THE GENERALIZED FERMI INTERACTION

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Konopinski and Uhlenbeck have shown that the empirical β-ray spectra seem to require a modification of Fermi's original expression for the interaction leading to β decay‡. Their expression includes a derivative of the neutrino wave function. The purpose of the present paper is to investigate the most general type of such an interaction term and the possibility of introducing second-order derivatives. We have to deal with the transition§

neutron + neutrino → proton + electron

or proton + anti-neutrino → neutron + positron.

The whole system has the equations of motion

$$\left(i\hbar\frac{\partial}{\partial t}+H_1\right)\psi_1=\mathbf{A}\psi_2,$$

$$\left(i\hbar\frac{\partial}{\partial t}+H_2\right)\psi_2=\mathbf{A}^*\psi_1,$$

where ψ_1 and ψ_2 are the wave functions of the initial and final states in configuration space, H_1 and H_2 are the Hamiltonians of the particles in the initial and final states, and $\mathbf{A}$ is a small interaction operator.

If initially we take a neutron and a neutrino in the stationary states m and σ, and obtain finally a proton and an electron in the stationary states n and s, then, if we adjust the energy of the neutrino so as to conserve the total energy in the disintegration, the transition probability is proportional to $|\int\psi_s^* u_u^* \mathbf{A} v_m \phi_\sigma d\tau|^2$, where ψ_s, u_n, v_m and ϕ_σ are the wave functions of the electron, proton, neutron and neutrino respectively. In the Fermi theory $\mathbf{A}$ operates upon the spin components of the wave functions. The equations of motion are relativistic and the whole theory is consistent provided that $\psi_s^* u_n^* \mathbf{A} v_m \phi_\sigma$ is relativistically invariant. Konopinski and Uhlenbeck modified the theory by making $\mathbf{A}$ an operator which can also produce derivatives. Assuming that this is allowable, we wish to consider the possible forms for $\mathbf{A}$, using first- and second-order derivatives, compatible with the experimental evidence.

‡ Fermi, *Zeit. f. Physik*, 88 (1934), 161; Konopinski and Uhlenbeck, *Phys. Rev.* 48 (1935), 7. The latter paper will be quoted as K.–U.

§ Following K.–U. we choose this more symmetric description of the process, in which the emission of an anti-neutrino is replaced by the absorption of a neutrino from a state of negative energy and vice versa.

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1. EVALUATION OF THE POSSIBLE ENERGY DISTRIBUTIONS

We require $\psi_s^* u_n^* \mathbf{A} v_m \phi_\sigma$ to be relativistically invariant. The transformation properties can be conveniently discussed by writing this expression in the form $(\psi_s^* \mathbf{A}' \phi_\sigma)(u_n^* \mathbf{A}'' v_m)$, where $\mathbf{A}'$, $\mathbf{A}''$ are operators so chosen that both $(\psi_s^* \mathbf{A}' \phi_\sigma)$ and $(u_n^* \mathbf{A}'' v_m)$ are scalar, pseudo-scalar, vector, pseudo-vector or tensor quantities, such that their product is invariant‡. The most general form for $\psi_s^* u_n^* \mathbf{A} v_m \phi_\sigma$ is then obtained by taking a linear combination of such products. We might also have considered the combination $(\psi_s^* \mathbf{A}' v_m)(u_n^* \mathbf{A}'' \phi_\sigma)$, but this does not give any new result, since it can be expressed as a sum of products of the previous kind§. A convenient classification of the possible forms for $\psi_s^* u_n^* \mathbf{A} v_m \phi_\sigma$ is obtained by noting that forms of different orders in the derivatives of the light particles will, in general, have coefficients of different orders of magnitude. We can therefore discuss the terms of each kind separately. We evaluate the energy distributions resulting from each of them.

For this purpose we assume that the nuclei concerned are light and that the neutrino mass is zero unless otherwise stated. It may be noted that, if we average over the spin and direction of the electron,

$\psi_s^* \psi_s$ gives 1,

$\psi_s^* \alpha_m \psi_s$ gives $-mc^2/|E|$,

$\psi_s^* \alpha_i \psi_s$, $\psi_s^* \alpha_m \alpha_i \psi_s$, $\psi_s^* \alpha_i \alpha_j \psi_s$, $\psi_s^* \alpha_m \alpha_i \alpha_j \psi_s$, $\psi_s^* \alpha_i \alpha_j \alpha_k \psi_s$ all give zero,

where s denotes a stationary state of energy E. For each term we obtain a transition probability proportional to $|\int (\psi_s^* \mathbf{A}' \phi_\sigma)(u_n^* \mathbf{A}'' v_m) d\tau|^2$ taken over the nucleus. This may be written approximately as $|(\psi_s^* \mathbf{A}' \phi_\sigma) \int (u_n^* \mathbf{A}'' v_m) d\tau|^2$, provided that the remaining integral does not vanish, where $(\psi_s^* \mathbf{A}' \phi_\sigma)$ is evaluated at some point in the nucleus. We may write $\mathbf{A}_{ij\ldots}$ for the components of $\int (u_n^* \mathbf{A}'' v_m) d\tau$ to give the transition probability in the form $g^2 |(\psi_s^* \mathbf{A}' \phi_\sigma) \mathbf{A}_{ij\ldots}|^2$, where $(\psi_s^* \mathbf{A}' \phi_\sigma)$ is one of the possible terms, g is a constant and $\mathbf{A}_{ij\ldots}$ is a corresponding tensor for the heavy particles.

The expressions for $(\psi_s^* \mathbf{A}' \phi_\sigma)$ containing no derivatives all give essentially the Fermi probability distribution. We consider these first.

(A) The general interaction term is

$$\psi_s^\dagger (A + a\gamma_5 + A_i \gamma_i + a_i \gamma_i \gamma_5 + A_{ij} \gamma_i \gamma_j) \phi_\sigma.$$

We use the double suffix notation with Latin suffixes taking the values 0, 1, 2, 3 and Greek ones the values 1, 2, 3. Also we write $\psi^\dagger$ for $(\alpha_m \psi)^*$, γ_ν for $\alpha_m \alpha_\nu$; $\gamma_0 = -\alpha_m$

‡ Bethe and Bacher, *Rev. Mod. Phys.* 8 (1936), 190. By pseudo-vector or pseudo-scalar we mean quantities which transform like vectors or scalars under Lorentz transformations of determinant $+1$ but change sign if the transformation determinant is -1.

§ Such an expression was actually proposed by Weizäcker, *Zeit. f. Physik*, 102 (1936), 572, and it seems therefore rather too special to exclude linear combinations as in the treatment of Bethe and Bacher.

