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A NOTE ON INEQUALITY CONSTRAINTS IN THE GARCH MODEL

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We consider the parameter restrictions that need to be imposed to ensure that the conditional variance process of a GARCH(p,q) model remains nonnegative. Previously, Nelson and Cao (1992, *Journal of Business & Economic Statistics* 10, 229–235) provided a set of necessary and sufficient conditions for the aforementioned nonnegativity property for GARCH(p,q) models with $p \leq 2$ and derived a sufficient condition for the general case of GARCH(p,q) models with $p \geq 3$. In this paper, we show that the sufficient condition of Nelson and Cao (1992) for $p \geq 3$ actually is also a necessary condition. In addition, we point out the linkage between the absolute monotonicity of the generalized autoregressive conditional heteroskedastic (GARCH) generating function and the nonnegativity of the GARCH kernel, and we use it to provide examples of sufficient conditions for this nonnegativity property to hold.

1. INTRODUCTION AND MAIN RESULTS

The generalized autoregressive conditional heteroskedastic (GARCH) model (Engle, 1982; Bollerslev, 1986) has become popular for modeling volatility over the past two decades. An important problem of volatility modeling concerns the identification of necessary and sufficient conditions for the model to have nonnegative conditional variances almost surely. Nelson and Cao (1992) derived some necessary and sufficient conditions for the nonnegativity of GARCH(p,q) models with $p \leq 2$ and a sufficient condition for $p > 2$. In this note, we show that the sufficient condition of Nelson and Cao (1992) for $p > 2$ actually is also a necessary condition (see Theorem 1 in this section).

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Requiring the nonnegativity of the conditional variances imposes an infinite number of inequality constraints on the parameters. For practical purposes (e.g., in estimation), it is necessary to reduce this to a finite number of inequalities. In Theorem 1(ii), we derive a set of verifiable necessary and sufficient conditions for the nonnegativity of the GARCH kernel in terms of a finite number of inequalities under the weak condition that the characteristic equation $1 - \beta(z) = 0$ (the definition of $\beta(z)$ is given subsequently) has distinct roots and the root of the smallest magnitude is unique. The equivalence of the absolute monotonicity of the GARCH generating function and the nonnegativity of the GARCH kernel is established in Theorem 2 at the end of this section. In Section 2, we prove Theorem 1, and we apply Theorem 2 to derive some simple sufficient conditions for the nonnegativity of higher order GARCH models from known results of lower order GARCH models.

The GARCH(p,q) model is defined as

$$\epsilon_t = \sigma_t z_t, \tag{1}$$

$$\sigma_t^2 = \omega + \beta(L)\sigma_t^2 + \alpha(L)\epsilon_t^2, \tag{2}$$

where $\{z_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean and unit variance, L is the backshift operator, $\alpha(L) = \sum_{i=1}^{q} \alpha_i L^i$, and $\beta(L) = \sum_{i=1}^{p} \beta_i L^i$. We now make the following assumption.

(A1) All the roots of $1 - \beta(z) = 0$ lie outside the unit circle.

Under Assumption (A1), equation (2) can be rewritten as an ARCH(∞) form:

$$\sigma_t^2 = \omega^* + \Psi(L)\epsilon_t^2, \tag{3}$$

where $\omega^* = \{1 - \beta(1)\}^{-1}\omega$ and

$$\Psi(z) = \sum_{k=1}^{\infty} \psi_k z^k = \frac{\alpha(z)}{1 - \beta(z)}. \tag{4}$$

We also make the following assumption, which is needed for model identifiability; see equation (9) in Nelson and Cao (1992).

(A2) The polynomials $1 - \beta(z)$ and $\alpha(z)$ have no common roots.

For the GARCH(p,q) model to be well defined, having a nonnegative conditional variance almost surely for all t, it is sufficient that $\omega^* \geq 0$, and $\psi_k \geq 0$, for $k = 1, 2, \ldots$. These conditions are also necessary under some mild regularity condition, for example, if the marginal distribution of z_t admits a probability density that is positive everywhere.

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Let $\lambda_j, 1 \le j \le p$, be the roots of $1 - \beta(z) = 0$. With no loss of generality, we can and shall henceforth assume the following convention that

$$|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_p|. \tag{5}$$

Let $B(z) = 1 - \beta(z)$ and $B^{(1)}$ be the first derivative of B; then we have the following result.

