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The Interpolation Method for Random Graphs with Prescribed Degrees

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We consider large random graphs with prescribed degrees, as generated by the configuration model. In the regime where the empirical degree distribution approaches a limit μ with finite mean, we establish the systematic convergence of a broad class of graph parameters that includes the independence number, the maximum cut size, the logarithm of the Tutte polynomial, and the free energy of the anti-ferromagnetic Ising and Potts models. Contrary to previous works, our results are not *a priori* limited to the free energy of some prescribed graphical model. They apply more generally to any additive, Lipschitz and concave graph parameter. In addition, the corresponding limits are shown to be Lipschitz and concave in the degree distribution μ. This considerably extends the applicability of the celebrated interpolation method, introduced in the context of spin glasses, and recently related to the challenging question of right-convergence of sparse graphs.

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1. Background

A decade ago, Guerra and Toninelli [16] introduced a powerful method to prove the existence of an infinite volume limit for the free energy of the classical *Sherrington–Kirkpatrick model*. The argument is based on an ingenious interpolation scheme which allows a system of size n to be compared with two similar but independent systems of sizes n_1 and n_2 respectively, where $n_1 + n_2 = n$. The quantity of interest turns out to be superadditive with respect to n, hence convergent once divided by n. We refer to the recent books [21, 20] for more details. This technique was then transferred from the fully connected regime (complete graph) to its diluted counterpart (sparse random graphs), where each particle only interacts with a bounded number of neighbours. See in particular [13] for the Erdős–Rényi case and [14] for arbitrary degree distributions.

In a recent breakthrough [2], the applicability of the interpolation method was extended to a variety of important models including, among others, the *Ising model*, the *Potts model*

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and the *hard-core model*. As a special case, this confirmed the long-conjectured convergence of the independence ratio of sparse Erdős–Rényi and d-regular random graphs (see also the recent preprint [12], where the limit is explicitly determined when d is large enough). The superadditivity inherent to all those models was subsequently shown to follow from a single convexity property [15], thereby shedding new light on the challenging question of *right-convergence* (*i.e.*, generic convergence of free energies) of sparse random graphs [10, 7, 9].

In the present paper we extend to random graphs with an arbitrary degree sequence the results obtained in [2, 15] for the Erdős–Rényi and d-regular random graphs. This allows us to investigate general properties of infinite volume limits when regarded as functions of the asymptotic degree distribution (continuity, concavity). In addition, and contrary to previous works, we do not restrict our attention to the free energy of some prescribed graphical model: our results apply much more generally to any additive, Lipschitz and concave graph parameter. This shift provides generalization and proof simplification at the same time.

2. Setting

Throughout the paper, the word *graph* refers to a finite undirected multigraph (loops and multiple edges are allowed). A *graph parameter* is a real-valued function $\mathfrak{f}$ defined on graphs, that is invariant under graph isomorphism. We shall restrict our attention to graph parameters satisfying the following properties.

- *Additive*. If G is the vertex-disjoint union of G_1 and G_2, then
$$\mathfrak{f}(G) = \mathfrak{f}(G_1) + \mathfrak{f}(G_2).$$
- *Lipschitz*. There is $\kappa < \infty$ such that, for each graph $G = (V, E)$ and each $(i, j) \in V \times V$,
$$|\mathfrak{f}(G + ij) - \mathfrak{f}(G)| \leqslant \kappa,$$
where $G + ij$ is the graph obtained from G by adding a new edge between i and j.
- *Concave*. For every graph $G = (V, E)$, the $V \times V$ matrix $\{\Delta_{ij}^G\}$ defined by
$$\Delta_{ij}^G = \mathfrak{f}(G + ij) - \mathfrak{f}(G)$$
is *conditionally negative semidefinite* (CND), that is,
$$\left(\sum_{i \in V} x_i = 0\right) \Longrightarrow \left(\sum_{(i,j) \in V \times V} \Delta_{ij}^G x_i x_j \leqslant 0\right).$$

This relaxed form of negative semi-definiteness is less restrictive than the one imposed in [15]. The class of CND matrices happens to be well studied, due in particular to its intimate connection with infinite divisible matrices. We mention here a few useful properties, and refer the reader to [1, Chapter 4] for more details.

(i) $\{\Delta_{ij}\}$ is CND if and only if $\{\alpha_i + \alpha_j - \Delta_{ij}\}$ is positive semidefinite for some $\{\alpha_i\} \in \mathbb{R}^V$. Another equivalent condition is the infinite divisibility of $\{e^{-\Delta_{ij}}\}$, that is, $\{e^{-\lambda\Delta_{ij}}\}$ is positive definite for all $\lambda > 0$: see [1, Theorem 4.1.3].

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(ii) If $\{\Delta_{ij}\}_{(i,j)\in S\times S}$ is CND then so is $\{\Delta_{\sigma(i),\sigma(j)}\}_{(i,j)\in V\times V}$ for any $\sigma : V \to S$.
(iii) The CND matrices $\{\Delta_{ij}\}_{(i,j)\in V\times V}$ form a convex cone.
(iv) A sufficient condition for $\{\Delta_{ij}\}$ to be CND is that $\{e^{\Delta_{ij}}\}$ is CND (combine [1, Theorem 4.4.4] with [1, Corollary 4.1.5]).
(v) If $\leftrightarrow$ is any equivalence relation on a finite set V, then its incidence matrix $\{1_{i\leftrightarrow j}\}$ is positive semidefinite, so the incidence matrix of its negation $\{1_{i\not\leftrightarrow j}\}$ is CND.

