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Some Ramsey Schur Numbers

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In memory of Walter Deuber

The Ramsey Schur number $RS(s,t)$ is the smallest n such that every 2-colouring of the edges of K_n with vertices $1,2,\ldots,n$ contains a green K_s or there are vertices $x_1,x_2,\ldots,x_t$ fulfilling the equation $x_1+x_2+\cdots+x_{t-1}=x_t$ and all edges (x_i,x_j) are red. We prove $RS(3,3)=11$, $RS(3,t)=t^2-3$ for $t\equiv 1 \pmod 6$ and $t=8$, and $RS(3,t)\geqslant t^2-3$.

1. Introduction

In October 1995, during his last visits to the universities of Bielefeld (W. Deuber, D. Gunderson) and Braunschweig (H. Harborth) the following question was posed by Paul Erdős. Does there exist a complete graph K_n with the numbers $1,2,\ldots,n$ as vertices such that every 2-colouring (green/red) of the edges of K_n contains a given green subgraph or all edges between vertices fulfilling a given equation are red? This can be interpreted as a combination of Ramsey numbers and generalized Schur numbers where edges are coloured instead of numbers.

Here we will consider the special Ramsey Schur numbers $RS(3,t)$ defined as follows. For $s,t\geqslant 3$, the Ramsey Schur number $RS(s,t)$ is the smallest n such that every 2-colouring of the edges of K_n with vertices $1,2,\ldots,n$ contains a green K_s or there are vertices $x_1,x_2,\ldots,x_t$ fulfilling the equation $x_1+x_2+\cdots+x_{t-1}=x_t$ and all edges (x_i,x_j), $1\leqslant i<j\leqslant t$, $x_i\neq x_j$, are red. It may be remarked that we do not restrict ourselves to those solutions of the sum equations having pairwise distinct values x_i. The value $RS(3,3)=11$ is determined in [4] and the proof is repeated in this paper (Theorem 3.4)

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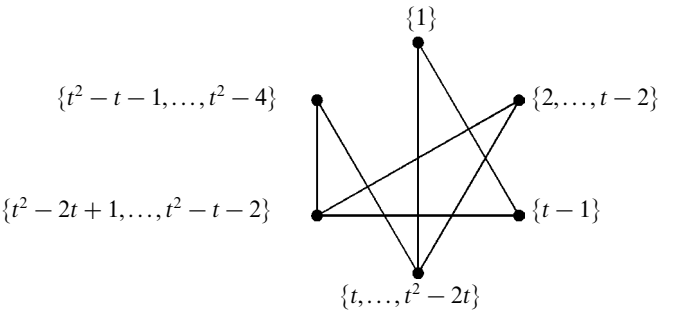


Figure 1. Green edges in K_{t^2-4}

establishing our shorter notation. We will give a general lower bound for $RS(3,t)$ and exact values for $t \equiv 1 \pmod 6$ and $t = 3, 4, 5, 6, 8, 9, 10, 11$, and 15.

Originally, the question of Erdős arose from discussions with Hajnal and Pach [2] in the summer of 1995. In this context only sum equations have been considered having pairwise distinct summands and some general results have been published in [1, 3, 5]. The existence of $RS^{\neq}(s,t)$ differing from $RS(s,t)$ only in the condition that the summands have to be pairwise distinct, follows from a more general existence theorem in [5] and the existence of $RS(s,t)$ follows from $RS(s,t) \leqslant RS^{\neq}(s,t)$. We have determined $RS^{\neq}(3,3) = 18$ by computer answering a question of Erdős.

2. Lower bound for $RS(3,t)$

By a special 2-colouring the following lower bound is obtained.

Theorem 2.1. *$RS(3,t) \geqslant t^2 - 3$ for $t \geqslant 3$.*

Proof. As green edges of a 2-colouring of the complete graph K_{t^2-4} we use the edges

(1) $(1,j)$ for $t-1 \leqslant j \leqslant t^2-2t$,
(2) (i,j) for $2 \leqslant i \leqslant t-2$ and $t \leqslant j \leqslant t^2-t-2$,
(3) $(t-1,j)$ for $t^2-2t+1 \leqslant j \leqslant t^2-t-2$,
(4) (i,j) for $t \leqslant i \leqslant t^2-t-2$ and $t^2-t-1 \leqslant j \leqslant t^2-4$.

