*J. Aust. Math. Soc.* **120** (2026), 325–372
doi:10.1017/S1446788725101377

# A PANORAMIC VIEW OF GROUPOIDS AND MRAs

MARIUS IONESCU and PAUL MUHLY

(Received 3 May 2025; accepted 5 November 2025; first published online 12 January 2026)

Communicated by Aidan Sims

*To the memory of Iain Raeburn who was a dear friend, a wonderful collaborator, and an inspiring mathematician to all who knew him*

**Abstract**

The primary objective of this paper is to show how the theory of groupoids and their $C^*$-algebras provide new proofs and extensions of Mallat's famous theorems on the construction of multiresolution analyses that form the basis of wavelet theory. This work was inspired in large part by the research of Iain Raeburn and co-authors.

2020 *Mathematics subject classification*: primary 42C40, 46L55.

*Keywords and phrases*: groupoids, $C^*$-algebras, wavelets, multi-resolution analyses, irreversible dynamical systems, induced representations.

## 1. Introduction

In [17], we began to investigate how groupoids might be used to study wavelets. We were inspired by the interesting mix of harmonic analysis with the 'Cuntz Relations' that had appeared recently in the literature (see Ola Bratteli and Palle Jorgensen's important book [7] and paper [6] and the references cited therein), but we were also discomfited by a lack of evident unifying structure that behaved with some semblance of functoriality. The relations among the parameters involved were unclear to us and needed clarification. The pioneering paper of Stephane Mallat [23] offered (and continues to offer) inspiration on how to develop a functorial setting for wavelets and fractal analysis. So, we begin there.

Recall the definition of a multiresolution analysis (MRA) that forms the foundation of the subject.

DEFINITION 1.1. An MRA is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ such that:

(MRA1) $V_j \subset V_{j+1}, j \in \mathbb{Z}$;
(MRA2) $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $L^2(\mathbb{R})$;

© The Author(s), 2026. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.



https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

(MRA3) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$;
(MRA4) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(MRA5) if $f(x) \in V_j$, then $f(x - 2^{-j}k) \in V_j$ for all $k \in \mathbb{Z}$; and
(MRA6) translation by $\mathbb{Z}$ on $\mathbb{R}$ restricted to $V_0$ is unitarily equivalent to translation by $\mathbb{Z}$ on $\ell^2(\mathbb{Z})$.

To the uninitiated, it may be surprising that MRAs exist. In fact, they exist in superabundance. The following theorem, which is distilled from [23, Theorems 1 and 2] begins to exhibit what lies at the heart of the proof of the existence of an MRA.

THEOREM 1.2. *Each MRA may be constructed from a function, $H$, on $\mathbb{T}$ that satisfies:*

(1) $|H(1)| = 1$;
(2) $|H(z)|^2 + |H(-z)|^2 = 1$ *for all* $z$;
(3) $H(z) \neq 0$, *when* $\Re(z) > 0$.

*The correspondence between such functions and MRAs is essentially bijective.*

Each such $H$ is called a *Quadrature Mirror Filter (QMF)*. Of course, it may seem that we have just shifted one difficult problem to another. In a sense, we have. However, the theorem gives an important change of focus.

A brief outline of the proof reveals the relation between MRAs and QMFs. Given $H$, the product $\Pi_{k=1}^{\infty} H(e^{2\pi i(2^{-k}\omega)})$, $\omega \in \mathbb{R}$, converges almost everywhere to a function, $\widehat{\phi}$, in $L^2(\mathbb{R})$, whose Fourier transform, $\phi$, lies in the space $V_0$ of an MRA and the sequence $\{\sqrt{2^j}\phi(2^j x - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $V_j$ for each $j$. Here, $\phi$ is called the *scaling function* for the MRA. Conversely, given an MRA, Item (MRA6) guarantees the existence of a scaling function, whose Fourier transform, $\widehat{\phi}$, satisfies

$$\widehat{\phi}(2\omega) = M(\omega)\widehat{\phi}(\omega), \quad \omega \in \mathbb{R}, \tag{1-1}$$

where $M$ is a $2\pi$-periodic function on $\mathbb{R}$. This $M$ passes to a QMF $H$ on $\mathbb{T}$.

The structure of an $H$ is revealed through its interaction with the irreversible dynamical system on $\mathbb{T}$

$$z \to z^2 \quad z \in \mathbb{T}.$$

In many, if not all, of the myriad variations on, or generalizations of, Mallat's analysis, there are local homeomorphisms in the background that play fundamental roles similar to that of $z \to z^2$. We are thus led to study these, which we do through the lens of groupoid harmonic analysis that is focused on the *Deaconu–Renault groupoid* of a local homeomorphism.

Apart from this introduction, our paper is divided into 12 sections, and these, in turn, are partitioned into three parts. Part I consists of Sections 2–4, in which we develop some general background about groupoid harmonic analysis. The key feature about groupoids that we emphasize is that they provide useful coordinates for $C^*$-algebras of interest. Further, algebraic operations on groupoid $C^*$-algebras are often mirrored directly in the groupoids. In particular, Rieffel's theories of induced

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

representations and Morita equivalence are reflected in the notion of equivalence of groupoids first developed in [25]. Further, still, what proves especially important for this paper is the recent contribution of Rohit Dilip Holkar [14] in which he advances the notion of a *topological correspondence* and the attendant concept of an *adjoining function*. The reader should bear in mind that $C^*$-algebras form a category in which a morphism from a $C^*$-algebra $A$ to a $C^*$-algebra $B$ is a certain type of $A$–$B$-bimodule called a *$C^*$-correspondence*. This is a *directed* construct: if $\mathcal{X}$ is such a bimodule, one says that $\mathcal{X}$ is a $C^*$-correspondence *from $A$ to $B$*. In a sense that we make clear, topological correspondences are the morphisms in the category whose objects are groupoids with Haar systems. The equivalences from [25] are the isomorphisms. Holkar's adjoining functions parameterize the noninvertible correspondences.

Part II, which covers Sections 5–9, begins in Section 5 with some review materials from [17]. In that paper, we showed how to express much of the wavelet analysis in terms of the $C^*$-algebra of the Deaconu–Renault groupoid $G(X,\sigma)$ of a generic local homeomorphism $\sigma$ of a compact Hausdorff space $X$. Here, we want to show how 'transfer operators' from statistical mechanics parameterize the left inverses of the endomorphism of $C(X)$ induced by composition with $\sigma$. This parameterization is used in Section 13 to identify those adjoining functions of Holkar that give rise to scaling functions. Section 6 is devoted to building a new groupoid from $G(X,\sigma)$, denoted $G_\infty(X,\sigma)$, which is the 'pullback' of $G(X,\sigma)$ by the limit, $X_\infty$, of the canonical projective system built from iterates of $\sigma$.

The groupoid $G_\infty(X,\sigma)$ is equivalent to $G(X,\sigma)$ in the sense of [25], but, by virtue of being a pullback, the $C^*$-algebra of $G_\infty(X,\sigma)$ carries a copy of $C^*(G(X,\sigma))$ inside it and, as we show in Section 7, there is a conditional expectation from $C^*(G_\infty(X,\sigma))$ onto the copy of $C^*(G(X,\sigma))$. We use this conditional expectation coupled with Holkar's analysis to build a $C^*$-correspondence $\mathcal{X}$ from $C^*(G_\infty(X,\sigma))$ to $C^*(G(X,\sigma))$ that *is not* invertible in the collection of $C^*$-correspondences from $C^*(G_\infty(X,\sigma))$ to $C^*(G(X,\sigma))$. Further, in the space of bounded adjointable operators on $\mathcal{X}$, $\mathcal{L}(\mathcal{X})$, we build a sequence of projections $\{\mathbb{E}_j\}_{j=-\infty}^{\infty}$ and a unitary operator $U$ with the following properties:

(PMRA1) $\mathbb{E}_j \leq \mathbb{E}_i$ when $j \leq i$;
(PMRA2) $I_{\mathcal{X}} = \bigvee_{-\infty<j<\infty} \mathbb{E}_j$;
(PMRA3) $U\mathbb{E}_j U^* = \mathbb{E}_{j-1}$ for all $j$; and
(PMRA4) the copy of $C^*(G(X,\sigma))$ in $C^*(G_\infty(X,\sigma))$ leaves the range of $\mathbb{E}_j \ominus \mathbb{E}_{j-1}$ invariant, while $U^{j-i}$ implements a unitary equivalence between the restriction of the copy of $C^*(G(X,\sigma))$ to the range of $\mathbb{E}_j \ominus \mathbb{E}_{j-1}$ and the restriction of the copy to the range of $\mathbb{E}_i \ominus \mathbb{E}_{i-1}$.

We call the pair $(\{\mathbb{E}_j\}_{j=-\infty}^{\infty}, U)$ a *proto-multiresolution analysis* because of the evident relations with Definition 1.1. Property (PMRA1) is a clear variant of Item (MRA1), while Property (PMRA2) corresponds to Item (MRA2). Items (MRA4), (MRA5), and (MRA6) are captured in part by Properties (PMRA3) and (PMRA4).

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

What is conspicuously absent from the defining properties of a proto-multiresolution analysis is anything that relates to Item (MRA3). To understand what may happen, we choose a $C^*$- representation $\pi$ of $C^*(G(X,\sigma))$ acting on a Hilbert space $\mathcal{H}_\pi$ and form the induced representation $\mathrm{Ind}_\pi$ of $C^*(G_\infty(X,\sigma))$, which acts on $\mathcal{X}\otimes_\pi H_\pi$. An application of $\mathrm{Ind}_\pi$ to the proto-multiresolution analysis yields a system of operators, $\{\mathbb{E}_j\otimes I\}_{j=-\infty}^{\infty}$ and $U\otimes I$, which live in $B(\mathcal{X}\otimes H_\pi)$. When we form the spaces $V_j := \mathbb{E}_j\otimes I(\mathcal{X}\otimes H_\pi)$, $-\infty < j < \infty$, we obtain a family of *Hilbert spaces* that satisfy the evident analogues of Properties (PMRA1)–(PMRA4). However, also, as we show in Theorem 9.5, we can determine, in terms of $\pi$, when $\bigcap_j V_j = \{0\}$. When this intersection is $\{0\}$, we have a *generalized multiresolution analysis* in the sense of [4].

In Part III, which comprises Sections 10–13, we analyze when representations $\pi$ of $C^*(G(X,\sigma))$ give rise to generalized multiresolution analyses. Our analysis culminates in Theorem 12.1, which shows that $\pi$ gives rise to a generalized MRA if and only if the endomorphism of $C(X)$ determined by $\sigma$ admits a transfer operator with properties that are fairly easy to determine. We apply Theorem 12.1 to $\sigma$'s, which are local homeomorphisms of the circle $\mathbb{T}$ determined by finite Blaschke products in Section 13. In Theorem 13.7, we give our proof of Mallat's theorem from the paragraph above Theorem 1.2. Our proof reveals the role of a scaling function: it implements a Hilbert space isomorphism between $\mathcal{X}\otimes H_\pi$ and $L^2(\mathbb{R})$ that maps the $\{V_j\}_{-\infty<j<\infty}$ to a complete nest in $L^2(\mathbb{R})$ that faithfully reflects the features of $\mathrm{Ind}_\pi$. Thus, the scaling function is the link between the dynamical system $(X,\sigma)$ and $L^2(\mathbb{R})$. So, in the event that there is a scaling function, we refer to all the structure it reveals simply as a *multiresolution analysis* in $L^2(\mathbb{R})$.

Scaling functions are constructed by carefully punching holes in the potentials that are used to generate transfer operators for $\sigma$. (See Lemma 13.4.) These potentials, in turn, are built from Holkar's adjoining functions.

It may seem that our version of Mallat's analysis is unnecessarily complicated. However, as we show, it has three advantages. First, and foremost, it puts into evidence precisely the roles that the various parameters play. Second, it provides new proofs of other results in the literature. We have in mind, in particular, [6, Theorem 3.1], [2, Theorem 8], and [3, Theorem 3.1]. Third, it reveals a whole new world of examples that deserves to be explored.

# Part I
# Selected generalities about groupoids

## 2. Groupoids and Haar systems

We use Dana Williams's book [36] as our basic reference about groupoids. We supplement it with Jean Renault's thesis [26] and his treatise on amenable groupoids, written with Claire Anantharaman-Delaroche [1]. A generic second countable, locally compact, Hausdorff groupoid is denoted by $G$. Its unit space is denoted $G^{(0)}$ and $r$ and $s$ denote the *range* and *source* maps, mapping $G$ to $G^{(0)}$. It is part of the definition of a 'locally compact groupoid' that these maps are continuous and *open*.

The important role of 'open-ness' is revealed through 'duality'.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

DEFINITION 2.1. [1, Definition 1.1.1] Let $\pi : Y \to X$ be a continuous map between locally compact Hausdorff spaces. A *continuous $\pi$-system* is a family $\{\alpha^x \mid x \in X\}$ of positive Radon measures on $Y$ such that:

(1) the support of each $\alpha^x$, $\operatorname{supp} \alpha^x$, is contained in $\pi^{-1}(x)$; and
(2) the map $\alpha$ defined on $C_c(Y)$ by the formula $\alpha(f)(x) = \int f \, d\alpha^x$ carries $C_c(Y)$ into $C_c(X)$.

We write $\alpha = \{\alpha^x\}_{x \in X}$ and call $\alpha$ *proper* provided $\alpha^x \neq 0$ for all $x \in X$, while we call $\alpha$ *full* in the case where the support of each $\alpha^x$ is all of $\pi^{-1}(x)$.

REMARK 2.2. The map $\pi : Y \to X$ makes $C_c(Y)$ a right module over $C_c(X)$ via the formula

$$\phi \cdot \psi(y) := \phi(y)\psi \circ \pi(y), \quad \phi \in C_c(Y), \ \psi \in C_c(X), \ y \in Y.$$

Also, one may view $C(X)$ as a right module over itself induced by the identity map on $X$. So, from the perspective of these modules, a continuous $\pi$-system, $\alpha$, defines a $C_c(X)$-module map from $C_c(Y)_{C_c(X)}$ to $C_c(X)_{C_c(X)}$ that maps nonnegative functions on $Y$ to nonnegative functions on $X$. Conversely, the Riesz representation theorem implies that every such module map defines a continuous $\pi$-system. Further, it is a consequence of what we like to call the Williams–Blanchard theorem [36, Corollary B.18] that *full* continuous $\pi$-systems exist if and only if $\pi$ is an *open* map.

A key feature of groupoid theory is the superabundance of fibered products. Suppose $X$, $Y$, and $Z$ are three sets, and $f : X \to Z$ and $g : Y \to Z$ are two maps, then the *fibered product* of $X$ and $Y$ determined by $f$ and $g$ is

$$X_f *_g Y := X * Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

DEFINITION 2.3. A groupoid $G$ *acts* on a set $X$ (on the *left*) in the case where there is a map $\rho : X \to G^{(0)}$, called the *moment map* or *anchor map*, and a composition map $\cdot$, from the fibered product $G *_\rho X$ to $X$ satisfying:

(1) $\rho(x) \cdot x = x$ for all $x \in X$;
(2) $\rho(\gamma \cdot x) = r(\gamma)$ for all $(\gamma, x) \in G *_\rho X$; and
(3) $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$ for all $\alpha, \beta$, and $x$ such that $s(\alpha) = r(\beta)$, $s(\beta) = \rho(x)$.

The action is *continuous* if all the maps involved are continuous and the moment map is open.

Right actions of groupoids are defined similarly. The corresponding moment map is denoted by $\sigma$.

DEFINITION 2.4. Let $G$ act continuously on $X$ and on $Y$, and let $\pi : Y \to X$ be an invariant continuous map, that is, $\pi$ intertwines the two actions—$\pi(\gamma \cdot x) = \gamma \cdot \pi(x)$, then a continuous $\pi$-system of measures $\alpha = \{\alpha^x\}_{x \in X}$ is *invariant* (or equivariant) for the actions in the case where the map $\alpha$ viewed as a map from $C_c(Y)$ to $C_c(X)$ is

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

invariant, that is,

$$\int f(\gamma \cdot x)\, d\alpha^{s(\gamma)}(x) = \int f(x)\, d\alpha^{r(\gamma)}(x).$$

If $X = Y = G$, while $\pi$ is the identity map, and if $\rho$ is the range map, $r$, then a *full* invariant system of measures is a *Haar* system on $G$. Typically, we denote Haar systems by $\lambda$ and write also $\lambda = \{\lambda^u\}_{u \in G^0}$. Together with a Haar system $\lambda$, the space $C_c(G)$, consisting of continuous complex-valued functions on $G$ endowed with the inductive limit topology, becomes a locally convex topological $*$-algebra, called the *convolution algebra* of $G$ and $\lambda$, and denoted $C_c(G, \lambda)$. The product is defined by the formula

$$f * g(\gamma) := \int_G f(\alpha) g(\alpha^{-1}\gamma)\, d\lambda^{r(\gamma)}(\alpha), \quad f, g \in C_c(G, \lambda),$$

and the involution is defined by

$$f^*(\gamma) := \overline{f(\gamma^{-1})}, \quad f \in C_c(G, \lambda).$$

The $C^*$-algebra of $G$ and a Haar system $\lambda$ is the enveloping $C^*$-algebra of $C_c(G, \lambda)$, and is denoted $C^*(G, \lambda)$ (see [26, Section 2.1] and [36, Section 1.4]).

Here, we provide a telegraphic account of groupoid representations that involve Hilbert *spaces*. Later, we discuss representations of groupoids on certain Hilbert *$C^*$-modules*.

Representations of groupoids take place on Hilbert bundles that are fibred over their unit spaces. The representations of the associated convolution algebras take place on the Hilbert spaces obtained by integrating the fibres of the bundles using quasi-invariant measures on the unit spaces [36, Ch. 3]. How to build quasi-invariant measures for our groupoids is the focus of much of Parts II and III of this paper.

If $G$ is a locally compact groupoid with Haar system $\lambda := \{\lambda^u\}_{u \in G^{(0)}}$ and if $\mu$ is a (positive) measure on $G^{(0)}$, then one may form the measure on $G$ that is denoted $\mu \circ \lambda$ and is defined by the formula:

$$\int_G f(\gamma)\, d\mu \circ \lambda(\gamma) := \int_{G^{(0)}} \int_G f(\gamma)\, d\lambda^{r(\gamma)}(\gamma)\, d\mu(r(\gamma)), \quad f \in C_c(G).$$

Composing $\mu \circ \lambda$ with the inverse map on $G$ creates a new measure, denoted $(\mu \circ \lambda)^{-1}$. The measure $\mu$ is called *quasi-invariant* in the case where $\mu \circ \lambda$ and $(\mu \circ \lambda)^{-1}$ are mutually absolutely continuous, in which case, the function $\Delta_\mu := (d(\mu \circ \lambda))/(d(\mu \circ \lambda)^{-1})$ is called the *modular function of $\mu$ and $\lambda$*. The modular function $\Delta_\mu$, while defined only almost everywhere with respect to $\mu \circ \lambda$, can be chosen so that it is a homomorphism from $G$ to the positive real numbers [36, Proposition 7.6].

A *(unitary) representation* of a groupoid $G$ is a *triple* $(\mu, G^{(0)} * \mathcal{H}, \hat{L})$, where $\mu$ is a quasi-invariant measure on $G^{(0)}$, $G^{(0)} * \mathcal{H}$ is a Borel Hilbert bundle on $G^{(0)}$, and $\hat{L} : G \to \mathrm{Iso}(G^{(0)} * \mathcal{H})$ is a homomorphism of $G$ into the unitary groupoid, $\mathrm{Iso}(G^{(0)} * \mathcal{H})$, of the bundle $G^{(0)} * \mathcal{H}$. By definition, $\mathrm{Iso}(G^{(0)} * \mathcal{H}) := \{(u, U, v) \mid U \in B_{\mathrm{iso}}(H_v, H_u)\}$, where $B_{\mathrm{iso}}(H_v, H_u)$ is the space of all Hilbert space isomorphisms from $H_v$ to $H_u$. It

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

carries a Borel structure induced by that of $G^{(0)} * \mathcal{H}$. The product of $(u, U, v)$ and $(w, V, z)$ is defined only when $v = w$ and in that case, the product is $(u, UV, z)$. Since $G$ carries a Borel structure that is generated by its topology, it makes perfectly good sense to insist that $\hat{L}$ be a Borel homomorphism from $G$ to $\mathrm{Iso}(G^{(0)} * \mathcal{H})$ as described in [36, Proposition 3.38]. We write

$$\hat{L}(\gamma) := (r(\gamma), L_\gamma, s(\gamma)), \quad \gamma \in G, \tag{2-1}$$

where $L_\gamma$ lies in $B_{\mathrm{iso}}(H_{s(\gamma)}, H_{r(\gamma)})$.

Given a Borel Hilbert bundle $G^{(0)} * \mathcal{H}$ and (sigma-finite) Borel measure $\mu$, one can form the Hilbert *space* consisting of all its Borel sections $\mathfrak{h}$ such that $\int_{G^{(0)}} \langle \mathfrak{h}(u), \mathfrak{h}(u) \rangle_u \, d\mu(u)$ is finite. This space is denoted $L^2(G^{(0)} * \mathcal{H}, \mu)$ and the inner product of two sections $\mathfrak{h}$ and $\mathfrak{k}$ is given by the expected integral

$$(\mathfrak{h}, \mathfrak{k}) := \int_{G^{(0)}} (\mathfrak{h}(u), \mathfrak{k}(u))_{H_u} \, d\mu(u).$$

When the dimensions of the fibres are one, we may choose a nonvanishing cross-section $\mathfrak{k}$ of norm one in $L^2(G^{(0)} * \mathcal{H}, \mu)$. Then, if $\mathfrak{h}$ is another cross-section, it can be written as $\mathfrak{h} = c \cdot \mathfrak{k}$, where $c$ is a uniquely determined complex-valued Borel function on $G^{(0)}$. The map $\mathfrak{h} \to c$ is a Hilbert space isomorphism from $L^2(G^{(0)} * \mathcal{H}, \mu)$ onto the space of scalar-valued, square-integrable functions on $G^{(0)}$, $L^2(G^{(0)}, \mu)$. It enables us to replace $L^2(G^{(0)} * \mathcal{H}, \mu)$ with $L^2(G^{(0)}, \mu)$ and to carry out computations there.

Let $(\mu, G^{(0)} * \mathcal{H}, \hat{L})$ be a unitary representation of $G$ in the unitary groupoid, $\mathrm{Iso}(G^{(0)} * \mathcal{H})$. The *integrated form*, $\mathfrak{L}$, of $L$ represents the convolution algebra of $G$, $C_c(G, \lambda)$, on $L^2(G^{(0)} * \mathcal{H}, \mu)$ via the formula

$$\mathfrak{L}(f)\mathfrak{h}(r(\gamma)) := \int_G f(\gamma) L_\gamma(\mathfrak{h}(s(\gamma)) \Delta(\gamma)^{-1/2} \, d\lambda^{r(\gamma)}(\gamma), \quad f \in C_c(G, \lambda), \mathfrak{h} \in L^2(G^{(0)} * \mathcal{H}, \mu). \tag{2-2}$$

This formula extends to all Borel functions $f$ such that $\mathfrak{L}(f)$ makes sense as a bounded operator on $L^2(G^{(0)} * \mathcal{H}, \mu)$.