Many important graph parameters (or their negative when indicated by the symbol $\star$) belong to the above class. Here are a few examples.

- $\star$ *Number of connected components.* The increment matrix is $\Delta^G_{ij} = \mathbf{1}_{i\leftrightarrow j} - 1$, where $\leftrightarrow$ denotes the connectedness relation in G.
- *Independence number.* Here $\Delta^G_{ij} = 1 - (\mathbf{1}_S \mathbf{1}_S^\top)_{ij}$, where S is the intersection of all maximum independent sets on G, and $\mathbf{1}_S$ denotes its incidence vector.
- *Maximum cut size.* $\Delta^G_{ij} = 1 - \prod(\mathbf{1}_S \mathbf{1}_S^\top + \mathbf{1}_{\overline{S}} \mathbf{1}_{\overline{S}}^\top)_{ij}$, where Π runs over all maximum cuts $(S, \overline{S})$ (recall that the entry-wise product preserves positive semidefiniteness).
- $\star$ *Decycling number.* The decycling number $\mathfrak{f}(G)$ of a graph $G = (V, E)$ is the smallest possible size of a set $S \subseteq V$ whose removal makes the graph acyclic [3]. In this case, $\Delta^G_{ij} = \mathbf{1}_{i\leftrightarrow j}$, where $i \leftrightarrow j$ if and only if i and j belong to the same component of every maximum-size induced forest of G.
- *Free energies.* Fix a finite set S, a map $h : S \to (0, +\infty)$ and a symmetric map $J : S \times S \to (0, +\infty)$, and consider the graph parameter

$$\mathfrak{f}(G) := \log\left(\sum_{\sigma\in S^V} w(\sigma)\right) \text{ where } w(\sigma) = \prod_{i\in V} h(\sigma_i) \prod_{ij\in E} J(\sigma_i, \sigma_j).$$

  Here $\mathfrak{f}$ is easily seen to be additive and Lipschitz, and a sufficient condition for it to be concave is that the interaction matrix J is CND. Indeed, this implies that $J^\sigma := \{J(\sigma_i, \sigma_j)\}_{(i,j)\in V\times V}$ is CND for each $\sigma \in S^V$ by (ii), and then (iii) ensures that so is

$$\frac{\sum_{\sigma\in S^V} w(\sigma) J^\sigma}{\sum_{\sigma\in S^V} w(\sigma)}.$$

  But this matrix is exactly $\{e^{\Delta^G_{ij}}\}$, and (iv) allows us to conclude. In particular, the free energy of the *Ising model* ($S = \{-1, +1\}$, $J(s,t) = e^{\beta st}$) and the *Potts model* ($S = \{1, \dots, q\}$, $J(s,t) = \mathbf{1}_{s\neq t} + e^\beta \mathbf{1}_{s=t}$) are additive, Lipschitz and concave graph parameters for any $\beta \leqslant 0$ (anti-ferromagnetic regime).
- *Tutte polynomial.* Fix $x, y > 0$ and consider the graph parameter

$$\mathfrak{f}(G) := \log\left(\sum_{A\subseteq E} x^{c(A)} y^{|A|}\right),$$

  where $c(A)$ denotes the number of connected components of the graph (V, A). If $x \leqslant 1$, then $\mathfrak{f}$ is additive, Lipschitz and concave, as can be deduced from the expression

$$e^{\Delta^G_{ij}} = \frac{\sum_{A\subseteq E} x^{c(A)} y^{|A|} \left(1 + y + y(x^{-1} - 1)\mathbf{1}_{i\not\leftrightarrow^A j}\right)}{\sum_{A\subseteq E} x^{c(A)} y^{|A|}},$$

  using (iii), (iv) and (v) (here, $i \leftrightarrow^A j$ means connectedness in (V, A)).

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- *Spanning forests.* Fix $x \geqslant 0$ and consider the graph parameter

$$\mathfrak{f}(G) := \log\Bigg(\sum_{A \subseteq E \text{ acyclic}} x^{|A|} \Bigg),$$

where the sum runs over all subsets $A \subseteq E$ such that (V, A) is acyclic. Then $\mathfrak{f}$ is additive, Lipschitz and concave, by a similar argument.