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and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_0$, α_ν and α_m being the Dirac matrices. The capital letters A, A_i, ... denote tensors of a rank indicated by the number of suffixes, while small letters denote pseudo-tensors.

The transition probability is proportional to the square of the interaction term. In averaging, for example, the square of the first term with respect to spin and direction of emission of the neutrino we obtain $|A|^2\psi_s^*\psi_s$, and after averaging also with respect to the spin and direction of the electron we have $|A|^2$.

The statistical weight, given by the number of states for a free electron with energy between W and $W+dW$ and the corresponding number of neutrino states, is

$$\text{constant} \times W(W^2-1)^{\frac{1}{2}}(W_0-W)^2\,dW,$$

where W_0 denotes the total disintegration energy. W and W_0 are measured in units of mc^2. Thus the first term in the interaction gives rise to an electron distribution $N(W)\,dW$, where

$$N = |A|^2 W(W^2-1)^{\frac{1}{2}}(W_0-W)^2.$$

Similarly we can work out the other square terms and cross-products, obtaining finally

$$N = [(|A|^2+|a|^2+|A_0|^2+|A_\nu|^2+|a_0|^2+|a_\nu|^2+|A_{0\nu}|^2+|A_{\mu\nu}|^2) + 2W^{-1}\mathscr{R}(AA_0^*+aa_0^*+A_\nu A_{0\nu}^*+A_{\mu\nu}a_\tau^*\delta_{\mu\nu\tau})]\,W(W^2-1)^{\frac{1}{2}}(W_0-W)^2,$$

where $\mathscr{R}$ denotes the real part, and $\delta_{\mu\nu\tau}$ is zero if two suffixes are equal and otherwise it is equal to the sign of the permutation $(\mu\nu\tau)$.

First derivatives

If we assume a linear combination of all the first derivative terms, the interaction is

$$\left(A_i^{(1)}\psi_s^\dagger\frac{\partial\phi_\sigma}{\partial x_i}+a_i^{(2)}\psi_s^\dagger\gamma_5\frac{\partial\phi_\sigma}{\partial x_i}+B_i^{(3)}\frac{\partial\psi_s^\dagger}{\partial x_i}\phi_\sigma + b_i^{(4}\frac{\partial\psi_s^\dagger}{\partial x_i}\gamma_5\phi_\sigma+A_{ijk}^{(5)}\psi_s^\dagger\gamma_i\gamma_j\frac{\partial\phi_\sigma}{\partial x_k}+B_{ijk}^{(6)}\frac{\partial\psi_s^\dagger}{\partial x_i}\gamma_j\gamma_k\phi_\sigma\right) + \left(A_{ij}^{(1)}\psi_s^\dagger\gamma_i\frac{\partial\phi_\sigma}{\partial x_j}+a_{ij}^{(2)}\psi_s^\dagger\gamma_i\gamma_5\frac{\partial\phi_\sigma}{\partial x_j}+B_{ij}^{(3)}\frac{\partial\psi_s^\dagger}{\partial x_i}\gamma_i\phi_\sigma+b_{ij}^{(4)}\frac{\partial\psi_s^\dagger}{\partial x_i}\gamma_j\gamma_5\phi_\sigma\right).$$

For convenience in evaluating the resulting distribution we divide this sum into two separate groups, those terms containing α_m and those terms not containing α_m. We give first a summary of the results for the direct products, the exact combinations of the coefficients being omitted.

Table I

Resulting distributions	First group			Second group	
$W(W^2-1)^{\frac{1}{2}}(W_0-W)^4$	1	2	5	1	2
$W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^2$ $W(W^2-1)^{\frac{3}{2}}(W_0-W)^2$	3	4	6	3	4

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The direct product of the term represented by any particular number in the table gives the distribution or a linear combination of the distributions specified in the first column. For example, the direct product of term 6 of the first group gives a combination of $W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^2$ and $W(W^2-1)^{\frac{1}{2}}(W_0-W)^2$.

In evaluating the cross-products between the two groups we use the fact that the average value of $\psi_s^* \alpha_m \psi_s$ over the spin and direction of the electron is $-mc^2/|E|$. The following table summarizes the results between the two separate groups:

Table II

Resulting distributions	First group						Second group					
$(W^2-1)^{\frac{1}{2}}(W_0-W)^4$	1	2	5				1	2	1, 2			
$W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^2$ $(W^2-1)^{\frac{3}{2}}(W_0-W)^2$	3	4	6				3	4	3, 4			
$W(W^2-1)^{\frac{1}{2}}(W_0-W)^3$	1	2	3	4	5	6	3	4	1	2	3, 4	1, 2

Any number representing a term of the first group, combined with the terms of the second group whose numbers occupy a similar position in the table, gives the distribution, or a linear combination of the distributions, specified in the first column. Here the different components of the vectors and tensors A_i, a_{ij}, etc. have to be combined in certain ways, which again we do not specify. For example, term 6 of the first group combined with either of terms 3 or 4 of the second group gives a linear combination of the distributions

$$W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^2 \quad \text{and} \quad (W^2-1)^{\frac{3}{2}}(W_0-W)^2.$$

Members of the same group may also give combination terms. These given in tabular form are:

Table III

Resulting distributions	First group					First group				
$W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^3$ $(W^2-1)^{\frac{3}{2}}(W_0-W)^3$	1	2	3	4	5	6, 3	6, 4	5	5	6
	Second group					Second group				
$W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^3$ $(W^2-1)^{\frac{3}{2}}(W_0-W)^3$	1	2				3, 4	3, 4			

To see how these tables are formed and how to find the exact combinations of coefficients, we consider the combination of the first term of the first group and the third term of the second group.

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We obtain $\left(A_i\psi_s^\dagger\frac{\partial\phi_\sigma}{\partial x_i}\right)\left(\phi_\sigma^\dagger\gamma_j\frac{\partial\psi_s}{\partial x_k}B_{kj}^*\right)+\left(B_{kj}\frac{\partial\psi_s^\dagger}{\partial x_k}\gamma_j\phi_\sigma\right)\left(A_i^*\frac{\partial\phi_\sigma^\dagger}{\partial x_i}\psi_s\right)$.