Turning to groupoid representations on Hilbert $C^*$-modules and Holkar's ideas [14, 15], suppose $A$ and $B$ are $C^*$-algebras and $\mathcal{X}$ is a (right) Hilbert module over $B$ with a $B$-valued inner product $\langle \cdot, \cdot \rangle$ in the sense of Lance [20, Ch. 1]. We say that $\mathcal{X}$ is a *$C^*$-correspondence from $A$ to $B$* in the case where $A$ is represented in the algebra of all bounded adjointable operators on $\mathcal{X}$, $\mathcal{L}(\mathcal{X})$. In this event, following Rieffel [33], one may induce $C^*$-representations of $A$ from $C^*$-representations of $B$ by forming balanced tensor products: for a $C^*$-representation of $B$, $\pi$, on the Hilbert space $\mathcal{H}$, $\mathcal{X} \otimes_\pi \mathcal{H}$ becomes a Hilbert space in the inner product that is defined on decomposable tensors by $\langle \xi \otimes h, \eta \otimes k \rangle := \langle h, \pi(\langle \xi, \eta \rangle) k \rangle$. The representation, $\mathrm{Ind}_\pi : A \to B(\mathcal{X} \otimes_\pi \mathcal{H})$, then, is defined by $\mathrm{Ind}_\pi(a)(\xi \otimes h) := (a \cdot \xi) \otimes h, a \in A, \xi \otimes h \in \mathcal{X} \otimes \mathcal{H}$, that is, $\mathrm{Ind}_\pi(a) = L(a) \otimes I_{\mathcal{H}}$.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

Suppose that $A = C^*(H,\beta)$ and $B = C^*(G,\lambda)$ for groupoids $H$ and $G$ endowed with Haar systems $\beta$ and $\lambda$. Holkar showed how to build a correspondence from $A$ to $B$ using what he calls a topological correspondence from $(H,\beta)$ to $(G,\lambda)$.

DEFINITION 2.5. A *topological correspondence* from $(H,\beta)$ to $(G,\lambda)$ is a pair $(Z,\alpha)$ where we have the following.

(1) $Z$ is a locally compact $H$–$G$ bispace, meaning that $Z$ is a right $G$-space with moment map $\sigma$ and a left $H$-space with moment map $\rho$ such that $\eta\cdot(z\cdot\gamma) = (\eta\cdot z)\cdot\gamma$ for all $\eta\in H$, $\gamma\in G$, and $z\in Z$, for which the formula is defined.
(2) The action of $G$ on $Z$ is proper in the sense of [36, Definition 2.15].
(3) $\alpha = \{\alpha_u\}_{u\in G^{(0)}}$ is a $G$-invariant, continuous $\sigma$-system of measures on $Z$ in the sense of Definition 2.4 that is also proper in the sense of Definition 2.1.
(4) There is a continuous positive function $\Delta_a$ defined on $H * Z$ such that

$$\int_Z\int_H F(\eta^{-1},z)\,d\beta^{\rho(z)}(\eta)d\alpha_u(z) = \int_Z\int_H F(\eta,\eta^{-1}z)\Delta_a(\eta,\eta^{-1}z)\,d\beta^{\rho(z)}(\eta)d\alpha_u(z)$$

for all $u\in G^{(0)}$ and $F\in C_c(H*Z)$. The function $\Delta_a$ is called the *adjoining function* for $(Z,\alpha)$.

We emphasize that the system of measures $\alpha$ must be proper but *not* necessarily full. We use the notation $\Delta_a$ for the adjoining function to distinguish it from the modular function, $\Delta$ or $\Delta_\mu$, of a quasi-invariant measure $\mu$. The main theorem in [14, Theorem 2.10] proves that $C_c(Z)$ can be endowed with operations that give rise to a $C^*$-correspondence from $C^*(H,\beta)$ to $C^*(G,\lambda)$.

An equivalence $(Z,\alpha)$, where $\alpha$ is a full $\sigma$-system, between $(H,\beta)$ and $(G,\lambda)$ in the sense of [25, Definition 2.1] is a particular example of a topological correspondence. The adjoining function $\Delta_a$ is the constant function 1 in this situation [14, Example 3.9]. The $C^*$-algebras $C^*(H,\beta)$ and $C^*(G,\lambda)$ are Morita equivalent [25], and the $C^*$-correspondence defined above becomes an imprimitivity bimodule. We specialize this setting further in Sections 5 and 9 by setting $H = G$, $Z = G$, and $\alpha_x = \lambda_x$ for all $x\in X$, where $\{\lambda_u\}_{u\in G^{(0)}}$ is the *right* Haar system determined by $\lambda$, $\int f(\gamma)\,d\lambda_u(\gamma) := \int f(\gamma^{-1})\,d\lambda^u(\gamma)$. Then, $(G,\alpha)$ is an equivalence between $(G,\beta)$ and $(G,\lambda)$, where $\beta$ can be a different Haar system on $G$. When we view $(G,\alpha)$ as a topological correspondence from $(G,\lambda)$ to $(G,\lambda)$, we denote it by $\mathcal{X}_0$.

## 3. Pullbacks, imprimitivity groupoids, and Haar systems

Throughout this section, $G$ is a groupoid with Haar system $\lambda$; $Y$ is a second countable, locally compact Hausdorff space; and $\Phi : Y\to G^{(0)}$ is a continuous, open, and surjective map. We explore Haar systems on two related groupoids, $G^Z$ and $Y * G * Y$, that are built from $G$, $\lambda$, $\Phi$, and continuous $\Phi$-systems of measures on $Y$.

The space $Z$ is defined to be the fibered product,

$$Z := Y\,{}_\Phi{*}_r\,G := \{(y,\gamma)\in Y\times G \mid \Phi(y) = r(\gamma)\}.$$

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

It carries the evident right $G$ action through right translation on $G$:

$$(z,\alpha),\beta \to (z,\alpha\beta), \quad (z,\alpha) \in Z * G,\ \beta \in G.$$

This action of $G$ on $Z$ is free and proper because the action of $G$ on $G$ via right translation is free and proper.

To define the imprimitivity groupoid determined by $G$ and $Z$, $G^Z$, let $\sigma : Z \to G$ be the map, $\sigma(y,\gamma) := s(\gamma)$ and set

$$Z * Z := \{(x,y) \in Z \times Z \mid \sigma(x) = \sigma(y)\}.$$

Then, $G$ acts on the right on $Z * Z$ by the diagonal action $(x,y)\cdot\gamma = (x\gamma, y\gamma)$. The *imprimitivity groupoid* determined by $G$ and $Z$ is $G^Z := Z *_G Z := (Z * Z)/G$. The unit space of $G^Z$ can be identified with $Z/G$, which in turn may be identified with $Y$, and the range and source maps on $G^Z$ are given by

$$r([x,y]) = x \cdot G \quad \text{and} \quad s([x,y]) = y \cdot G.$$

The space $Z$ becomes a free and proper left $G^Z$-space as follows. The moment map $\rho : Z \to Z/G$ is the quotient map; so, $G^Z * Z$ is $\{([x,y],z) \in G^Z \times Z \mid y \cdot G = z \cdot G\}$. The left action of $G^Z$ on $Z$ is defined via the formula

$$[x,y]\cdot z := x\gamma,$$

where $\gamma$ is the unique element in $G$ such that $z = y\cdot\gamma$. The space $Z$ is a free and proper left $G^Z$-space and, in fact, $Z$ is an equivalence between $G^Z$ and $G$ [25, Section 2].

PROPOSITION 3.1. *The imprimitivity groupoid $G^Z$ is isomorphic to the pullback groupoid determined by $Y$ and $\Phi$, $Y * G * Y$, where*

$$Y * G * Y := Y {}_{\Phi}*_{r} G {}_{s}*_{\Phi} Y := \{(x,\gamma,y) \in Y \times G \times Y \mid \Phi(x) = r(\gamma) \quad \textit{and} \quad \Phi(y) = s(\gamma)\}$$

*is endowed with the operations*

$$(x,\gamma,y)\cdot(y,\delta,z) := (x,\gamma\delta,z)$$

*and*

$$(x,\gamma,y)^{-1} := (y,\gamma^{-1},x).$$

PROOF. It is easy to check that $\Psi : G^Z \to Y * G * Y$ defined via

$$\Psi([(x,\alpha),(y,\beta)]) = (x,\alpha\beta^{-1},y)$$

is an isomorphism with the desired properties. □

We next want to discuss how the Haar system $\lambda := \{\lambda^u\}_{u\in G^{(0)}}$ on $G$ may be 'induced' to Haar systems on $G^Z$ and $Y * G * Y$. We start by choosing an arbitrary continuous $\Phi$-system of measures $\nu := \{\nu_u\}_{u\in G^{(0)}}$ on $Y$ (as per Definition 2.1). While, as the following lemma shows, we need $\nu$ to be a full $\Phi$-system to obtain a Haar system on $G^Z$, we proceed with arbitrary systems. We use proper nonfull systems to define

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

groupoid correspondences in later sections. We then promote $\nu$ to a system of measures $\mathfrak{t} := \{\mathfrak{t}_u\}_{u\in G^{(0)}}$ on $Z$ defined by the formula:

$$\mathfrak{t}_u(f) := \int_{G_u}\int_Y f(y,\gamma)\,d\nu_{r(\gamma)}(y)\,d\lambda_u(\gamma), \quad u \in G^{(0)},$$

for $f \in C_c(Z)$.

The Haar system on $Y * G * Y$ defined by $\mathfrak{t}$ was studied in [36, Proposition 6.27], which is reproduced here as the following proposition.

PROPOSITION 3.2. *Assume that $\nu$ is a full $\Phi$-system. The equation*

$$\beta(f)(x) := \int_{G^{\Phi(x)}}\int_Y f(x,\gamma,y)\,d\nu_{s(\gamma)}(y)\,d\lambda^{\Phi(x)}(\gamma), \quad x \in Y, \tag{3-1}$$

*defines a Haar system, $\beta$, on $Y * G * Y$.*

It is important for us that we can induce representations from $(G,\lambda)$ to $(G^Z,\beta)$ as in Section 4 using $\Phi$-systems of measures that are not full by using the theory of groupoid correspondences. We provide here the main set-up in a slightly more general setting. Assume that $G$ is a locally compact Hausdorff groupoid endowed with a Haar system $\lambda$. Assume that $Z$ is a right $G$-space with a continuous, open, and surjective anchor map $\sigma : Z \to G^{(0)}$ and let $\alpha := \{\alpha_u\}_{u\in G^{(0)}}$ be a full $\sigma$-system that is invariant under the action of $G$ on $Z$ in the sense of Definition 2.4. Since $\sigma(z\cdot\gamma) = s(\gamma)$, this means that $\int f(z\cdot\gamma)\,d\alpha_{r(\gamma)}(z) = \int f(z)\,d\alpha_{s(\gamma)}(z)$. Let $G^Z := Z_\sigma *_\sigma Z/G$ be the imprimitivity groupoid determined by $Z$. Then, $\alpha$ determines a Haar system $\beta$ on $G^Z$ via

$$\int_{G^Z} f([x,y])\,d\beta^{xG}([x,y]) = \int_Z f([x,y])\,d\alpha_{s(x)}(y) \tag{3-2}$$

for all $xG \in Z/G$ [18, Proposition 5.2]. This Haar system on $G^Z$ is fixed for the remainder of the section.

Let $\delta$ be a nonnegative function on $Z$ that is invariant under the action of $G$. This implies, of course, that the pointwise product $\delta\cdot\alpha$ is an invariant system of measures. In addition, we assume that $\delta\cdot\alpha$ is a *proper* invariant system, which we denote by $\alpha_\delta$. This assumption places constraints on how the support of $\delta$ is distributed. We write

$$\int_Z f(x)\,d\alpha_{\delta,u}(x) = \int_Z f(x)\,d(\alpha_\delta)_u(x) = \int_Z f(x)\delta(x)\,d\alpha_u(x).$$

LEMMA 3.3. *$(Z,\alpha_\delta)$ is a topological correspondence in the sense of Definition 2.5 with the adjoining function $\Delta_a([x,y],z) = \delta(y)/\delta(x)$.*

REMARK 3.4. Note that if $Z = \{x \in X \mid \delta(x) = 0\}$, then $\alpha_\delta(Z) = 0$. Hence, $\Delta_a$ is well defined on the support of $\alpha_{\delta,u}$.

PROOF OF LEMMA 3.3. Conditions (i), (ii), and (iii) of Definition 2.5 are satisfied by our assumptions. We check the fourth condition of the definition. Let $F \in C_c(G^Z * Z)$.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

For $u \in G^{(0)}$, we have

$$\begin{aligned}
&\int_Z \int_{G^Z} F([x,y]^{-1},z)\, d\beta^{zG}([x,y])\, d\alpha_{\delta,u}(z) \\
&\quad = \int_Z \int_Z F([z,y]^{-1},z)\, d\alpha_u(y)\, d\alpha_{\delta,u}(z) \\
&\quad = \int_Z \int_Z F([y,z],z)\delta(z)\, d\alpha_u(y)\, d\alpha_u(z) \\
&\quad = \int_Z \int_Z F([z,y],[y,z]z)\delta([y,z]z)\, d\alpha_u(y)\, d\alpha_u(z) \\
&\quad = \int_Z \int_Z F([z,y],y)\delta(y)\, d\alpha_u(y)\, d\alpha_u(z) \\
&\quad = \int_Z \int_Z F([z,y],y)(\delta(y)/\delta(z))\, d\alpha_u(y)\, \delta(z)\, d\alpha_u(z) \\
&\quad = \int_Z \int_Z F([z,y],y)\Delta_a([z,y],y)\, d\alpha_u(y)\, d\alpha_{\delta,u}(z) \\
&\quad = \int_Z \int_{G^Z} F([z,y],y)\Delta_a([z,y],y)\, d\beta^{zG}([z,y])\, d\alpha_{\delta,u}(z).
\end{aligned}$$

The first line follows from (3-2), the second line from the definition of $\alpha_\delta$, and the third line follows since $(Z,\alpha)$ is an equivalence between $G^Z$ and $G$ (see [14, Examples 3.9 and 3.8]). The remaining lines follow by defining $\Delta_a([x,y],z) = \delta(y)/\delta(x)$, from (3-2), and from the definition of $\alpha_\delta$. □

EXAMPLE 3.5. Returning to the main case we study, assume that $\Phi : Y \to G^{(0)}$ is a continuous, open, surjective map. Let $\nu = \{\nu_u\}_{u\in G^{(0)}}$ be a full $\Phi$-system of measures. Let $Z = Y * G$ and $\alpha = \{\alpha_u\}_{u\in G^{(0)}}$ be the invariant full $s$-system defined via

$$\int_Z f(x,\gamma)\, d\alpha_u(x,\gamma) = \int_G \int_Y f(x,\gamma)\, d\nu_{r(\gamma)}(x)\, d\lambda_u(\gamma).$$

That is, $\alpha_u = \natural_u$ of Definition 3. Let $\mathbb{d} : Y \to \mathbb{R}^+$ be a continuous function. Then, $\delta : Z \to \mathbb{R}^+$ defined via $\delta(x,\gamma) = \mathbb{d}(x)$ satisfies the hypothesis that $\delta((x,\gamma)\eta) = \delta(x,\gamma)$. Lemma 3.3 implies that $(Z,\alpha_\delta)$ is a topological correspondence, where

$$\int_Z f(x,\gamma)\, d\alpha_{\delta,u}(x,\gamma) = \int_G \int_Y f(x,\gamma)\mathbb{d}(x)\, d\nu_{r(\gamma)}(x)\, d\lambda_u(\gamma) \tag{3-3}$$

and the adjoining function is defined via $\Delta_a((x,\gamma,y),(y,\eta)) = \mathbb{d}(y)/\mathbb{d}(x)$.

## 4. Inducing representations to the pullback groupoid

Continuing with the notation from Section 3, observe that the title of the current section may appear confusing because the groupoid $G$ is not a subgroupoid of the pullback groupoid, $Y * G * Y$. Rather, it is a *quotient* of $Y * G * Y$. However, as we have shown in Proposition 3.1, $Y * G * Y$ is isomorphic to $G^Z$, which is the imprimitivity

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

groupoid of the right $G$-space $Z := Y * G$. So, it is natural to try to use the theory developed by Rieffel in [33] coupled with the equivalence theorem of [25, Theorem 2.8] to induce representations of $G$ to $G^Z$ and relate them to representations of $Y * G * Y$ using the isomorphism $\Psi$. One of the difficulties we face when trying to do this is the necessity in this paper to use nonfull $\Phi$-systems of measures on $Y$. This leads to groupoid correspondences from $Y * G * Y$ to $G$, and we can use [14] to induce representations from $C^*(G,\lambda)$ to $C^*(Y * G * Y,\beta)$ and [32] to induce representations from $(G,\lambda)$ to $(Y * G * Y,\beta)$. Our analysis leads to two different Hilbert space representations of the convolution algebra $C_c(Y * G * Y,\beta)$ that one may induce from a Hilbert space representation of the convolution algebra $C_c(G,\lambda)$. These are denoted $\mathrm{Ind}_{\mathrm{J}}\,\mathfrak{L}$ and $\mathrm{Ind}_{\mathrm{M}}\,\mathfrak{L}$, where $\mathfrak{L}$ is a prescribed Hilbert space representation of $C_c(G,\lambda)$ that comes from a representation $\hat{L}$ of the groupoid $G$ in the isomorphism groupoid of a Hilbert bundle built over $G^{(0)}$. Our purpose in this section is to define $\mathrm{Ind}_{\mathrm{J}}\,\mathfrak{L}$ and $\mathrm{Ind}_{\mathrm{M}}\,\mathfrak{L}$, and to show that they are unitarily equivalent. (See Theorem 4.1.)

We begin with a unitary representation, $\hat{L} = (\mu, G^{(0)} * \mathcal{H}, \hat{L})$, of $(G,\lambda)$ in the unitary groupoid $\mathrm{Iso}(G^{(0)} * \mathcal{H})$ and form its integrated form $\mathfrak{L}$, representing the convolution algebra, $C_c(G,\lambda)$, on $L^2(G^{(0)} * \mathcal{H},\mu)$ (see text following Definition 2.4). These ingredients are fixed throughout this section, as are the modular function $\Delta_\mu$ determined by $\mu$ and the Haar system $\lambda$ on $G$. We also fix a full $\Phi$-system $\nu$ on $Y$ that determines a full invariant $s$-system $\alpha$ on $Z = Y * G$. The Haar system $\beta$ on $Y * G * Y$ is fixed and given as in (3-1).

We let $\mathrm{d} : Y \to \mathbb{R}^+$ be a continuous function such that $\mathrm{d}|_{\Phi^{-1}(u)} \neq 0$ for all $u \in G^{(0)}$. Then, $\mathrm{d}(x)$ defines a proper $\Phi$-system of measures $\nu_{\mathrm{d}}$ via

$$\int_Y f(x)\, d\nu_{\mathrm{d},u}(x) = \int_Y f(x)\mathrm{d}(x)\, d\nu_u(x) \tag{4-1}$$

for all $u \in G^{(0)}$. Let $\delta(x,\gamma) := \mathrm{d}(x)$ and let $\alpha_\delta$ be the proper invariant $\sigma$-system defined as in (3-3). Lemma 3.3 implies that $(Z,\alpha_\delta)$ is a topological correspondence. We describe next the process of inducing the groupoid representation $\hat{L} = (\mu, G^{(0)} * \mathcal{H}, \hat{L})$, of $(G,\lambda)$ to a groupoid representation of $(Y * G * Y,\beta)$ via $(Z,\alpha_\delta)$ following Renault's ideas presented in [32]. The result is a triple, $\mathrm{Ind}\,\hat{L} := (\mu_{\mathrm{Ind}}, Y * \mathcal{K}, \mathrm{Ind}\,\hat{L})$, which we call the *(unitary) induced representation of* $(Y * G * Y,\beta)$. The measure, $\mu_{\mathrm{Ind}}$, is defined on $Y$, the unit space of $Y * G * Y$, through the formula

$$\int_Y f(x)\, d\mu_{\mathrm{Ind}}(x) := \int_{G^{(0)}} \int_Y f(x)\, d\nu_{\mathrm{d},u}(x)\, d\mu(u)$$

for all $f \in C_c(Y)$. The following equation, (4-2), shows that $\mu_{\mathrm{Ind}}$ is indeed quasi-invariant on $Y$ when $Y * G * Y$ is endowed with the Haar system $\beta$ defined in (3-1). It also shows that the modular function $\Delta_{\mu_{\mathrm{Ind}}}$ is given by the equation $\Delta_{\mu_{\mathrm{Ind}}}(x,\gamma,y) = \Delta_\mu(\gamma)\mathrm{d}(x)/\mathrm{d}(y)$ for all $(x,\gamma,y) \in Y * G * Y$. The computation, itself, is a conscientious application of Fubini's theorem coupled with the quasi-invariance

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

of the measure $\mu$ on $G^{(0)}$:

$$\begin{aligned}\mu_{\mathrm{Ind}} \circ \beta(f) &= \int_Y \int_{Y*G*Y} f(x,\gamma,y)\, d\beta^x(x,\gamma,y)\, d\mu_{\mathrm{Ind}}(x) \\ &= \int_{G^{(0)}} \int_Y \int_G \int_Y f(x,\gamma,y)\Delta_\mu(\gamma)(\mathrm{d}(x)/\mathrm{d}(y))\, d\nu_{r(\gamma)}(x)\, d\lambda_u(\gamma)\mathrm{d}(y)\, d\nu_u(y)\, d\mu(u) \\ &= \int_Y \int_{Y*G*Y} f(x,\gamma,y)\Delta_\mu(\gamma)\mathrm{d}(x)/\mathrm{d}(y)\, d\beta_y(x,\gamma,y)\, d\mu_{\mathrm{Ind}}(y) \\ &= \Delta_{\mu_{\mathrm{Ind}}}(\mu_{\mathrm{Ind}} \circ \beta)^{-1}(f) \end{aligned} \tag{4-2}$$

for all $f \in C_c(Y * G * Y)$.

The Hilbert space bundle $Y * \mathcal{K}$ is the pull-back Hilbert space bundle $Y * \mathcal{H} := \{(x,h) \mid x \in Y \quad \text{and} \quad h \in \mathcal{H}(\Phi(x))\}$ over $Y$. That is, $\mathcal{K}(y) = \mathcal{H}(\Phi(y))$. The induced action of the pullback groupoid $Y * G * Y$ on $Y * \mathcal{H}$ is given by

$$(\mathrm{Ind}\,\hat{L})_{(x,\gamma,y)}(y,h) := (x, \hat{L}_\gamma h) \quad (x,\gamma,y) \in Y * G * Y,\ h \in \mathcal{H}(\Phi(y)).$$

The integrated form of $\mathrm{Ind}\,\hat{L}$ is denoted $\mathrm{Ind}_{\mathrm{J}}\,\mathfrak{L}$ and is defined on $L^2(Y * \mathcal{H}, \mu_{\mathrm{Ind}})$ via the equation

$$\begin{aligned}&((\mathrm{Ind}_{\mathrm{J}}\,\mathfrak{L})(f)\mathfrak{h},\mathfrak{k}) \\ &= \int_Y \int_{Y*G*Y} f(x,\gamma,y)((\mathrm{Ind}\,L)_{(x,\gamma,y)}\mathfrak{h}(\Phi(y)), \mathfrak{k}(\Phi(x)))\Delta_{\mu_{\mathrm{Ind}}}(x,\gamma,y)^{-1/2} d\beta^x(x,\gamma,y) \\ &\quad \cdots d\mu_{\mathrm{Ind}}(x) \\ &= \int_{G^{(0)}} \int_Y \int_G \int_Y f(x,\gamma,y)(L_\gamma(\mathfrak{h}(\Phi(y)))\,, \mathfrak{k}(\Phi(x)))\Delta_\mu(\gamma)^{-1/2}\mathrm{d}(x)^{1/2}\mathrm{d}(y)^{1/2} \\ &\quad \cdots d\nu_{s(\gamma)}(y)\, d\lambda^u(\gamma)\, d\nu_u(x)\, d\mu(u) \end{aligned} \tag{4-3}$$

for all $f \in C_c(Y * G * Y)$, $\mathfrak{h}, \mathfrak{k} \in L^2(Y * \mathcal{H}, \mu_{\mathrm{Ind}})$.