3. Result

The present paper is concerned with the asymptotic behaviour of such graph parameters when evaluated on large random graphs with prescribed degrees. Specifically, for each $n \geqslant 1$ we let $\mathbb{G}_{d_n}$ denote a random graph generated by the *configuration model* [5, 17] with degrees $d_n = \{d_n(i)\}_{1 \leqslant i \leqslant n}$ (see Section 5 for the precise definition). We assume that $\{d_n\}_{n \geqslant 1}$ approaches a probability measure μ on $\mathbb{N}$ with mean $\overline{\mu} < +\infty$, in the sense that,

$$\text{for all } k \in \mathbb{N}, \quad \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{d_n(i)=k\}} \xrightarrow[n \to \infty]{} \mu(k), \tag{3.1}$$

$$\frac{1}{n} \sum_{i=1}^{n} d_n(i) \xrightarrow[n \to \infty]{} \overline{\mu}. \tag{3.2}$$

In other words, the empirical degree distribution $\frac{1}{n} \sum_{i=1}^{n} \delta_{d_n(i)}$ converges to μ in the so-called *Wasserstein space* $\mathcal{P}_1(\mathbb{N})$. This is the space of probability measures on $\mathbb{N}$ with finite mean, equipped with the Wasserstein (or Kantorovich–Rubinstein) distance

$$\mathcal{W}(\mu, \mu') = \sum_{i=1}^{\infty} \Bigg| \sum_{k=i}^{\infty} (\mu(k) - \mu'(k)) \Bigg|.$$

We refer the reader to the books [4, Chapter 2] or [22, Chapter 6] for more details on Wasserstein spaces and many alternative expressions for $\mathcal{W}$.

Theorem 3.1. *Every additive, Lipschitz, concave graph parameter* $\mathfrak{f}$ *admits an 'infinite volume limit'* $\Psi : \mathcal{P}_1(\mathbb{N}) \to \mathbb{R}$ *in the following sense: for any* $\mu \in \mathcal{P}_1(\mathbb{N})$ *and any* $\{d_n\}_{n \geqslant 1}$ *satisfying* (3.1)–(3.2)*, we have the almost sure convergence*

$$\frac{\mathfrak{f}(\mathbb{G}_{d_n})}{n} \xrightarrow[n \to \infty]{} \Psi(\mu). \tag{3.3}$$

Moreover, Ψ *is Lipschitz and concave: for any* $\mu, \mu' \in \mathcal{P}_1(\mathbb{N})$ *and* $\theta \in [0, 1]$*,*

$$|\Psi(\mu) - \Psi(\mu')| \leqslant 2\kappa \mathcal{W}(\mu, \mu'), \tag{3.4}$$

$$\theta \Psi(\mu) + (1 - \theta)\Psi(\mu') \leqslant \Psi(\theta\mu + (1 - \theta)\mu'). \tag{3.5}$$

Remark 1 (random degrees). A common setting consists in taking $d_n = (\Delta_1, \ldots, \Delta_n)$, where $\{\Delta_i\}_{i \geqslant 1}$ are IID samples from a target degree distribution $\mu \in \mathcal{P}_1(\mathbb{N})$. We denote by $\mathbb{G}_{\mu,n}^{\text{IID}}$ the resulting doubly random graph. The result (3.3) applies, since the strong law of large

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numbers guarantees that, almost surely,

$$\mathcal{W}\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{\Delta_i},\mu\right)\xrightarrow[n\to\infty]{}0. \tag{3.6}$$

This convergence actually also holds in L^1: see [4, Theorem 2.14] (and [4, Theorem 3.5] for a quantitative version). This fact will be used in our proof of Theorem 3.1.

Remark 2 (simple graphs). Assuming (3.1), a simple sufficient condition for (3.2) is

$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{i=1}^{n}d_n^2(i)<\infty. \tag{3.7}$$

Under this condition and if $\sum_{i=1}^{n}d_n(i)$ is even, the probability that $\mathbb{G}_{d_n}$ is simple remains bounded away from 0 as $n\to\infty$: see [18, 19]. Moreover, conditionally on being simple, $\mathbb{G}_{d_n}$ is uniformly distributed over all simple graphs with degrees d_n. Consequently, the convergence (3.3) also applies to uniform simple graphs with a given degree sequence. For example, in the sparse regime, the Erdős–Rényi random graph and the more general *rank-one inhomogeneous random graph* [6, 17] have degree sequences which satisfy (3.1) and (3.7) almost surely. Moreover, conditionally on their degree sequence, they are uniformly distributed. Thus, the convergence (3.3) applies to those models as well.

Remark 3 (extension). By linearity, the convergence (3.3) extends to linear combinations of additive, Lipschitz and concave graph parameters. Since the resulting parameters remain additive and Lipschitz, it is perhaps natural to raise the following question:

Does (3.3) hold for any additive Lipschitz graph parameter?

A positive answer would constitute a substantial strengthening of [15, Conjecture 1].

4. Proof outline

Let $\mathfrak{f}$ be an additive, κ-Lipschitz and concave graph parameter. Our main ingredient is the following inequality, in which $\varphi(x)=7\kappa\sqrt{x\ln(1+x)}$.

Proposition 4.1. *Let A,B be finite disjoint sets and let $d\colon A\cup B\to\mathbb{N}$. Write $d\restriction A$, $d\restriction B$ for the restrictions of d to A, B. Then,*

$$\mathbb{E}[\mathfrak{f}(\mathbb{G}_{d\restriction A})]+\mathbb{E}[\mathfrak{f}(\mathbb{G}_{d\restriction B})]\leqslant\mathbb{E}[\mathfrak{f}(\mathbb{G}_d)]+\varphi\left(\frac{1}{2}\sum_{i\in A\cup B}d(i)\right).$$