This may be written as

$$A_i\psi_s^*\alpha_m p_i\left(1+\frac{\alpha_\tau p_\tau}{W_0-W}\right)\phi_\sigma\phi_\sigma^*(B_{k\nu}^*\alpha_\nu-B_{k0}^*)\,P_k\left(1-\frac{\alpha_\mu P_\mu}{W}-\frac{\alpha_m}{W}\right)\psi_s$$
$$+\text{the conjugate complex},$$

where p_i is the energy and momentum vector of the neutrino, and P_μ gives the momentum components of the electron. If we average over the spin and directions of both particles, we obtain

$$(W_0-W)\{(A_0B_{00}^*+A_0^*B_{00})-\tfrac{1}{3}(A_\tau B_{0\tau}^*+A_\tau^*B_{0\tau})\},$$

since the average value of any momentum component of the neutrino is $\frac{1}{3}(W_0-W)^2$.

Comparison of the expressions in Table I with the experimental evidence indicates that, if terms containing an electron derivative are used, they occur only with small coefficients. It is therefore sufficient to consider the complete expression given by the terms with one neutrino derivative. This is

$$\left(A_i\psi_s^\dagger\frac{\partial\phi_\sigma}{\partial x_i}+a_i\psi_s^\dagger\gamma_5\frac{\partial\phi_\sigma}{\partial x_i}+A_{ijk}\psi_s^\dagger\gamma_i\gamma_j\frac{\partial\phi_\sigma}{\partial x_k}+A_{ij}\psi_s^\dagger\gamma_i\frac{\partial\phi_\sigma}{\partial x_j}+a_{ij}\psi_s^\dagger\gamma_i\gamma_5\frac{\partial\phi_\sigma}{\partial x_j}\right).$$

The transition probability may be written in the form

$$\psi_s^*(A_i-a_i\gamma_5-A_{\mu\nu i}\alpha_\mu\alpha_\nu+A_{0\nu i}\alpha_\nu-A_{\nu i}\alpha_m\alpha_\nu-A_{0i}\alpha_m+a_{\nu i}\alpha_m\alpha_\nu\gamma_5$$
$$+a_{0i}\alpha_m\gamma_5)\,p_i\left(1-\frac{\alpha_\tau p_\tau}{W_0-W}\right)p_j\phi_\sigma\phi_\sigma^*(A_j^*+a_j^*\gamma_5-A_{\mu\nu j}^*\alpha_\nu\alpha_\mu+A_{0\nu j}^*\alpha_\nu$$
$$-A_{\nu j}^*\alpha_\nu\alpha_m-A_{0j}^*\alpha_m-a_{\nu j}^*\gamma_5\alpha_\nu\alpha_m-a_{0j}^*\gamma_5\alpha_m)\,(1-\alpha_m/W)\,\psi_s.$$

If $p_i=p_j$, we obtain the direct products

$$\overline{p}_i^2(|A_i|^2+|a_i|^2+|A_{\mu\nu i}|^2+|A_{0\nu i}|^2+|A_{\nu i}|^2+|A_{0i}|^2+|a_{\nu i}|^2+|a_{0i}|^2),$$

and the cross-products

$$\frac{2\overline{p}_i^2}{W}\mathscr{R}(A_iA_{0i}^*+a_ia_{0i}^*+A_{\mu\nu i}a_{\tau i}^*\delta_{\mu\nu\tau}+A_{0\nu i}A_{\nu i}^*).$$

If i or $j=0$, we obtain

$$-\frac{2p_0\overline{p}_\tau^2}{W_0-W}\mathscr{R}\{A_\tau(A_{0\tau0}^*+A_{\tau0}^*/W)+a_\tau(A_{\mu\nu0}^*\delta_{\mu\nu\tau}+a_{\tau0}^*/W)$$
$$+A_{\mu\nu\tau}\delta_{\mu\nu\tau}(a_0^*+a_{00}^*/W)-A_{\mu\tau\tau}(A_{0\mu0}^*+A_{\mu0}^*/W)+A_{0\nu\tau}(A_{\tau\nu0}^*+\delta_{\mu\tau\nu}a_{\mu0}^*/W)$$
$$+A_{0\tau\tau}(A_0^*+A_{00}^*/W)+A_{\nu\tau}(a_{\mu0}^*\delta_{\mu\tau\nu}+A_{\tau\nu0}^*/W)+A_{\tau\tau}(A_{00}^*+A_0^*/W)$$
$$+A_{0\tau}(A_{\tau0}^*+A_{0\tau0}^*/W)+a_{\tau\tau}(a_{00}^*+a_0^*/W)+a_{\nu\tau}\delta_{\mu\nu\tau}(A_{\mu0}^*+A_{0\mu0}^*/W)$$
$$+a_{0\tau}(a_{\tau0}^*+\delta_{\mu\nu\tau}A_{\mu\nu0}/W)\}.$$

The total expression may be written in the form $(W^{-1}A+B)(W_0-W)^2$, where $-B\leqslant A\leqslant B$.

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(B) Taking a general linear combination of second order terms, we have

$$\left|\left(A^{(1)}\frac{\partial\psi_s^\dagger}{\partial x_\alpha}\frac{\partial\phi_\sigma}{\partial x_\alpha}+a^{(2)}\frac{\partial\psi_s^\dagger}{\partial x_\alpha}\gamma_5\frac{\partial\phi_\sigma}{\partial x_\alpha}+A_{ij}^{(3)}\frac{\partial\psi_s^\dagger}{\partial x_i}\frac{\partial\phi_\sigma}{\partial x_j}+a_{ij}^{(4)}\frac{\partial\psi_s^\dagger}{\partial x_i}\gamma_5\frac{\partial\phi_\sigma}{\partial x_j}\right.\right.$$
$$+B_{ij}^{(5)}\psi_s^\dagger\frac{\partial^2\phi_\sigma}{\partial x_i\partial x_j}+b_{ij}^{(6)}\psi_s^\dagger\gamma_5\frac{\partial^2\phi_\sigma}{\partial x_i\partial x_j}+C_{ij}^{(7)}\frac{\partial^2\psi_s^\dagger}{\partial x_i\partial x_j}\phi_\sigma+c_{ij}^{(8)}\frac{\partial^2\psi_s^\dagger}{\partial x_i\partial x_j}\phi_\sigma$$
$$+D_{ij}^{(9)}\frac{\partial\psi_s^\dagger}{\partial x_\alpha}\gamma_i\gamma_j\frac{\partial\phi_\sigma}{\partial x_\alpha}+A_{ijkl}^{(10)}\frac{\partial\psi_s^\dagger}{\partial x_i}\gamma_j\gamma_k\frac{\partial\phi_\sigma}{\partial x_l}+B_{ijkl}^{(11)}\psi_s^\dagger\gamma_i\gamma_j\frac{\partial^2\phi_\sigma}{\partial x_k\partial x_l}$$
$$\left.+C_{ijkl}^{(12)}\frac{\partial^2\psi_s^\dagger}{\partial x_i\partial x_j}\gamma_k\gamma_l\phi_\sigma\right)+\left(A_i^{(1)}\frac{\partial\psi_s^\dagger}{\partial x_\alpha}\gamma_i\frac{\partial\phi_\sigma}{\partial x_\alpha}+a_i^{(2)}\frac{\partial\psi_s^\dagger}{\partial x_\alpha}\gamma_i\gamma_5\frac{\partial\phi_\sigma}{\partial x_\alpha}\right.$$
$$+A_{ijk}^{(3)}\frac{\partial\psi_s^\dagger}{\partial x_i}\gamma_j\frac{\partial\phi_\sigma}{\partial x_k}+a_{ijk}^{(4)}\frac{\partial\psi_s^\dagger}{\partial x_i}\gamma_j\gamma_5\frac{\partial\phi_\sigma}{\partial x_k}+B_{ijk}^{(5)}\psi_s^\dagger\gamma_i\frac{\partial^2\phi_\sigma}{\partial x_j\partial x_k}$$
$$\left.\left.+b_{ijk}^{(6)}\psi_s^\dagger\gamma_i\gamma_5\frac{\partial^2\phi_\sigma}{\partial x_j\partial x_k}+C_{ijk}^{(7)}\frac{\partial^2\psi_s^\dagger}{\partial x_i\partial x_j}\gamma_k\phi_\sigma+c_{ijk}^{(8)}\frac{\partial^2\psi_s^\dagger}{\partial x_i\partial x_j}\gamma_k\gamma_5\phi_\sigma\right)\right|^2,$$