We abbreviate $L^2(Y * \mathcal{H}, \mu_{\mathrm{Ind}})$ with $H_{\mathrm{Ind}_{\mathrm{J}}\,\mathfrak{L}}$.

We turn now to a description of Rieffel's perspective on induced representations. Recall from Lemma 3.3 that $(Z, \alpha_\delta)$, where $\delta(x,\gamma) = \mathrm{d}(x)$, is topological correspondence from $(Y * G * Y, \beta)$ to $(G, \lambda)$. Reference [14, Theorem 2.10] (see Definition 2.5) implies that $C_c(Z) = C_c(Y * G)$ can be completed to a $C^*$-correspondence from $C^*(Y * G * Y, \beta)$ to $C^*(G, \lambda)$ via the formulas listed below. The right action of $C_c(G, \lambda)$ on $C_c(Z)$ is given by the integral

$$(\xi \cdot b)(x,\gamma) = \int_G \xi(x,\gamma\eta) b(\eta^{-1})\, d\lambda^{s(\gamma)}(\eta), \quad \xi \in C_c(Z),\ b \in C_c(G,\lambda). \tag{4-4}$$

The inner product on $C_c(Z)$ with values in $C_c(G, \lambda)$ is defined by the integral

$$\langle \xi, \eta \rangle_{C_c(G)}(\gamma) = \int_G \int_Y \overline{\xi(x,\zeta)}\eta(x,\zeta\gamma)\, d\nu_{\mathrm{d},r(\zeta)}(x)\, d\lambda_{r(\gamma)}(\zeta), \tag{4-5}$$

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

and the left action of $C_c(Y * G * Y, \beta)$ on $C_c(Z)$ is given by the integral

$$(a \cdot \xi)(x, \gamma) = \int_G \int_Y a(x, \zeta, y)\xi(y, \zeta^{-1}\gamma)\mathbb{d}(y)^{1/2}/\mathbb{d}(x)^{1/2}\, d\nu_{s(\zeta)}(y)\, d\lambda^{r(\gamma)}(\zeta), \tag{4-6}$$

where $a \in C_c(Y * G * Y, \beta)$ and $\xi \in C_c(Z)$.

So, we may define the representation of $C_c(Y * G * Y, \beta)$ that is induced in the sense of Rieffel from the representation $\mathfrak{L}$ which, recall, is the integrated form of the representation $\hat{L} = (\mu, G^{(0)} * \mathcal{H}, \hat{L})$ of the groupoid $G$. The Hilbert space for $\mathfrak{L}$ is $H_{\mathfrak{L}} := L^2(G^{(0)} * \mathcal{H}, \mu)$.

The Hilbert space $H_{\mathrm{Ind}_M \mathfrak{L}}$ of Rieffel's induced representation, $\mathrm{Ind}_M \mathfrak{L}$, is the completion of the algebraic tensor product $C_c(Z) \odot H_{\mathfrak{L}}$ under the inner product

$$(\xi \otimes \mathfrak{h}, \eta \otimes \mathfrak{k}) = (\mathfrak{L}(\langle \eta, \xi \rangle_{C^*(G)})\mathfrak{h}, \mathfrak{k})$$

for $\xi, \eta \in C_c(Z)$ and $\mathfrak{h}, \mathfrak{k} \in H_{\mathfrak{L}}$. The induced representation $\mathrm{Ind}_{\mathrm{M}} \mathfrak{L}$ acts on $H_{\mathrm{Ind}_{\mathrm{M}} \mathfrak{L}}$ via the formula

$$(\mathrm{Ind}_{\mathrm{M}} \mathfrak{L})(f)\xi \otimes \mathfrak{h} := (f \cdot \xi) \otimes \mathfrak{h}$$

for all $f \in C_c(Y * G * Y)$, $\xi \in C_c(Z)$, and $\mathfrak{h} \in H_{\mathfrak{L}}$.

The following theorem is anticipated in the discussion at the bottom of the first page of [32]. We need a precise statement.

THEOREM 4.1. *Define* $\mathfrak{U} : C_c(Z) \odot H_{\mathfrak{L}} \to H_{\mathrm{Ind}_J \mathfrak{L}}$ *via*

$$\mathfrak{U}(\xi \otimes \mathfrak{h})(x) = \int_{G^{\Phi(x)}} \xi(x, \gamma) L_\gamma \mathfrak{h}(s(\gamma))\Delta_\mu(\gamma)^{-1/2} d\lambda^{\Phi(x)}(\gamma). \tag{4-7}$$

*Then,* $\mathfrak{U}$ *extends to a Hilbert space isomorphism from* $H_{\mathrm{Ind}_M \mathfrak{L}}$ *to* $H_{\mathrm{Ind}_J \mathfrak{L}}$ *that intertwines* $\mathrm{Ind}_M \mathfrak{L}$ *and* $\mathrm{Ind}_J \mathfrak{L}$.

PROOF. The proof that $\mathfrak{U}$ is an isometry and that it intertwines $\mathrm{Ind}_{\mathrm{M}} \mathfrak{L}$ and $\mathrm{Ind}_{\mathrm{J}} \mathfrak{L}$ is a tedious but straightforward computation using the invariance of the Haar system $\lambda$, the homomorphism property of $\Delta_\mu$, and Fubini's theorem. Arguing as in the proof of [27, Lemma 2.3], one can prove that $\mathfrak{U}$ has dense range. Thus, $\mathfrak{U}$ is a unitary, and $\mathrm{Ind}_{\mathrm{M}} \mathfrak{L}$ and $\mathrm{Ind}_{\mathrm{J}}$ are unitarily equivalent. □

# Part II
# From Deaconu–Renault groupoids to proto-multiresolution analyses

## 5. The Deaconu–Renault groupoid

The principal player in this paper is the *Deaconu–Renault groupoid* of a local homeomorphism. Our goal in this section is to review some of its salient features and to show how it leads organically to the theory of transfer operators.

We begin with the following definition.

DEFINITION 5.1. Let $X$ be a second countable, compact Hausdorff space and let $\sigma : X \to X$ be a surjective local homeomorphism. The *Deaconu–Renault Groupoid*

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

defined by $X$ and $\sigma$ is

$$G(X,\sigma) := \{(x, k-l, y) \in X \times \mathbb{Z} \times X \mid k, l \in \mathbb{Z}_+,\ \sigma^k(x) = \sigma^l(y)\}.$$

The product $(x, k-l, y)(w, m-n, z)$ is defined only when $y = w$ and then

$$(x, k-l, y)(y, m-n, z) := (x, (k+m)-(l+n), z).$$

The inverse of $(x, k-l, y)$ is $(y, l-k, x)$.

The unit space of $G(X,\sigma)$ is identified with $X$, and the maps $r : G(X,\sigma) \to X$ and $s : G(X,\sigma) \to X$ given by $r(x, k-l, y) := x$ and $s(x, k-l, y) := y$ are the *range* and *source* maps. Because $\sigma$ is assumed to be a surjective local homeomorphism, it is easy to see that $G(X,\sigma)$ viewed as a subset of $X \times \mathbb{Z} \times X$ with the relative topology is locally compact and Hausdorff, and that the maps $r$ and $s$ are local homeomorphisms. Consequently, $G(X,\sigma)$ is an example of what nowadays is universally known as an *étale groupoid*.

It is useful at times to 'change variables' and rewrite the defining formulas for $G(X,\sigma)$ as Valentin Deaconu did in [9]:

$$G(X,\sigma) = \{(x, n, y) \in X \times \mathbb{Z} \times X \mid \text{ there exists } k, l \geq 0,\ n = l-k, \sigma^k(x) = \sigma^l(y)\}.$$

Two triples, $(x, n, y)$ and $(w, m, z)$, can be multiplied if and only if $y = w$, in which case $(x, n, y)(y, m, z) = (x, m+n, z)$; $(x, n, y)^{-1} = (y, -n, x)$.

The fact that $G(X,\sigma)$ is étale implies that the family of counting measures on the fibers of $r$, taken together, constitute a Haar system, $\lambda = \{\lambda^x\}_{x \in X}$, on $G(X,\sigma)$. The formula for convolution becomes

$$\begin{aligned} f * g(x, k-l, y) &:= \int_{G(X,\sigma)} f(\alpha) g(\alpha^{-1}\gamma)\, d\lambda^{r(\gamma)}(\alpha) \\ &= \sum f(x, m-n, z) \cdot g(z, (n+k)-(m+l), y), \end{aligned}$$

where $\alpha := (z, n-m, w)$, $\gamma := (x, k-l, y)$, and where the sum ranges over all $m, n$, and $z$ such that $\sigma^m(x) = \sigma^n(z)$. The adjoint of a function $f$ is defined by

$$f^*(x, k-l, y) := \overline{f(y, l-k, x)}.$$

To get a feeling for this algebra, let $U$ and $V$ be open subsets of $X$, suppose $k$ and $l$ are such that the restrictions, $\sigma^l|U$ and $\sigma^k|V$, are homeomorphisms with common range, i.e., $\sigma^k(U) = \sigma^l(V)$, and let

$$Z(U, V, k, l) := \{(x, k-l, y) \in G(X,\sigma) \mid x \in U,\ y \in V\}.$$

Then, $Z(U, V, k, l)$ is essentially the graph of $(\sigma^l|_V)^{-1} \circ (\sigma^k|_U)$ and is a *G-set* in the sense of Renault [26, Definition 1.10]. It is also an open subset of $G(X,\sigma)$. Under an evident notion of composition discussed in [26, page 10], these open $G$-sets form an inverse semigroup which also is a basis for the topology of $G(X,\sigma)$. (The germs of such $G$-sets are explored in [28].) If a $G$-set $Z := Z(U, V, k, l)$ happens also to be

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

closed, then its characteristic function $1_Z$ is an element of $C_c(G(X,\sigma),\lambda)$ which, when properly scaled, yields a partial isometry in $C_c(G(X,\sigma),\lambda)$.

Another important class of subsets of $G(X,\sigma)$ is defined by the equations

$$R_{n,m} := \{(x, n-m, y) \in G(X,\sigma) \mid \sigma^n(x) = \sigma^m(y)\}, \quad n,m \geq 0. \tag{5-1}$$

Each $R_{n,m}$ is compact and open in $G(X,\sigma)$, so its characteristic function $1_{R_{n,m}}$ lies in $C_c(G(X,\sigma),\lambda)$. Also, $1^*_{R_{n,m}} = 1_{R_{m,n}}$; $R_{n,m}$ may be viewed as the graph of the multi-valued function $\sigma^{-m}\circ\sigma^n$; if $m=0$, then $R_{n,0}$ is the graph of $\sigma^n$; and $\{R_{n,n}\}_{n\geq 1}$ is a nested family of equivalence relations on $X$ such that $R_{n,n} \subseteq R_{m,m}$ when $n \leq m$.

Each $R_{n,n}$ is a subgroupoid of $G(X,\sigma)$ whose $C^*$-algebra, $C^*(R_{n,n},\lambda)$, is the cross-sectional $C^*$-algebra of a matrix bundle over $X$ and so is a $C^*$-algebra with continuous trace. The union $\bigcup_n R_{n,n} := R_\infty$ is also a subgroupoid of $G(X,\sigma)$ whose $C^*$-algebra, $C^*(R_\infty,\lambda)$, is the inductive limit, $\varinjlim C^*(R_{n,n},\lambda)$. These facts about the equivalence relations $R_{n,n}$ and their $C^*$-algebras are proved and developed further in [30].

The set $R_{1,0} = Z(X,X,1,0) = \{(x,1,y) \mid y = \sigma(x)\}$ plays a central role in the theory because it is essentially the graph of $\sigma$. This observation leads to the straightforward calculation:

$$1_{R_{1,0}} * (1_{R_{1,0}})^*(x,k-l,y) = |\sigma^{-1}(\sigma(x))| 1_{R_{1,1}}(x,k-l,y), \tag{5-2}$$

where $|\sigma^{-1}(\sigma(x))|$ is the cardinality of $\sigma^{-1}(\sigma(x))$. Evidently, $x \to |\sigma^{-1}(\sigma(x))|$ is an integer-valued function in $C(X)$, and so is finitely valued and constant on the connected components of $X$. Further, it is strictly positive since $\sigma$ is surjective. Its *reciprocal* is denoted $D_0$. Equation (5-2) shows that $S$ is an isometry with range projection $1_{R_{1,1}}$.

DEFINITION 5.2. The function $S$ in $C_c(G(X,\sigma))$ defined by

$$S(x,k-l,y) := D_0^{\frac{1}{2}}(x)\cdot 1_{R_{1,0}}(x,k-l,y), \quad (x,k-l,y) \in G(X,\sigma),$$

is called the *master isometry* in $C_c(G(X,\sigma))$ and $D_0$ is called its *potential*.

To explain where this terminology comes from, we specialize Remark 2.2 and the discussion around it to the case where $Y = X$ and $\pi = \sigma$. This frees us to use $\pi$ to denote the *unital endomorphism* of $C(X)$ given by composition with $\sigma$. Since $\sigma$ is surjective, $\pi$ is injective. Observe that $C(X)$ may be viewed as a module over itself in two different ways: the *regular module*, $C(X)_{C(X)}$, in which $C(X)$ acts on itself via pointwise multiplication—denoted simply by a \cdot, ·—and the $\pi$-*induced module*, $C(X)_{\pi(C(X))}$, whose multiplication, $*$, is defined through composition with $\pi$: $f * a := f\cdot\pi(a)$, $f,a \in C(X)$. We call an algebraic homomorphism from the module $C(X)_{C(X)}$ to the module $C(X)_{\pi(C(X))}$ that maps nonnegative functions to nonnegative functions a $\sigma$-*module map* on $C(X)$.

The following proposition may well be known to some, but since we do not know a precise reference and since it is an essential part of our analysis, we supply a proof.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

PROPOSITION 5.3. *Every $\sigma$-module map $\mathcal{L}$ on $C(X)$ is determined by a unique nonnegative continuous function $\psi$ on $X$ through the formula*

$$\mathcal{L}(a)(x) = \sum_{\sigma(y)=x} \psi(y)a(y), \quad a \in C(X). \tag{5-3}$$

PROOF. Every positive linear map $\mathcal{L}$ on $C(X)$ is given by a unique family of measures $\alpha = \{\alpha_x\}_{x\in X}$ via the formula $\mathcal{L}(a)(x) = \int a(y)\, d\alpha_x(y)$. If $\mathcal{L}$ is a module map, then $\operatorname{supp}\alpha_x \subseteq \sigma^{-1}(x)$. Indeed, assume to the contrary that there are $x, z \in X$ such that $z \in \operatorname{supp}\alpha_x$ and $\sigma(z) \neq x$. We can then find an open set $U_z$ such that $\sigma|_{U_z}$ is a homeomorphism onto $\sigma(U_z)$ and $x \notin \sigma(U_z)$. Let $a \in C(X)$ be such that $a(\sigma(z)) = 1$ and $a(y) = 0$ if $y \notin \sigma(U_z)$. It follows that $\mathcal{L}(\pi(a))(x) = \mathcal{L}(a \circ \sigma)(x) > 0$ since $z \in \operatorname{supp}\alpha_x$. However,

$$\mathcal{L}(\pi(a))(x) = \mathcal{L}(1 \cdot \pi(a))(x) = (\mathcal{L}(1) \cdot a)(x) = \mathcal{L}(1)(x)a(x) = 0.$$

This is a contradiction. Hence, $\operatorname{supp}\alpha_x \subseteq \sigma^{-1}(x)$ for all $x \in X$. Observe next that since $X$ is compact, $\sigma^{-1}(x)$ is a finite set for each $x \in X$. Consequently, there is a nonnegative function $\Delta$ on $X \times X$ such that

$$\mathcal{L}(f)(x) = \int_X f(y)\, d\alpha_x(y) = \sum_{\sigma(y)=x} f(y)\Delta(x, y). \tag{5-4}$$

To prove that $\Delta$ is continuous, let $U$ be an open subset of $X$ on which $\sigma$ is a homeomorphism and let $V$ be an open subset of $U$ such that $\overline{V} \subsetneq U$. Let $f \in C(X)$ such that $f \equiv 1$ on $\overline{V}$ and $f \equiv 0$ on the complement of $U$. Then, $\mathcal{L}(f) \in C(X)$ and (5-4) yields

$$\mathcal{L}(f)(x) = \Delta(x, y) \quad \text{if } x \in \sigma(V) \text{ and } y = \sigma^{-1}(x) \in V.$$

Therefore, $\Delta$ is continuous on open sets of the form $\sigma(V) \times V$. Since the cograph of $\sigma$ is covered by such sets, $\Delta$ is supported and continuous on the cograph of $\sigma$. Therefore, the function $\psi$ defined via the equation $\psi(y) := \Delta(\sigma(y), y)$ is continuous and nonnegative. Moreover, (5-4) implies that

$$\mathcal{L}(f)(x) = \sum_{\sigma(y)=x} f(y)\Delta(x, y) = \sum_{\sigma(y)=x} f(y)\Delta(\sigma(y), y) = \sum_{\sigma(y)=x} \psi(y)f(y).$$

The converse assertion and the uniqueness of $\psi$ are immediate. □

REMARKS 5.4.

(1) In the statistical mechanics literature (cf. [5, 35]), an operator of the form defined in (5-3) is called a *transfer operator* (associated with the dynamical system defined by $\sigma$) and $\psi$ is referred to as its *potential*. We adopt that terminology here. However, because of certain conventions in the groupoid theory literature, we have occasions to use capital $D$ as well as $\psi$ to denote potentials.

(2) The theory of transfer operators plays a large role in groupoid harmonic analysis thanks to the pioneering work of Ola Bratteli and Palle Jorgensen [6], Ruy Exel

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

[12], Jean Renault [30, 31], and many others. Our objective here is simply to observe that from the defining properties of a module map, the theory of transfer operators arrives fully formed, much like Athena from the forehead of Zeus.

(3) Analysis of transfer operators is the key to building the *adjoining functions* of Holkar [14, Definition 2.1] that parameterize wavelets from our groupoid perspective.

(4) Observe that $\mathcal{L}(1)(x) = \sum_{\sigma(y)=x} \psi(y) = \alpha_x(X)$. So, $\{\alpha_x\}_{x\in X}$ is a proper system of measures on $X$ in the sense of Definition 2.1 if and only if $\mathcal{L}(1)$ is a strictly positive function, in which case, we say $\mathcal{L}$ *is proper*. All transfer operators considered in this paper are assumed to be proper.

PROPOSITION 5.5. *Let $\mathcal{L}_\psi$ be the transfer operator with potential $\psi$ associated to the local homeomorphism $\sigma$ and the endomorphism $\pi$ of $C(X)$ it determines. Then, the following assertions are equivalent.*

(1) $\mathcal{L}_\psi(1) = 1$, *that is, $\mathcal{L}_\psi$ is a unital map.*
(2) *$\mathcal{L}_\psi$ is a left inverse for $\pi$.*
(3) *$\pi \circ \mathcal{L}_\psi$ is a conditional expectation onto the range of $\pi$, that is, $\pi \circ \mathcal{L}_\psi$ acts like the identity on $\pi(C(X))$.*

If $\mathcal{L}_\psi$ satisfies any of these three equivalent conditions, we call $\psi$ a *normalized* potential.

PROOF. If $\mathcal{L}_\psi(1) = 1$, then for all $f \in C(X)$,

$$(\mathcal{L}_\psi \circ \pi)(f)(x) = \sum_{\sigma(y)=x} \psi(y)\pi(f)(y) = \sum_{\sigma(y)=x} \psi(y) f(\sigma(y))$$
$$= \sum_{\sigma(y)=x} \psi(y) f(x) = f(x)\Big( \sum_{\sigma(y)=x} \psi(y)\Big) = f(x),$$

so $\mathcal{L}_\psi$ is a left inverse of $\pi$. Write $\iota$ for the identity map on $C(X)$. Then, the assertion that $\mathcal{L}_\psi$ is a left inverse of $\pi$ is equivalent to the equation $\mathcal{L}_\psi \circ \pi = \iota$. In this event, $(\pi \circ \mathcal{L}_\psi) \circ (\pi \circ \mathcal{L}_\psi) = \pi \circ (\mathcal{L}_\psi \circ \pi) \circ \mathcal{L}_\psi = \pi \circ \iota \circ \mathcal{L}_\psi = \pi \circ \mathcal{L}_\psi$. So, $\pi \circ \mathcal{L}_\psi$ is a positive idempotent map on $C(X)$ such that $(\pi \circ \mathcal{L}_\psi) \circ \pi = \pi \circ (\mathcal{L}_\psi \circ \pi) = \pi$. Thus, $\pi \circ \mathcal{L}_\psi$ is an idempotent map that acts like the identity on the range of $\pi$, that is, $\pi \circ \mathcal{L}_\psi$ is a conditional expectation onto the range of $\pi$. Finally, if $\pi \circ \mathcal{L}_\psi$ is a conditional expectation onto the range of $\pi$, $(\pi \circ \mathcal{L}_\psi) \circ \pi = \pi$. However, then, $\pi \circ (\mathcal{L}_\psi \circ \pi) = \pi$. Since $\pi$ is injective, $\mathcal{L}_\psi \circ \pi = \iota$, that is, $\mathcal{L}_\psi$ is a left inverse of $\pi$. However, this manifestly implies $\mathcal{L}_\psi(1) = 1$. □

DEFINITION 5.6. A potential is called *full* in the case where it is nonvanishing.

The following proposition is part of Jean Renault's [30, Proposition 3.1].

PROPOSITION 5.7. *Every full normalized potential $D$ satisfies an equation of the form*

$$D = D_0 \cdot b,$$

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

*where $b$ is a nonvanishing function in $\pi(C(X))$ and $D_0$ is the potential of the master isometry from Definition 5.2.*

REMARK 5.8. For our purposes, every (proper) potential may be taken to be normalized. For if $\psi$ is a potential and if

$$D := \psi/\mathcal{L}_\psi(1) \circ \sigma,$$

then $D$ is a normalized potential that vanishes precisely at the same points as $\psi$. Full normalized potentials may always be taken to be $D_0$. Sections 10 and 11 are devoted to constructing nonfull normalized potentials from full nonnormalized potentials. This is how we build *scaling functions* and associated *mother wavelets*, that is, *quadrature mirror filters*.