There is no green triangle since all green edges excluded those of case (2) induce a bipartite graph and all vertices of K_{t^2-4} are adjacent to at most one vertex of each of the edges in case (2). This construction is depicted in Figure 1, where every edge represents all edges pairwise connecting the vertices of the two sets at its endpoints.

To see that for every solution $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_t$ of $x_1 + \cdots + x_{t-1} = x_t$ there is an edge (x_i, x_j) in green we distinguish three intervals for x_t.

For $t-1 \leqslant x_t \leqslant t^2-2t$ we have $1 \leqslant x_1 \leqslant t-2$ and by (1) and (2) the edge (x_1, x_t) is green.

For $t^2-2t+1 \leqslant x_t \leqslant t^2-t-2$ we have $x_1 \leqslant t-1$. If $x_1 \geqslant 2$ then (x_1, x_t) is green by (2) and (3). If $x_1 = 1$ then $t \leqslant x_{t-1} \leqslant t^2-2t$ and (x_1, x_{t-1}) is green by (1).

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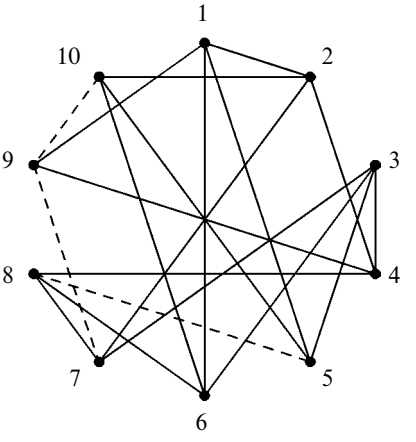


Figure 2. $RS(3,3) \geqslant 11$

For $t^2 - t - 1 \leqslant x_t \leqslant t^2 - 4$ we have $t \leqslant x_{t-1} \leqslant t^2 - t - 2$ and then (x_{t-1}, x_t) is green by (4). □

For $t = 3$, the larger lower bound $RS(3,3) \geqslant 11$ is obtained using the 2-colouring of the edges of K_{10} in Figure 2 where the green edges are drawn.

Up to the 3 dashed edges which may be independently coloured in green or in red, this 2-colouring is unique.

3. Exact values of $RS(3,t)$

The lower bound in Theorem 2.1 is not attained for $t = 3$; however, for small values $t \geqslant 4$ and for $t \equiv 1 \pmod 6$ we have proofs for equality in Theorem 2.1. For the presentation of some of the proofs we use two lemmas.

Lemma 3.1. *In a 2-colouring of K_n, four vertices a, b, c, d with edges (a,b) and (a,c) in green force (a,d) to be in red if*

(i) *there is a solution of $x_1 + \cdots + x_{t-1} = x_t$ with $\{x_1, \ldots, x_t\} = \{b,c,d\}$,*
(ii) *there is no green K_3, and*
(iii) *there is no solution of $x_1 + \cdots + x_{t-1} = x_t$ with all edges (x_i, x_j) in red.*

Proof. Assume (a,d) is green. By (ii) the triangle (b,c,d) is red, contradicting (i) and (iii). □

Lemma 3.2. *In a 2-colouring of K_n, four vertices a, b, c, d with (a,b) and (a,c) in red and (b,c) in green force (a,d) to be green if*

(i) *there are two solutions of $x_1 + \cdots + x_{t-1} = x_t$ with $\{x_1, \ldots, x_t\} = \{a,b,d\}$ and $\{x_1, \ldots, x_t\} = \{a,c,d\}$,*
(ii) *there is no green K_3, and*

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(iii) *there is no solution of* $x_1 + \cdots + x_{t-1} = x_t$ *with all edges* (x_i, x_j) *in red.*