As this inequality holds for any degrees $\{d(i)\}_{i\in A\cup B}$, we may fix $\mu\in\mathcal{P}_1(\mathbb{N})$ and average it against $\mu^{\otimes A\cup B}$. Since φ is concave, Jensen's inequality yields

$$\mathbb{E}[\mathfrak{f}(\mathbb{G}^{\mathrm{IID}}_{\mu,|A|})]+\mathbb{E}[\mathfrak{f}(\mathbb{G}^{\mathrm{IID}}_{\mu,|B|})]\leqslant\mathbb{E}[\mathfrak{f}(\mathbb{G}^{\mathrm{IID}}_{\mu,|A|+|B|})]+\varphi\left(\frac{\overline{\mu}}{2}(|A|+|B|)\right).$$

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By a classical result of de Bruijn and Erdős [11, Theorem 23], this near-superadditivity suffices to guarantee the existence of the limit

$$\Psi(\mu) := \lim_{n\to\infty} \frac{\mathbb{E}[\mathfrak{f}(\mathbb{G}^{\text{IID}}_{\mu,n})]}{n} \in \mathbb{R} \cup \{+\infty\}. \tag{4.1}$$

In our case we must have $\Psi(\mu) < +\infty$, since the additive and Lipschitz properties of $\mathfrak{f}$ easily imply that $\mathfrak{f}(G) = O(|V| + |E|)$ uniformly over all graphs $G = (V, E)$.

Our second ingredient is the following simple inequality, which quantifies the intuition that $\mathbb{E}[\mathfrak{f}(\mathbb{G}_{d'})]$ should be close to $\mathbb{E}[\mathfrak{f}(\mathbb{G}_d)]$ whenever d' is close to d.

Proposition 4.2. *For any $d, d' : \{1, \dots, n\} \to \mathbb{N}$,*

$$\left| \frac{\mathbb{E}[\mathfrak{f}(\mathbb{G}_d)]}{n} - \frac{\mathbb{E}[\mathfrak{f}(\mathbb{G}_{d'})]}{n} \right| \leqslant 2\kappa \mathcal{W}\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{d(i)}, \frac{1}{n}\sum_{i=1}^{n} \delta_{d'(i)} \right).$$

Let us apply this when $\{d'(i)\}_{1\leqslant i\leqslant n}$ are IID samples from μ. In view of the L^1 convergence (3.6) and the assumption that $\{d_n\}_{n\geqslant 1}$ approaches μ, we obtain by the triangle inequality

$$\frac{\mathbb{E}[\mathfrak{f}(\mathbb{G}_{d_n})]}{n} - \frac{\mathbb{E}[\mathfrak{f}(\mathbb{G}^{\text{IID}}_{\mu,n})]}{n} \xrightarrow[n\to\infty]{} 0. \tag{4.2}$$

Finally, since $\mathfrak{f}$ is Lipschitz, a now-standard application of the Azuma–Hoeffding inequality ensures that $\mathfrak{f}(\mathbb{G}_d)$ is exponentially concentrated: for any $\varepsilon > 0$,

$$\mathbb{P}\big(|\mathfrak{f}(\mathbb{G}_d) - \mathbb{E}[\mathfrak{f}(\mathbb{G}_d)]| \geqslant \varepsilon\big) \leqslant \exp\left(-\frac{\varepsilon^2}{4\kappa^2 \sum_i d(i)} \right).$$

See [23, Theorem 2.19] for a proof in the case where d is constant, and [8, Corollary 3.27] for the general case. Thus, the Borel–Cantelli lemma ensures that

$$\frac{\mathfrak{f}(\mathbb{G}_{d_n})}{n} - \frac{\mathbb{E}[\mathfrak{f}(\mathbb{G}_{d_n})]}{n} \xrightarrow[n\to\infty]{} 0, \tag{4.3}$$

almost surely under any coupling of the graphs $\{\mathbb{G}_{d_n}\}_{n\geqslant 1}$ (complete convergence). Combining (4.1), (4.2) and (4.3) proves the convergence (3.3). The Lipschitz continuity (3.4) follows by passing to the limit in Proposition 4.2 along sequences $\{d_n\}_{n\geqslant 1}, \{d'_n\}_{n\geqslant 1}$ that approach μ, μ' in the sense of (3.1)–(3.2). Since the concatenation of d_n and d'_n approaches $(\mu + \mu')/2$, we may also pass to the limit in Proposition 4.1 to obtain (3.5) for $\theta = 1/2$. This mid-point concavity suffices to guarantee concavity, since Ψ is continuous. The remainder of the paper is devoted to the proofs of Propositions 4.1 and 4.2.

5. Proof of Proposition 4.1

Throughout this section, we fix a finite set V and a function $d : V \to \mathbb{N}$. The set $\mathcal{H}$ of half-edges is formed by 'attaching' $d(i)$ half-edges with each end-point $i \in V$:

$$\mathcal{H} := \bigcup_{i\in V} \{(i, 1), \dots, (i, d(i))\}.$$

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A (partial) *matching* $\mathfrak{m}$ on $\mathcal{H}$ is a collection of pairwise disjoint two-element subsets of $\mathcal{H}$. Such a matching naturally induces a graph $G[\mathfrak{m}]$ on V by interpreting a pair of matched half-edges as an edge between the corresponding end-points. By definition, the configuration model $\mathbb{G}_d$ is the random graph induced by a uniformly chosen random maximal matching on $\mathcal{H}$. Now fix a bipartition $V = A \cup B$, and define

$$d(A) = \sum_{i \in A} d(i) \quad \text{and} \quad d(B) = \sum_{i \in B} d(i). \tag{5.1}$$

For $(\alpha, \beta, \gamma) \in \mathbb{N}^3$, let $\mathcal{M}(\alpha, \beta, \gamma)$ denote the set of all matchings of $\mathcal{H}$ containing

- α edges with both end-points in A,
- β edges with both end-points in B,
- γ edges with one end-point in A and the other in B (called cross-edges).