We want to draw attention, also, to some additional relations that hold between $\pi$ and one of its left inverses $\mathcal{L}$.

(1) First, recall that by Proposition 5.5, $\pi \circ \mathcal{L}$ is a conditional expectation of $C(X)$ onto the range of $\pi$ and every conditional expectation of $C(X)$ onto the range of $\pi$ is of this form for a suitable $\mathcal{L}$. So, $\pi^n \circ \mathcal{L}^n := \boldsymbol{E}_n$ is a conditional expectation onto the range of $\pi^n$ for every nonnegative integer $n$.

(2) Further still, we have the relations

$$\boldsymbol{E}_n\boldsymbol{E}_m = \boldsymbol{E}_m\boldsymbol{E}_n = \boldsymbol{E}_n, \quad m \le n.$$

That is, the sequence $\{\boldsymbol{E}_n\}_{n=0}^{\infty}$ is a decreasing sequence of projections *on* the $C^*$-algebra $C(X)$, sometimes called a *reverse martingale*. We want to understand their relationship with the representation theory of $C_c(G(X,\sigma),\lambda)$ that we develop later in this paper.

(3) Observe that computations like those in (5-3) show *every* normalized potential, full or not, leads to an isometry in the convolution algebra of $G(X,\sigma)$. In fact, one may do a bit better. If $\mathfrak{u}$ is a continuous function on $X$ such that $|\mathfrak{u}|^2$ is a normalized potential $D$, and if

$$S_{\mathfrak{u}}(x,k-l,y) := \mathfrak{u}(x)1_{R_{1,0}}(x,k-l,y), \tag{5-5}$$

then (5-3) shows that $S_{\mathfrak{u}}$ satisfies

$$f = S_{\mathfrak{u}}^* * (f \circ \sigma) * S_{\mathfrak{u}} = S_{\mathfrak{u}}^* * \pi(f) * S_{\mathfrak{u}} = \mathcal{L}_D \circ \pi(f)$$

for $f \in C(X), |\mathfrak{u}|^2 = D$. Additionally, in general, $\mathcal{L}_D^n(f) = (S_{\mathfrak{u}}^*)^n * f * (S_{\mathfrak{u}})^n$.

(4) Finally, we have the easily proved fact that if the $\mathcal{L}$ leaves a probability measure $\mu$ invariant, the conditional expectations $\boldsymbol{E}_n$ satisfy the equations

$$\int_X \boldsymbol{E}_n(f)g\,d\mu = \int_X f\boldsymbol{E}_n(g)\,d\mu, \quad f,g \in C(X). \tag{5-6}$$

It is convenient to regard the isometry $S_{\mathfrak{u}}$ as an element of $\mathcal{L}(\mathcal{X}_0)$, where $\mathcal{X}_0$ is the $C^*$-correspondence defined by the equivalence $(G,\alpha)$ between $(G,\lambda)$ and $(G,\lambda)$

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

as described in the paragraph preceding Section 3. There, the operator $S_{\mathfrak{u}} \in \mathcal{L}(\mathcal{X}_0)$ is defined via

$$S_{\mathfrak{u}}\xi(x,t,y) = \mathfrak{u}(x)\xi(\sigma(x),t-1,y). \tag{5-7}$$

It may seem that we have introduced confusion by using '$S_{\mathfrak{u}}$' to denote both an isometry in $C^*(G,\lambda)$ and the isometry from (5-7) on $\mathcal{L}(\mathcal{X}_0)$. However, this ambiguity should cause no significant problem. The two isometries are the same, up to a unitary equivalence between the two Hilbert modules involved.

## 6. Pullbacks by projective limits

In nearly every operator-theoretic approach to wavelets that we have encountered, there are isometries like our $S_{\mathfrak{u}}$ and efforts are made to understand unitary extensions of them. We want to do that here as well. However, we also want to emphasize that the extensions we produce do not live in $C^*(G(X,\sigma))$, but in larger $C^*$-algebras that we reveal as certain pullbacks. Their description requires several ingredients.

The first ingredient is a pullback of $G(X,\sigma)$ in the sense of Proposition 3.1 that is determined by a pair consisting of a space $X_\infty$ and map $p : X_\infty \to X$. The space $X_\infty$ is the projective limit space induced by $\sigma$:

$$X_\infty := \{(x_0,x_1,x_2,\ldots) \in X^\infty \mid x_i = \sigma(x_{i+1})\}.$$

Here, of course, $X^\infty$ is the infinite product of copies of $X$ indexed by $\mathbb{N}$. Since $X$ is compact, and $\sigma$ is continuous and surjective, $X_\infty$ is a nonempty closed, and therefore compact, subset of $X^\infty$. We write its elements as $\underline{x} := (x_0,x_1,\ldots)$. The map $p$ is simply the projection onto the first coordinate: $p(\underline{x}) := x_0$. Since $\sigma$ is open, $p$ is also open, as well as continuous, and it is surjective because $\sigma$ is surjective.

So, using the notation from Proposition 3.1, we write the pullback of $G(X,\sigma)$ by $X_\infty$ as $X_\infty * G(X,\sigma) * X_\infty$. To shorten the notation, we usually write $G_\infty(X,\sigma)$ for this groupoid. We also use some self-evident variations of this notation.

The main purpose of $X_\infty$ is to carry an invertible 'extension' of $\sigma$. Define $\sigma_\infty$ on $X_\infty$ by the formula

$$\sigma_\infty(\underline{x}) := (\sigma(x_0),x_0,x_1\cdots), \quad \underline{x} = (x_0,x_1\cdots) \in X_\infty.$$

The inverse of $\sigma_\infty$ is given by the equation

$$\sigma_\infty^{-1}(x_0,x_1,x_2\cdots) := (x_1,x_2\cdots).$$

The map $\sigma_\infty$ is not really an *extension* of $\sigma$, but it does satisfy the important 'intertwining' equation

$$\sigma \circ p = p \circ \sigma_\infty.$$

By Proposition 3.1, $G_\infty(X,\sigma)$ is a locally compact Hausdorff groupoid with unit space $X_\infty$. The range and source maps are the obvious projections that are, however, not local homeomorphisms. Consequently, $G_\infty(X,\sigma)$ is not étale. So, we must find

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

Haar systems that are not given by counting measures. These are built from potentials for transfer operators associated to $\sigma$ (see Remark 5.4).

Using a normalized potential $D$ on $X$, one can construct a system of probability measures on $G_\infty(X,\sigma)$, indexed by $X_\infty$, that is denoted $\nu^D * \lambda := \{(\nu^D * \lambda)^{\underline{x}}\}_{\underline{x}\in X_\infty}$. The symbol $\nu^D$ by itself refers to a system of measures $\{\nu^D_{x_0}\}_{x_0\in X}$ on $X_\infty$ that is parameterized by $X$ and is defined as follows.

Recall that the product topology on $X^\infty$ is such that the collection of complex-valued functions of the form $f = f_0 \cdot f_1 \cdot f_2 \cdots f_n$, where the $f_i$ lie in $C(X)$ and

$$f(\underline{x}) := f_0(x_0)\cdot f_1(x_1)\cdot f_2(x_2)\cdots f_n(x_n), \quad \underline{x} = (x_0, x_1, x_2, \cdots) \in X^\infty,$$

generate $C(X^\infty)$. Consequently, their restrictions to $X_\infty$ generate $C(X_\infty)$. Recall, too, that Kolmogorov's extension theorem [34, Appendix A.3] guarantees that the formula

$$\begin{aligned}&\int_{X_\infty} f_0(x_0)\cdot f_1(x_1)\cdots f_n(x_n)\, d\nu^D_{x_0}(\underline{x})\\ &\quad := f_0(x_0)\Big(\sum_{\sigma(x_1)=x_0} D(x_1)f_1(x_1)\Big(\sum_{\sigma(x_2)=x_1} D(x_2)f_2(x_2)\cdots\Big(\sum_{\sigma(x_n)=x_{n-1}} D(x_n)f_n(x_n)\Big)\cdots\Big)\Big),\end{aligned} \tag{6-1}$$

which is valid for all $f = f_0 f_1 \cdots f_n$ with $f_i \in C(X)$, yields a well-defined probability measure supported on $p^{-1}(x_0)$ that we have denoted $\nu^D_{x_0}$.

Equation (6-1) becomes more manageable when one uses the transfer operator $\mathcal{L}_D$ to abbreviate the sums. The result is

$$\begin{aligned}&\int_{X_\infty} f_0(x_0)\cdot f_1(x_1)\cdots f_n(x_n)\, d\nu^D_{x_0}(\underline{x})\\ &\quad := (f_0\mathcal{L}_D(f_1\mathcal{L}_D(f_2\cdots\mathcal{L}_D(f_{n-1}\mathcal{L}_D(f_n))\cdots)))(x_0),\end{aligned} \tag{6-2}$$

which not only makes it clear that $\nu^D_{x_0}$ is a probability measure on $X_\infty$, it shows also that the family $\{\nu^D_{x_0}\}_{x_0\in X}$ may be viewed as an element of $C(X, \mathfrak{M}_1(X_\infty))$, where $\mathfrak{M}_1(X_\infty)$ is the space of all probability measures on $X_\infty$ with the weak-$*$ topology. Thus, $\{\nu^D_{x_0}\}_{x_0\in X}$ is a continuous family of probability measures on $X_\infty$ indexed by $X$. Because the family $\nu^D = \{\nu^D_{x_0}\}_{x_0\in X}$ is a system of measures on $X_\infty$ that is fibered over $X$ by $p$ (that is, each $\nu^D_{x_0}$ is supported on $p^{-1}(x_0), x_0 \in X$), we refer to $\nu^D$ as the *p-system of measures on* $X_\infty$ *determined by* $D$.

Finally, letting $\lambda := \{\lambda^x\}_{x\in X}$ denote the counting measure Haar system on $G(X,\sigma)$ (paragraphs following Definition 5.1), the formula

$$\begin{aligned}&\int_{G_\infty(X,\sigma)} f(\underline{x},(p(\underline{x}),t,p(\underline{y})),\underline{y})\, d(\nu^D * \lambda)^{\underline{x}}\\ &\quad := \int_{G(X,\sigma)}\int_{X_\infty} f(\underline{x},(p(\underline{x}),t,p(\underline{y})),\underline{y})\, d\nu^D_{p(\underline{y})}(\underline{y})\, d\lambda^{p(\underline{x})}(p(\underline{x}),t,p(\underline{y})),\end{aligned} \tag{6-3}$$

where $f \in C_c(G_\infty(X,\sigma))$, defines the system of measures $\nu^D * \lambda := \{(\nu^D * \lambda)^{\underline{x}}\}_{\underline{x}\in X_\infty}$ on $G_\infty(X,\sigma)$. Close inspection reveals that (6-3) is a special case of (3-1), that is,

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

$\nu^D * \lambda = \beta$ specialized to $G_\infty(X, \sigma)$. Thus, $\nu^D * \lambda$ is a Haar system on $G_\infty(X, \sigma)$ if $\nu^D$ is full. The following proposition describes when this happens.

PROPOSITION 6.1. *$\nu^D$ is full if and only if $D$ is nonvanishing, that is, if and only if $D$ is a full potential.*

PROOF. Observe that (6-2) implies that $\nu^D_{x_0}((U_0 \times U_1 \times X \times \cdots) \cap p^{-1}(x_0)) = 1_{U_0}(x_0) \times \mathcal{L}_D(U_1 \cap p^{-1}(x_0))$. So, if $(U_0 \times U_1) \times X \times \cdots) \cap p^{-1}(x_0)$ is nonempty, then $x_0 \in U_0$ and there must exist a $y \in \sigma^{-1}(x_0) \cap U_1$. It follows that $\nu^D_{x_0}(U_0 \times U_1 \times X \cdots) \cap p^{-1}(x_0) \geq D(y) > 0$. Thus, $\nu^D_{x_0}$ gives positive measure to each open set that meets $p^{-1}(x_0)$, which means that $\nu^D$ is full. For the converse, suppose $D(y) = 0$ for some $y \in \sigma^{-1}(x_0)$. Then, there is an open $U_1$ such that $y \in U_1$ and $\sigma$ restricted to $U_1$ is a homeomorphism onto an open set $U_0$. Then, $\nu^D_{x_0}(U_0 \times U_1 \times X \times X \cdots) = D(y) = 0$. □

From now on, we usually drop the subscript, 0, on $\nu^D_{x_0}$.

## 7. $p$-systems of measures and conditional expectations

We begin with a continuation of the paragraph following Definition 5.2.

First, note that the defining formula for $\nu^D$ reveals it to be a map from $C(X_\infty)$ to $C(X)$ that is linear, positive, and unital:

$$\nu^D(f)(x) := \int_{X_\infty} f(\underline{y})\, d\nu^D_x(\underline{y}), \quad f \in C(X_\infty). \tag{7-1}$$

Most important for our purposes, (7-1) makes it clear that $\nu^D$ is a left inverse of the map $p_* : C(X) \to C(X_\infty)$ that is dual to $p$:

$$p_*(f)(\underline{x}) := f(p(\underline{x})), \quad f \in C(X).$$

Indeed,

$$\begin{aligned}[(\nu^D \circ p_*)(f)](x) = [\nu^D(p_*(f))](x) &= \int_{X_\infty} p_*(f)(\underline{y})\, d\nu^D_x(\underline{y}) \\ &= \int_{X_\infty} f(p(\underline{y}))\, d\nu^D_x(\underline{y}) = \int_{X_\infty} f(x)\, d\nu^D_x(\underline{y}) = f(x).\end{aligned} \tag{7-2}$$

The penultimate equation reflects the fact that each $\nu^D_x$ is a probability measure on $X_\infty$ supported on $p^{-1}(x)$. Consequently, also, (7-2) shows that $\nu^D$ is a surjective map from $C(X_\infty)$ onto $C(X)$.

In analogy with the notation following Remark 5.8, we find that $p_* \circ \nu^D$ is a conditional expectation, $\mathcal{E}_0$, on $C(X_\infty)$, whose definition expressed in terms of integrals is

$$\begin{aligned}\mathcal{E}_0(f)(\underline{x}) = [(p_* \circ \nu^D)(f)](\underline{x}) &= [p_*(\nu^D(f))](\underline{x}) \\ &= (\nu^D(f))(p(\underline{x})) = \int_{X_\infty} f(\underline{y})\, d\nu^D_{p(\underline{x})}(\underline{y}), \quad f \in C(X_\infty).\end{aligned} \tag{7-3}$$

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

This equation shows that $\mathcal{E}_0$ is a unital, positive map from $C(X_\infty)$ to $C(X_\infty)$, while the fact that $\mathcal{E}_0$ is idempotent is immediate from (7-2).

Finally, the fact that the range of $\mathcal{E}_0$ consists of all functions in $C(X_\infty)$ that depend only on the first variable is clear from (7-3). Evidently, such functions form an algebra isomorphic to $C(X)$, and $\mathcal{E}_0$ is a bimodular map over it. Consequently, we informally, but unambiguously, write $\mathcal{E}_0(a \cdot f \cdot b) = a \cdot \mathcal{E}_0(f) \cdot b$ for all $f \in C(X_\infty)$ and $a, b \in C(X)$.

The following equation arises later in our analysis (see (9-2)). We believe it is helpful to parse it now. For $f \in C(X_\infty)$,

$$\sum_{\sigma(z)=x} D(z) \int_{X_\infty} f(\sigma_\infty(\underline{z}))\, d\nu_z^D(\underline{z}) = \int_{X_\infty} f(\underline{x})\, d\nu_x^D(\underline{x}) \tag{7-4}$$

for all $x \in X$. This equation may be easier to understand if it is rewritten in terms of the maps that are implicit in it. First, let $\pi_\infty$ be defined by the formula

$$\pi_\infty(f) := f \circ \sigma_\infty, \quad f \in C(X_\infty).$$

Thus, $\pi_\infty$ is an analog of the endomorphism $\pi$ of $C(X)$ defined in Proposition 5.7. This time, however, $\pi_\infty$ is an automorphism of $C(X_\infty)$, and its inverse is given by composing with $\sigma_\infty^{-1}$, that is, $\pi_\infty^{-1}(f) = f \circ \sigma_\infty^{-1}$ for all $f \in C(X_\infty)$. So, now we can parse (7-4) in terms of maps:

$$\mathcal{L}_D \circ \nu^D \circ \pi_\infty = \nu^D, \tag{7-5}$$

or a bit more symmetrically, we may write

$$\mathcal{L}_D \circ \nu^D = \nu^D \circ \pi_\infty^{-1}. \tag{7-6}$$

Keep in mind that both sides of each of these equations are maps from $C(X_\infty)$ to $C(X)$.

For our purposes here, the most important role of the $p$-system of measures $\nu^D$ is revealed in Equation (7-10), below, that is based on the following lemma.

LEMMA 7.1. *The map $\Phi : G_\infty(X, \sigma) \to G(X, \sigma)$ defined by the equation*

$$\Phi((\underline{x}, k-l, \underline{y})) := (p(\underline{x}), k-l, p(\underline{y})), \quad (\underline{x}, k-l, \underline{y}) \in G_\infty(X, \sigma),$$

*is a continuous, open, proper and surjective groupoid homomorphism mapping the pullback groupoid $G_\infty(X, \sigma)$ onto $G(X, \sigma)$.*

PROOF. Since $p$ is continuous and open, it is clear that $\Phi$ is also. Furthermore, it is clear that $\Phi$ is a surjective groupoid homomorphism mapping $G_\infty(X, \sigma)$ onto $G(X, \sigma)$. Since $G_\infty(X, \sigma)$ and $G(X, \sigma)$ are locally compact and Hausdorff, to show that $\Phi$ is proper, it suffices to show that for each compact subset $K \subset G(X, \sigma)$, $\Phi^{-1}(K)$ is compact in $G_\infty(X, \sigma)$. For this purpose, simply note that since each of the sets $R_{n,m}$ from (5-1) is compact and open in $G(X, \sigma)$, and since $G(X, \sigma)$ is covered by the family of them, any compact subset of $G(X, \sigma)$ is covered by a finite number of them. However, also note that $\Phi^{-1}(R_{n,m})$ is clearly compact in $G_\infty(X, \sigma)$. It follows that $\Phi$ is proper. □

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

As a consequence of this lemma, $\Phi_*$, defined by the equation

$$\Phi_*(f) := f \circ \Phi, \quad f \in C_c(G, \sigma), \tag{7-7}$$

is a linear map from $C_c(G, \sigma)$ into $C_c(G_\infty, \sigma)$ that is continuous with respect to the inductive limit topologies on each space. It is also clear from (6-3) that for all $f \in C_c(G(X, \sigma))$,

$$\int_{G_\infty(X,\sigma)} \Phi_*(f)(\underline{x}, t, \underline{y})\, d(\nu^D * \lambda)^{\underline{x}} := \int_{G(X,\sigma)} f(x_0, t, y_0)\, d\lambda^{x_0}(x_0, t, y_0). \tag{7-8}$$

A succinct way to express (7-8) is to write everything in terms of the maps determined by the systems of measures involved:

$$p_* \circ \lambda = (\nu^D * \lambda) \circ \Phi_*, \tag{7-9}$$

where $\lambda := \{\lambda^x\}_{x \in X}$ is viewed as a map from $C_c(G(X, \sigma))$ to $C(X)$, $p_* : C(X) \to C(X_\infty)$ is the map dual to $p$, $\nu^D * \lambda$ is the map determined by the system $\{\nu^D_{p(\underline{x})} * \lambda^{p(\underline{x})}\}_{\underline{x} \in X_\infty}$, and $\Phi_*$ is the map dual to $\Phi$ acting on functions.

The image of $\Phi_*$ thus consists of those functions in $C_c(G_\infty(X, \sigma), \nu^D * \lambda)$ that depend only on the zeroth-coordinate. The equations in the paragraphs following the proof of Lemma 7.1, in particular, (7-9), make it clear that $\Phi_*$ defines an *algebra isomorphism* from $C_c(G(X, \sigma), \lambda)$ onto its image in $C_c(G_\infty(X, \sigma), \nu^D * \lambda)$ that is a homeomorphism in the inductive limit topologies involved.

Recall that $1_{R_{0,0}}$ is the identity of $C_c(G(X, \sigma), \lambda)$ (see the paragraphs following (5-1)). Consequently, $\Phi_*(1_{R_{0,0}})$ is a projection *in* $C_c(G_\infty(X, \sigma), \nu^D * \lambda)$. However, a projection *in* a $*$-algebra determines a projection *on* the algebra via 'cornering'. So, in our setting, $f \to \Phi_*(1_{R_{0,0}}) * f * \Phi_*(1_{R_{0,0}})$ defines an idempotent, positivity-preserving, linear map, which we denote by $\mathbb{E}$, whose range is the image of $\Phi_*$. In fact, it is a straightforward calculation to see that

$$\begin{aligned} \mathbb{E}(f)(\underline{x}, k-l, \underline{y}) &:= \Phi_*(1_{R_{0,0}}) * f * \Phi_*(1_{R_{0,0}})(\underline{x}, k-l, \underline{y}) \\ &= \int_{X_\infty} \int_{X_\infty} f(\underline{u}, k-l, \underline{v})\, d\nu^D_{p(\underline{x})}(\underline{u})\, d\nu^D_{p(\underline{y})}(\underline{v}) \end{aligned} \tag{7-10}$$

for all $f \in C_c(G_\infty(X, \sigma), \nu^D * \lambda)$. This equation, in turn, coupled with (7-3), justifies our writing

$$\mathbb{E} = \mathcal{E}_0 \otimes \iota \otimes \mathcal{E}_0,$$

restricted to $C_c(G_\infty(X, \sigma))$, where we are regarding $G_\infty(X, \sigma)$ as a locally closed subset of $X_\infty \times \mathbb{Z} \times X_\infty$ and where $\iota$ denotes the identity map on $C_c(\mathbb{Z})$.

It is also easy to see that $\mathbb{E}$ is bimodular in the sense that $\mathbb{E}(\Phi_*(b_1) * f * \Phi_*(b_2)) = \Phi_*(b_1) * \mathbb{E}(f) * \Phi_*(b_2)$ for all $f \in C_c(G_\infty(X, \sigma), \nu^D * \lambda)$ and $b_i \in C_c(G(X, \sigma), \lambda)$, $i = 1, 2$.

Thus, $\mathbb{E}$ has all the trappings of a conditional expectation, except one: $\mathbb{E}$ is not unital. That is not important here. The defining equation (7-10) means that $\mathbb{E}$ is a 'cornering'

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

projection and so extends to a completely positive, contractive, and idempotent map on any $C^*$-algebra that is generated by an $*$-representation of $C_c(G_\infty(X,\sigma), \nu^D * \lambda)$.

## 8. Representations of $G_\infty(X,\sigma)$ induced from $G(X,\sigma)$

We fix a full normalized potential $D$ and we consider the Haar system $\nu^D * \lambda$ on $G_\infty(X,\sigma)$ determined by $D$. This Haar system on $G_\infty(X,\sigma)$ is fixed throughout the section. We induce representations of $C^*(G(X,\sigma),\lambda)$ to representations of $C^*(G_\infty(X,\sigma),\nu^D * \lambda)$ using nonfull normalized potentials derived from $D$ via the techniques developed in Section 4. Since $X$ and $\sigma$ are fixed, we write $G$ for $G(X,\lambda)$ and $G_\infty$ for $G_\infty(X,\sigma)$ when the notation becomes cluttered.

