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Non-K-exact uniform Roe C*-algebras

by

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Abstract

We prove that uniform Roe C*-algebras $C^*_u X$ associated to some expander graphs X coming from discrete groups with property (τ) are not K-exact. In particular, we show that this is the case for the expander obtained as Cayley graphs of a sequence of alternating groups (with appropriately chosen generating sets).

Key Words: uniform Roe C*-algebras, K-exactness, expanders
Mathematics Subject Classification 2000: Primary 46L80

1. Introduction

Uniform Roe C*-algebras (also called uniform translation C*-algebras) provide, among other things, a link between coarse geometry and C*-algebra theory via the following theorem, which connects a coarse–geometric property of a discrete group Γ with a purely analytic property of its reduced C*-algebra $C^*_r\Gamma$:

Theorem 1 (Guentner–Kaminker [5] and Ozawa [12]) *Let Γ be a finitely generated discrete group. Then the following are equivalent:*

- *Γ has property A (see [23]),*
- *the reduced group C*-algebra $C^*_r\Gamma$ is exact,*
- *the uniform Roe C*-algebra $C^*_u|\Gamma|$ is nuclear.*

The only known examples of groups which do not have property A are Gromov's random groups [4]. The theorem also characterizes nuclearity of uniform Roe C*-algebras of discrete groups.

More generally, one may ask if an analogue of the above theorem for general bounded geometry metric spaces X is true, i.e. if property A of X is equivalent to nuclearity of $C^*_u X$. One proof was obtained by representing $C^*_u X$ as a groupoid C*-algebra (see [15]) and referring to the general groupoid C*-algebra theory (see [1]). For a more elementary argument for one direction see [14].

In a further attempt to generalize, we may ask about a K-theoretic analogue of the C*-algebraic property of nuclearity. Namely, we may seek some (coarse)

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geometric conditions of X that would imply K-nuclearity of uniform Roe C*-algebras. The first step beyond the realm of property A is coarse embeddability into a Hilbert space. Using groupoid language, Ulgen proved that if X admits a coarse embedding into a Hilbert space, then C_u^*X is K-nuclear (see [19, subsection 3.2]). The argument uses the fact that groupoids with the Haagerup property are K-amenable, and that crossed products of K-nuclear algebras with K-amenable groupoids are again K-nuclear (see [16] and [17]).

On the other side, we may look for examples of spaces, whose uniform Roe C*-algebras are not nuclear. An unpublished result of Higson asserts that uniform Roe C*-algebras of expander graphs constructed from groups with property (T) are not exact, and therefore not nuclear.

Wandering into K-theoretic territory, we can ask for examples of spaces, whose uniform Roe C*-algebras are not K-nuclear. Ulgen defined K-exactness in [19] as a generalization of exactness in the context of K-theory. She also proved that if a separable C*-algebra is K-nuclear, then it is also K-exact. Unfortunately, this cannot be applied to uniform Roe C*-algebras, which are usually not separable. In this paper, we show that for some spaces X, C_u^*X is not even K-exact. Our examples are expander graphs, constructed out of groups with property (τ) with respect to a family of subgroups $\mathscr{L}$ (see [9]). A certain assumption on the family $\mathscr{L}$ is required at this point:

Theorem 2 *Let Γ be a finitely generated discrete group with property $\tau(\mathscr{L})$. Assume that*

($\star$) Γ has $\tau(\mathscr{L}')$, where $\mathscr{L}' = \{N_1 \cap N_2 \mid N_1, N_2 \in \mathscr{L}\}$.

Then C_u^*X is not K-exact, where $X = \sqcup_{N\in\mathscr{L}}\Gamma/N$.

Using results of Kassabov [7], we obtain the following corollary:

Corollary 3 *There is a sequence $(n_i)_{i\in\mathbb{N}}$, such that the uniform Roe C*-algebra of the expander obtained as a coarse disjoint union of Cayley graphs of the alternating groups* $\mathrm{Alt}(n_i)$ *(with appropriately chosen generating sets) is not K-exact.*

The question of K-exactness for uniform Roe C*-algebras of expander graphs is closely related to the same question for C*-algebras of the type $\prod_q \mathcal{M}_q(\mathbb{C})$. This has been settled negatively in various contexts by Ozawa [13] and by Manuilov–Thomsen [11]. Both constructions extend the work of Wassermann [22].