where $\dfrac{\partial}{\partial x_\alpha}\dfrac{\partial}{\partial x_\alpha}$ stands for $\dfrac{\partial}{\partial x_1}\dfrac{\partial}{\partial x_1}+\dfrac{\partial}{\partial x_2}\dfrac{\partial}{\partial x_2}+\dfrac{\partial}{\partial x_3}\dfrac{\partial}{\partial x_3}-\dfrac{\partial}{\partial x_0}\dfrac{\partial}{\partial x_0}$.

As before we divide the terms into separately numbered groups, the first from 1 to 12 and the second from 1 to 8. The results are shown in Tables IV—VI.

Table IV. *Direct products*

Resulting distributions	First group			Second group	
$W(W^2-1)^{\frac{1}{2}}(2W^2+1)(W_0-W)^4$	1	2	9	1	2
$W(W^2-1)^{\frac{1}{2}}(W_0-W)^6$	5	6	11	5	6
$W^5(W^2-1)^{\frac{1}{2}}(W_0-W)^2$ $W^3(W^2-1)^{\frac{3}{2}}(W_0-W)^2$ $W(W^2-1)^{\frac{5}{2}}(W_0-W)^2$	7	8	12	7	8
$W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^4$ $W(W^2-1)^{\frac{3}{2}}(W_0-W)^4$	3	4	10	3	4

We now work out the complete expression for

$$\left|\left(A_{ij}\frac{\partial\psi_s^\dagger}{\partial x_i}\frac{\partial\phi_\sigma}{\partial x_j}+B_{ij}\psi_s^\dagger\frac{\partial^2\phi_\sigma}{\partial x_i\partial x_j}\right)\right|^2.$$

The cross-product is

$$A_{ij}B_{kl}^*P_iP_j\alpha_m\left(1+\frac{\alpha_\tau p_\tau}{W_0-W}\right)\alpha_m p_k p_l\left(1-\frac{\alpha_\nu P_\nu}{W}-\frac{\alpha_m}{W}\right)(\psi_s^*\phi_\sigma\phi_\sigma^*\psi_s)$$
$$+\text{the conjugate complex},$$

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Table V. *Cross-products*

Resulting distributions	First group						Second group					
$(W^2-1)^{\frac{1}{2}}(2W^2+1)(W_0-W)^4$	1	2	9				1	2	1, 2			
$(W^2-1)^{\frac{1}{2}}(W_0-W)^6$	5	6	11				5	6	5, 6			
$W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^4$	7	8	9	10	11	12	5	6	3, 4	1, 2, 3, 4	7, 8	5, 6
$(W^2-1)^{\frac{3}{2}}(W_0-W)^4$	1	2	3	4	5	6	3	4	1, 3	2, 4	7	8
$W^4(W^2-1)^{\frac{1}{2}}(W_0-W)^2$ $W^2(W^2-1)^{\frac{3}{2}}(W_0-W)^2$ $(W^2-1)^{\frac{5}{2}}(W_0-W)^2$	7	8	12				7	8	7, 8			
$W(W^2-1)^{\frac{1}{2}}(W_0-W)^5$	9	10	11				5, 6	5, 6	1, 2, 3, 4			
$W^{-1}(W^2-1)^{\frac{3}{2}}(W_0-W)^5$	1	2	3	4	5	6	5	6	5	6	1, 3	2, 4
$W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^3$	9	10	12				7, 8	7, 8	1, 2, 3, 4			
$W(W^2-1)^{\frac{3}{2}}(W_0-W)^3$	1	2	3	4	7	8	7	8	7	8	1, 3	2, 4

Table VI

Resulting distributions	First group						First group					
$W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^5$	9	10					11	11				
$(W^2-1)^{\frac{3}{2}}(W_0-W)^5$	1	2	3	4	5	6	5, 11	6, 11	5, 11	6, 11	9, 10	9, 10
$W^4(W^2-1)^{\frac{1}{2}}(W_0-W)^3$	9	10					12	12				
$W^2(W^2-1)^{\frac{3}{2}}(W_0-W)^3$ $(W^2-1)^{\frac{5}{2}}(W_0-W)^3$	1	2	3	4	7	8	7, 12	8, 12	7, 12	8, 12	9, 10	9, 10
$W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^4$ $W(W^2-1)^{\frac{3}{2}}(W_0-W)^4$	1	2	5	6	9	11	3	4	7	8	10	12
	Second group						Second group					
$W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^5$ $(W^2-1)^{\frac{3}{2}}(W_0-W)^5$	1	2	3	4			5, 6	5, 6	5, 6	5, 6		
$W^4(W^2-1)^{\frac{1}{2}}(W_0-W)^3$ $W^2(W^2-1)^{\frac{3}{2}}(W_0-W)^3$ $(W^2-1)^{\frac{5}{2}}(W_0-W)^3$	1	2	3	4			7, 8	7, 8	7, 8	7, 8		
$W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^4$ $W(W^2-1)^{\frac{3}{2}}(W_0-W)^4$	1	2	5	6			3	4	7	8		

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which gives on averaging

$$\left\{W B^*_{kl} A_{0j} \bar{p}_j \bar{p}_k \bar{p}_l + \frac{B^*_{kl} A_{\tau j}}{W(W_0 - W)} \bar{P}^2_\tau \bar{p}_j \bar{p}_\tau \bar{p}_k \bar{p}_l\right\}$$

$$+ \text{the conjugate complex},$$

where bars indicate averages over direction. This average for any component of the momentum of the electron is $\frac{1}{3}(W^2-1)$.