Proof. Assume (a, d) is red. By (i) and (iii) the edges (b, d) and (c, d) are green, contradicting (ii). □

In the proofs of the upper bounds for $RS(3, t)$ we use the following abbreviations for the forced colourings of edges, as in the following examples:

$(\mathbf{2, 8})\mathbf{r}: 4$ the edge $(2, 8)$ has to be red since $(2, 4)$ and $(4, 8)$ are known to be green;
$(\mathbf{3, 5})\mathbf{g}: 2 + 3 = 5$ the edge $(3, 5)$ has to be green since all other edges between vertices occurring in the sum equation, $(2, 3)$ and $(2, 5)$, are known to be red;
$(\mathbf{4, 6})\mathbf{r}: (4, 2, 8)2 + 6 = 8$ the edge $(4, 6)$ is red due to Lemma 3.1 with $(4, 2, 8) = (a, b, c)$;
$(\mathbf{1, 15})\mathbf{g}: (1, 5, 35)5 * 1 + 2 * 5 = 15, 5 * 1 + 2 * 15 = 35$ the edge $(1, 15)$ is green due to Lemma 3.2 with $(1, 5, 35) = (a, b, c)$. We used $5 * 1$ for $1 + 1 + 1 + 1 + 1$.

Theorem 3.3. $RS(3, 3) = 11$.

Proof. Figure 2 proves $RS(3, 3) \geqslant 11$. To prove $RS(3, 3) \leqslant 11$ we assume the existence of a 2-colouring of K_{11} without a green K_3 and without a solution of $x_1 + x_2 = x_3$ with all edges (x_i, x_j) in red.

Then $(\mathbf{1, 2})\mathbf{g}: 1 + 1 = 2$, $(\mathbf{2, 4})\mathbf{g}: 2 + 2 = 4$, $(\mathbf{3, 6})\mathbf{g}: 3 + 3 = 6$, $(\mathbf{4, 8})\mathbf{g}: 4 + 4 = 8$, $(\mathbf{5, 10})\mathbf{g}: 5 + 5 = 10$, $(\mathbf{1, 4})\mathbf{r}: 2$, $(\mathbf{2, 3})\mathbf{r}: (2, 1, 4)1 + 3 = 4$, $(\mathbf{2, 5})\mathbf{r}: (2, 1, 4)1 + 4 = 5$, $(\mathbf{3, 5})\mathbf{g}: 2 + 3 = 5$, $(\mathbf{4, 6})\mathbf{r}: (4, 2, 8)2 + 6 = 8$, $(\mathbf{4, 10})\mathbf{r}: (4, 2, 8)2 + 8 = 10$, $(\mathbf{5, 6})\mathbf{r}: 3$, $(\mathbf{1, 3})\mathbf{r}: (3, 5, 6)1 + 5 = 6$, $(\mathbf{3, 4})\mathbf{g}: 1 + 3 = 4$, $(\mathbf{4, 5})\mathbf{r}: 3$, $(\mathbf{1, 5})\mathbf{g}: 1 + 4 = 5$, $(\mathbf{1, 7})\mathbf{r}: (1, 2, 5)2 + 5 = 7$, $(\mathbf{5, 11})\mathbf{r}: (5, 1, 10)1 + 10 = 11$, $(\mathbf{6, 10})\mathbf{g}: 4 + 6 = 10$, $(\mathbf{6, 7})\mathbf{r}: (6, 3, 10)3 + 7 = 10$, $(\mathbf{1, 6})\mathbf{g}: 1 + 6 = 7$, $(\mathbf{6, 11})\mathbf{r}: (6, 1, 10)1 + 10 = 11$, and $(\mathbf{6, 11})\mathbf{g}: 5 + 6 = 11$, a contradiction. □

Theorem 3.4. $RS(3, t) = t^2 - 3$ *for* $t \equiv 1 \pmod 6$, $t \geqslant 7$.