Projections in (uniform) Roe C*-algebras similar to the one that is used in the construction in this paper are called *ghosts* and were studied in the context of the ideal structure of Roe C*-algebras [2, 21].

The paper is organized as follows: In the next section, we recall the definitions of properties and objects involved. Section 3 is devoted to the proof of Theorem 2

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and in the last section we show how to deduce Corollary 3.

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2. Definitions

2.1. K-exactness

Recall that a C*-algebra A is exact, if $\cdot \otimes_{\min} A$ is an exact functor, i.e. if we min–tensor every term in a short exact sequence with A, the sequence stays exact. If it does, then we obtain a 6-term exact sequence in K-theory (as below). It may happen that the tensored short exact sequence is not exact for a non–exact C*-algebra, but there is always an exact 6-term sequence in K-theory. Let us be more precise:

Proposition 4 ([18, Theorem 2.3.2]) *For a C*-algebra A, the following are equivalent:*

- *for any exact sequence of C*-algebras $0 \to I \to B \to B/I \to 0$, there is a cyclic 6-term exact sequence in K-theory:*

$$\begin{array}{ccccc} K_0(I \otimes_{\min} A) & \longrightarrow & K_0(B \otimes_{\min} A) & \longrightarrow & K_0(B/I \otimes_{\min} A) \\ \uparrow & & & & \downarrow \\ K_1(B/I \otimes_{\min} A) & \longleftarrow & K_1(B \otimes_{\min} A) & \longleftarrow & K_1(I \otimes_{\min} A). \end{array}$$

  (Note that in general, there might be no such 6-term K-theory sequence at all.),

- *for any exact sequence of C*-algebras $0 \to I \to B \to B/I \to 0$, the sequences*

$$K_i(I \otimes_{\min} A) \to K_i(B \otimes_{\min} A) \to K_i(B/I \otimes_{\min} A),$$

  are exact in the middle for both $i = 0, 1$.

Definition 5 We say that a C*-algebra A is *K-exact*, if it satisfies the conditions in the previous proposition.

For separable C*-algebras, a sufficient condition for K-exactness is K-nuclearity. The argument [19, Proposition 3.3] can be summarized as follows: the

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max–tensor product always preserves exact sequences, and if a C*-algebra is K-nuclear, then min–tensor products and max–tensor products with it are KK–equivalent. However, this relies on the key properties of KK-theory (existence and associativity of Kasparov product), which have been proved only for separable C*-algebras in general.

2.2. *Uniform Roe C*-algebras*

A countable metric space X has bounded geometry, if for each $r > 0$ the number of points in any ball of radius r is uniformly bounded. We say that X is uniformly discrete, if there exists $c > 0$, such that any two distinct points of X are at least c apart.

Definition 6 Let X be a uniformly discrete metric space with bounded geometry. We say that an X-by-X matrix $(t_{yx})_{x,y\in X}$ with complex entries has *finite propagation*, if there exists $R \geq 0$, such that $t_{yx} = 0$ whenever $d(x,y) \geq R$. We say that such a matrix is *uniformly bounded*, if there exists $T \geq 0$, such that $|t_{yx}| \leq T$ for all $x, y \in X$.

Let $A(X)$ be the algebra of all finite propagation matrices which are uniformly bounded. It is easy to see that each element of $A(X)$ represents a bounded operator on $\ell^2(X)$ (see [14, lemma 4.27]). This yields a representation $\lambda : A(X) \to \mathscr{B}(\ell^2(X))$.

Definition 7 The *uniform Roe C*-algebra* $C^*_u X$ of X is defined to be the norm closure of $\lambda(A(X)) \subset \mathscr{B}(\ell^2(X))$.

2.3. *Expanders and property* (τ)

Definition 8 An *expander* is a sequence X_n of finite graphs with the properties:

- The maximum number of edges emanating from any vertex is uniformly bounded.
- The number of vertices of X_n tends to infinity as n increases.
- The first nonzero eigenvalue of the Laplacian, $\lambda_1(X_n)$, is uniformly bounded away from zero, say by $\lambda > 0$.