We consider a nonnegative continuous function $b : X \to \mathbb{R}_+$ such that the function $D_b(x) := b(x)D(x)$ is another normalized potential for $\sigma$, not necessarily full: $\sum_{\sigma(y)=x} D_b(y) = 1$ for all $x \in X$. The potential $D_b$ defines a $p$-system of measure $\nu^{D_b}$ on $X_\infty$ via (6-1) or, equivalently, (6-2). Moreover, $\nu_u^{D_b}$ is absolutely continuous with respect to $\nu_u^D$ since

$$\int_{X_\infty} f_0(x_0)f_1(x_1)\cdots f_n(x_n)\, d\nu_{x_0}^{D_b}(\underline{x}) = \int_{X_\infty} f_0(x_0)f_1(x_1)\cdots f_n(x_n)\prod_{i=1}^{n} b(x_i)\, d\nu_{x_0}^{D}(\underline{x})$$

for all $f = f_0 f_1 \cdots f_n$ with $f_i \in C(X)$ and $n \geq 0$. We let $\mathrm{d}$ be the Radon–Nykodim derivative such that $\nu_u^{D_b} = \mathrm{d}\nu_u^D$ for all $u \in X$. In the notation of (4-1), one would write $\nu_{\mathrm{d}}^{D_b}$ for $\nu^{D_b}$. We prefer the new notation since $\nu_{\mathrm{d}}^{D_b}$ is hard to read and $\mathrm{d}$ is determined by $b$. We define a groupoid correspondence $(Z,\alpha_b)$ from $(G_\infty(X,\sigma),\nu^D * \lambda)$ to $(G,\lambda)$ using the framework described in the paragraphs preceding Lemma 3.3, where

$$Z := X_\infty {}_p*_r G(X,\sigma) = \{(\underline{x},(p(\underline{x}),t,y)) \mid \underline{x} \in X_\infty, (p(\underline{x}),t,y) \in G(X,\sigma)\},$$

and $\alpha_b = \nu^{D_b} * \lambda$.

Our goal now is to relate the representation theory of $G(X,\sigma)$ to the representation theory of $G_\infty(X,\sigma)$ using the potential $D_b$. It is fixed for the remainder of this section.

Recall that $C_c(Z)$ can be completed to a $C^*$-correspondence that we denote $\mathcal{X}_\infty$ from $C^*(G_\infty(X,\sigma),\nu^D * \lambda)$ to $C^*(G(X,\sigma),\lambda)$ via (4-4), (4-5), and (4-6). In particular, $C^*(G_\infty(X,\sigma),\nu^D * \lambda)$ acts as multipliers on $\mathcal{X}_\infty$ via

$$\begin{aligned}\mathbb{L}(a)\xi(\underline{x},(p(\underline{x}),t,y)) &:= a\cdot\xi(\underline{x},(p(\underline{x}),t,y))\\ &= \int_{G_\infty} a(\underline{x},s,\underline{w})\xi(\underline{w},(p(\underline{w}),t-s,y))\mathrm{d}(\underline{w})^{1/2}/\mathrm{d}(\underline{x})^{1/2}\, d(\nu^D * \lambda)^{\underline{x}}(\underline{x},s,\underline{w}).\end{aligned} \tag{8-1}$$

Fix a representation $\hat{L} = (\mu, X * \mathcal{H}, \hat{L})$ of the groupoid $G$ and let $\mathfrak{L}$ be its integrated form representing $C^*(G,\lambda)$ on $L^2(X * \mathcal{H},\mu)$ as described in the paragraph containing (2-1). We want to explain how to induce $\hat{L}$, $\mathfrak{L}$, and all related structures to the pullback groupoid, $G_\infty(X,\sigma)$. The relevance of the normalized potential, $D_b$, is tracked in the process.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

The induced Hilbert bundle determined by $G^{(0)} * \mathcal{H} = X * \mathcal{H}$ is the bundle $X_\infty * \mathcal{H} = G_\infty(X,\sigma)^{(0)} * \mathcal{H}$ that is the pull back of $X * \mathcal{H}$ by $p$, i.e., the fiber $H_{\underline{x}} := H_{p(\underline{x})}$. This bundle is the same for all the other induced structures we consider. The isomorphism groupoid of $X_\infty * \mathcal{H}$, $\mathrm{Iso}(X_\infty * \mathcal{H})$, is similarly identified:

$$\mathrm{Iso}(X_\infty * \mathcal{H}) := \{(\underline{x}, U, \underline{y}) \mid U \in B_{\mathrm{Iso}}(H_{\underline{y}}, H_{\underline{x}})\} = \{(\underline{x}, U, \underline{y}) \mid U \in B_{\mathrm{Iso}}(H_{p(\underline{y})}, H_{p(\underline{x})})\}.$$

The representation $\hat{L}_\infty$ of $G_\infty(X,\sigma)$ on $X_\infty * \mathcal{H}$ induced by $\hat{L}$ is given by the formula

$$\hat{L}_\infty(\underline{x}, t, \underline{y}) = (\underline{x}, L_{\infty,(\underline{x},t,\underline{y})}, \underline{y}),$$

where

$$L_{\infty,(\underline{x},t,\underline{y})}(\mathfrak{h}(\underline{y})) := L_{(p(\underline{x}),t,p(\underline{y}))}\mathfrak{h}(p(\underline{y})), \quad (\underline{x}, t, \underline{y}) \in G_\infty, \mathfrak{h} \in X_\infty * \mathcal{H}.$$

Its action on $X_\infty * \mathcal{H}$, too, is independent of the potential $D_b$.

What depends upon $D_b$ is the quasi-invariant measure on $G_\infty^{(0)} = X_\infty$. Viewed as a map acting from $C(G_\infty^{(0)})$ to $\mathbb{C}$, it is defined by the equation

$$\mu_\infty^{D_b} := \mu \circ \nu^{D_b}.$$

Note that $\mu_\infty^{D_b}$ is well defined: $\nu^{D_b}$ is a map from $C(G_\infty^{(0)})$ to $C(G^{(0)}(X,\sigma)) = C(X)$ and $\mu$ is a map from $C(X)$ to $\mathbb{C}$. Recall from (4-2) that the modular function is

$$\Delta_\infty(\underline{x}, k-l, \underline{y}) = \Delta_\mu(p(\underline{x}), k-l, p(\underline{y}))\mathfrak{d}(\underline{x})/\mathfrak{d}(\underline{y}).$$

We write $\mathfrak{Q}_\infty$ for the integrated form of $L_\infty$, which is a $C^*$-representation of $C^*(G_\infty(X,\sigma), \nu^D * \lambda)$ on the space $L^2(X_\infty * \mathcal{H}_\infty, \mu_\infty^{D_b})$. For $f \in C_c(G_\infty)$, $\mathfrak{Q}_\infty(f)$ acts on $L^2(X_\infty * \mathcal{H}_\infty, \mu_\infty^{D_b})$ via

$$\begin{aligned}&\mathfrak{Q}_\infty(f)\mathfrak{f}(\underline{x})\\ &\quad = \int_{G_\infty} f(\underline{x}, k-l, \underline{y})L_{(p(\underline{x}),k-l,p(\underline{y}))}\mathfrak{f}(\underline{y})\Delta_\infty(\underline{x}, k-l, \underline{y})^{-1/2}\, d(\nu^D * \lambda)^{\underline{x}}(\underline{x}, k-l, \underline{y}) \quad (8\text{-}2)\end{aligned}$$

for all $\mathfrak{f} \in L^2(X_\infty * \mathcal{H}_\infty, \mu_\infty)$.

## 9. Proto-multiresolution analyses and their representations

In this section, we return to the elements of Rieffel's theory that are developed preceding Theorem 4.1. We continue with the same notation and we let $\mathfrak{u} \in C(X)$ be a function such that $|\mathfrak{u}|^2 = D_b$. Our first goal is to define a unitary operator $U_\mathfrak{u}$ on $\mathcal{X}_\infty$ that conjugates the image of $C^*(G_\infty(X,\sigma), \nu^D * \lambda)$ in $\mathcal{L}(\mathcal{X}_\infty)$ into itself and is a unitary extension of the isometry $S_\mathfrak{u}$ defined in (5-7). This unitary is used to generate what we call a proto-multiresolution analysis. In a sense we make clear, constructing a proto-multiresolution analysis constitutes the first step when building an MRA. Proto-multiresolution analyses put into evidence precisely the variables that can be 'tuned' to produce MRAs.

Again, we abbreviate $G(X,\sigma)$ to $G$ and $G_\infty(X,\sigma)$ to $G_\infty$.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

The putative unitary extension of $S_{\mathfrak{u}}$, $U_{\mathfrak{u}}$, is defined initially on $C_c(X_\infty * G)$ via

$$(U_{\mathfrak{u}}\xi)(\underline{x}, (p(\underline{x}), n, y)) := \mathfrak{u}(p(\underline{x}))\xi(\sigma_\infty(\underline{x}), (p(\sigma_\infty(\underline{x})), n-1, y)) \tag{9-1}$$

for all $\xi \in C_c(X_\infty * G)$ and $(\underline{x}, (p(\underline{x}), n, y))) \in X_\infty * G$.

Before proving that $U_{\mathfrak{u}}$ is a unitary extension of $S_{\mathfrak{u}}$, we first show that $U_{\mathfrak{u}}$ is an isometry and compute its adjoint. Let $\xi \in C_c(X_\infty * G)$ and compute, using (4-5), (7-4), and its compact form (7-5):

$$\begin{aligned}
&\langle U_{\mathfrak{u}}\xi, U_{\mathfrak{u}}\xi\rangle_{C_c(G)}(x, n, y)\\
&\quad := \int_G \int_{X_\infty} \overline{U_{\mathfrak{u}}\xi(\underline{z}, (p(\underline{z}), m, x))} U_{\mathfrak{u}}\xi(\underline{z}, (p(\underline{z}), m+n, y))\, d\nu_z^{D_b}(\underline{z})\, d\lambda_x(z, m, x)\\
&\quad = \int_G \int_{X_\infty} \overline{\mathfrak{u}(z)\xi(\sigma_\infty(\underline{z}), (p(\sigma_\infty(\underline{z})), m-1, x))}\\
&\qquad \cdots \mathfrak{u}(z)\xi(\sigma_\infty(\underline{z}), (p(\sigma_\infty(\underline{z})), m+n-1, y))\, d\nu_z^{D_b}(\underline{z})\, d\lambda_x(z, m, x)\\
&\quad = \int_G \sum_{\sigma(z)=t} D_b(z) \int_{X_\infty} \overline{\xi(\sigma_\infty(\underline{z}), (p(\sigma_\infty(\underline{z})), m-1, x))}\\
&\qquad \cdots \xi(\sigma_\infty(\underline{z}), (p(\sigma_\infty(\underline{z})), m+n-1, y))\, d\nu_z^{D_b}(\underline{z})\, d\lambda_x(t, m-1, x)\\
&\quad = \int_G \mathcal{L}_{D_b} \circ \nu_t^{D_b} \circ \pi_\infty(\overline{\xi(\cdot, (\cdot, m-1, x))}\xi(\cdot, (\cdot, m+n-1, y)))\, d\lambda_x(t, m-1, x)\\
&\quad = \int_G \nu_t^{D_b}(\overline{\xi(\cdot, (\cdot, m-1, x))}\xi(\cdot, (\cdot, m+n-1, y)))\, d\lambda_x(t, m-1, x)\\
&\quad = \int_G \int_{X_\infty} \overline{\xi(\underline{t}, (p(\underline{t}), m, x))}\xi(\underline{t}, (p(\underline{t}), m+n, y))\, d\nu_z^{D_b}(\underline{t})\, d\lambda_x(t, m, x)\\
&\quad = \langle \xi, \xi\rangle_{C_c(G)}(x, n, y),
\end{aligned} \tag{9-2}$$

where the penultimate equality follows by relabeling the integers, that is, by the invariance of the Haar system $\lambda$.

To complete the proof that $U_{\mathfrak{u}}$ is a unitary, we must examine first how the zero set of $D_b$, $Z_{D_b}$, affects the calculation of the $C_c(G(X, \sigma))$-norm of elements in $C_c(X_\infty * G)$. Let $z_1 \in Z_{D_b}$ and let $z_0 := \sigma(z_1)$. Consider an open neighborhood $A$ of $z_1$ such that $\sigma|_A : A \to \sigma(A)$ is a homeomorphism onto the open set $\sigma(A)$. Let $f \in C(X_\infty)$ be supported in $\sigma(A) * A * X_\infty$. Then,

$$\nu_{z_0}^{D_b}(f) = D_b(z_1)\nu_{z_1}^{D_b}(f(z_0, z_1, \cdot)) = 0.$$

Assume that $\xi \in C_c(X_\infty * G)$ is of the form $\xi(\underline{x}, (p(\underline{x}), n, y)) = f_1(\underline{x})f_2(p(\underline{x}), n, y)$ such that $f_1$ is supported in $\sigma(A) * A * X_\infty$. Let

$$Z_{2,D_b} = \{\underline{x} \in X_\infty \mid x_1 \in Z_{D_b}\}.$$

If $\tilde{f}_1 = f_1 1_{Z_{2,D}^c}$ and $\tilde{\xi} := \tilde{f}_1 f_2$, then $\langle \xi - \tilde{\xi}, \xi - \tilde{\xi}\rangle_{C_c(G)} = 0$. That is, in the inner product, an element $\tilde{\xi} \in C_c(X_\infty * G)$ depends only on the values outside the set $Z_{2,D}$. Hence, the

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

operator $U_{\mathfrak{u}}^*$ defined on $C_c(X_\infty * G)$ via

$$U_{\mathfrak{u}}^*\xi(\underline{x},(p(\underline{x}),n,y)) = \frac{1}{\mathfrak{u}(p(\sigma_\infty^{-1}(\underline{x})))}\xi(\sigma_\infty^{-1}(\underline{x}),(p(\sigma_\infty^{-1}(\underline{x})),n+1,y))$$

for all $(\underline{x},(p(\underline{x}),n,y)) \in X_\infty * G$ such that $\mathfrak{u}(x_1) \neq 0$ is well defined in the $C_c(G)$-inner product (4-5) since $p(\sigma_\infty^{-1}(\underline{x})) = x_1$. One can check that $U_{\mathfrak{u}}^*$ is an isometry using version (7-6) of (7-4). We leave the details to the reader. The fact that $U_{\mathfrak{u}}^*$ is the inverse of $U_{\mathfrak{u}}$ is a straightforward computation.

Recall that $\mathcal{X}_0$ is the $C^*$-correspondence defined by the $(G,\lambda)-(G,\lambda)$ equivalence $(G,\alpha)$, where $\alpha_x = \lambda_x$, as described in the paragraph following (5-6). The isometry $S_{\mathfrak{u}} \in \mathcal{K}(\mathcal{X}_0)$ is defined via (5-7). To prove that $U_{\mathfrak{u}}$ is a unitary extension of $S_{\mathfrak{u}}$, we define $V_{\infty,0} : \mathcal{X}_0 \to \mathcal{X}_\infty$ via

$$V_{\infty,0}\xi(\underline{x},(p(\underline{x}),t,y)) = \xi(p(\underline{x}),t,y) \quad \text{for all } \xi \in C_c(Z).$$

PROPOSITION 9.1. *The map $V_{\infty,0}$ extends to an adjointable isometry that intertwines $U_{\mathfrak{u}}$ and $S_{\mathfrak{u}}$, $U_{\mathfrak{u}}V_{\infty,0} = V_{\infty,0}S_{\mathfrak{u}}$, such that*

$$V_{\infty,0}^*\eta(x,t,y) = \int_{X_\infty} \eta(\underline{x},(x,t,y))\, d\nu_x^{D_b}(\underline{x})$$

*for all $\eta \in C_c(X_\infty * G)$. Moreover, $V_{\infty,0}(\xi\cdot a) = V_{\infty,0}\xi \cdot a$ for all $\xi \in \mathcal{X}_0$ and $a \in C^*(G)$.*

PROOF. The fact that $V_{\infty,0}$ is an isometry follows from the following computation that uses the fact that $D_b$ is unital and, thus, $\nu_x^{D_b}(1) = 1$ for all $x \in X$:

$$\begin{aligned}\langle V_{\infty,0}\xi, V_{\infty,0}\eta\rangle_{C_c(G)}(x,t,y) &= \sum_{(z,s,x)\in G}\int_{X_\infty} \overline{V_{\infty,0}\xi(\underline{z},(z,s,x))}V_{\infty,0}\eta(\underline{z},(z,s+t,y))\, d\nu_z^{D_b}(\underline{z})\\ &= \sum_{(z,s,x)\in G}\int_{X_\infty} \overline{\xi(z,s,x)}\eta(z,s+t,y)\, d\nu_z^{D_b}(\underline{z})\\ &= \sum_{(z,s,x)\in G} \overline{\xi(z,s,x)}\eta(z,s+t,y) = \langle\xi,\eta\rangle_{C_c(G)}(x,t,y).\end{aligned}$$

The fact that $V_{\infty,0}$ intertwines $S_{\mathfrak{u}}$ and $U_{\mathfrak{u}}$ is also an easy verification:

$$\begin{aligned}U_{\mathfrak{u}}V_{\infty,0}\xi(\underline{x},(p(\underline{x}),t,y)) &= \mathfrak{u}(p(\underline{x}))V_{\infty,0}\xi(\sigma_\infty(\underline{x}),(p(\sigma_\infty(\underline{x})),t-1,y))\\ &= \mathfrak{u}(p(\underline{x}))\xi(\sigma(p(\underline{x})),t-1,y) = V_{\infty,0}S_{\mathfrak{u}}(\underline{x},(p(\underline{x}),t,y)).\end{aligned}$$

The last statement of the proposition follows from the following computation:

$$\begin{aligned}\langle V_{\infty,0}\xi,\eta\rangle_{C_c(G)}(x,t,y) &= \sum_{(z,s,x)\in G}\int_{X_\infty} \overline{V_{\infty,0}\xi(\underline{z},(z,s,x))}\eta(\underline{z},(z,s+t,y))\, d\eta_z^{D_b}(\underline{z})\\ &= \sum_{(z,s,x)\in G} \overline{\xi(z,s,x)}\int_{X_\infty}\eta(\underline{z},(z,s+t,y))\, d\nu_z^{D_b}(\underline{z}) = \langle\xi,V_{\infty,0}^*\eta\rangle_{C_c(G)}(x,t,y).\end{aligned}$$

□

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

Recall that $\Phi_*(f)$ is the image of $f \in C_c(G(X,\sigma),\lambda)$ in $C_c(G_\infty(X,\sigma),\nu^D * \lambda)$ under the map $\Phi_*$ defined in (7-7). Define $\mathbb{E}_0 := \mathbb{L}(\Phi_*(1_{R_{0,0}}))$, where $\mathbb{L}$ is the left action of $C^*(G_\infty(X,\sigma),\nu^D * \lambda)$ on $\mathcal{X}_\infty$ defined in (8-1). Then, $\mathbb{E}_0$ is a projection since $\Phi_*(1_{R_{0,0}})$ is a projection (see paragraph above (7-10)).

DEFINITION 9.2. The sequence of projections $\{\mathbb{E}_j\}_{j=-\infty}^{\infty}$ on $\mathcal{L}(\mathcal{X}_\infty)$ defined by the formula

$$\mathbb{E}_j := U_{\mathfrak{u}}^j \mathbb{E} U_{\mathfrak{u}}^{-j}, \quad j \in \mathbb{Z},$$

together with $U_{\mathfrak{u}}$, is called the *proto-multiresolution analysis* determined by $(X,\sigma,D,b)$.

THEOREM 9.3. *The proto-multiresolution analysis* $(\{\mathbb{E}_j\}_{j=-\infty}^{\infty}, U_{\mathfrak{u}})$ *satisfies Properties (PMRA1), (PMRA2), (PMRA3), and (PMRA4) of Section 1:*

(1) $\mathbb{E}_j \geq \mathbb{E}_i$ *when* $j \geq i$;
(2) $I_{\mathcal{X}} = \bigvee_{-\infty<j<\infty} \mathbb{E}_j$;
(3) $U_{\mathfrak{u}}^* \mathbb{E}_j U_{\mathfrak{u}} = \mathbb{E}_{j-1}$ *for all* $j$; *and*
(4) *the copy of* $C^*(G(X,\sigma),\lambda)$ *in* $C^*(G_\infty(X,\sigma),\nu^D * \lambda)$ *leaves the range of* $\mathbb{E}_j \ominus \mathbb{E}_{j-1}$ *invariant, and* $U_{\mathfrak{u}}^{j-i}$ *implements a unitary equivalence between the restriction of the copy of* $C^*(G(X,\sigma),\lambda)$ *to the range of* $\mathbb{E}_j \ominus \mathbb{E}_{j-1}$ *and the restriction of the copy to the range of* $\mathbb{E}_i \ominus \mathbb{E}_{i-1}$.

PROOF. The proof is a consequence of the definition of the projections $\mathbb{E}_j$, the fact that $1_{R_{0,0}}$ is the unit of $C^*(G(X,\sigma),\lambda)$, and the fact that $\mathbb{E}_j = \mathbb{L}(\Phi_*(1_{R_{j,j}}))$ for all $j \geq 0$. We check this property for $j = 1$ for simplicity:

$$\begin{aligned}
U_{\mathfrak{u}}\mathbb{E}_0 U_{\mathfrak{u}}^{-1}\xi(\underline{x},(p(\underline{x}),t,y))) &= \mathfrak{u}(p(\underline{x}))\mathbb{E}_0 U^{-1}\xi(\sigma_\infty(\underline{x}),(p(\sigma_\infty(\underline{x})),t-1,y)) \\
&= \mathfrak{u}(p(\underline{x}))\int_G\int_{X_\infty} \Phi_*(1_{R_{0,0}})(\sigma_\infty(\underline{x}),s,\underline{w})(U_{\mathfrak{u}}^{-1}\xi)(\underline{w},(w,t-s-1,y))\frac{\mathrm{d}(\underline{w})^{1/2}}{\mathrm{d}(\sigma_\infty(\underline{x}))^{1/2}} \\
&\quad \cdots d\nu_w^D(\underline{w})\, d\lambda^{\sigma(x)}(\sigma(x),s,w) \\
&= \mathfrak{u}(p(\underline{x}))\int_{X_\infty} \frac{1}{\mathfrak{u}(p(\sigma_\infty^{-1}(\underline{x})))}\xi(\sigma_\infty^{-1}(\underline{x}),(p(\sigma_\infty^{-1}(\underline{x})),t,y))\frac{\mathrm{d}(\underline{w})^{1/2}}{\mathrm{d}(\sigma_\infty(\underline{x}))^{1/2}}\, d\nu_{\sigma(x)}^D(\underline{w}) \\
&= \sum_{\sigma(u)=\sigma(x)}\int_{X_\infty} \xi(\underline{u},(p(\underline{u}),t,y))\frac{\mathrm{d}(\underline{u})^{1/2}}{\mathrm{d}(\underline{x})^{1/2}}\, d\nu_u^D(\underline{u}) = \mathbb{L}(1_{R_{1,1}})\xi(\underline{x},(p(\underline{x}),t,y)).
\end{aligned}$$

For example, the last property follows from the fact that if $a \in C^*(G(X,\sigma),\lambda)$, then

$$\mathbb{L}(\Phi_*(a)) = \mathbb{L}(\Phi_*(1_{R_{0,0}} * a * 1_{R_{0,0}})) = \mathbb{E}_0\mathbb{L}(\Phi_*(a))\mathbb{E}_0. \qquad \square$$

REMARK 9.4. There is a strong connection between the proto-resolution analysis and the projective-resolution analysis of Larsen and Raeburn [21, 22]. The groupoid $G_\infty(X,\sigma)$ is the projective limit of the groupoids $G(X,\sigma^n)$ with the appropriate maps. Therefore, $\mathcal{X}_\infty$ is the inductive limit of Hilbert modules $\mathcal{X}_n$ defined by $D_b^n$. In this

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

picture, $\mathbb{E}_n$ can be viewed as the projection onto the image of $\mathcal{X}_n$ into $\mathcal{X}_\infty$. The advantage of using the groupoid $G_\infty(X,\sigma)$ to define the proto-resolution analysis is that we can use the powerful theory of induced representations of groupoid $C^*$-algebras, as in the next paragraph.