THEOREM 1. *Consider a GARCH(p,q) model where $p \ge 2$. Let Assumptions (A1) and (A2) be satisfied. Then the following results hold:*

(i) $\omega^* \ge 0$ *if and only if* $\omega \ge 0$*;*

(ii) Assuming that the roots of $1 - \beta(z) = 0$ *are distinct, and* $|\lambda_1| < |\lambda_2|$*, then conditions (6)–(8) are necessary and sufficient for* $\psi_k \ge 0$ *for all positive integer k:*

$$\lambda_1 \quad \text{is real,} \quad \text{and } \lambda_1 > 1, \tag{6}$$

$$\alpha(\lambda_1) > 0, \tag{7}$$

$$\psi_k \ge 0, \quad \text{for } k = 1, \ldots, k^*, \tag{8}$$

where k^* *is the smallest integer greater than or equal to* $\max\{0, \gamma\}$*, where* $\gamma = \{\log r_1 - \log((p-1)r^*)\}/(\log|\lambda_1| - \log|\lambda_2|)$*,* $r^* = \max_{2 \le j \le p} |r_j|$*, and* $r_j = -\alpha(\lambda_j)/B^{(1)}(\lambda_j)$*,* $1 \le j \le p$*.*

Similar to Tsai and Chan (2007), we can characterize the nonnegativity of $\{\psi_i\}_{i=1}^{\infty}$, the GARCH$(p,q)$ kernel, in terms of its generating function (see Feller, 1968, Ch. XI). For the kernel $\{\psi_j\}$ defined by (3), its generating function is given by equation (4). It is well known that a sequence of numbers is nonnegative if and only if its generating function is absolutely monotonic (Feller, 1971, Ch. VII.2, Thm. 2). See chapter VII of Feller (1971) and chapter IV of Widder (1946) for a review of absolute monotonicity.

Tsai and Chan (2007) exploited some properties of absolutely monotonic functions to derive some necessary and sufficient conditions for an autoregressive moving average (ARMA) model to be nonnegative. Now we state the nonnegativity of $\{\psi_k\}$ in terms of the absolute monotonicity of its generating function in the following theorem.

THEOREM 2. *Let Assumptions (A1) and (A2) be satisfied. Then* $\psi_k \ge 0$ *for all positive integer k if and only if* $\Psi(z) = \{1 - \beta(z)\}^{-1}\alpha(z), 0 \le z < 1$*, is absolutely monotonic.*

2. PROOFS AND DISCUSSION

Proof of Theorem 1. We first prove part (i). By equation (3), $\omega^* = \{1 - \beta(1)\}^{-1}\omega$. Furthermore, condition (A1) on the roots of $1 - \beta(z)$ implies that

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$1 - \beta(1) > 0$. Thus, $\omega^* \geq 0$ if and only if $\omega \geq 0$. This proves part (i). For part (ii), the necessity of (8) is obvious. The necessity of (6) and (7) can be proved as follows. By equations (4.8) and (4.9) of Feller (1968, pp. 276, 277), we have, for $n \geq \max\{p,q\} + 1$, $\psi_n = \sum_{i=1}^{p} r_i \lambda_i^{-n-1} \sim r_1 \lambda_1^{-n-1}$, where $\sim$ means that the ratio of the two sides tends to 1, as $n \to \infty$. Thus, λ_1 must be real and >1. Moreover, $r_1 = -\alpha(\lambda_1)/B^{(1)}(\lambda_1)$ must be ≥ 0. Note also that $-B^{(1)}(\lambda_1) = \prod_{j=2}^{p}(1 - \lambda_1/\lambda_j)/\lambda_1$, and by (5), $-B^{(1)}(\lambda_1) > 0$. Hence, $\alpha(\lambda_1) \geq 0$. But $\alpha(\lambda_1) \neq 0$ by Assumption (A2). This proves the necessity of (6) and (7).

If $\gamma \geq 0$, then the proof of (ii) for the sufficiency of conditions (6)–(8) is given in Nelson and Cao (1992). If γ is negative, then it can be shown that conditions (6) and (7) entail that $\psi_k \geq 0$ for all positive k, as follows. First note that $\gamma < 0$ implies $r_1 > (p - 1)r^*$. Thus, $\psi_n = \sum_{i=1}^{p} r_i \lambda_i^{-n-1} \geq r_1 \lambda_1^{-n-1} - (p - 1)r^*|\lambda_2|^{-n-1}$, for all $n \geq 0$. Therefore, $\lambda_1^{n+1}\psi_n \geq r_1 - (p - 1)r^*|\lambda_1|^{n+1}/|\lambda_2|^{n+1} \geq (p - 1)r^*(1 - |\lambda_1|^{n+1}/|\lambda_2|^{n+1}) \geq 0$. Consequently, $\psi_n \geq 0$ for all $n \geq 0$. This completes the proof of Theorem 1. ■