We obtain finally for the cross-product

$$2(W_0 - W)\bar{p}^2_k \mathscr{R}\left\{W(2A_{0k}B^*_{0k} - A_{00}B^*_{kk}) + \frac{W^2-1}{9W}(2A_{\tau k}B^*_{\tau k} - A_{\tau\tau}B^*_{kk})\right\}.$$

We wish to consider this expression for the special case

$$-aB_{ij} = bA_{ij} = \text{constant} \times \left(u^*_n \gamma_i \frac{\partial v_m}{\partial x_j} + u^*_n \gamma_j \frac{\partial v_m}{\partial x_i}\right).$$

Since the velocity of the heavy particles is small compared with that of light, we see that only the 00 components are not small. Thus we obtain

$$C\,|\,A_{00}\,|^2\, 2ab\, W^2(W^2-1)^{\frac{1}{2}}(W_0 - W)^5\, dW$$

for the cross-product, where C is some constant. The direct product gives

$$C\,|\,A_{00}\,|^2[a^2W^3(W^2-1)^{\frac{1}{2}}(W_0 - W)^4 + b^2W(W^2-1)^{\frac{1}{2}}(W_0 - W)^6]\,dW,$$

and we have the total distribution

$$g^2[a^2W^3(W^2-1)^{\frac{1}{2}}(W_0 - W)^4 + 2abW^2(W^2-1)^{\frac{1}{2}}(W_0 - W)^5 + b^2W(W^2-1)^{\frac{1}{2}}(W_0 - W)^6]\,dW.$$

2. Effect of a neutrino mass

The results given above hold only for a neutrino mass zero. It is found necessary to assume a neutrino mass in fitting the experimental results. We work out the distribution, assuming a neutrino mass μ, for terms containing one neutrino derivative in the special case in which

$$A_i = u^\dagger_n \frac{\partial v_m}{\partial x_i}, \quad a_i = u^\dagger_n \gamma_5 \frac{\partial v_m}{\partial x_i}, \quad A_{ijk} = u^\dagger_n \gamma_i \gamma_j \frac{\partial v_m}{\partial x_k},$$

$$A_{ij} = u^\dagger_n \gamma_i \frac{\partial v_m}{\partial x_j}, \quad a_{ij} = \frac{\partial u^\dagger_n}{\partial x_i}\frac{\partial v_m}{\partial x_j}.$$

The large components of these tensors are A_0, a_0, $A_{\mu\nu 0}$, a_{00}. The average value of the square of the momentum of the neutrino in any direction is now

$$\tfrac{1}{3}\{(W_0 - W)^2 - \mu^2/m^2\},$$

and the statistical weight is

$$\text{constant} \times W(W^2-1)^{\frac{1}{2}}(W_0 - W)\{(W_0 - W)^2 - \mu^2/m^2\}^{\frac{1}{2}}.$$

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We have to consider

$$(A_0-a_0\gamma_5-A_{\mu\nu0}\alpha_\mu\alpha_\nu-A_{00}\alpha_m+a_{00}\alpha_m\gamma_5)\left\{1+\frac{\alpha_m\mu}{m(W_0-W)}\right\}$$
$$\times(A_0^*+a_0^*\gamma_5-A_{\mu\nu0}^*\alpha_\nu\alpha_\mu-A_{00}^*\alpha_m-a_{00}^*\gamma_5\alpha_m)(1-\alpha_m/W),$$

which gives

$$(|A_0|^2+|A_{00}|^2+|a_0|^2+|a_{00}|^2+|A_{\mu\nu0}|^2)+2/W\mathscr{R}(A_0A_{00}^*+a_0a_{00}^*)$$
$$+\frac{2\mu}{m(W_0-W)}\mathscr{R}(a_0a_{00}^*-A_0A_{00}^*)+\frac{\mu}{m(W_0-W)W}$$
$$\times(|a_0|^2+|a_{00}|^2-|A_{00}|^2-|A_0|^2-|A_{\mu\nu0}|^2).$$

Thus the total distribution can be written

$$\left\{(A+A')\frac{1}{W}+(B+B'+B'')+\frac{\mu(A-A')}{m(W_0-W)}+\frac{\mu}{m(W_0-W)W}(B-B'-B'')\right\}$$
$$\times W(W^2-1)^{\frac{1}{2}}(W_0-W)^3\{(W_0-W)^2-\mu^2/m^2\}^{\frac{1}{2}},$$

where $-B\leqslant A\leqslant B$, $-B'\leqslant A'\leqslant B'$, and B, B', $B''\geqslant 0$.

3. Comparison with the experimental results

In this section we give a summary of the electron energy spectra as far as they are similar to the experimental curves.

No derivatives

The only possible distribution is the one given by Fermi, except for the possibility of varying the low-energy end of the distribution.

First derivatives

The terms involving the derivatives of the electron wave function give distributions of the wrong shape. The others, separately, give identical distributions which coincide with that given by Konopinski and Uhlenbeck. If we admit linear combinations of terms which contain only one neutrino derivative, we have to consider the interaction given on p. 281. This gives the total distribution

$$(A+BW)(W^2-1)^{\frac{1}{2}}(W_0-W)^4dW,$$

where $-1\leqslant A/B\leqslant 1$. A, B are certain combinations of the time components of $A_i^{(1)}$, $a_i^{(2)}$....

For a comparison with experiment we use the value $W_0=5{\cdot}7$ extrapolated by fitting a curve known as the K.–U. curve to the results for oxygen by Kurie, Richardson and Paxton‡. The results are expressed by plotting

$$F=N^{\frac{1}{2}}/\{W(W^2-1)^{\frac{1}{2}}\}^{\frac{1}{2}}$$

‡ Kurie, Richardson and Paxton, *Phys. Rev.* 49 (1936), 368.

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against W, where N is the number of electron tracks observed per unit energy range. On this diagram the K.–U. distribution gives a straight line, which is fitted through the point $F=2$, $W=2$ to give the most symmetrical distribution of experimental points about the line. These points lie close to the K.–U. line for $W>2$, but, for $W<2$, the K.–U. distribution predicts too many slow electrons.