We think of X_n as a discrete metric space, where the points are vertices of the graph, and the metric is given by the path distance in the graph. We understand the sequence as one metric space $\sqcup_n X_n$ via the coarse disjoint union construction.

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Let us recall one possibility of how to construct a coarse disjoint union of finite spaces: Given a sequence $(X_q)_{q\in\mathbb{N}}$ of finite metric spaces, we define their coarse disjoint union $\sqcup_q X_q$ to be the set $\cup_q X_q$ endowed with the metric inherited from individual X_q's together with the condition $d(X_q, X_{q'}) = \max(q, q')$ for $q \neq q'$.

The first explicit examples of expanders were constructed by Margulis as $\sqcup_q \Gamma/\Gamma_q$, where Γ is a finitely generated group with property (T) (with a fixed generating set), and $\Gamma_q \leq \Gamma$ is a decreasing sequence of normal subgroups with finite index, such that $\bigcap_q \Gamma_q = \{1\}$. This construction eventually led to Lubotzky's property (τ) [9].

Definition 9 Let Γ be a finitely generated group and $\mathscr{L}$ a countable family of finite index normal subgroups of Γ. We also assume that $\mathscr{L}$ is infinite, and that $[\Gamma : N] \to \infty$ as $N \to \infty$[2], $N \in \mathscr{L}$. We say that Γ has *property (τ) with respect to the family $\mathscr{L}$* (written also $\tau(\mathscr{L})$) if the trivial representation is isolated in the set of all unitary representations of Γ, which factor through Γ/N, $N \in \mathscr{L}$. We say that Γ has *property (τ)* if it has this property with respect to the family of all finite index normal subgroups.

Property $\tau(\mathscr{L})$ is equivalent to $\sqcup_{N\in\mathscr{L}} \Gamma/N$ being an expander [9, Theorem 4.3.2].

3. Proof of Theorem 2

We describe the construction starting with Γ, a finitely generated discrete group and a countable family $\mathscr{L}$ of normal subgroups of Γ with finite index. We also fix a finite symmetric generating set S of Γ.

For each $N \in \mathscr{L}$, we denote $G_N = \Gamma/N$ and by $q_N : \Gamma \to G_N$ the quotient map. We let X to be a coarse disjoint union of the Cayley graphs of G_N's with respect to the generating sets $\{q_N(g) \mid g \in S\}$.

Let $\lambda_N : G_N \to \mathscr{B}(\ell^2 G_N)$ be the left regular representation of G_N. Denote also $\tilde{\lambda}_N = \lambda_N \circ q_N : \Gamma \to \mathscr{B}(\ell^2 G_N)$ and $\Lambda = \oplus_{N\in\mathscr{L}} \tilde{\lambda}_N : \Gamma \to \mathscr{B}(\ell^2 X)$.

Claim 1 For each $N \in \mathscr{L}$, we can choose an irreducible representation $\pi_N : G_N \to \mathscr{B}(H_N)$, so that

$(\star\star)$ $\dim(H_N) \to \infty$ as $N \to \infty$.

Proof: We shall use the fact that if Γ has $\tau(\mathscr{L})$, then for each fixed $d > 0$, there are only finitely many non-equivalent irreducible d'-dimensional $(d' \leq d)$ representations of Γ factoring through some G_N, $N \in \mathscr{L}$. The same conclusion

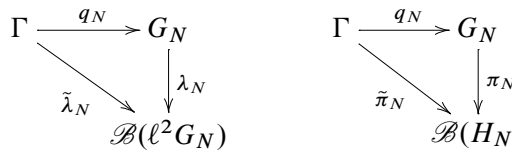
[2]By $N \to \infty$ we mean "N leaves each finite subset of $\mathscr{L}$".

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is known for groups with property (T) [20, 3], the argument for groups with (τ) is outlined also in [10, Theorem 3.11].