Note that $\mathcal{M}(\alpha, \beta, \gamma) \neq \emptyset$ if and only if $2\alpha + \gamma \leqslant d(A)$ and $2\beta + \gamma \leqslant d(B)$. When this condition holds, we call the triple (α, β, γ) *feasible*, and we define

$$F(\alpha, \beta, \gamma) = \frac{1}{|\mathcal{M}(\alpha, \beta, \gamma)|} \sum_{\mathfrak{m} \in \mathcal{M}(\alpha, \beta, \gamma)} \mathfrak{f}(G[\mathfrak{m}]).$$

In other words, $F(\alpha, \beta, \gamma)$ is the expectation of $\mathfrak{f}(G[\mathfrak{m}])$ when $\mathfrak{m}$ is uniform on $\mathcal{M}(\alpha, \beta, \gamma)$. To connect this with the configuration model, observe that conditionally on its number γ of cross-edges, a uniformly chosen maximal matching $\mathfrak{m}$ on $\mathcal{H}$ is uniformly distributed in

$$\mathcal{M}\left(\left\lfloor \frac{d(A) - \gamma}{2} \right\rfloor, \left\lfloor \frac{d(B) - \gamma}{2} \right\rfloor, \gamma\right).$$

Thus

$$F\left(\left\lfloor \frac{d(A) - \gamma}{2} \right\rfloor, \left\lfloor \frac{d(B) - \gamma}{2} \right\rfloor, \gamma\right)$$

is the conditional expectation of $\mathfrak{f}(\mathbb{G}_d)$ given the number γ of cross-edges. On the other hand, since $\mathfrak{f}$ is additive,

$$F\left(\left\lfloor \frac{d(A)}{2} \right\rfloor, \left\lfloor \frac{d(B)}{2} \right\rfloor, 0\right) = \mathbb{E}[\mathfrak{f}(\mathbb{G}_{d\restriction A})] + \mathbb{E}[\mathfrak{f}(\mathbb{G}_{d\restriction B})].$$

Thus, the parameter γ allows us to *interpolate* between the original model $\mathbb{G}_d$ and the vertex-disjoint union $\mathbb{G}_{d\restriction A} \sqcup \mathbb{G}_{d\restriction B}$. In particular, Proposition 4.1 is a clear consequence of the following stronger result, to the proof of which this whole section is devoted. We use the standard notation $a \wedge b$ for the minimum between a and b.

Proposition 5.1. *For any* $\gamma \in \{0, \dots, d(A) \wedge d(B)\}$,

$$F\left(\left\lfloor \frac{d(A)}{2} \right\rfloor, \left\lfloor \frac{d(B)}{2} \right\rfloor, 0\right) \leqslant F\left(\left\lfloor \frac{d(A) - \gamma}{2} \right\rfloor, \left\lfloor \frac{d(B) - \gamma}{2} \right\rfloor, \gamma\right) + \varphi(\gamma).$$

Given a matching $\mathfrak{m}$ on $\mathcal{H}$, one can create a larger matching $\mathfrak{m}' \supset \mathfrak{m}$ by adding to $\mathfrak{m}$ a uniformly chosen pair of distinct unmatched half-edges (if any). Restricting the choice to half-edges whose end-point lies in A, or in B, or to pairs in which one end-point lies in A

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and the other in B, defines what we call a random A-pairing, B-pairing or cross-pairing. As we shall now see, these can be performed sequentially to sample $\mathcal{M}(\alpha, \beta, \gamma)$ uniformly.

Lemma 5.2. *Let $\mathfrak{m}$ be uniformly distributed on $\mathcal{M}(\alpha, \beta, \gamma)$. Conditionally on $\mathfrak{m}$, make a random A-pairing (resp. B-pairing, resp. cross-pairing). Then the result $\mathfrak{m}'$ is uniformly distributed on $\mathcal{M}(\alpha+1, \beta, \gamma)$ (resp. $\mathcal{M}(\alpha, \beta+1, \gamma)$, resp. $\mathcal{M}(\alpha, \beta, \gamma+1)$).*

Proof. Every $m \in \mathcal{M}(\alpha, \beta, \gamma)$ admits $\binom{d(A)-2\alpha-\gamma}{2}$ allowed A-pairings, each producing a distinct $m' \supset m$ in $\mathcal{M}(\alpha+1, \beta, \gamma)$. By uniformity, it follows that $\mathbb{P}(\mathfrak{m}' = m')$ is proportional to the number of matchings $m \in \mathcal{M}(\alpha, \beta, \gamma)$ such that $m \subset m'$. But this number is exactly $\alpha+1$ for every $m' \in \mathcal{M}(\alpha+1, \beta, \gamma)$. The argument for B-pairings and cross-pairings is similar. □

Let us now exploit this elementary observation to establish two properties of F. The second one is really the crux of our entire interpolation argument.