Proof. Theorem 2.1 proves $RS(3, t) \geqslant t^2 - 3$. To prove $RS(3, t) \leqslant t^2 - 3$ we assume the existence of a 2-colouring of K_{t^2-3} without a green K_3 and without a solution of $x_1 + x_2 + \cdots + x_{t-1} = x_t$ with all edges (x_i, x_j) in red.

Then $(\mathbf{1, t-1})\mathbf{g}: (t-1) * 1 = t - 1$, $(\mathbf{t-1, t^2-2t+1})\mathbf{g}: (t-1) * (t-1) = t^2 - 2t + 1$, $(\mathbf{1, t^2-2t+1})\mathbf{r}: t - 1$, $(\mathbf{t-1, t})\mathbf{r}: (t-1, 1, t^2-2t+1)1 + (t-2) * t = t^2 - 2t + 1$, $(\mathbf{t-1, 2t-3})\mathbf{r}: (t-1, 1, t^2-2t+1)\frac{t-1}{2} * 1 + \frac{t-1}{2} * (2t-3) = t^2 - 2t + 1$, $(\mathbf{t-1, 3t-5})\mathbf{r}: (t-1, 1, t^2-2t+1)\frac{2t-2}{3} * 1 + \frac{t-1}{3} * (3t-5) = t^2 - 2t + 1$, $(\mathbf{t-1, t^2-t-1})\mathbf{r}: (t-1, 1, t^2-2t+1)(t-2) * 1 + (t^2-2t+1) = t^2 - t - 1$, $(\mathbf{t, t^2-t-1})\mathbf{g}: (t-1) + (t-2) * t = t^2 - t - 1$, $(\mathbf{2t-3, t^2-t-1})\mathbf{g}: (t-2) * (t-1) + (2t-3) = t^2 - t - 1$, $(\mathbf{t^2-t-1, t^2-3})\mathbf{r}: (t^2 - t - 1, t, 2t-3)(t-2) * t + (2t-3) = t^2 - 3$.

Assume $(\mathbf{1, t})\mathbf{g}$. Then $(\mathbf{1, t^2-t-1})\mathbf{r}: t$, $(\mathbf{1, t^2-3})\mathbf{g}: (t-2) * 1 + (t^2 - t - 1) = t^2 - 3$, $(\mathbf{t-1, t^2-3})\mathbf{r}: 1$, $(\mathbf{2t-3, t^2-3})\mathbf{g}: (t-3) * (t-1) + 2 * (2t-3) = t^2 - 3$, $(\mathbf{1, 2t-3})\mathbf{r}: t^2 - 3$, $(\mathbf{2t-3, t^2-2t+1})\mathbf{g}: \frac{t-1}{2} * 1 + \frac{t-1}{2} * (2t-3) = t^2 - 2t + 1$, $(\mathbf{1, 3t-5})\mathbf{g}: (1, 2t-3, t^2-2t+1)(t-2) * 1 + (2t-3) = 3t - 5$, $\frac{2t-2}{3} * 1 + \frac{t-1}{3} * (3t-5) = t^2 - 2t + 1$, $(\mathbf{3t-5, t^2-3})\mathbf{r}: 1$, and $(\mathbf{3t-5, t^2-3})\mathbf{g}: (t-2) * (t-1) + (3t-5) = t^2 - 3$, a contradiction yielding $(\mathbf{1, t})\mathbf{r}$.