Let $\hat{L} = (\mu, X * \mathcal{H}, \hat{L})$ be a representation of $G(X,\sigma)$. As before, we write $\mathfrak{L}$ for the integrated representation of $C^*(G(X,\sigma),\lambda)$ on $L^2(X*\mathcal{H},\mu)$ defined by $\hat{L}$ via (4-3). The normalized potential $D_b$ defines a topological correspondence from $G_\infty(X,\sigma)$ to $G(X,\sigma)$ (see Section 8). Theorem 4.1 implies that the induced $C^*(G_\infty(X,\sigma),\nu^D * \lambda)$-representation of $\mathfrak{L}$ is unitarily equivalent to the integrated form $\mathfrak{L}_\infty$ of the induced $G_\infty(X,\sigma)$ representation $\hat{L}_\infty = (\mu_\infty^{D_b}, X_\infty * \mathcal{H}_\infty, \hat{L}_\infty)$ defined in Section 8. Set $\tilde{S}_{\mathfrak{u}} := \mathfrak{L}(S_{\mathfrak{u}}) \in \mathcal{B}(L^2(X*\mathcal{H},\mu))$, $\tilde{U} := \mathfrak{L}_\infty(U_{\mathfrak{u}}) \in \mathcal{B}(L^2(X_\infty * \mathcal{H}_\infty, \mu_\infty^{D_b}))$, and $V_j := \mathfrak{L}_\infty(\mathbb{E}_j)L^2(X_\infty * \mathcal{H}_\infty, \mu_\infty^{D_b})$ for all $j \in \mathbb{Z}$. Theorem 9.3 implies the following result.

THEOREM 9.5. *The family $\{V_j\}$ of subspaces of $L^2(X_\infty * \mathcal{H}_\infty, \mu_\infty^{D_b})$ satisfies the properties:*

(1) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$;
(2) $\bigcup_{j=-\infty}^{\infty} V_j = L^2(X_\infty * \mathcal{H}_\infty, \mu_\infty^{D_b})$;
(3) *the copy of $C^*(G(X,\sigma))$ in $C^*(G_\infty(X,\sigma))$ leaves $V_j \ominus V_{j-1}$ invariant and $\tilde{U}^{j-i}$ implements a unitary equivalence between the restriction of the copy of $C^*(G(X,\sigma))$ to $V_j \ominus V_{j-1}$ and the restriction of the copy to $V_i \ominus V_{i-1}$.*

*If $\tilde{S}_{\mathfrak{u}}$ is a pure isometry, then $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$ and $\{V_j\}$ is a generalized multiresolution analysis determined by $(X,\sigma,D,b,\hat{L})$.*

PROOF. The numbered items follow immediately from the corresponding properties of the proto-resolution analysis $\{\mathbb{E}_j\}_{j\in\mathbb{Z}}$.

For the last part, note that the isometry $V_{\infty,0} : \mathcal{X}_0 \to \mathcal{X}_\infty$ determines an isometry $W_{\infty,0} : L^2(X*\mathcal{H},\mu) \to L^2(X_\infty * \mathcal{H}_\infty, \mu_\infty^{D_b})$, $W_{\infty,0}\mathfrak{f}(\underline{x}) = \mathfrak{f}(p(\underline{x}))$, such that $\tilde{U}_{\mathfrak{u}}W_{\infty,0} = W_{\infty,0}\tilde{S}_{\mathfrak{u}}$ and $V_0 = W_{\infty,0}(L^2(X*\mathcal{H},\mu))$. Hence, $\bigcap_{j=-\infty}^{\infty} V_j = \bigcap_{j\geq 0} W_{\infty,0}\tilde{S}_{\mathfrak{u}}^j(L^2(X*\mathcal{H},\mu))$. Therefore, $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$ if $\tilde{S}_{\mathfrak{u}}$ is a pure isometry. □

# Part III
# Mallat's theorem redux

## 10. Potentials, filters, and scaling functions

We fix a transfer operator $\mathcal{L}_\psi$ for $\sigma$ as in the paragraph following Definition 5.2 and we assume the potential $\psi$ is full. In this section, we define and develop the concept of a filter associated to the potential $\psi$. A *filter* is a function that determines a new potential that is normalized but not necessarily full. In fact, to produce an analog of the scaling function that appears in (1-1), the new potential is definitely not full (see the paragraph before Section 11). A filter defines a $p$-system of measures on the projective limit $X_\infty$, where, recall, $p : X_\infty \to X$ is the projection onto the first component. The main result of this section is Proposition 10.1, which shows that a scaling function for $(X,\sigma)$ exists if and only if the system of measures is atomic.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

The full potential $\psi$ is used to define the Haar system on the pullback groupoid $X_\infty * G * X_\infty$ via (3-1). To keep the notation shorter, we assume in the remainder of the paper that $\psi$ is an unital potential in addition to being a full potential. If $\psi$ is not unital, then one needs to replace $\mathcal{L}_\psi$ with $\mathcal{L}_{D_0}$, where $D_0(x) = \psi(x)/\mathcal{L}_\psi(1)(\sigma(x))$ in the formula below and in the construction of the $p$-system of measures on $X_\infty$ (see Remark 5.8).

A function $\boldsymbol{m} \in C(X)$ is called a *$\psi$-filter* if $\mathcal{L}_\psi(|\boldsymbol{m}|^2) = 1$. That is,

$$\sum_{\sigma(y)=x} \psi(y)|\boldsymbol{m}(y)|^2 = 1 \tag{10-1}$$

for all $x \in X$.

If $\boldsymbol{m}$ is a $\psi$-filter, we write $\mathfrak{u}$ for the continuous function $\mathfrak{u} : X \to \mathbb{C}$ defined via $\mathfrak{u}(x) = \sqrt{\psi(x)}\boldsymbol{m}(x)$ and we define $D : X \to \mathbb{R}_+$ via

$$D(x) = |\mathfrak{u}(x)|^2 = \psi(x)|\boldsymbol{m}(x)|^2 \quad \text{for all } x \in X.$$

Connecting the notation with the notation used in previous sections, we have $D(x) = D_b(x) = b(x)\psi(x)$, where $b(x) = |\boldsymbol{m}(x)|^2$.

By definition, $D$ is a normalized potential for a new unital transfer operator $\mathcal{L}_D$, but note that $D$ does not have to be full. In fact, the failure of $D$ being full provides our most important applications. For a given potential $\psi$, one can find different $\psi$-filters $\boldsymbol{m}$. As we see in the examples, some of them lead to full normalized potentials, while others lead to normalized potentials that are not full. In a sense, one can view the remainder of the paper as the study of such normalized potentials and how non-full potentials can lead to wavelets.

In the remainder of this section, we fix a $\psi$-filter $\boldsymbol{m}$ and the maps $\mathfrak{u}$ and $D$ defined by $\boldsymbol{m}$. Recall from (6-1) that $D$ defines a $p$-system of measures on $X_\infty$ denoted by $\{\nu_x^D\}_{x\in X}$. We prove that a scaling function, as defined in (10-2) below, exists if and only if this $p$-system of measures is atomic. The scaling function is a key ingredient in our extension of Mallat's theorem in Section 13. One should keep in mind that a scaling function provides a link or 'map' from analysis on $X$ that is controlled by $\sigma$ to classical harmonic analysis on $\mathbb{R}$ that is built on the Fourier transform.

We say that a function $\mathfrak{s} : X \to X_\infty$ is a *section* of the projective limit induced by $(X, \sigma)$ if $\mathfrak{s}$ is a (Borel) measurable function such that $p \circ \mathfrak{s}(x) = x$ for all $x \in X$. We then say that such a section $\mathfrak{s}$ is *positively supported* on $X_\infty$ with respect to $\boldsymbol{m}$ if $\nu_x^D(\{\mathfrak{s}(x)\}) > 0$ for all $x \in X$.

Evidently, a function $\mathfrak{s} : X \to X_\infty$ is a section if and only if there is $\underline{x} = (x_n)_{n\geq 0} \in X_\infty$ with $x_0 = x$ and $\sigma(x_{n+1}) = x_n$ for all $n \geq 0$ such that $\mathfrak{s}(x) = \underline{x}$. Moreover, $\mathfrak{s}$ is positively supported if and only if $\mathfrak{s}(x)$ is an atom for $\nu_x^D$ for all $x \in X$.

PROPOSITION 10.1. *Assume that $\boldsymbol{m}$ is a $\psi$-filter and $\mathfrak{s} : X \to X_\infty$ is a section of the projective limit induced by $(X, \sigma)$. Then, $\mathfrak{s}$ is positively supported with respect to $\boldsymbol{m}$ if and only if the function $\phi : X \to \mathbb{C}$,*

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

$$\phi(x) := \prod_{i=1}^{\infty} \mathfrak{u}(x_i), \tag{10-2}$$

*is defined for all* $x \in X$, *where* $(x_n)_{n\geq 0} = \mathfrak{s}(x)$. *The function* $\phi$ *satisfies the scaling property*

$$\phi(\sigma(x)) = \mathfrak{u}(x)\phi(x)$$

*and if* $\mathfrak{s}$ *is positively supported with respect to* $\boldsymbol{m}$ *and if* $\mu$ *is an arbitrary probability measure on* $X$, *then* $\phi \in L^2(X, \mu)$.

PROOF. Let $x \in X$ and $(x_n)_{n=0}^{\infty} := \mathfrak{s}(x)$. It follows from the definition of $\nu_x^D$ that the product $\prod_{i=1}^{\infty} D(x_i) = \prod_{i=1}^{\infty} |\mathfrak{u}(x_i)|^2$ converges if and only if

$$\nu_x^D(\{\mathfrak{s}(x)\}) = \prod_{i=1}^{\infty} D(x_i) = |\phi(x)|^2 > 0.$$

Therefore, $\prod_{i=1}^{\infty} \mathfrak{u}(x_i)$ converges for all $x \in X$ if and only if $\mathfrak{s}$ is positively supported.

For the scaling property, note that $\sigma_\infty(\mathfrak{s}(x)) = \mathfrak{s}(\sigma(x))$ for all $x \in X$. Therefore,

$$\phi(\sigma(x)) = \prod_{i=1}^{\infty} \mathfrak{u}(\sigma(x_i)) = \mathfrak{u}(x) \prod_{i=1}^{\infty} \mathfrak{u}(x_i) = \mathfrak{u}(x)\phi(x).$$

The function defined in (10-2) is measurable since $\mathfrak{s}$ is measurable. Moreover,

$$\int_X |\phi(x)|^2 \, d\mu(x) = \int_X \nu_x(\mathfrak{s}(x)) \, d\mu(x) \leq \int_X \int_{X_\infty} 1 \, d\nu_x(\underline{x}) \, d\mu(x) = 1.$$

Therefore, $\phi \in L^2(X, \mu)$. □

DEFINITION 10.2. We call the function $\phi$ of (10-2) the *scaling function* of the system $(X, \sigma, \psi, \boldsymbol{m}, \mathfrak{s})$.

Using the fact that $\mathfrak{u}(x) = \sqrt{\psi(x)}\boldsymbol{m}(x)$, we can rewrite the scaling property as

$$\frac{1}{\sqrt{\psi(x)}}\phi(\sigma(x)) = \boldsymbol{m}(x)\phi(x).$$

So, for example, if $\psi(x) = 1/N$ for all $x \in X$ for some $N \geq 2$, we recover the well-known $N$-scaling property:

$$\sqrt{N}\phi(\sigma(x)) = \boldsymbol{m}(x)\phi(x).$$

Note that a necessary condition for a section $\mathfrak{s} : X \to X_\infty$ to be positively supported is that $\lim_{n\to\infty} |\mathfrak{u}(x_n)| = 1$ for all $x \in X$, where $(x_n)_{n\geq 0} = \mathfrak{s}(x)$. This implies that there is at least one $z \in X$ such that $\mathfrak{u}(z) = 1$ since $\{x_n\}_{n\geq 0}$ is a sequence in the compact space $X$ and $\mathfrak{u}$ is assumed to be continuous. Hence, $D(z) = 1$ and, thus, $D(x) = 0$ for all $x \in \sigma^{-1}(\sigma(z)) \setminus \{z\}$. Therefore, a *necessary* condition for the existence of a positively supported section is that the normalized potential $D$ is not full.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

## 11. $\psi$-filters and isometries in $C_c(G(X,\sigma),\lambda)$

Recall from Theorem 9.5 that to build generalized multiresolution analyses, it is natural first to build isometries $S_{\mathfrak{u}}$ in $C^*(G(X,\sigma),\lambda)$ and to seek conditions under which there are representations $\mathfrak{L}$ of $C^*(G(X,\sigma),\lambda)$ such that $\mathfrak{L}(S_{\mathfrak{u}})$ is a pure isometry. In this section, we study in more detail the fundamental building blocks: the isometry $S_{\mathfrak{u}}$ defined in (5-5) and the projections in $C^*(G(X,\sigma),\lambda)$ determined by its iterates. Recall also that the key ingredients in the definition of $S_{\mathfrak{u}}$ are, in addition to the local homeomorphism $\sigma$, a full potential $\psi$ and a $\psi$-filter $\boldsymbol{m}$. We first explain how full potentials show up naturally in the context of the Deaconu–Renault groupoid once one chooses a unitary representation of the groupoid.

Recall that a unitary representation of a groupoid requires a quasi-invariant measure on its unit space. In the setting of the Deaconu–Renault groupoid $G(X,\sigma)$, Renault gives a detailed analysis in [31] that connects quasi-invariant measures on $X$, the unit space of $G(X,\sigma)$, with full, not-necessarily normalized potentials. His Proposition 3.4.1 proves that any such quasi-invariant measure $\mu$ is invariant for the transpose of a transfer operator given by a full, not-necessarily normalized potential $\psi$. (We use the terms 'transpose', 'adjoint', and 'dual' interchangeably to refer to the operator on measures induced by a transfer's action on functions.) That is, writing $\mathcal{L}_\psi$ for the transfer operator given by $\psi$, $\mu$ satisfies

$$\int_X \mathcal{L}_\psi(f)(x)\,d\mu = \int_X \sum_{\sigma(y)=x} \psi(y)f(y)\,d\mu = \int_X f(x)\,d\mu(x)$$

for all $f \in C(X)$. In general, a transfer operator $\mathcal{L}_\psi$ may have many invariant probability measures. However, there are general conditions, which we discuss in Section 12, under which the correspondence between transfer operators and quasi-invariant measures for $(X,\sigma)$ is one-to-one. Since all of our examples satisfy these conditions, we proceed to write $\mu_\psi$ for the 'pairing' between quasi-invariant measures $\mu$ and full potentials $\psi$.

We continue to assume that $\psi$ is a *unital* full potential to keep our notation shorter. Our results are valid in the nonunital case. However, one needs to adjust slightly the computations involving $\mu$ using the fact that

$$\int_X \mathcal{L}_{D_0}(f)(x)\,d\mu(x) = \int_X \mathcal{L}_\psi(1)(\sigma(x))f(x)\,d\mu(x),$$

where $D_0(x) = \psi(x)/\mathcal{L}_\psi(1)(\sigma(x))$. The reader is encouraged to consult, for example, [11] and the papers cited therein for computations involving similar formulas.

Additionally, [31, Proposition 3.4.1] also proves that the Radon–Nykodim derivative of $\mu_\psi$ is given by

$$\Delta_\psi(x,k-l,y) = \frac{\psi(x)\psi(\sigma(x))\cdots\psi(\sigma^{k-1}(x))}{\psi(y)\psi(\sigma(y))\cdots\psi(\sigma^{l-1}(y))}.$$

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

Note that $\mu_\psi$ is also invariant for the dual of the transfer operator $\mathcal{L}_\psi^n$ whose potential is given by $\psi_n(x) := \prod_{i=0}^{n-1} \psi(\sigma^i(x))$.

Note that in the notation following Remark 5.8, $\boldsymbol{E}_n^\psi := \pi^n \circ \mathcal{L}_\psi^n$, (5-6) can be written as

$$\int_X \boldsymbol{E}_n^\psi(f) g \, d\mu_\psi = \int_X f \boldsymbol{E}_n^\psi(g) \, d\mu_\psi. \tag{11-1}$$

From now on, we fix a probability measure $\mu_\psi$ that is invariant for the dual of a transfer operator with full potential $\psi$ and view it as a quasi-invariant measure for the Deaconu–Renault groupoid $G = G(X, \sigma)$.

Next, we consider a $\psi$-filter $\boldsymbol{m}$ as defined in (10-1), and we continue to use the notation $\mathfrak{u}(x) = \sqrt{\psi(x)}\boldsymbol{m}(x)$ and $D(x) = |\mathfrak{u}(x)|^2 = \psi(x)|\boldsymbol{m}(x)|^2$ for all $x \in X$ of the previous section. We stress that $D$ is a normalized not necessarily full potential for its transfer operator $\mathcal{L}_D$. Recall from (5-5) that the function $\mathfrak{u}$ defines an isometry $S_\mathfrak{u}$ in $C_c(G(X, \sigma), \lambda)$. It is important to keep in mind that $\mu_\psi$ is not invariant for $\mathcal{L}_D$ unless $|\boldsymbol{m}(x)| = 1$ for $\mu_\psi$-almost every (a.e.) $x$. As we see in Theorem 12.1, choosing $\boldsymbol{m}$ such that $\mu_\psi$ is not invariant for $\mathcal{L}_D$ is the condition one needs to obtain a pure Hilbert space isometry from $S_\mathfrak{u}$.

To simplify the notation in the remainder of the paper, we write $\mathfrak{u}_n(x) := \prod_{i=0}^{n-1} \mathfrak{u}(\sigma^i(x))$, $D_n(x) := \prod_{i=0}^{n-1} D(\sigma^i(x))$ and $\boldsymbol{m}_n(x) := \prod_{i=0}^{n-1} \boldsymbol{m}(\sigma^i(x))$ for all $n \geq 0$ and $x \in X$. Additionally, we use the notation $\psi_n$ defined above as the potential of $\mathcal{L}_\psi^n$ and write $D_n$ for the potential of $\mathcal{L}_D^n$.

The unitary extension $U_\mathfrak{u}$ of $S_\mathfrak{u}$ defined in (9-1) is one of the pillars of the proto-multiresolution analysis (see Theorem 9.3). Theorem 9.5 shows that a choice of a representation of $G(X, \sigma)$ defines a generalized multiresolution analysis in the induced Hilbert space. The same theorem proves that the proto-multiresolution analysis becomes a multiresolution analysis if the image of $S_\mathfrak{u}$ under the integrated form of the original representation is a pure isometry. To facilitate the study, we undertake in the next section, which is devoted to determining when the image of $S_\mathfrak{u}$ under the integrated form of a unitary representation is a pure isometry, to describe the projections, $E_n := S_\mathfrak{u}^n * (S_\mathfrak{u}^*)^n$, onto the ranges of the $S_\mathfrak{u}^n$ and develop some of their useful properties. An easy computation shows that

$$E_n(x, k-l, y) = \mathfrak{u}_n(x)\overline{\mathfrak{u}_n(y)} 1_{R_{n,n}}(x, k-l, y). \tag{11-2}$$

Point (4) under Remark 5.8 implies that the following equation holds in $C_c(G)$ for all $f \in C(X)$:

$$E_n * f * E_n = (\pi^n \circ \mathcal{L}_D^n(f)) * E_n = \boldsymbol{E}_n^D(f) * E_n, \tag{11-3}$$

where recall $\pi : C(X) \to C(X)$ is the composition with $\sigma$ and $\boldsymbol{E}_n^D$ is the projection $\pi^n \circ \mathcal{L}_D^n$ defined under Remark 5.8.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

## 12. Unitary representations of $G(X,\sigma)$: deciding when the image of $S_{\mathfrak{u}}$ is a pure isometry

In this section, we consider a representation $\hat{L}$ of $G(X,\sigma)$ and provide conditions that guarantee that the image of $S_{\mathfrak{u}}$ under the integrated form of $\hat{L}$ is a pure isometry. Our results generalize numerous previous results in the literature. See, in particular, [2, 3, 6].

We continue to fix a quasi-invariant measure $\mu_\psi$ that is invariant for the dual of a transfer operator $\mathcal{L}_\psi$ with unital full potential $\psi$. Let $\hat{L}$ be a representation of $G(X,\sigma)$ into a Hilbert bundle $X*\mathcal{H}$ over $X$. Recall that this means that $\hat{L}$ is a Borel homomorphism of $G(X,\sigma)$ into the unitary groupoid of $X*\mathcal{H}$, $\mathrm{Iso}(X*\mathcal{H})$. Hence, the triple $\hat{L} := (\mu_\psi, X*\mathcal{H}, \hat{L})$ constitutes a unitary representation of $G(X,\sigma)$ that may be integrated to a $C^*$-representation $\mathfrak{L}$ of $C_c(G(X,\sigma),\lambda)$ as in (2-2). Next, we fix a $\psi$-filter $\boldsymbol{m}$ and consider how the functions $\mathfrak{u}(x) := \sqrt{\psi(x)}\boldsymbol{m}(x)$ and $D(x) = |\mathfrak{u}(x)|^2$ interact with $\hat{L}$.

The main goal of this section is to determine whether the isometry $\tilde{S}_{\mathfrak{u}} := \mathfrak{L}(S_{\mathfrak{u}})$, where $S_{\mathfrak{u}}$ is defined in (5-5), is a pure isometry. To achieve our goal, we are going to use the reverse martingale theorem as adapted to the groupoid setting in [30, Proposition 3.5, Theorems 3.10 and 6.1]. We describe briefly how Renault's setting applies to our situation. Equation (11-2) implies that the range projection of $\tilde{S}_{\mathfrak{u}}$ depends only on the proper equivalence relation $R_{n,n}$. In the language of [30], $R_\infty = \bigcup_n R_{n,n}$ is an approximately proper (AP) equivalence relation.

Recall that we assume that $\psi$ is an unital full normalized potential. We also assume that the hypotheses of [30, Theorem 6.1] are satisfied. These hypotheses are:

(i) the AP equivalence $R_\infty$ is minimal;
(ii) there exists an integer $L \geq 1$ such that $\sigma^L$ admits a generator;
(iii) there exists an integer $M \geq 1$ such that $\psi_m$ satisfies Bowen's condition with respect to $\sigma^M$.