Now assume that the claim doesn't hold, that is, there is a sequence $N_n \in \mathscr{L}$, such that all irreducible representations of G_{N_n}'s are at most d-dimensional. By the above fact, all of them come from a finite set of irreducible representations $\{\rho_1,\dots,\rho_m\}$ of Γ. Consequently, each G_{N_n} embeds as a subgroup of $K = \text{image}(\rho_1 \oplus \cdots \oplus \rho_m)$. Since every ρ_i factors through some G_{N_k}, its image is a finite group. Hence K is finite, so we obtain a contradiction with the assumption that $|G_{N_n}| \to \infty$ as $n \to \infty$. □

Denote $\tilde{\pi}_N = \pi_N \circ q_N$, $H = \oplus_{N\in\mathscr{L}} H_N$ and $\pi = \oplus_{N\in\mathscr{L}} \tilde{\pi}_N : \Gamma \to \mathscr{B}(H)$. Let us summarize the notation in the following diagram:



Finally, we let $B = \prod_{N\in\mathscr{L}} \mathscr{B}(H_N)$ and $J = \oplus_{N\in\mathscr{L}} \mathscr{B}(H_N)$. We obtain an exact sequence of C*-algebras

$$0 \to J \to B \to B/J \to 0.$$

We shall use this sequence to show that $C^*_u X$ is not K-exact. We construct a projection $e \in C^*_u X \otimes B$, whose K_0-class will violate the exactness of the K-theory sequence

$$K_0(C^*_u X \otimes J) \to K_0(C^*_u X \otimes B) \to K_0(C^*_u X \otimes (B/J)).$$

To construct such e, we let

$$T = \frac{1}{|S|} \sum_{g\in S} (\Lambda \otimes \pi)(g) \in C^*_u X \otimes B \subset \mathscr{B}(\ell^2 X \otimes H).$$

A few remarks are in order:

- If we denote $s = \frac{1}{|S|}\sum_{g\in S} g \in \mathbb{C}\Gamma$, then $T = (\Lambda \otimes \pi)(s)$.
- T is "diagonal" with respect to the decomposition

$$\ell^2 X \otimes H = \oplus_{M,N\in\mathscr{L}} \ell^2 G_N \otimes H_M.$$

  This is clear from the fact that each $\ell^2 G_N \otimes H_M$ is a $\Lambda \otimes \pi$–invariant subspace. We denote its "entries" by $T_{NM} \in \mathscr{B}(\ell^2 G_N \otimes H_M)$.

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- With our choice of metric on X, $\Lambda(g) \in \mathscr{B}(\ell^2 X)$ has propagation 1 for each $g \in S$.

The construction is finished by proving three claims, which we state and give some remarks about them. The proofs are spelled out afterward.

Claim 2 $1 \in \operatorname{spec}(T)$ is an isolated point.

For proving this, we need to use some form of property τ. The necessary condition for Claim 2 is that Γ has $\tau(\mathscr{L})$. However, this by itself is not sufficient, since the Claim requires a uniform bound on the spectral gap for all $\lambda_N \otimes \pi_M$, not just λ_N's. The condition $(\star)$ ensures this.

Claim 2 allows us to define the projection $e \in C_u^* X \otimes B$ to be the spectral projection of T corresponding to $1 \in \operatorname{spec}(T)$. Note that e is also "diagonal" as T, hence we can decompose it into projections $e_{NM} \in \mathscr{B}(\ell^2 G_N \otimes H_M)$. It is clear from the definition of T that each e_{NM} is in fact the projection onto the subspace of Γ-invariant vectors in $\ell^2 G_N \otimes H_M$. In fact, if Γ has property (T), then e is the image of the Kazhdan projection $p_0 \in C^*_{\max}\Gamma$ under $\Lambda \otimes \pi$.

Claim 3 e maps to $0 \in C_u^* X \otimes (B/J)$.

The key observation here is that if for a fixed N, e_{NM} are eventually 0, then Claim 3 holds. This is where the condition $(\star\star)$ is used.

Claim 4 $[e] \in K_0(C_u^* X \otimes B)$ does not come from any class in $K_0(C_u^* X \otimes J)$.

This claim is proved by "detecting the diagonal" of e. More precisely, observe that $e_{NN} \neq 0$ for every $N \in \mathscr{L}$, since π_N is conjugate to a subrepresentation of λ_N and hence $\lambda_N \otimes \pi_N$ has nonzero invariant vectors. However, any element coming from $C_u^* X \otimes J$ will have the NN-entries eventually 0. The construction of a *-homomorphism that detects this is essentially due to Higson [6].