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Then $(\boldsymbol{t}, \boldsymbol{t^2-2t+1})\mathbf{g}: 1+(t-2)*t = t^2-2t+1$, $(\boldsymbol{t^2-2t+1}, \boldsymbol{t^2-t-1})\mathbf{r}: t$, $(\mathbf{1}, \boldsymbol{t^2-t-1})\mathbf{g}: (t-2)*1+(t^2-2t+1) = t^2-t-1$, $(\mathbf{1}, \boldsymbol{2t-3})\mathbf{r}: t^2-t-1$, $(\boldsymbol{2t-3}, \boldsymbol{t^2-3})\mathbf{g}: (2t-3, 1, t-1)\frac{t-3}{2}*1+\frac{t+1}{2}*(2t-3) = t^2-3, (t-3)*(t-1)+2*(2t-3) = t^2-3$, $(\boldsymbol{2t-3}, \boldsymbol{t^2-2t+1})\mathbf{g}: \frac{t-1}{2}*1+\frac{t-1}{2}*(2t-3) = t^2-2t+1$, $(\mathbf{1}, \boldsymbol{3t-5})\mathbf{g}: (1, 2t-3, t^2-2t+1)(t-2)*1+(2t-3) = 3t-5, \frac{2t-2}{3}*1+\frac{t-1}{3}*(3t-5) = t^2-2t+1$, $(\mathbf{1}, \mathbf{2})\mathbf{r}: (1, t-1, 3t-5)(t-2)*2+(t-1) = 3t-5$, $(\mathbf{2}, \boldsymbol{t})\mathbf{g}: (t-2)*1+2 = t$, $(\mathbf{2}, \boldsymbol{2t-3})\mathbf{g}: 1+(t-2)*2 = 2t-3$, $(\mathbf{2}, \boldsymbol{t^2-2t+1})\mathbf{r}: t$, $(\mathbf{2}, \boldsymbol{t^2-3})\mathbf{r}: 2t-3$, $(\boldsymbol{t^2-2t+1}, \boldsymbol{t^2-3})\mathbf{g}: (t-2)*2+(t^2-2t+1) = t^2-3$, and $(\boldsymbol{t^2-2t+1}, \boldsymbol{t^2-3})\mathbf{r}: 2t-3$, a contradiction. □

Theorem 3.5. $RS(3, 8) = 61$.

Proof. Theorem 2.1 proves $RS(3, 8) \geqslant 61$. To prove $RS(3, 8) \leqslant 61$ we assume the existence of a 2-colouring of K_{61} without a green K_3 and without a solution of $x_1 + x_2 + \cdots + x_7 = x_8$ with all edges (x_i, x_j) in red.

Then $(\mathbf{1}, \mathbf{7})\mathbf{g}: 7*1 = 7$, $(\mathbf{2}, \mathbf{14})\mathbf{g}: 7*2 = 14$, $(\mathbf{3}, \mathbf{21})\mathbf{g}: 7*3 = 21$, $(\mathbf{5}, \mathbf{35})\mathbf{g}: 7*5 = 35$, $(\mathbf{7}, \mathbf{49})\mathbf{g}: 7*7 = 49$, $(\mathbf{8}, \mathbf{56})\mathbf{g}: 7*8 = 56$, $(\mathbf{1}, \mathbf{49})\mathbf{r}: 7$, $(\mathbf{7}, \mathbf{8})\mathbf{r}: (7, 1, 49)1+6*8 = 49$, $(\mathbf{7}, \mathbf{55})\mathbf{r}: (7, 1, 49)6*1+49 = 55$, $(\mathbf{8}, \mathbf{55})\mathbf{g}: 7+6*8 = 55$.