Under these hypotheses, the stationary cocycle $\Delta_\psi$ on $R_\infty$ is uniquely ergodic and we may invoke the main theorems of [30]. While we do not go into details, we note that if $X$ is a compact metric space and $\sigma$ is a positively expansive surjective local homeomorphism, then the above hypotheses are satisfied [30, Example 6.2]. The local homeomorphisms in the examples that we consider in this paper are positively expansive and, thus, satisfy these hypotheses.

The next theorem, which is the main result of this section, shows that $\tilde{S}_{\mathfrak{u}}$ is a pure isometry provided that $\mu_\psi$ is not invariant for the dual of the transfer operator $\mathcal{L}_D$. To ease the notation a bit, we are going to write $\tilde{E}_n$ for $\mathfrak{L}(E_n)$ throughout the proof.

THEOREM 12.1. *Assume that $\sigma : X \to X$ is a surjective local homeomorphism such that the above hypotheses are satisfied. Let $\mu_\psi$ be a quasi-invariant measure for $G(X,\sigma)$ that is invariant for the dual of the transfer operator $\mathcal{L}_\psi$ with unital full potential $\psi$. Let $\boldsymbol{m}$ be a $\psi$-filter such that $\mu_\psi(\{x \in X \mid |\boldsymbol{m}(x)| \neq 1\}) > 0$. Then, the isometry $\tilde{S}_{\mathfrak{u}} = \mathfrak{L}(S_{\mathfrak{u}})$ is a pure isometry.*

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

PROOF. Assume, by way of contradiction, that $\tilde{S}_u$ is not pure. Then, there is a unit vector $\mathfrak{f} \in \bigcap_{n\geq 0} \tilde{S}_u^n L^2(X * \mathcal{H}, \mu_\psi)$ and, thus, $\tilde{E}_n \mathfrak{f} = \mathfrak{f}$ for all $n \geq 0$. Using the notation under Remark 5.8, we write $\boldsymbol{E}_n^\psi := \pi^n \circ \mathcal{L}_\psi^n$ and $\boldsymbol{E}_n^D := \pi^n \circ \mathcal{L}_D^n$. We claim that

$$\|\mathfrak{f}(x)\|^2 = |\boldsymbol{m}_n(x)|^2 \boldsymbol{E}_n^\psi(\|\mathfrak{f}(\cdot)\|^2)(x), \tag{12-1}$$

for $\mu_\psi$-a.e. $x \in X$ and for all $n \geq 0$. Equation (12-1) is a generalization of [6, (3.26)] and [2, Lemma 10]. We provide a shorter proof using (11-3) and (11-1).

First, note that $\boldsymbol{E}_n^\psi$ satisfies (11-1) since $\mu_\psi$ is invariant for $\mathcal{L}_\psi$. Moreover, $\boldsymbol{E}_n^D(f) = \boldsymbol{E}_n^\psi(|\boldsymbol{m}_n|^2 f)$ since $D_n = \psi_n \cdot |\boldsymbol{m}_n|^2$. Let $f \in C(X)$. Then,

$$\begin{aligned}
\int_X f(x)\|\mathfrak{f}(x)\|^2 \, d\mu_\psi(x) &= \int_X \langle f(x)\mathfrak{f}(x), \mathfrak{f}(x)\rangle \, d\mu_\psi(x) = \langle f\mathfrak{f}, \mathfrak{f}\rangle \\
&= \langle f\tilde{E}_n\mathfrak{f}, \tilde{E}_n\mathfrak{f}\rangle = \langle \tilde{E}_n f \tilde{E}_n \mathfrak{f}, \mathfrak{f}\rangle = \langle \boldsymbol{E}_n^D(f)\mathfrak{f}, \mathfrak{f}\rangle \\
&= \int_X \boldsymbol{E}_n^D(f)(x)\|\mathfrak{f}(x)\|^2 \, d\mu_\psi(x) = \int_X \boldsymbol{E}_n^\psi(|\boldsymbol{m}_n|^2 f)\|\mathfrak{f}(x)\|^2 \, d\mu_\psi(x) \\
&= \int_X |\boldsymbol{m}_n(x)|^2 f(x) \boldsymbol{E}_n^\psi(\|\mathfrak{f}\|^2)(x) \, d\mu_\psi(x).
\end{aligned}$$

Hence, (12-1) holds.

Recall from under Remark 5.8 that the sequence of $\boldsymbol{E}_n^\psi$ is a reversed martingale: $\boldsymbol{E}_m^\psi \boldsymbol{E}_n^\psi = \boldsymbol{E}_n^\psi \boldsymbol{E}_m^\psi = \boldsymbol{E}_n^\psi$ for $m \leq n$ (see [30, Proposition 3.5]). Under our hypotheses, [30, Theorem 3.10] implies that $\boldsymbol{E}_n^\psi(\|\mathfrak{f}(\cdot)\|)$ converges uniformly to a constant function. Part (iii) of [30, Corollary A.3] implies that the limit of $\boldsymbol{E}_n^\psi(\|\mathfrak{f}(\cdot)\|)$ equals $\int_X \|\mathfrak{f}(x)\|^2 \, d\mu_\psi(x) = 1$. Therefore,

$$\|\mathfrak{f}(x)\|^2 = \lim_{n\to\infty} |\boldsymbol{m}_n(x)|^2$$

for $\mu_\psi$-a.e. $x \in X$. In particular, $\boldsymbol{m}_\infty(x) := \lim_{n\to\infty} \boldsymbol{m}_n(x)$ exists for $\mu_\psi$-a.e. $x$. Note that $|\boldsymbol{m}_\infty(x)|^2 = \|\mathfrak{f}(x)\|^2$ for $\mu_\psi$-a.e. $x \in X$. Hence, $\int_X |\boldsymbol{m}_\infty(x)|^2 \, d\mu_\psi(x) = 1$ since $\mathfrak{f}$ is a unit vector.

Let $k, n \in \mathbb{N}$ satisfy the inequality $n > k$. Recall that $\mathcal{L}_\psi^k(|\boldsymbol{m}_k|^2) = 1$. Using the fact that $\boldsymbol{m}_n(x) = \boldsymbol{m}_k(x) \prod_{i=k}^{n-1} \boldsymbol{m}(\sigma^i(x))$ and the fact that $\mathcal{L}_\psi^k$ is a transfer operator with range equal to that of $\pi^k$, we obtain

$$\mathcal{L}_\psi^k(|\boldsymbol{m}_n|^2)(x) = \mathcal{L}_\psi^k\Big(|\boldsymbol{m}_k|^2 \prod_{i=k}^{n-1} |\boldsymbol{m} \circ \sigma^i|(x)\Big) = \mathcal{L}_\psi^k(|\boldsymbol{m}_k|^2)(x)|\boldsymbol{m}_{n-k}(x)|^2 = |\boldsymbol{m}_{n-k}(x)|^2.$$

Taking the limit as $n \to \infty$, we obtain that $\mathcal{L}_\psi^k(|\boldsymbol{m}_\infty|^2) = |\boldsymbol{m}_\infty|^2$ for all $k \geq 1$. Therefore, $\lim_{k\to\infty} \mathcal{L}_\psi^k(|\boldsymbol{m}_\infty|^2) = |\boldsymbol{m}_\infty|^2$. However, [30, Corollary A.3] implies that

$$\lim_{k\to\infty} \mathcal{L}_\psi^k(|\boldsymbol{m}_\infty|^2) = \int_X |\boldsymbol{m}_\infty(x)|^2 \, d\mu_\psi(x) = 1.$$

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

Hence, $|\boldsymbol{m}_\infty(x)| = 1$ for $\mu_\psi$-a.e. $x$. To finish the proof, we note that $\boldsymbol{m}_\infty(x) = \boldsymbol{m}(x)\boldsymbol{m}_\infty(\sigma(x))$ for $\mu_\psi$-a.e. $x$. Therefore, $|\boldsymbol{m}(x)| = 1$ for $\mu$-a.e. $x$. This is a contradiction and, hence, $\tilde{S}_{\mathfrak{u}}$ is a pure isometry. □

To relate our result with those in the literature, note that under the assumption that $\psi$ is unital, the isometry $\tilde{S}_{\mathfrak{u}}$ is given by the formula

$$\tilde{S}_{\mathfrak{u}}(\mathfrak{h})(x) = \boldsymbol{m}(x)L_{(x,1,\sigma(x))}\mathfrak{h}(\sigma(x))$$

for all $\mathfrak{h} \in L^2(G^{(0)} * \mathcal{H}, \mu)$. In particular, if $\mathcal{H} = X \times \mathbb{C}$ is the trivial bundle and $L = \iota$ is the fundamental representation on $\mu$, $\iota_{(x,t,y)}z = z$, then $\tilde{S}_{\mathfrak{u}} \in B(L^2(X, \mu))$ (see paragraph above (2-2)) and it is given by the formula

$$\tilde{S}_{\mathfrak{u}}(h)(x) = \boldsymbol{m}(x)h(\sigma(x))$$

for all $h \in L^2(X, \mu)$. Thus, our result recovers [6, Theorem 3.1], [3, Theorem 3.1], and, by choosing a different representation, [2, Theorem 8].

## 13. The conclusion: MRAs with nonconstant scaling functions and Mallat's theorem

We apply the techniques developed in the previous sections to a class of examples that provide a far reaching generalization of Mallat's multiresolution analysis. The examples that we study are based on the general $n$-fold map of $\mathbb{T}$ studied in [16, Section 4.5] and include, among others, finite Blaschke products.

Let $\varphi : [0, 1] \to \mathbb{R}$ be a continuous positive function such that $\varphi(0) = \varphi(1)$ and $N := \int_0^1 \varphi(x)\, dx$ is a positive integer greater than or equal to 2. We extend $\varphi$ to be defined on $\mathbb{R}$ via periodicity. We assume in the following that $\varphi := \min_{x\in[0,1]} \varphi(x) > 1$. Let $X = \mathbb{T}$ be viewed as $\mathbb{R}/\mathbb{Z}$ with the corresponding metric. Define $\sigma : \mathbb{T} \to \mathbb{T}$ via

$$\sigma(e^{2\pi i x}) = e^{2\pi i \int_0^x \varphi(t)\, dt} \quad \text{for all } x \in [0, 1].$$

Then, $\sigma$ is an $N$-fold covering map of $\mathbb{T}$ and, in particular, it is a local homeomorphism.

EXAMPLE 13.1.

(1) Let $\varphi : [0, 1] \to \mathbb{R}$ be defined via $\varphi(t) = N \geq 2$ for all $t \in [0, 1]$. Then, $\sigma : \mathbb{T} \to \mathbb{T}$ is given by $\sigma(z) = z^N$ for all $z \in \mathbb{T}$.

(2) Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ and let $b : \mathbb{D} \to \mathbb{D}$ be a finite Blaschke product

$$b(z) = C \prod_{i=1}^{N} \frac{z - a_k}{1 - \overline{a_k} z} \quad \text{for all } z \in \mathbb{D}, \tag{13-1}$$

where $C \in \mathbb{T}$ and $a_i \in \mathbb{D}$ are such that $|a_i| < 1$ for all $i = 1, \ldots, N$ and $b(1) = 1$. Let $\sigma$ be the restriction of $b$ to $\mathbb{T}$. It is known that $\sigma$ is an $N$-to-1 local homeomorphism on $\mathbb{T}$ (see, for example, [13, Theorem 3.4.10]). Reference [8, Lemma 4.2] implies that there is a function $\varphi : [0, 1] \to [0, 1]$ such that $\sigma(e^{2\pi i x}) = e^{2\pi i \int_0^x \varphi(t)\, dt}$.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

Let $\mathbb{T}_\infty$ be the projective limit induced by $(\mathbb{T}, \sigma^n)_{n\geq 0}$. We define next a section $\mathfrak{s} : \mathbb{T} \to \mathbb{T}_\infty$ such that $\lim_{n\to\infty} z_n = e^{2\pi i 0} = 1$ for all $z \in \mathbb{T}$, where $\mathfrak{s}(z) = (z_n)_{n\geq 0}$. To accomplish this, we describe explicitly the preimage under $\sigma$ of a point $z \in \mathbb{T}$. Let $x \in [0, 1)$ be such that $z = e^{2\pi i x}$. If $x \notin [0, 1)$, we replace it by $x - k$, where $k \in \mathbb{Z}$ is such that $x - k \in [0, 1)$. There are $N$ points $y_0, y_1, \ldots, y_{N-1} \in [0, 1)$ such that $\int_0^{y_k} \varphi(t)\, dt = x + k$ for all $k = 0, \ldots, N - 1$. Therefore, $\sigma(e^{2\pi i y_k}) = e^{2\pi i(x+k)} = e^{2\pi i x}$ for all $k = 0, \ldots, N - 1$. We say that $e^{2\pi i y_0}, \ldots, e^{2\pi i y_{N-1}}$ are the $N$ roots of $e^{2\pi i x}$ under $\sigma$. With a slight abuse of notation, we also say that $y_0, \ldots, y_{N-1}$ are the $N$ roots of $x$ under $\sigma$. Moreover, even though the $N$ roots of $x$ under $\sigma$ depend on $x$, we do not explicitly write this dependence to ease the notation. Note that $y_0 < y_1 < \cdots < y_{N-1}$ for all $x \in [0, 1)$ since $\varphi(t) > 0$ for all $t \in [0, 1]$. We say that $y_0$ is the smallest root of $x$ under $\sigma$ and that $y_{N-1}$ is the largest root of $x$ under $\sigma$. For $x = 0$, we use the special notation $r_0, r_1, \ldots, r_{N-1}$ for the roots of 0 under $\sigma$: $r_0 = 0$ and $\int_0^{r_k} \varphi(t)\, dt = k$ for all $k = 1, \ldots, N - 1$. We write $r := \min\{r_1, 1 - r_{N-1}\}$.

The following lemma captures a few basic properties of the roots under $\sigma$ that help us define the section $\mathfrak{s}$.

LEMMA 13.2. *Let $x \in [0, 1)$ and let $y_0, \ldots, y_{N-1}$ be the roots of $x$ under $\sigma$. Then,*

$$0 \leq y_0 \leq \frac{x}{\underline{\varphi}} < x \quad \text{and} \quad 0 < 1 - y_{N-1} \leq \frac{1 - x}{\underline{\varphi}} < 1 - x. \tag{13-2}$$

PROOF. The first formula follows using an easy computation:

$$x = \int_0^{y_0} \varphi(t)\, dt \geq \int_0^{y_0} \underline{\varphi}\, dt = y_0 \underline{\varphi}.$$

Thus, $y_0 < x/\underline{\varphi}$. The fact that $x/\underline{\varphi} < x$ follows since $\underline{\varphi} > 1$ by our assumption. The second formula follows from a similar computation:

$$1 - x = N - (x + (N - 1)) = \int_0^1 \varphi(t)\, dt - \int_0^{y_{N-1}} \varphi(t)\, dt = \int_{y_{N-1}}^1 \varphi(t)\, dt \geq (1 - y_{N-1})\underline{\varphi}.$$

□

Equation (13-2) implies that if $x \in [0, 1/2)$, then $y_0 \in [0, 1/2)$ and if $x \in [1/2, 1)$, then $y_{N-1} \in [1/2, 1)$. We call $y_0$ if $x \in [0, 1/2)$ and $y_{N-1}$ if $x \in [1/2, 1)$ the *standard root* of $x$ under $\sigma$ and denote it by $\tilde{x}$.

It is convenient for our purposes to parameterize the unit circle starting at the point (-1,0) instead of (1,0). That is, we view $z = e^{2\pi i x}$ with $x \in [-1/2, 1/2)$ instead of $x \in [0, 1)$. Using this perspective, if $x \in [-1/2, 1/2)$, the standard root $\tilde{x}$ of $x$ under $\sigma$ satisfies $\int_0^{\tilde{x}} \varphi(t)\, dt = x$.

We can now define the promised section $\mathfrak{s} : \mathbb{T} \to \mathbb{T}_\infty$. Let $z = e^{2\pi i x}$ with $x \in [-1/2, 1/2)$. Set $x_0 = x$ and define $x_1 := \widetilde{x_0}$, the standard root of $x_0$ under $\sigma$. We proceed then inductively: $x_{n+1} := \widetilde{x_n}$, the standard root of $x_n$ under $\sigma$ for all $n \geq 1$. Hence, by definition, $\sigma(e^{2\pi i x_{n+1}}) = e^{2\pi i x_n}$ for all $n \geq 0$.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

LEMMA 13.3. *Let $x \in [-1/2, 1/2)$ and let $(x_n)_{n\geq 0}$ be the sequence defined above. Then, $(e^{2\pi i x_n}) \in \mathbb{T}_\infty$ and, thus, the function $\mathfrak{s} : \mathbb{T} \to \mathbb{T}_\infty$ defined via $\mathfrak{s}(e^{2\pi i x}) := (e^{2\pi i x_n})_{n\geq 0}$ for all $x \in [-1/2, 1/2)$ is a section. Moreover, $\lim_{n\to\infty} x_n = 0$ and, thus, $\lim_{n\to\infty} e^{2\pi i x_n} = 1$ for all $x \in [-1/2, 1/2)$.*

PROOF. The fact that $(e^{2\pi i x_n})_{n\geq 0} \in \mathbb{T}_\infty$ follows from the construction of the sequence: $\sigma(e^{2\pi i x_{n+1}}) = e^{2\pi i x_n}$ for all $n \geq 0$. If $x \in [-1/2, 1/2)$, (13-2) implies that $|x_n| \leq |x|/\underline{\varphi}^n$ for all $n \geq 1$. Therefore, $\lim_{n\to\infty} x_n = 0$ since $\underline{\varphi} > 1$. □

We use tacitly the following notation in the remainder of this section: for each $x \in [-1/2, 1/2)$, we let $(x_n)_{n\geq 0}$ be the sequence defined in Lemma 13.3 and $z_n = e^{2\pi i x_n}$ for all $n \geq 0$. Therefore, $\mathfrak{s}(z) = (z_n)_{n\geq 0}$. Moreover, we do not write explicitly the dependence of the sequence $(x_n)$ on $x$ to keep the notation short.

Let $\psi : \mathbb{T} \to \mathbb{R}$ be the full potential defined via $\psi(e^{2\pi i x}) = 1/\varphi(x)$ for all $x \in [0, 1]$, and let $\mathcal{L}_\psi$ be its transfer operator. With respect to the induced metric, the Hausdorff dimension of $\mathbb{T}$ is 1 and the Hausdorff measure $\mu := \mu^1$ is invariant under the transpose of $\mathcal{L}_\psi$. Consequently, it is a quasi-invariant measure for $G(\mathbb{T}, \sigma)$ [16, Section 4.5]. Moreover, if $\varphi(x) > 1$ for all $x \in [0, 1]$ and is continuously differentiable, then $\sigma$ is expansive and exact. Thus, $\mu$ is the unique measure that is a fixed point of the transpose of $\mathcal{L}_\psi$ ([19, Theorem 2.8], [29, Proposition 4.2], [16, Proposition 3.5]). The Hausdorff measure $\mu$ is the only measure that we consider in this example.

As discussed in the paragraph above (10-1), we assume that $\mathcal{L}_\psi$ is unital to keep the notation manageable. The results below remain true in the nonunital case if one defines $\mathfrak{u}(x) = \boldsymbol{m}(x)\sqrt{\psi(x)}/\mathcal{L}_\psi(1)(\sigma(x))$ and uses the invariance of $\mu$ as described in the paragraph containing (11).

Fix a $\psi$-filter $\boldsymbol{m} \in C(\mathbb{T})$ and set, as usual, $\mathfrak{u}(z) = \boldsymbol{m}(z)\sqrt{\psi(z)}$ and $D(z) = |\mathfrak{u}(z)|^2 = |\boldsymbol{m}(z)|^2\psi(z)$ for all $z \in \mathbb{T}$. Therefore, $\mathfrak{u}(e^{2\pi i x}) = \boldsymbol{m}(e^{2\pi i x})/\sqrt{\varphi(x)}$ for all $x \in [0, 1]$. Since we extended $\varphi$ to $\mathbb{R}$ by periodicity, $\mathfrak{u}(e^{2\pi i x}) = \boldsymbol{m}(e^{2\pi i x})/\sqrt{\varphi(x)}$ for all $x \in \mathbb{R}$.

Consider the $p$-system of measures $\{\nu_z^D\}_{z\in\mathbb{T}}$ on $\mathbb{T}_\infty$ defined by $D$ as in (6-1), where, recall, $p : \mathbb{T}_\infty \to \mathbb{T}$ is the projection onto the first component. Recall also that $r_0 = 0, r_1, \ldots, r_{N-1} \in [0, 1)$ are the roots of 0 under $\sigma$ and $r = \min\{r_1, 1 - r_{N-1}\} \in (0, 1)$.

The next lemma is key to our analysis. It provides conditions that guarantee that the section $\mathfrak{s}$ defined before is positively supported and, thus, a scaling function $\phi$ exists. This lemma is the first step towards our generalization of [23, Theorem 2]. We note that even though our set-up is more general than the one that Mallat considered, the hypotheses of our lemma are virtually identical to Mallat's hypotheses.

LEMMA 13.4. *With the notation as above, assume that $\varphi(x) > 1$ for all $x \in [0, 1]$. Assume that $\mathfrak{u} : \mathbb{T} \to \mathbb{C}$, $\mathfrak{u}(e^{2\pi i x}) := \boldsymbol{m}(e^{2\pi i x})/\sqrt{\varphi(x)}$ for $x \in [-1/2, 1/2)$, satisfies the following hypotheses:*

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

$$|\mathfrak{u}(1)| = 1, \tag{13-3}$$

$$|\mathfrak{u}(e^{2\pi ix})| \neq 0 \quad \textit{if } |x| < r, \tag{13-4}$$

$$\ln D(e^{2\pi ix}) = \ln |\mathfrak{u}(e^{2\pi ix})|^2 = O(x) \quad \textit{for } x \textit{ near } 0. \tag{13-5}$$

*Then, the section $\mathfrak{s} : \mathbb{T} \to \mathbb{C}$ defined above is positively supported. That is, the scaling function $\phi : \mathbb{T} \to \mathbb{C}$ given via*

$$\phi(z) = \prod_{n=1}^{\infty} \mathfrak{u}(z_n) = \prod_{n=1}^{\infty} \frac{1}{\sqrt{\varphi(x_n)}} \boldsymbol{m}(e^{2\pi ix_n})$$

*for all $z = e^{2\pi ix} \in \mathbb{T}$ with $x \in [-1/2, 1/2)$ converges for all $z \in \mathbb{T}$, where $(x_n)_{n\geq 0}$ is the sequence defined in Lemma 13.3.*

REMARK 13.5.

(1) If the hypothesis (13-3) is satisfied, then $D(e^{2\pi ir_k}) = 0$ for all $k = 1, \dots, N-1$ since $\sum_{k=0}^{N-1} D(e^{2\pi ir_k}) = 1$. Thus, $\boldsymbol{m}(e^{2\pi ir_k}) = 0$ for all $k = 1, \dots, N-1$. The converse is also true: if $\boldsymbol{m}(e^{2\pi ir_k}) = 0$ for all $k = 1, \dots, N-1$, then $\mathfrak{u}$ satisfies (13-3). If the hypothesis (13-4) is satisfied as well, then $e^{2\pi ir_1}, \dots, e^{2\pi ir_{N-1}}$ are the only points where $\boldsymbol{m}$ vanishes.