Lemma 5.3 (Lipschitz continuity). *For any feasible (α, β, γ) and $(\alpha', \beta', \gamma')$,*

$$|F(\alpha, \beta, \gamma) - F(\alpha', \beta', \gamma')| \leqslant \kappa(|\alpha - \alpha'| + |\beta - \beta'| + |\gamma - \gamma'|).$$

Proof. It is sufficient to prove this when the triples differ by 1 at a single coordinate. Let us treat only the case $(\alpha', \beta', \gamma') = (\alpha+1, \beta, \gamma)$, the other cases being similar. Choose $\mathfrak{m}$ uniformly in $\mathcal{M}(\alpha, \beta, \gamma)$, and let $\mathfrak{m}'$ be obtained from $\mathfrak{m}$ by a random A-pairing. Then $G[\mathfrak{m}']$ differs from $G[\mathfrak{m}]$ by exactly one edge, so the Lipschitz assumption guarantees that

$$\mathfrak{f}(G[\mathfrak{m}]) - \kappa \leqslant \mathfrak{f}(G[\mathfrak{m}']) \leqslant \mathfrak{f}(G[\mathfrak{m}]) + \kappa.$$

Taking expectations concludes the proof, since $\mathfrak{m}'$ is uniform on $\mathcal{M}(\alpha+1, \beta, \gamma)$ by Lemma 5.2. □

Lemma 5.4 (local superadditivity). *For $\delta \geqslant 2$, if $(\alpha, \beta, \gamma+\delta)$ is feasible then*

$$\frac{F(\alpha+1, \beta, \gamma) + F(\alpha, \beta+1, \gamma)}{2} \leqslant F(\alpha, \beta, \gamma+1) + \frac{2\kappa}{\delta}.$$

Proof. Fix $\mathfrak{m} \in \mathcal{M}(\alpha, \beta, \gamma)$. Write $c(i)$ for the number of unpaired half-edges attached to $i \in V$ in $\mathfrak{m}$, and define $c(A), c(B)$ as in (5.1). Let $\mathfrak{m}'$ be obtained from $\mathfrak{m}$ by a random cross-pairing, and let $\mathfrak{m}''$ be obtained from $\mathfrak{m}$ by flipping a fair coin and making a random A-pairing or B-pairing accordingly. We will prove that

$$\mathbb{E}[\mathfrak{f}(G[\mathfrak{m}''])] - \mathbb{E}[\mathfrak{f}(G[\mathfrak{m}'])] \leqslant \frac{\kappa}{c(A)} + \frac{\kappa}{c(B)}.$$

The assumption ensures that the right-hand side is at most $2\kappa/\delta$, and averaging over all $\mathfrak{m} \in \mathcal{M}(\alpha, \beta, \gamma)$ implies the result, by Lemma 5.2. Setting $\Delta_{ij} = \mathfrak{f}(G[\mathfrak{m}] + ij) - \mathfrak{f}(G[\mathfrak{m}])$, we

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have

$$\mathbb{E}[\mathfrak{f}(G[\mathfrak{m}'']) - \mathfrak{f}(G[\mathfrak{m}])] = \sum_{(i,j)\in A\times A} \Delta_{ij} \frac{c(i)(c(j)-\mathbf{1}_{i=j})}{2c(A)(c(A)-1)} + \sum_{(i,j)\in B\times B} \Delta_{ij} \frac{c(i)(c(j)-\mathbf{1}_{i=j})}{2c(B)(c(B)-1)},$$

$$\mathbb{E}[\mathfrak{f}(G[\mathfrak{m}']) - \mathfrak{f}(G[\mathfrak{m}])] = \sum_{(i,j)\in A\times B} \Delta_{ij} \frac{c(i)c(j)}{2c(A)c(B)} + \sum_{(i,j)\in B\times A} \Delta_{ij} \frac{c(i)c(j)}{2c(B)c(A)}.$$

We may thus decompose the difference $\mathbb{E}[\mathfrak{f}(G[\mathfrak{m}''])] - \mathbb{E}[\mathfrak{f}(G[\mathfrak{m}'])]$ as

$$\frac{1}{2}\sum_{(i,j)\in V\times V} \Delta_{ij}x_i x_j + \frac{1}{2}\sum_{(i,j)\in A\times A} \Delta_{ij} z_{ij} + \frac{1}{2}\sum_{(i,j)\in B\times B} \Delta_{ij} z_{ij}, \tag{5.2}$$

where we have set

$$x_i = \begin{cases} \dfrac{c(i)}{c(A)} & \text{if } i \in A, \\ -\dfrac{c(i)}{c(B)} & \text{if } i \in B, \end{cases}$$

$$z_{ij} = \begin{cases} \dfrac{c(i)(c(j)-c(A)\mathbf{1}_{i=j})}{c(A)c(A)(c(A)-1)} & \text{if } (i,j) \in A\times A, \\ \dfrac{c(i)(c(j)-c(B)\mathbf{1}_{i=j})}{c(B)c(B)(c(B)-1)} & \text{if } (i,j) \in B\times B. \end{cases}$$