Proof of Claim 2: Taking $N \in \mathscr{L}$, the G_N-action on $\ell^2 G_N \otimes H_N$ is via $\lambda_N \otimes \pi_N$. This representation contains the trivial representation (since λ_N contains the conjugate of π_N, as it does any irreducible representation of G_N), so there are nonzero G_N-invariant vectors in $\ell^2 G_N \otimes H_N$. Therefore, $1 \in \operatorname{spec}(T_{NN})$.

To show that 1 is actually isolated in each $\operatorname{spec}(T_{NM})$ with the uniform bound on the size of the gap, we shall use the condition $(\star)$. Property $\tau(\mathscr{L}')$ says that we have such a uniform bound on the size of the spectral gap of the image of $\rho(T)$ for all the representations ρ of Γ which factor through some of G_L, $L \in \mathscr{L}'$ [9, Theorem 4.3.2]. Using that $\tilde{\pi}_M$ is contained in $\tilde{\lambda}_M$, we obtain

$$\ker(\tilde{\lambda}_N \otimes \tilde{\pi}_M) \supseteq \ker(\tilde{\lambda}_N) \cap \ker(\tilde{\pi}_M) \supseteq \ker(\tilde{\lambda}_N) \cap \ker(\tilde{\lambda}_M) = N \cap M.$$

This shows that $\tilde{\lambda}_N \otimes \tilde{\pi}_M$ factors through $\Gamma/(N \cap M)$ and the proof is finished. □

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Proof of Claim 3: Denote $A = \prod_{N\in\mathscr{L}} \mathscr{B}(\ell^2 G_N)$, a product of matrix algebras. It is clear that $T \in (C_u^* X \otimes B) \cap (A \otimes B)$, and so also $e \in A \otimes B \subset \mathscr{B}(\ell^2 X \otimes H)$.

For $N \in \mathscr{L}$, let us examine the $\mathscr{B}(\ell^2 G_N) \otimes \mathscr{B}(H)$–component of e. Denote by $P_N \in \mathscr{B}(\ell^2 X)$ the projection onto $\ell^2 G_N$. It suffices to show that $e_N = (P_N \otimes 1_B)e(P_N \otimes 1_B) \in \mathscr{B}(\ell^2 G_N \otimes H)$ actually belongs to $\mathscr{B}(\ell^2 G_N) \otimes J$, since then e maps to 0 in $\prod_{N\in\mathscr{L}} \big[\mathscr{B}(\ell^2 G_N) \otimes (B/J)\big]$, into which $A \otimes (B/J)$ embeds.

Further decompose e_N into $e_{NM} \in \mathscr{B}(\ell^2 G_N \otimes H_M)$. Recall that $e_{NM} \neq 0$ if and only if $\ell^2 G_N \otimes H_M$ has nonzero invariant vectors. Since the representation $\tilde{\pi}_M$ is irreducible, this is further equivalent to $\tilde{\pi}_M$ being conjugate to a subrepresentation of $\tilde{\lambda}_N$. But by $(\star\star)$, this can only happen for finitely many M's, since $\tilde{\lambda}_N$ is fixed and $\dim(H_M) \to \infty$. □

Proof of Claim 4: For $N \in \mathscr{L}$, denote $C_N = \mathscr{B}(\ell^2 G_N \otimes H_N)$. We construct a *-homomorphism $f : C_u^* X \otimes B \to \prod_N C_N \big/ \oplus_N C_N$, such that $f_*([e]) \neq 0 \in K_0(\prod_N C_N \big/ \oplus_N C_N)$, but $f_*([x]) = 0$ for any $[x] \in K_0(C_u^* X \otimes J)$.

We first embed $C_u^* X$ into a direct limit of C*-algebras A_k, defined below. We enumerate $\mathscr{L} = \{N_k \mid k \in \mathbb{N}\}$ and put $A^q = \mathscr{B}(\ell^2 G_{N_q})$, $A^{0k} = \mathscr{B}(\ell^2(\sqcup_{q\leq k} G_{N_q}))$ and finally $A_k = A^{0k} \oplus \prod_{q>k} A^q$, $k \geq 1$. There are obvious inclusion maps $A_k \hookrightarrow A_l$ for $k < l$, so we can form a direct limit $A_0 = \lim_k A_k$. It follows from the condition on distances $d(G_{N_q}, G_{N_p})$ that each finite propagation operator on $\ell^2 X$ is a member of some $A_k \subset \mathscr{B}(\ell^2 X)$, hence $C_u^* X \hookrightarrow A_0$.