Several remarks follow. Conditions (6) and (7) can be easily checked, whereas condition (8) reduces an infinite number of inequality constraints to a finite number of inequalities. The assumption that $|\lambda_1| < |\lambda_2|$ is needed for the constant $\{\log r_1 - \log((p - 1)r^*)\}/(\log|\lambda_1| - \log|\lambda_2|)$ to be well defined. The distinct-root assumption for $1 - \beta(z)$ is not needed for the necessity of conditions (6) and (7). This is because ψ_k in (4) admits an integral representation:

$$\psi_k = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Psi(w)}{w^{k+1}}\, dw, \qquad k = 0, 1, \ldots,$$

where Γ is some circle on the complex plane that centers at zero and of a positive radius >1; cf. p. 83 of Titchmarsh (1939). Thus, the ARCH(∞) representation (3) of σ_t^2 is continuous with respect to the parameters α's and β's, subject to Assumptions (A1) and (A2), and hence the desired results upon passing to the limit. Moreover, except for a parametric set of zero Lebesgue measure, the roots of $1 - \beta(z) = 0$ are distinct, and $|\lambda_1| < |\lambda_2|$ (see also Tsai and Chan, 2007, Thm. 1(c)).

Theorem 2 and the fact that the product of two absolutely monotonic functions is again absolutely monotonic (Widder, 1946, Thm. 2a, p. 145) can be used to construct simple sufficient conditions for higher order GARCH(p,q) models from known results of GARCH($1,q$) and GARCH($2,q$) models.

Example 1

If the ARCH coefficients (α's) of a GARCH(p,q) model are all nonnegative, the model has nonnegative conditional variances if the nonnegativity property holds for the associated GARCH($p,1$) models.

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Example 2

Consider a GARCH(4,1) model for which $\alpha_1 \geq 0$, λ_1 and λ_4 are real numbers, $\lambda_4 > 1$, $\lambda_2 = a + bi$, where a and b are real numbers, $a \geq \lambda_1 > 1$, and λ_3 is the complex conjugate of λ_2. Then by Theorem 3(a), (b), and (d) of Tsai and Chan (2006), $\{\psi_i\}_{i=1}^{\infty}$ is nonnegative for this particular GARCH(4,1) model.

Example 3

Consider a GARCH(3,3) model with $1 - \beta(z) = (1 - \beta_{2,1}z)(1 - \beta_{1,1}z - \beta_{1,2}z^2)$ and

$$\frac{\alpha(z)}{1-\beta(z)} = z\,\frac{\alpha_{1,1} + \alpha_{1,2}z}{(1 - z/\lambda_1)(1 - z/\lambda_2)}\,\frac{\alpha_{2,1} + \alpha_{2,2}z}{1 - z/\lambda_3}.$$

Now, consider the following two conditions: (i) $|\lambda_1| \leq |\lambda_2|$, $\lambda_1 > 1$, $\alpha_{1,1} + \alpha_{1,2}\lambda_1 > 0$, $\alpha_{1,1} \geq 0$, and $\alpha_{1,2} + \alpha_{1,1}\beta_{1,1} \geq 0$, (ii) $\lambda_3 > 1$, $\alpha_{2,1} + \alpha_{2,2}\lambda_3 > 0$, $\alpha_{2,1} \geq 0$, and $\alpha_{2,2} + \alpha_{2,1}\beta_{2,1} \geq 0$. If conditions (i) and (ii) are satisfied, then by Theorems 1 and 2 of Nelson and Cao (1992) and the aforementioned result of Widder (1946, Thm. 2a, p. 145), $\{\psi_i\}_{i=1}^{\infty}$ is nonnegative for the GARCH(3,3) model.

These examples complement the well-known necessary and sufficient condition for the nonnegativity of GARCH$(1,q)$ and GARCH$(2,q)$ models obtained by Nelson and Cao (1992).

Recently, Conrad and Haag (2006) derived necessary and sufficient conditions for the nonnegativity of the conditional variance in the fractionally integrated generalized autoregressive conditional heteroskedastic (FIGARCH) (p,d,q) model of the order $p \leq 2$ and two sets of sufficient conditions for $p \geq 3$. The second set of sufficient conditions stated in Theorem 4 of Conrad and Haag (2006) is analogous to the sufficient condition of Nelson and Cao (1992) for the GARCH(p,q) model. It might be possible to adopt the idea of the proofs of Theorem 1 to show that the sufficient condition stated in Theorem 4 of Conrad and Haag (2006) is also a necessary condition for the FIGARCH (p,d,q) model, which is an interesting future research problem.

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