Now, the first term in (5.2) is non-positive since $\sum_i x_i = 0$ and $\{\Delta_{ij}\}$ is CND ($\mathfrak{f}$ is concave). For the second term, recall that $|\Delta_{ij}| \leqslant \kappa$ ($\mathfrak{f}$ is Lipschitz), so that

$$\begin{aligned} \frac{1}{2}\sum_{(i,j)\in A\times A} \Delta_{ij} z_{ij} &\leqslant \frac{\kappa}{2}\sum_{(i,j)\in A\times A} |z_{ij}| \\ &\leqslant \kappa \sum_{i\in A} \frac{c(i)(c(A)-c(i))}{c(A)c(A)(c(A)-1)} \\ &\leqslant \frac{\kappa}{c(A)}, \end{aligned}$$

where we have used the inequality $k(n-k) \leqslant k(n-1)$ valid for any integers $0 \leqslant k \leqslant n$. Replacing A with B yields the bound $\kappa/c(B)$ for the third term. □

We may now deduce Proposition 5.1 from the above two properties of F.

Proof of Proposition 5.1. The claim is trivial when γ is small. Indeed,

$$\left|\left\lfloor \frac{d(A)}{2}\right\rfloor - \left\lfloor \frac{d(A)-\gamma}{2}\right\rfloor\right| + \left|\left\lfloor \frac{d(B)}{2}\right\rfloor - \left\lfloor \frac{d(B)-\gamma}{2}\right\rfloor\right| + |0-\gamma| \leqslant 2\gamma + 1,$$

so the Lipschitz continuity of F (Lemma 5.3) guarantees that

$$F\left(\left\lfloor \frac{d(A)}{2}\right\rfloor, \left\lfloor \frac{d(B)}{2}\right\rfloor, 0\right) \leqslant F\left(\left\lfloor \frac{d(A)-\gamma}{2}\right\rfloor, \left\lfloor \frac{d(B)-\gamma}{2}\right\rfloor, \gamma\right) + \kappa(2\gamma+1).$$

This implies the claim as long as $2\gamma + 1 \leqslant 7\sqrt{\gamma \ln(1+\gamma)}$, that is, $\gamma \in \{1,\dots,46\}$. We now assume that $\gamma \geqslant 47$. Let $\delta \in \mathbb{N}$ to be chosen later, such that $2 \leqslant \delta \leqslant \gamma/2$. Let $\{S_t\}_{t\in\mathbb{N}}$

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denote the successive positions of a simple random walk on $\mathbb{Z}$ started at 0. Set $\tau = \gamma - 2\delta$, and for every $t \in \{0, \dots, \tau\}$ consider the random triple $(\boldsymbol{\alpha}_t, \boldsymbol{\beta}_t, \boldsymbol{\gamma}_t)$ defined by

$$\boldsymbol{\alpha}_t = \left\lfloor \frac{d(A) - \gamma}{2} \right\rfloor + \frac{t + S_t}{2}, \quad \boldsymbol{\beta}_t = \left\lfloor \frac{d(B) - \gamma}{2} \right\rfloor + \frac{t - S_t}{2}, \quad \boldsymbol{\gamma}_t = \tau - t.$$

In other words, conditionally on $\mathcal{F}_t = \sigma(S_0, \dots, S_t)$, the triple $(\alpha_{t+1}, \beta_{t+1}, \gamma_{t+1})$ is obtained from $(\alpha_t, \beta_t, \gamma_t)$ by decrementing the last coordinate and incrementing either the first or the second coordinate, with probability half each. It is immediate to check that $(\boldsymbol{\alpha}_t, \boldsymbol{\beta}_t, \boldsymbol{\gamma}_t + \delta)$ is feasible for all $t < \tau \wedge T$, where

$$T := \inf\{t \in \mathbb{N} : |S_t| > \delta\}.$$

Therefore, Lemma 5.4 guarantees that $X_t := F(\boldsymbol{\alpha}_{t\wedge T}, \boldsymbol{\beta}_{t\wedge T}, \boldsymbol{\gamma}_{t\wedge T})$ satisfies

$$\mathbb{E}[X_{t+1} | \mathcal{F}_t] \leqslant X_t + \frac{2\kappa}{\delta} \mathbf{1}_{\{t<T\}} \quad (0 \leqslant t < \tau).$$

Taking expectations and summing over all $0 \leqslant t < \tau$, we deduce that

$$\mathbb{E}[X_\tau] - X_0 \leqslant \frac{2\kappa\tau}{\delta}. \tag{5.3}$$

Now, since

$$X_0 = F\left(\left\lfloor \frac{d(A) - \gamma}{2} \right\rfloor, \left\lfloor \frac{d(B) - \gamma}{2} \right\rfloor, \gamma - 2\delta \right),$$

The Lipschitz continuity of F (Lemma 5.3) yields

$$X_0 - F\left(\left\lfloor \frac{d(A) - \gamma}{2} \right\rfloor, \left\lfloor \frac{d(B) - \gamma}{2} \right\rfloor, \gamma \right) \leqslant 2\kappa\delta. \tag{5.4}$$

