl), and (*$\Diamond$*-Refl).*

THEOREM 8.7. *The modal set theory MS is interpretable in the nonmodal theory ZF and is therefore consistent provided that ZF is.*

Proof. We begin by providing a recursive definition of a translation $\phi \mapsto [\phi]^{V_0}$ from $\mathcal{L}^{\Diamond}_{P\in}$ to $\mathcal{L}_{\in}$. The translation is trivial on atomic formulas, commutes with the truth-functional connective, and is otherwise as follows:

$$[u \prec xx]^{V_\alpha} = u \in xx \tag{4}$$

$$[\forall x\, \phi]^{V_\alpha} = (\forall x \in V_\alpha)[\phi]^{V_\alpha} \tag{5}$$

$$[\forall xx\, \phi]^{V_\alpha} = (\forall xx \in V_{\alpha+1})[\phi]^{V_\alpha} \tag{6}$$

$$[\Box \phi]^{V_\alpha} = (\forall \beta \geq \alpha)[\phi]^{V_\beta} \tag{7}$$

where the occurrences of 'xx' on the right-hand sides of (4) and (6) are understood as occurrences of an ordinary singular variable written in an unusual way. We now need to verify that all the axioms of MS are mapped to theorems of the nonmodal theory and that logical relations are preserved.

The axioms of S4.2 (i.e., $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$, $\Box\phi \rightarrow \phi$, $\Box\phi \rightarrow \Box\Box\phi$, $\Diamond\Box\phi \rightarrow \Box\Diamond\phi$) are easily seen to be mapped to truths of first-order logic. Next we consider the axioms of plural logic. The instances of (P-Comp) translate as formulas of the form:

$$(\forall \alpha)(\exists xx \in V_{\alpha+1})(\forall u \in V_\alpha)(u \in xx \leftrightarrow [\phi]^{V_\alpha}),$$

which are easily seen to be theorems of Z—Infinity. The axioms (STB^{+} $\prec$), (STB^{-} $\prec$), and (INEXT-$\prec$) are easily seen to be mapped to theorems of Z—Infinity. The axioms of NS—that is, (Ext), (ED-$\in$), Foundation, and (INEXT-$\subseteq$)—are easily seen to be mapped to theorems of Z—Infinity. Further, (C) is mapped to the following (trivial but awkward) theorem:

$$(\forall \alpha)(\forall xx \in V_{\alpha+1})(\exists \beta \geq \alpha)(\exists y \in V_\beta)(\forall \gamma \geq \beta)(\forall u \in V_\gamma)(u \in y \leftrightarrow u \in xx)$$

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Next, (ED-Repl) is easily seen to translate to the ordinary replacement scheme. Finally, the translation of (the necessitation of) ($\Diamond$-Refl) is easily seen to be equivalent to:

$$\forall\alpha(\exists\beta \geq \alpha)(\phi \rightarrow [\phi]^{V_\beta}).$$

When ϕ is nonmodal and purely singular, it is not hard to see that this can be simplified to:

$$\forall\alpha(\exists\beta \geq \alpha)(\phi \rightarrow \phi^{V_\beta}), \tag{Refl}$$

where ϕ^{V_β} is just the ordinary relativization of ϕ to V_β. But this is just the ordinary nonmodal reflection principle, which is known to be a theorem scheme of ZF.

It remains to prove that every step licensed by an inferences rule of the modal theory gets mapped to a step that is licensed by the nonmodal theory. In the case of the introduction and elimination rules for the quantifiers, this is a straightforward but tedious verification. The only other rule is the rule of necessitation from S4.2: if $\vdash \phi$, then $\vdash \Box\phi$. The nonmodal translation of this rule says that if we can prove $[\phi]^{V_0}$, then we can prove $\forall\alpha[\phi]^{V_\alpha}$. We establish this claim by induction on proofs: all the axioms have this property, and all the inference rules preserve it. □

§9. Comparison and discussion. The most important investigation to date of the potential hierarchy of sets is in my opinion Parsons (1983b). In particular, Parsons proves versions of my Theorems 8.2, 8.5, and 8.7.[29]

However, there are some important differences between Parsons' approach and mine. Firstly, Parsons uses a different translation from the language of nonmodal set theory to that of modal set theory and accordingly has a different view of the relation between the two theories. His translation is a version of the standard translation of intuitionistic logic into S4 (pp. 321–322), whereas mine is based on the idea that in set theory the quantifiers $\forall$ and $\exists$ are used with an implicit modal character and should therefore be translated as $\Box\forall$ and $\Diamond\exists$, respectively. My translation is technically simpler and, I contend, philosophically more natural.[30] Secondly, where I use plural logic, Parsons uses second-order logic. My choice of plural logic enables a better motivation of (C) and of the principles governing the higher-order variables. Thirdly, where I use the reflection principle ($\Diamond$-Refl), Parsons uses the following principle:

$$\Box\forall u\forall v(\psi(u, v) \rightarrow \Box\psi(u, v)) \wedge \Box\forall u\,\Diamond\exists v\,\psi(u, v) \rightarrow \Diamond\forall u\exists v\,\psi(u, v) \tag{8}$$

As he explains, this principle can be regarded as a device for turning a potential infinity into a completed one. I believe the motivation that I offer for ($\Diamond$-Refl) is significantly better than that which Parsons offers for (8). In particular, Parsons provides no account of why higher-order variables must be banned from (8), whereas the corresponding restriction on ($\Diamond$-Refl) is a natural consequence of my account.

Another important earlier investigation is Fine (1981), which develops a modal set theory based on the modal logic S5 and which thus contrasts with Parsons' and my own choice of the modal logic S4.2. However, this difference is not one of substance but merely reflects what the parties are attempting to model. Fine is interested in metaphysical modality. On this interpretation of the modal operators, the modal logic S5 is quite natural. The same goes for his principle that necessarily if the elements of a set exist, then so

[29] See his Theorems 4, 5, and 6 respectively.

[30] A view on the set theoretic quantifiers similar to mine is suggested in (Putnam, 1967) and developed in Hellman (1989, pp. 73–79).

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does the set itself. Since this principle implies that all pure sets exist of necessity, there is nothing potential about the hierarchy of sets as far as metaphysical modality is concerned. By contrast, Parsons and I seek to model a sense in which the existence of each individual set is potential relative to the existence of its elements and consequently the hierarchy of sets itself is inherently potential.[31] Although further work is no doubt needed to clarify the kind of modality involved in these ideas, both authors are very clear that the relevant modality is not the metaphysical one.

It is interesting to observe that some principles of modal set theory are valid on both the mentioned interpretations. In particular, Fine's analysis of the nature of sets is essentially the same as Parsons' and mine. Fine too endorses the strong transworld principle of extensionality which is encapsulated in my axioms (Ext) and (ED-$\in$).

A more recent alternative is the bimodal theory of Studd (forthcoming), which supplements the 'forward-looking' modal operators of the present work with an analogous pair of 'backward-looking' operators. The resulting grain in expressive power makes possible a better approach to the axiom of Foundation (see footnote 17) as well as the elimination of all plural resources. The price one pays is a complication of the underlying modal logic and an expressive power that is dangerously strong, as it can be seen to be equivalent to that provided by outright quantification over worlds.

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[31] Similar ideas are developed in the more recent 'procedural postulationism' of Fine (2005).

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