In our distribution,

$$F=(AW^{-1}+B)^{\frac{1}{2}}(W_0-W).$$

F was plotted against W for this distribution, putting

$$A=-B, \quad \log B=\bar{1}\cdot 2151, \quad W_0=5\cdot 7.$$

The curve then passes through $F=1\cdot 75$, $W=2\cdot 5$, and leaves the K.–U. line at $W=2$ (cf. Fig. 1). We thus see that it is possible to obtain even fewer slow electrons than are observed.

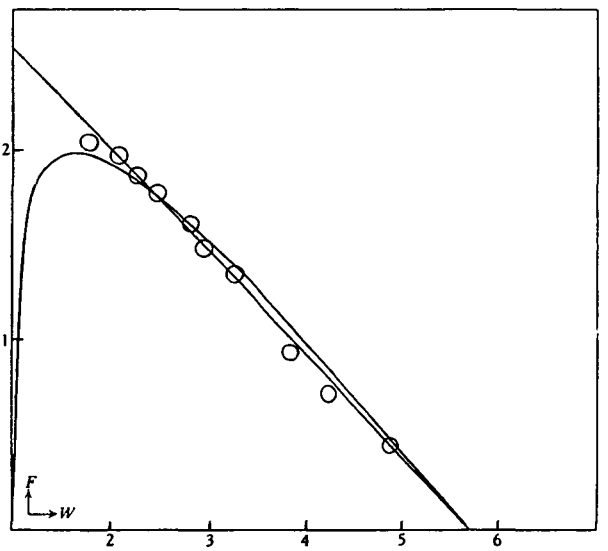


Fig. 1.

$F\equiv\{N/W(W^2-1)^{\frac{1}{2}}\}^{\frac{1}{4}}=(AW^{-1}+B)^{\frac{1}{2}}(W_0-W)$, where $A/B=-1$, $\log B=\bar{1}\cdot 2151$, $W_0=5\cdot 7$. The circles indicate the approximate position of the experimental points.

New results have recently appeared in a paper by Lyman‡. The distribution curve for P^{32} is given, and has a sharp maximum at low energies, while the curvature is large for $W\sim 4$. The K.–U. end-point is 5·10 and the end-point obtained from the experimental curve is 4·33. First derivatives cannot account for this curve unless we assume a neutrino mass, since otherwise the curvature must decrease as the energy increases.

‡ Lyman, *Phys. Rev.* 51 (1937), 1.

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We have seen that the distribution obtained from terms containing one neutrino derivative, and a neutrino mass μ, can be written in the form

$$\left\{(A+A')\frac{1}{W}+(B+B'+B'')+\frac{\mu(A-A')}{m(W_0-W)}+\frac{\mu(B-B'-B'')}{mW(W_0-W)}\right\}$$
$$\times W(W^2-1)^{\frac{1}{2}}(W_0-W)^3\{(W_0-W)^2-\mu^2/m^2\}^{\frac{1}{2}},$$

where $-B\leqslant A\leqslant B$, $-B'\leqslant A'\leqslant B'$. The quantities A, B contain A_0, A_{00}; A', B'

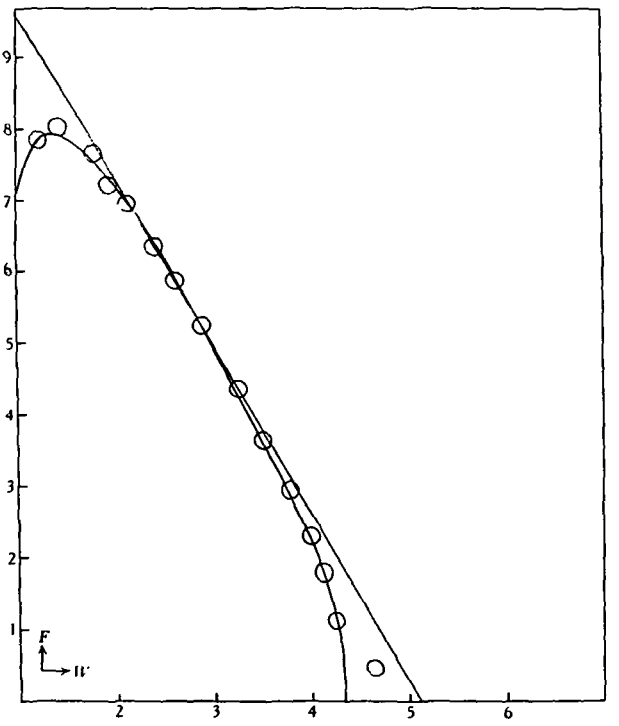


Fig. 2.

$F=\{(AW^{-1}+B)(W_0-W)-\mu/m(A+BW^{-1})\}^{\frac{1}{2}}(W_0-W)^{\frac{1}{2}}\{(W_0-W)^2-\mu^2/m^2\}^{\frac{1}{2}}$,
where $\mu/m=0{\cdot}47$, $W_0=4{\cdot}8$, $A/B=-0{\cdot}8$, $\log B=1{\cdot}8364$.

The circles again indicate the approximate position of a sample of the experimental points.

contain a_0, a_{00}; and B'' contains $A_{\mu\nu 0}$. At low energies the shape of the distribution curve for given μ/m and W_0 depends upon $(A+A')/(B+B'+B'')$, but at high energies it depends upon $(A-A')/(B+B'+B'')$.

At high energies, however, the curve is also considerably affected by our choice of μ/m and W_0.

The experimental data are not sufficiently detailed to justify a discussion of the possible values of $(A-A')/(B+B'+B'')$, μ/m, W_0.

The curve of Fig. 2 was plotted assuming $A'=B'=B''=0$ and $A=-0{\cdot}8B$,

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$\mu = 0{\cdot}47m$, $W_0 = 4{\cdot}8$, and follows Lyman's curve very closely. The high energy divergence from the K.–U. line takes place at $W \sim 3{\cdot}7$. If we decrease the neutrino mass by making W_0 smaller, then the curve will diverge from the K.–U. line for $W < 3{\cdot}7$, where B is again so chosen that the new curve passes through $F = 6{\cdot}15$, $W = 2{\cdot}5$. It is difficult to estimate the limit of agreement between the experimental and theoretical distributions, but it is probable that we have almost reached this limit in the above case. We may regard 0·4 as a suitable lower limit for μ/m. Thus in calculating the distribution curve for some other substance we must take W_0 to be at least 0·4 higher than the observed end-point.