Assume $(\mathbf{1}, \mathbf{8})\mathbf{g}$. Then $(\mathbf{1}, \mathbf{52})\mathbf{r}: (1, 7, 8)4*7+3*8 = 52$, $(\mathbf{1}, \mathbf{55})\mathbf{r}: 8$, $(\mathbf{9}, \mathbf{55})\mathbf{g}: (55, 1, 7)6*9+1 = 55, 3*9+4*7 = 55$, $(\mathbf{13}, \mathbf{55})\mathbf{g}: (55, 1, 7)4*13+3*1 = 55, 13+6*7 = 55$, $(\mathbf{8}, \mathbf{61})\mathbf{r}: (8, 1, 55)6*1+55 = 61$, $(\mathbf{8}, \mathbf{9})\mathbf{r}: 55$, $(\mathbf{9}, \mathbf{61})\mathbf{g}: 2*8+5*9 = 61$, $(\mathbf{8}, \mathbf{13})\mathbf{r}: 55$, $(\mathbf{13}, \mathbf{61})\mathbf{g}: 6*8+13 = 61$, $(\mathbf{55}, \mathbf{61})\mathbf{r}: 9$, $(\mathbf{1}, \mathbf{61})\mathbf{g}: 6*1+55 = 61$, $(\mathbf{1}, \mathbf{10})\mathbf{r}: (1, 7, 61)3*7+4*10 = 61$, $(\mathbf{1}, \mathbf{11})\mathbf{r}: (1, 7, 61)4*7+3*11 = 61$, $(\mathbf{10}, \mathbf{52})\mathbf{g}: 2*1+5*10 = 52$, $(\mathbf{1}, \mathbf{16})\mathbf{g}: (1, 10, 52)6*1+10 = 16, 4*1+3*16 = 52$, $(\mathbf{1}, \mathbf{19})\mathbf{r}: (1, 7, 61)6*7+19 = 61$, $(\mathbf{1}, \mathbf{9})\mathbf{r}: 61$, $(\mathbf{3}, \mathbf{61})\mathbf{r}: (61, 1, 9)6*1+3 = 9$, $(\mathbf{23}, \mathbf{61})\mathbf{r}: (61, 1, 9)5*1+2*9 = 23$, $(\mathbf{1}, \mathbf{13})\mathbf{r}: 61$, $(\mathbf{13}, \mathbf{19})\mathbf{g}: 6*1+13 = 19$, $(\mathbf{1}, \mathbf{3})\mathbf{g}: (1, 13, 19)4*1+3*3 = 13, 1+6*3 = 19$, $(\mathbf{1}, \mathbf{4})\mathbf{g}: (1, 13, 19)5*1+2*4 = 13, 3*1+4*4 = 19$, $(\mathbf{1}, \mathbf{43})\mathbf{g}: (1, 13, 19)4*1+3*13 = 43, 5*1+2*19 = 43$, $(\mathbf{1}, \mathbf{23})\mathbf{r}: (1, 3, 4)5*3+2*4 = 23$, $(\mathbf{1}, \mathbf{27})\mathbf{r}: (1, 3, 4)3+6*4 = 27$, $(\mathbf{1}, \mathbf{2})\mathbf{r}: (1, 3, 16)5*2+2*3 = 16$, $(\mathbf{2}, \mathbf{9})\mathbf{g}: 5*1+2*2 = 9$, $(\mathbf{2}, \mathbf{11})\mathbf{g}: 3*1+4*2 = 11$, $(\mathbf{1}, \mathbf{21})\mathbf{r}: 3$, $(\mathbf{1}, \mathbf{5})\mathbf{r}: (1, 7, 43)3*5+4*7 = 43$, $(\mathbf{5}, \mathbf{11})\mathbf{g}: 6*1+5 = 11$, $(\mathbf{9}, \mathbf{23})\mathbf{g}: 5*1+2*9 = 23$, $(\mathbf{21}, \mathbf{27})\mathbf{g}: 6*1+21 = 27$, $(\mathbf{11}, \mathbf{17})\mathbf{r}: (11, 2, 5)6*2+5 = 17$, $(\mathbf{1}, \mathbf{17})\mathbf{g}: 6*1+11 = 17$, $(\mathbf{1}, \mathbf{35})\mathbf{r}: (1, 3, 17)6*3+17 = 35$, $(\mathbf{1}, \mathbf{15})\mathbf{g}: (1, 5, 35)5*1+2*5 = 15, 5*1+2*15 = 35$, $(\mathbf{1}, \mathbf{39})\mathbf{r}: (1, 4, 15)6*4+15 = 39$, $(\mathbf{9}, \mathbf{39})\mathbf{g}: 3*1+4*9 = 39$, $(\mathbf{9}, \mathbf{27})\mathbf{r}: (9, 2, 39)6*2+27 = 39$, $(\mathbf{3}, \mathbf{27})\mathbf{r}: 21$, $(\mathbf{3}, \mathbf{9})\mathbf{g}: 6*3+9 = 27$, $(\mathbf{3}, \mathbf{23})\mathbf{r}: 9$, and $(\mathbf{3}, \mathbf{23})\mathbf{g}: 5*3+2*23 = 61$, a contradiction yielding $(\mathbf{1}, \mathbf{8})\mathbf{r}$.