For $k \geq 1$, denote $B^k = \mathscr{B}(H_{N_k})$ (so that $B = \prod_{k\in\mathbb{N}} B^k$) and define f_k as the following composition:

$$A_k \otimes B = \Big(A^{0k} \oplus \prod_{q>k} A^q\Big) \otimes B \hookrightarrow \Big(A^{0k} \otimes B\Big) \oplus \prod_{q>k} \left([A^q \otimes B^q] \oplus \Big[A^q \otimes \Big(\prod_{p\neq q} B^p \Big) \Big] \right) \xrightarrow{\text{proj}}$$
$$\xrightarrow{\text{proj}} \prod_{q>k} A^q \otimes B^q = \prod_{q>k} C_{N_k} \xrightarrow{\text{quot}} \prod_q C_{N_q} \Big/ \sum_q C_{N_q}.$$

It is easy to see that f_k's commute with inclusions $A_k \otimes B \hookrightarrow A_l \otimes B$, $k < l$. Consequently, we obtain a *-homomorphism $f : C_u^* X \otimes (B \rtimes \Gamma) \to \prod_N C_N \big/ \oplus_N C_N$.

It is known that $K_0(\prod_N C_N \big/ \oplus_N C_N)$ embeds into $\prod_N \mathbb{Z} \big/ \oplus_N \mathbb{Z}$. Examining the construction of f, we see that $f_*([e])$ is the class of the sequence $k \mapsto \operatorname{rank}(e_{N_k N_k})$ in $\prod_{N_k} \mathbb{Z} \big/ \oplus_{N_k} \mathbb{Z}$. As already noted, every term of this sequence is nonzero.

On the other hand, any projection $p \in C_u^* X \otimes J$ has only finitely many nonzero C_N-components, and so $f_*([p]) = 0$ in $\prod_N \mathbb{Z} \big/ \oplus_N \mathbb{Z}$. This obviously extends to the whole of $K_0(C_u^* X \otimes J)$. □

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4. Expanders coming from finite groups

Start with a sequence $G_n = \langle S_n \rangle$, $n \in \mathbb{N}$, of finite groups with chosen generating sets S_n of some fixed size. Let Γ be a subgroup of $G = \prod_{n\in\mathbb{N}} G_n$ generated by a finite set $S \subset G$ that projects onto S_n in each factor. Denote by $q_n : \Gamma \to G_n$ the natural projection, $N_n = \ker(q_n) \subset \Gamma$ and $\mathscr{L} = \{N_n \mid n \in \mathbb{N}\}$.

Now Γ has $\tau(\mathscr{L})$ if and only if the Cayley graphs of G_n's with respect to S_n's constitute an expander. In order to apply Theorem 2, we need to verify the condition $(\star)$. It seems to be open whether $\tau(\mathscr{L})$ implies $(\star)$ in general. For instance, it would be sufficient to know a positive answer to [10, Question 1.14], that is, whether $\tau(\mathscr{L})$ implies $\tau(\mathscr{L}'')$, where $\mathscr{L}''$ is the closure of $\mathscr{L}$ under finite intersections. See also [8, Question 6] for a discussion when there is a finitely generated dense $\Gamma \subset \prod_{n\in\mathbb{N}} G_n$ which has property (τ) (with respect to all finite index subgroups).

To prove Corollary 3, we appeal to the result of Kassabov [7, p. 352], which says that there is a finitely generated dense subgroup $\Gamma \subset \prod_s \mathrm{Alt}\big((2^{3s}-1)^6\big)$ which has property (τ). Hence taking a finite symmetric generating set of Γ projects to generating sets of the individual factors, making their Cayley graphs into an expander. Moreover, the condition $(\star)$ is obviously satisfied. Consequently, we have shown Corollary 3.

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