Second-order derivatives

No single second-order term of § 1 will represent the experimental results. The only distributions which cannot be dismissed immediately are

$$W(W^2-1)^{\frac{1}{2}}(W_0-W)^6\,dW, \quad W(W^2-1)^{\frac{1}{2}}(W_0-W)^4\,dW, \quad W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^4\,dW,$$

and these can be shown to differ too widely from the K.–U. line by Fig. 3, since the last two behave similarly for $W > 2{\cdot}5$. However,

$$W(W^2-1)^{\frac{3}{2}}(W_0-W)^4\,dW, \quad W(W^2-1)^{\frac{1}{2}}(W_0-W)^6\,dW$$

give curves of opposite curvature, and by taking two suitable terms we may obtain a resultant distribution curve having small curvature for $W > 2{\cdot}5$. A suitable distribution is given by

$$\psi_s^* u_n^* \mathbf{A}\, v_m \phi_\sigma = \left(A_{ij}\frac{\partial \psi_s^\dagger}{\partial x_i}\frac{\partial \phi_\sigma}{\partial x_j} + B_{ij}\psi_s^\dagger \frac{\partial^2 \phi_\sigma}{\partial x_i \partial x_j}\right),$$

which gives a sum of the distributions

$$W(W^2-1)^{\frac{3}{2}}(W_0-W)^4\,dW, \quad W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^4\,dW, \quad W(W^2-1)^{\frac{1}{2}}(W_0-W)^6\,dW,$$

and the combination distributions

$$W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^5\,dW, \quad (W^2-1)^{\frac{3}{2}}(W_0-W)^5\,dW.$$

This combination is very favourable since both the combination distributions give curves which are of the type we require (cf. Fig. 3). The final distribution involving the coefficients A_i and B_{ij} is given in § 1. Since the velocities of the heavy particles are small compared with that of light, we obtain the total distribution

$$A[a^2W^2 + 2abW(W_0-W) + b^2(W_0-W)^2]\,W(W^2-1)^{\frac{1}{2}}(W_0-W)^4\,dW.$$

This reduces to the K.–U. distribution in the case $a = b$‡.

By making a larger it is possible to obtain fewer slow electrons, but it is found

‡ For $a = b$ the 00 component of the interaction term becomes

$$\text{const} \times \frac{\partial}{\partial t}\left(\psi_s^\dagger \frac{\partial \phi_\sigma}{\partial t}\right).$$

This is W_0 times the K.–U. interaction, and similarly we could introduce $n+1$ derivative such as $\frac{\partial^n}{\partial t^n}\left(\psi_s^\dagger \frac{\partial \phi_\sigma}{\partial t}\right)$.

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that we obtain a gradual deviation from the K.–U. line for $W \sim 2{\cdot}5$. This deviation is not shown by the experimental data, and it is thus not possible to increase a indefinitely. It may, however, just be possible to obtain a curve having a very flat maximum at low energies which does not show a deviation outside the error of experiment.

We may avoid this difficulty, however, by considering a more complicated interaction such as

$$\left|\left(A_{ij}\frac{\partial\psi_s^\dagger}{\partial x_i}\frac{\partial\phi_\sigma}{\partial x_j}+B_{ij}\psi_s^\dagger\frac{\partial^2\phi_\sigma}{\partial x_i\partial x_j}\right)+\left(A_{ijk}\frac{\partial\psi_s^\dagger}{\partial x_i}\gamma_j\frac{\partial\phi_\sigma}{\partial x_j}+B_{ijk}\psi_s^\dagger\gamma_i\frac{\partial^2\phi_\sigma}{\partial x_j\partial x_k}\right)\right|^2.$$

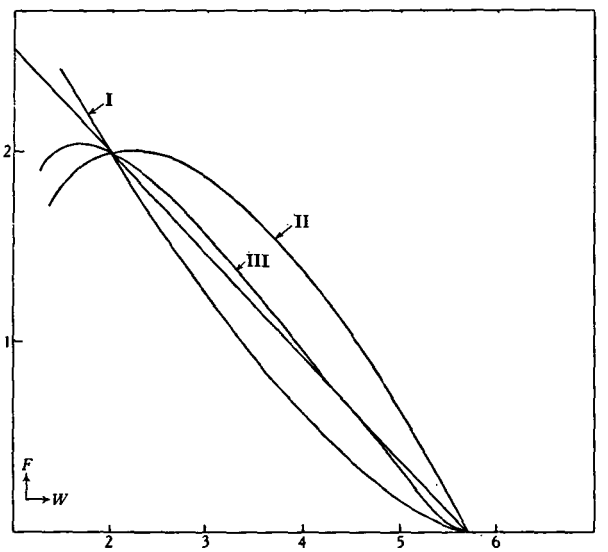


Fig. 3.

I. $F=A(W_0-W)^{\frac{3}{2}}$; $\log A=\bar{3}{\cdot}7949$.

II. $F=B(W^2-1)^{\frac{1}{2}}(W_0-W)$; $\log B=\bar{2}{\cdot}4542$.

III. $F=C(W^2-1)^{\frac{1}{2}}W^{-\frac{1}{2}}(W_0-W)^{\frac{5}{2}}$; $\log C=\bar{2}{\cdot}1870$.

When $W=1$, the value of F is zero for both II and III.

We restrict ourselves to the A_{000}, B_{000} components of A_{ijk}, B_{ijk}, and the direct products then give the same distribution. If we take $A_{00}=\alpha A_{000}$, $B_{00}=\alpha B_{000}$, where α is a constant, and put $a=b$, we obtain exactly as for first derivatives the final distribution

$$(A+BW)(W^2-1)^{\frac{1}{2}}(W_0-W)^4\,dW,$$

which, as we have seen, gives very good agreement with the empirical data at low energies. To give agreement at high energies it is again necessary to introduce a neutrino mass.

All substances examined by Kurie, Richardson and Paxton, with end-points

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higher than $W_0 = 5{\cdot}7$ have distribution curves which can be expressed as the sum of two K.–U. lines, one of high end-point ($> 5{\cdot}7$) and one of low end-point ($< 5{\cdot}7$). The distribution curve for low energies is dominated by the K.–U. line of low end-point. It is clearly not possible to obtain the observed curves with first derivatives as simple spectra. It might appear, however, that with second derivatives it would be possible to obtain the observed curve using only one disintegration energy equal to the high energy end-point. The results for chlorine are given by the addition of the K.–U. lines with end-points 12·8 and 4·2 both passing through $F = 1{\cdot}3$, $W = 3$.