On the other hand, observing that, almost surely,

$$\left| \left\lfloor \frac{d(A)}{2} \right\rfloor - \boldsymbol{\alpha}_{\tau\wedge T} \right| + \left| \left\lfloor \frac{d(B)}{2} \right\rfloor - \boldsymbol{\beta}_{\tau\wedge T} \right| + |\boldsymbol{\gamma}_{\tau\wedge T}| \leqslant 2\tau \mathbf{1}_{\tau \geqslant T} + 2\delta + 1,$$

we may invoke Lemma 5.3 again to obtain

$$F\left(\left\lfloor \frac{d(A)}{2} \right\rfloor, \left\lfloor \frac{d(B)}{2} \right\rfloor, 0 \right) - X_\tau \leqslant \kappa(2\tau \mathbf{1}_{\tau \geqslant T} + 2\delta + 1).$$

Taking expectations yields

$$F\left(\left\lfloor \frac{d(A)}{2} \right\rfloor, \left\lfloor \frac{d(B)}{2} \right\rfloor, 0 \right) - \mathbb{E}[X_\tau] \leqslant \kappa\left(4\tau e^{-\frac{(\delta+1)^2}{2\tau}} + 2\delta + 1 \right), \tag{5.5}$$

where we used the following classical consequence of Doob's maximal inequality:

$$\mathbb{P}(T \leqslant \tau) = \mathbb{P}\left(\max_{0 \leqslant t \leqslant \tau} |S_t| \geqslant \delta + 1 \right) \leqslant 2e^{-\frac{(\delta+1)^2}{2\tau}}.$$

Adding up (5.3)–(5.5) and recalling that $\tau = \gamma - 2\delta$, we finally arrive at

$$F\left(\left\lfloor \frac{d(A)}{2} \right\rfloor, \left\lfloor \frac{d(B)}{2} \right\rfloor, 0 \right) - F\left(\left\lfloor \frac{d(A) - \gamma}{2} \right\rfloor, \left\lfloor \frac{d(B) - \gamma}{2} \right\rfloor, \gamma \right) \leqslant \kappa\left(\frac{2\gamma}{\delta} + 4\delta + 4\gamma e^{-\frac{(\delta+1)^2}{2\gamma}} \right).$$

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The choice $\delta = \lfloor \sqrt{\gamma \ln(1+\gamma)} \rfloor$ yields the bound $c(\gamma)\kappa\sqrt{\gamma \ln(1+\gamma)}$, where

$$c(\gamma) = \frac{2}{\ln(1+\gamma) - \sqrt{\frac{\ln(1+\gamma)}{\gamma}}} + 4 + \frac{4}{\sqrt{\ln(1+\gamma)}}.$$

This quantity decreases with γ, so $c(\gamma) \leqslant c(47) \approx 6.59 < 7$. □

6. Proof of Proposition 4.2

Let us first establish that

$$|\mathbb{E}[\mathfrak{f}(\mathbb{G}_d)] - \mathbb{E}[\mathfrak{f}(\mathbb{G}_{d'})]| \leqslant 2\kappa \sum_{i=1}^{n} |d(i) - d'(i)|. \tag{6.1}$$

By an immediate induction, we may restrict our attention to the case where $d(i) = d'(i) + \mathbf{1}_{i=i_0}$ for some $i_0 \in \{1, \dots, n\}$. Recall that $\mathbb{G}_d, \mathbb{G}_{d'}$ can be realized as $G[\mathfrak{m}], G[\mathfrak{m}']$, where $\mathfrak{m}, \mathfrak{m}'$ are uniform maximal matchings on the corresponding sets of half-edges $\mathcal{H}, \mathcal{H}'$. But $\mathcal{H} = \mathcal{H}' \cup \{h\}$, where $h = (i_0, d(i_0))$ denotes the extra half-edge attached to i_0. We may thus couple $\mathfrak{m}'$ to $\mathfrak{m}$ as follows.

- If $|\mathcal{H}|$ is even, then the matching $\mathfrak{m}$ is perfect and we let $\mathfrak{m}'$ denote the matching obtained by simply removing from $\mathfrak{m}$ the pair containing h.
- If $|\mathcal{H}|$ is odd, then there must be an unpaired half-edge in $\mathfrak{m}$, and we let $\mathfrak{m}'$ denote the matching obtained by exchanging it with h.

In both cases, $\mathfrak{m}'$ is uniformly distributed over the maximal matchings of $\mathcal{H}'$. Moreover, $G[\mathfrak{m}]$ and $G[\mathfrak{m}']$ differ by at most two edges almost surely, so that

$$|\mathfrak{f}(G[\mathfrak{m}]) - \mathfrak{f}(G[\mathfrak{m}'])| \leqslant 2\kappa.$$

Taking expectations yields (6.1). Since $\mathfrak{f}$ is invariant under graph isomorphism, the left-hand side of (6.1) is invariant under reordering of d, d'. Consequently, we may choose a rearrangement that minimizes the right-hand side. It is classical that the choice $d(1) \leqslant \cdots \leqslant d(n)$ and $d'(1) \leqslant \cdots \leqslant d'(n)$ is optimal and satisfies

$$\sum_{i=1}^{n} |d(i) - d'(i)| = n\mathcal{W}\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{d(i)}, \frac{1}{n}\sum_{i=1}^{n} \delta_{d'(i)}\right).$$

Re-injecting this into (6.1) concludes the proof.

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