Then $(\mathbf{8}, \mathbf{49})\mathbf{g}: 1+6*8 = 49$, $(\mathbf{49}, \mathbf{55})\mathbf{r}: 8$, $(\mathbf{1}, \mathbf{55})\mathbf{g}: 6*1+49 = 55$, $(\mathbf{1}, \mathbf{13})\mathbf{r}: (1, 7, 55)6*7+13 = 55$.

Assume $(\mathbf{1}, \mathbf{2})\mathbf{g}$. Then $(\mathbf{1}, \mathbf{19})\mathbf{r}: (1, 2, 7)6*2+7 = 19$, $(\mathbf{1}, \mathbf{14})\mathbf{r}: 2$, $(\mathbf{2}, \mathbf{20})\mathbf{r}: (2, 1, 14)6*1+14 = 20$, $(\mathbf{8}, \mathbf{14})\mathbf{g}: 6*1+8 = 14$, $(\mathbf{13}, \mathbf{19})\mathbf{g}: 6*1+13 = 19$, $(\mathbf{1}, \mathbf{4})\mathbf{g}: (1, 13, 19)5*1+2*4 = 13, 3*1+4*4 = 19$, $(\mathbf{1}, \mathbf{26})\mathbf{r}: (1, 2, 4)2+6*4 = 26$, $(\mathbf{2}, \mathbf{8})\mathbf{r}: 14$, $(\mathbf{8}, \mathbf{20})\mathbf{g}: 6*2+8 = 20$, $(\mathbf{2}, \mathbf{32})\mathbf{g}: (2, 8, 20)4*2+3*8 = 32, 6*2+20 = 32$, $(\mathbf{1}, \mathbf{32})\mathbf{r}: 2$, $(\mathbf{2}, \mathbf{6})\mathbf{r}: (2, 1, 32)2*1+5*6 = 32$, $(\mathbf{2}, \mathbf{38})\mathbf{r}: (2, 1, 32)6*1+32 = 38$, $(\mathbf{26}, \mathbf{32})\mathbf{g}: 6*1+26 = 32$, $(\mathbf{6}, \mathbf{38})\mathbf{g}: 2+6*6 = 38$, $(\mathbf{2}, \mathbf{26})\mathbf{g}: (2, 6, 38)4*2+3*6 = 26, 6*2+26 = 38$, and $(\mathbf{2}, \mathbf{26})\mathbf{r}: 32$, a contradiction yielding $(\mathbf{1}, \mathbf{2})\mathbf{r}$.

Then $(\mathbf{2}, \mathbf{8})\mathbf{g}: 6*1+2 = 8$, $(\mathbf{8}, \mathbf{49})\mathbf{g}: 1+6*8 = 49$, $(\mathbf{8}, \mathbf{14})\mathbf{r}: 2$, $(\mathbf{1}, \mathbf{14})\mathbf{g}: 6*1+8 = 14$, $(\mathbf{1}, \mathbf{56})\mathbf{r}: (1, 7, 14)6*7+14 = 56$, $(\mathbf{2}, \mathbf{49})\mathbf{r}: 8$, $(\mathbf{2}, \mathbf{56})\mathbf{r}: 8$, $(\mathbf{49}, \mathbf{56})\mathbf{r}: 8$, and $(\mathbf{49}, \mathbf{56})\mathbf{g}: 5*1+2+49 = 56$, a contradiction. □

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In addition to the cases covered by Theorems 3.3 to 3.5 we have similar proofs of $RS(3,t) = t^2 - 3$ for $t = 4$, 5, 6, 9, 10, 11, and 15.

Conjecture. $RS(3,t) = t^2 - 3$ for $t \geqslant 4$.

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