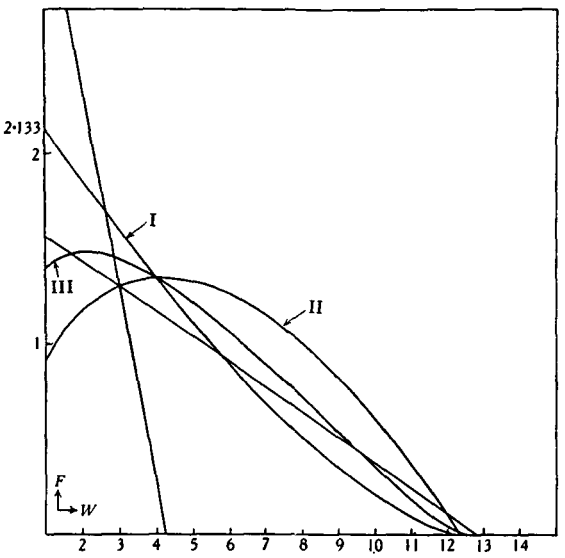


Fig. 4.

I. $F = A(W_0 - W)^{\frac{3}{2}}$; $\log A = \bar{6}{\cdot}9756$.

II. $F = BW^{\frac{1}{2}}(W_0 - W)$; $\log B = \bar{5}{\cdot}6201$.

III. $F = CW^{\frac{1}{2}}(W_0 - W)^{\frac{5}{2}}$; $\log C = \bar{5}{\cdot}2978$.

In Fig. 4, F is plotted against W for each of the distributions

$$W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^4\,dW,\quad W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^5\,dW,\quad W(W^2-1)^{\frac{1}{2}}(W_0-W)^6\,dW.$$

The gradients of these lines are all less than the gradient of the K.–U. line of end-point $W_0 = 4{\cdot}2$. It is therefore impossible to obtain a curve of the required steepness by any combination of these distributions.

We can increase the gradients by taking

$$(1+W^{-1})\,W^3(W^2-1)^{\frac{1}{2}}(W_0-W)^4\,dW,\quad (1-W^{-1})\,W^2(W^2-1)^{\frac{1}{2}}(W_0-W)^5\,dW,$$

$$(1-W^{-1})\,(W^2-1)\,(W_0-W)^6\,dW,$$

that is $A = \pm B$. If these distributions are again made to pass through the point $F = 1{\cdot}35$, $W = 4$ we have the values $2{\cdot}133\,(8/5)^{\frac{1}{2}}$, 0, 0 respectively for $W = 1$. Thus

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these modified distributions still give curves with gradients less steep than the K.–U. line. We conclude that it is not possible with second-order derivatives to predict this type of distribution using only one disintegration energy.

4. Forbidden transitions

Even if we assume a first-order interaction term, the discussion of second-order derivatives becomes of importance for forbidden transitions of the first order.

For such transitions,

$$\int (u_n^* \mathbf{A}'' v_m)\, d\tau = 0,$$

and, if we use a first-derivative form

$$\left| \left(A_i \psi_s^\dagger \frac{\partial \phi_\sigma}{\partial x_i} + B_{ij} \psi_s^\dagger \gamma_i \frac{\partial \phi_\sigma}{\partial x_j} \right) \right|^2,$$

the coefficients are all zero, if we neglect the variation of ψ_s, ϕ_σ over the extension of the nucleus.

We write $$\psi_s^* u_n^* \mathbf{A} v_m \phi_\sigma = \left(\psi_s^\dagger \frac{\partial \phi_\sigma}{\partial x_i} \right) (u_n^* \mathbf{A}_i v_m).$$

Then $$\int \left(\psi_s^\dagger \frac{\partial \phi_\sigma}{\partial x_i} \right) (u_n^* \mathbf{A}_i v_m)\, d\tau$$

is approximately $$\left(\frac{\partial \psi_s^\dagger}{\partial x_j} \frac{\partial \phi_\sigma}{\partial x_i} + \psi_s^\dagger \frac{\partial^2 \phi_\sigma}{\partial x_i \partial x_j} \right) \int x_j (u_n^* \mathbf{A}_i v_m)\, d\tau,$$

provided that

$$\int (u_n^* A_i v_m)\, d\tau = 0 \text{ and } \int x_j (u_n^* A_i v_m)\, d\tau \neq 0 \quad (j = 1, 2, 3).$$

Thus we may write

$$\int \left(\psi_s^\dagger \frac{\partial \phi_\sigma}{\partial x_i} \right) (u_n^* \mathbf{A}_i v_m)\, d\tau = A_{ij} \left(\frac{\partial \psi_s^\dagger}{\partial x_j} \frac{\partial \phi_\sigma}{\partial x_i} + \psi_s^\dagger \frac{\partial^2 \phi_\sigma}{\partial x_i \partial x_j} \right),$$

where the wave functions and their derivatives are evaluated at some point in the nucleus.

Similarly we may write

$$\int \left(\psi_s^\dagger \gamma_i \frac{\partial \phi_\sigma}{\partial x_j} \right) (u_n^* B_{ij} v_m)\, d\tau = B_{ijk} \left(\frac{\partial \psi_s^\dagger}{\partial x_k} \gamma_i \frac{\partial \phi_\sigma}{\partial x_j} + \psi_s^\dagger \gamma_i \frac{\partial^2 \phi_\sigma}{\partial x_j \partial x_k} \right),$$

where $k = 1, 2, 3$ and $i, j = 0, 1, 2, 3$. We obtain exactly the same distribution as the one used above except that $A_{ij} = B_{ij}$, $j \neq 0$; and $A_{ijk} = B_{ijk}$, $k \neq 0$.

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CONCLUSIONS

We see that besides the interaction term of Fermi and the modification suggested by Konopinski and Uhlenbeck there is a large number of possible expressions satisfying the requirements of Lorentz invariance. Of these the terms containing one first-order derivative, in so far as they are at all similar to the empirical curves, differ from the K.–U. distribution only at low energies (cf. Fig. 1). The experimental curves seem to show fewer slow electrons than the K.–U. distribution would predict. This feature could easily be accounted for by our more general expressions, but practically any other behaviour for low energies could be equally well described. If the neutrino has a mass, it is similarly possible to modify the distribution near the upper limit. The very extensive measurements of Lyman seem to suggest that the neutrino has indeed a mass of roughly half the electron mass and we can find a distribution to represent his curve (cf. Fig. 2).

From consideration of the connexion between the β-decay and nuclear forces it appears interesting to see whether one may introduce second-order derivatives*. We see that amongst the many possible terms of the second order we can find combinations which resemble the empirical curves equally well as those with first derivatives. Thus on the evidence of the energy distribution alone second, or even higher, derivatives cannot be ruled out.

I wish to express my thanks to Mr Peierls for introducing me to the problem and for many very helpful discussions.

* Bethe and Bacher, *Rev. Mod. Phys.* 8 (1936), 201.

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