(2) The scaling function $\phi$ satisfies the relation

$$\phi(z) = \Big( \prod_{i=1}^{n} \mathfrak{u}(z_i) \Big) \phi(z_n) \quad \text{for all } n \geq 1 \text{ and } z \in \mathbb{T}. \tag{13-6}$$

PROOF OF LEMMA 13.4. We continue to use the notation $\underline{\varphi} := \min_{x\in[0,1]} \varphi(x) > 1$. Note first that $|\phi(e^{2\pi i0})|^2 = |\phi(1)|^2 = 1$ by (13-3) since $\mathfrak{s}(1) = (\underline{1})$ by construction. Second, if $x \in [-1/2, 1/2)$, the product $\prod_{n=1}^{\infty} |\mathfrak{u}(z_n)|^2 = \prod_{n=1}^{\infty} D(z_n)$ converges since $0 \leq D(z) \leq 1$ for all $z \in \mathbb{T}$. Moreover, using (13-2) and $\underline{\varphi} > 1$, $|x_n| < r$ for all $n \geq 1$ and, thus, $D(x_n) > 0$ for all $n \geq 1$ by (13-4). Since

$$|\phi(e^{2\pi ix})|^2 = e^{\sum_{n\geq 1} \ln D(e^{2\pi ix_n})},$$

using (13-5) and the fact that $\lim_{n\to\infty} x_n = 0$, we obtain the formulas

$$\lim_{x\to 0} |\phi(e^{2\pi ix})|^2 = 1 = |\phi(1)|^2.$$

Let $0 < \varepsilon < 1$ and let $\delta > 0$ be such that $|\phi(x)|^2 > 1 - \varepsilon$ for all $|x| < \delta$. Let $x \in [-1/2, 1/2)$. Since $\lim_{n\to\infty} x_n = 0$, there is $M \geq 1$ such that $|x_n| < \delta$ for all $n \geq M$. Therefore, using (13-6), we have

$$|\phi(e^{2\pi ix})|^2 = \Big( \prod_{i=1}^{M} D(x_i) \Big) |\phi(e^{2\pi ix_M})|^2 > (1-\varepsilon) \Big( \prod_{i=1}^{M} D(x_i) \Big) > 0$$

for all $x \in [-1/2, 1/2)$. The conclusion follows. □

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

Next, we lift $\phi$ to a 'nonconstant' scale, scaling function $\tilde{\phi}$ on $\mathbb{R}$:

$$\tilde{\phi}\left(\int_0^x \varphi(t)\,dt\right) = \frac{1}{\sqrt{\varphi(x)}}\boldsymbol{m}(e^{2\pi ix})\tilde{\phi}(x)$$

or, equivalently,

$$\sqrt{\varphi(x)}\tilde{\phi}\left(\int_0^x \varphi(t)\,dt\right) = \boldsymbol{m}(e^{2\pi ix})\tilde{\phi}(x)$$

for all $x \in \mathbb{R}$. To do this, we extend the construction of the sequence $(x_n)$ from $x \in [-1/2, 1/2)$ to all $x \in \mathbb{R}$. Specifically, for $x \in \mathbb{R}$, let $\{x_n\}_{n\geq 0}$ be the sequence defined via: $x_0 = x$, and, $x_{n+1}$ is such that $\int_0^{x_{n+1}} \varphi(t)\,dt = x_n$ for $n \geq 0$. Equation (13-2) implies that $\lim_{n\to\infty} x_n = 0$. Moreover, $\sigma(e^{2\pi i x_{n+1}}) = e^{2\pi i x_n}$ for all $n \geq 0$. The sequence $\{x_n\}_{n\geq 0}$ depends on $x$; however, as before, we do not explicitly write the dependence of the sequence $\{x_n\}$ on $x$. Again, this should not lead to any confusion. Define

$$\tilde{\phi}(x) = \prod_{n\geq 1} \mathfrak{u}(e^{2\pi i x_n}) = \prod_{n\geq 1} \frac{\boldsymbol{m}(e^{2\pi i x_n})}{\sqrt{\varphi(x_n)}} \tag{13-7}$$

for all $x \in \mathbb{R}$. Hence, $|\tilde{\phi}(x)|^2 = \prod_{n\geq 1} D(e^{2\pi i x_n})$, $\tilde{\phi}(x) = \phi(e^{2\pi ix})$ for all $x \in [-1/2, 1/2)$, and $\tilde{\phi}(x) = (\prod_{i=1}^n \mathfrak{u}(e^{2\pi i x_i}))\tilde{\phi}(x_n)$ for all $n \geq 1$. Moreover, if (13-3) is satisfied, then $|\tilde{\phi}(0)| = 1$.

EXAMPLE 13.6. Assume that $\varphi : [0, 1] \to \mathbb{R}$ is the constant function $\varphi(t) = 2$ for all $x \in [0, 1]$. Then, the local homeomorphism $\sigma : \mathbb{T} \to \mathbb{T}$ is given by $\sigma(z) = z^2$. Moreover, $\psi(z) = 1/2$ and $\mathcal{L}_\psi$ is the standard transfer operator. Then, a $\psi$-filter is a classical filter and, if it satisfies the hypotheses of Lemma 13.4, it is a QMF filter in the sense of Mallat. If $x \in \mathbb{R}$, then $x_n = x/2^n$ for all $n \geq 1$. The scaling function $\tilde{\phi}$ is given by the same equation as in [23, Theorem 2]:

$$\tilde{\phi}(x) = \prod_{n\geq 1} \mathfrak{u}(e^{2\pi ix/2^n}) = \prod_{n\geq 1} \frac{\boldsymbol{m}(e^{2\pi ix/2^n})}{\sqrt{2}}$$

and it satisfies the 2-scaling property: $\tilde{\phi}(2x) = m(e^{2\pi ix})/\sqrt{2}\tilde{\phi}(x)$ for all $x \in \mathbb{R}$. This example generalizes immediately to the function $\varphi(t) := N \geq 2$. In this case, the scaling function has the well-known form

$$\tilde{\phi}(x) = \prod_{n\geq 1} \mathfrak{u}(e^{2\pi ix/N^n}) = \prod_{n\geq 1} \frac{\boldsymbol{m}(e^{2\pi ix/N^n})}{\sqrt{N}}$$

and it satisfies the $N$-scaling property: $\tilde{\phi}(Nx) = m(e^{2\pi ix})/\sqrt{N}\tilde{\phi}(x)$ for all $x \in \mathbb{R}$.

The following theorem is a generalization of Mallat's [23, Theorem 2] to our more general set-up.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

THEOREM 13.7. *Assume the hypotheses of Lemma 13.4. Then, the function* $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ *defined in (13-7) is a unit vector in* $L^2(\mathbb{R})$ *and*

$$\sum_{j\in\mathbb{Z}} |\tilde{\phi}(x+j)|^2 = 1 \quad \text{for all } x \in [-1/2, 1/2).$$

The proof is completed using a series of lemmas.

LEMMA 13.8. *For* $m \geq 1$, *let*

$$\tilde{\phi}_m(x) = \begin{cases} \displaystyle\prod_{i=1}^{m} \mathrm{u}(x_i) & \text{if } |x| \leq \dfrac{N^m}{2}, \\ 0 & \text{otherwise}, \end{cases}$$

*and let* $I_m := \int_{-((N^m)/2)}^{(N^m)/2} |\tilde{\phi}_m(x)|^2\, dx$. *Then,* $I_m = 1$ *for all* $m \geq 1$.

PROOF. Let $m \geq 1$. For $x \in [-1/2, 1/2)$, let

$$J_{m,x} = \{j \in \mathbb{Z} \mid x + j \in [-N^m/2, N^m/2)\}.$$

The cardinality of $J_{m,x}$ equals $N^m$ for all $x \in [-1/2, 1/2)$. For each $j \in J_{m,x}$, let $y_j \in [-1/2, 1/2)$ be the $m$th term in the sequence $(x_n)_{n\geq 0}$ corresponding to $x + j$. Then, $\sigma^m(e^{2\pi i y_j}) = e^{2\pi i x}$ for all $j \in J_{m,x}$. Therefore, $\{e^{2\pi i y_j}\}_{j\in J_{m,x}}$ is the set of pre-image points of $e^{2\pi i x}$ under $\sigma^m$. (Once again, while $y_j$ depends on $x$, we do not write explicitly the dependence in the notation.) Therefore,

$$\int_{-((N^m)/2)}^{(N^m)/2} |\tilde{\phi}_m(x)|^2\, dx = \int_{\mathbb{T}} \sum_{\sigma^m(w)=z} D_m(w)\, d\mu(z) = 1$$

since $D_m$ is the normalized potential of $\mathcal{L}_D^m$. □

LEMMA 13.9. *With the notation as above,* $|\tilde{\phi}(0)|^2 = 1$ *and* $|\tilde{\phi}(j)|^2 = 0$ *for all* $j \in \mathbb{Z} \setminus \{0\}$.

PROOF. The fact that $|\phi(0)| = 1$ was proved in Lemma 13.4. Let $j \in \mathbb{Z} \setminus \{0\}$. Assume first that $j \in [-N/2, N/2)$ (using the notation from the proof of Lemma 13.8, $j \in J_{1,0}$). Let $y_j \in [-1/2, 1/2)$ such that $\int_0^{y_j} \varphi(t)\, dt = j$. Therefore, $y_j$ equals the term $x_1$ of the sequence $\{x_n\}_{n\geq 0}$ used to define $\tilde{\phi}(j)$. Since $\sigma(e^{2\pi i y_j}) = e^{2\pi i j} = 1$, it follows that $D(e^{2\pi i y_j}) = 0$ (see the second part of Remark 13.5). Hence, $\tilde{\phi}(j) = 0$.

Assume that $j \in [-N/2 + kN, N/2 + kN)$ for some $k \in \mathbb{Z} \setminus \{0\}$. Then, $j - kN \in [-N/2, N/2)$. If $y_j \in [-1/2, 1/2)$ is such that $\int_0^{y_j} \varphi(t)\, dt = j - N$, then the term $x_1$ of the sequence $(x_n)_{n\geq 0}$ used to define $\tilde{\phi}(j)$ equals $y_j + k$ since $\int_0^1 \varphi(t)\, dt = N$. Therefore, $D(e^{2\pi i(y_j+k)}) = D(e^{2\pi i y_j}) = 0$ and, thus, $\tilde{\phi}(j) = 0$. □

PROOF OF THEOREM 13.7. The fact that $\tilde{\phi} \in L^2(\mathbb{R})$ and $\|\tilde{\phi}\|_2 = 1$ follows from Lemma 13.8, Proposition 10.1, and Fatou's lemma.

Let $x \in [-1/2, 1/2)$. For $m \geq 1$, we use the notation from the proof of Lemma 13.8: $J_{m,x} = \{j \in \mathbb{Z} \mid x + j \in [-N^m/2, N^m/2)\}$ and $y_j$ is the $m$th term of the sequence $\{x_n\}_{n\geq 0}$ that defines $x + j$ for all $j \in J_{m,x}$. Then,

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

$$\sum_{k=-N^m/2}^{N^m/2} |\tilde{\phi}(x+k)|^2 = \sum_{j\in J_{m,x}} D_m(y_j)|\tilde{\phi}(y_j)|^2.$$

One can prove inductively that $|y_j| \le (N/2)/\underline{\varphi}^m$ for all $j \in J_{m,x}$, where, as before, $\underline{\varphi} = \min_{t\in[0,1]} \varphi(t) > 1$. Recall from the proof of Lemma 13.4 that Condition (13-5) implies that $\tilde{\phi}$ is continuous at 0 and that $\tilde{\phi}(0) = 1$ by (13-3). Let $\varepsilon > 0$ and let $\delta > 0$ be such that $||\tilde{\phi}(t)|^2 - 1| < \varepsilon$ for all $t \in (-\delta, \delta)$. Then, there is $M \ge 1$ such that $|y_j| < \delta$ for all $m \ge M$ (recall that $y_j$ depends both on $x$ and on $m$, but we suppressed the dependence in the notation). Lemma 13.9 implies

$$\begin{aligned}\left|\sum_{k=-N^m/2}^{N^m/2} |\tilde{\phi}(x+k)|^2 - 1\right| &= \left|\sum_{j\in J_{m,x}} D_m(y_j)|\tilde{\phi}(y_j)|^2 - 1\right| \\ &= \left|\sum_{j\in J_{m,x}} D_m(y_j)(|\tilde{\phi}(y_j)|^2 - 1)\right| \le \sum_{j\in J_{m,x}} D_m(y_j)||\tilde{\phi}(y_j)|^2 - 1| < \sum_{j\in J_{m,x}} D_m(y_j)\varepsilon = \varepsilon\end{aligned}$$

for all $m \ge M$. Therefore, $\sum_{j\in\mathbb{Z}} |\tilde{\phi}(x+j)|^2 = 1$ for all $x \in [-1/2, 1/2)$. □

Next, we obtain the full generalization of Mallat's theorem in our context and we construct multiresolution analyses with scaling functions of nonconstant scale. For this, we consider a unitary representation $\hat{L} = (\mu, \mathbb{T} * \mathcal{H}, \hat{L})$ of the Deaconu–Renault groupoid $G(\mathbb{T}, \sigma)$, where, recall, $\mu$ is the Hausdorff measure on $\mathbb{T}$. We continue to assume that $\boldsymbol{m}$ is a $\psi$-filter that satisfies the hypotheses of Lemma 13.4. For simplicity and comparison with the previous results in the literature, assume that $\hat{L} = \hat{\iota}$ is the standard fundamental representation on $\mu$: $\hat{\iota} = (\mu, \mathbb{T} * \mathbb{C}, \hat{\iota})$. Therefore, the integrated form $\Im$ of $\hat{\iota}$ acts on $L^2(\mathbb{T}, \mu)$ and the isometry $\tilde{S}_{\mathfrak{u}} = \Im(S_{\mathfrak{u}})$ is given by $\tilde{S}_{\mathfrak{u}}\xi(z) = \boldsymbol{m}(z)\xi(\sigma(z))$. That is,

$$\tilde{S}_{\mathfrak{u}}\xi(e^{2\pi ix}) = \boldsymbol{m}(e^{2\pi ix})\xi(e^{2\pi i\int_0^x \varphi(t)\,dt}) \quad \text{for all } x \in [-1/2, 1/2).$$

We endow $G_\infty(\mathbb{T}, \sigma)$ with the Haar system $\nu^\psi * \lambda$ defined as in (6-3), where $\nu^\psi$ is the $p$-system of measures on $X_\infty$ defined by the full unital potential $\psi$ via (6-1). The unital potential $D = |\mathfrak{u}|^2$ defines, via the same formula, a $p$-system of measures $\nu^D$ on $X_\infty$ that is not full. Recall that $\nu_u^D$ is absolutely continuous with respect to $\nu_u^\psi$ since $D(z) = |\boldsymbol{m}(z)|^2\psi(z)$ for all $z \in \mathbb{T}$. Therefore, we can use the techniques of the paragraph after Proposition 3.2 and Section 8 to induce $\hat{\iota}$ to a representation $\hat{\iota}_\infty = (\mu_\infty, \mathbb{T}_\infty * \mathbb{C}, \hat{\iota}_\infty)$ of $G_\infty(\mathbb{T}, \sigma)$. Moreover $C^*(G_\infty(\mathbb{T}, \sigma), \nu^\psi * \lambda)$ acts on the completion of $C_c(X_\infty * G)$ under the $C^*(\mathbb{T})$-inner product defined in (4-5) via adjointable operators as in (4-6). Equations (8-2) and (4-7) imply that the unitary $\tilde{U}_{\mathfrak{u}} := \Im_\infty(U_{\mathfrak{u}})$, where $\Im_\infty$ is the integrated form of $\hat{\iota}$, acts on $L^2(\mathbb{T}_\infty, \mu_\infty)$ via

$$\tilde{U}_{\mathfrak{u}}(\xi)(\underline{z}) = \boldsymbol{m}(p(\underline{z}))\xi(\sigma_\infty(\underline{z})) \quad \text{for all } \underline{z} \in \mathbb{T}_\infty \text{ and } \xi \in L^2(\mathbb{T}_\infty, \mu_\infty).$$

THEOREM 13.10. *Using the above notation, assume the hypotheses of Lemma 13.4. Let $\phi : \mathbb{T} \to \mathbb{T}_\infty$ be the corresponding scaling function (10-2) and $\tilde{\phi} : \mathbb{R} \to \mathbb{T}_\infty$ its lift*

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

*defined via (13-7). Let $\hat{\iota} = (\mu, \mathbb{T} * \mathbb{C}, \hat{\iota})$ be the standard fundamental representation on $\mu$ of the Deaconu–Renault groupoid $G(\mathbb{T}, \sigma)$ and let $\hat{\iota}_\infty = (\mu_\infty, \mathbb{T}_\infty * \mathbb{C}, \hat{\iota}_\infty)$ be the induced representation of the pullback groupoid $G_\infty(\mathbb{T}, \sigma)$. Then, there is an isometric isomorphism $R : L^2(\mathbb{T}_\infty, \mu_\infty) \to L^2(\mathbb{R})$ that intertwines the unitary $\tilde{U}_{\mathfrak{u}}$ and the unitary $\mathfrak{U}_{\mathfrak{u}}$ on $L^2(\mathbb{R})$ defined via the formula*

$$\mathfrak{U}_{\mathfrak{u}}(\xi)(x) = \sqrt{\varphi(x)}\xi\Big(\int_0^x \varphi(t)\,dt\Big) \tag{13-8}$$

*for all $\xi \in L^2(\mathbb{R})$.*

REMARK 13.11. A proof similar to the proof of [3, Theorem 6.4] shows that the pair, $(L^2(\mathbb{T}_\infty, \mu_\infty), \tilde{U}_{\mathfrak{u}})$, is the inductive limit of the inductive system $(H_n, S_n)$, where $H_n = L^2(\mathbb{T}, \mu)$ and $S_n = \tilde{S}_{\mathfrak{u}}$ for all $n \geq 0$ (see also [21]).

PROOF. We prove the existence of the isometric isometry $R$. The remainder of the statements follow immediately. To prove the existence of $R$, we define a sequence of isometries $R_n : L^2(\mathbb{T}, \mu_n) \to L^2(\mathbb{R})$ such that $R_{n+1}\tilde{S}_{\mathfrak{u}} = R_n$ for all $n \geq 0$. The conclusion follows from the universal properties of the inductive limit of Hilbert spaces.

Recall from the definition of $\tilde{\phi}$, (13-7), that for each $x \in \mathbb{R}$, we define the sequence $\{x_n\}_{n\geq 0}$ where $x_0 = x$ and $\int_0^{x_{n+1}} \varphi(t)\,dt = x_n$ for all $n \geq 0$. Define $R_n : L^2(\mathbb{T}, \mu) \to L^2(\mathbb{R})$ via

$$R_n(f)(x) = \frac{1}{\sqrt{\varphi(x_n)\cdots\varphi(x_1)}} f(e^{2\pi i x_n})\tilde{\phi}(x_n)$$

for all $x \in \mathbb{R}$, $f \in L^2(\mathbb{T}, \mu)$, and $n \geq 0$. For $n = 0$, we assume that the denominator of the fraction is 1. It is relatively easy to check that $R_{n+1}\tilde{S}_{\mathfrak{u}} = R_n$:

$$\begin{aligned}(R_{n+1}\tilde{S}_{\mathfrak{u}})(f)(x) &= \frac{1}{\sqrt{\varphi(x_{n+1})\cdots\varphi(x_1)}}\tilde{S}_{\mathfrak{u}}(f)(e^{2\pi i x_{n+1}})\tilde{\phi}(x_{n+1}) \\ &= \frac{1}{\sqrt{\varphi(x_n)\cdots\varphi(x_1)}} f(e^{2\pi i x_n})\frac{m(e^{2\pi i x_{n+1}})}{\sqrt{\varphi(x_{n+1})}}\tilde{\phi}(x_{n+1}) \\ &= \frac{1}{\sqrt{\varphi(x_n)\cdots\varphi(x_1)}} f(e^{2\pi i x_n})\tilde{\phi}(x_n) = R_n(f)(x).\end{aligned}$$

We prove next that $R_n$ is an isometry for all $n \geq 1$. Let $f \in L^2(\mathbb{T}, \mu)$. Then,

$$\|R_n(f)\|_2^2 = \int_{-\infty}^{\infty} \frac{1}{\varphi(x_n)\cdots\varphi(x_1)} |f(e^{2\pi i x_n})|^2 |\tilde{\phi}(x_n)|^2\,dx.$$

We claim that

$$\|R_n(f)\|_2^2 = \int_{-1/2}^{1/2} |f(e^{2\pi i x})|^2 \Big(\sum_{j\in\mathbb{Z}} |\tilde{\phi}(x+j)|^2\Big)\,d\mu(x).$$

We prove this claim first for $n = 1$ for simplicity. We use the notation from the proof of Lemma 13.8. Let $x \in [-1/2, 1/2)$ and let $\{y_j\}_{j\in J_{1,x}} \in [-1/2, 1/2)$ be the roots of $x$ under

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

$\sigma$: $\int_0^{y_j} \varphi(t)\,dt = x + j$ for all $j \in J_{1,x}$. Then, since $\mu$ is invariant for the transpose of the transfer operator $\mathcal{L}_\psi$, we have

$$\int_{-N/2}^{N/2} \frac{1}{\varphi(x_1)} |f(e^{2\pi i x_1})|^2 |\tilde{\phi}(x_1)|^2\,dx = \int_{-1/2}^{1/2} \sum_{j \in J_{1,x}} \frac{1}{\varphi(y_j)} |f(e^{2\pi i y_j})|^2 |\phi(y_j)|^2\,d\mu(x)$$
$$= \int_{-1/2}^{1/2} \mathcal{L}_\psi(|f|^2|\phi|^2)(x)\,d\mu(x) = \int_{-1/2}^{1/2} |f(e^{2\pi i x})|^2 |\phi(x)|^2\,d\mu(x).$$

Let $k \in \mathbb{Z}$ such that $k \neq 0$ and let $x \in [-N/2 + kN, N/2 + kN)$. Then, $x - kN \in [-1/2, 1/2)$ and we let $\{y_j\}_{j \in J_{1,x-kN}}$ be the roots of $x - kN$ under $\sigma$. It follows that if $x_1$ is such that $\int_0^{x_1} \varphi(t)\,dt = x + j$, $j \in J_{1,x-kN}$, then $x_1 = k + y_j$. Therefore,

$$\int_{-N/2+kN}^{N/2+kN} \frac{1}{\varphi(x_1)} |f(e^{2\pi i x_1})|^2 |\tilde{\phi}(x_1)|^2\,dx$$
$$= \int_{-1/2}^{1/2} \sum_{j \in J_{1,x-kN}} \frac{1}{\varphi(y_j)} |f(e^{2\pi i y_j})|^2 |\tilde{\phi}(k + y_j)|^2\,d\mu(x) = \int_{-1/2}^{1/2} |f(e^{2\pi i x})|^2 |\tilde{\phi}(k + x)|^2\,d\mu(x).$$

The claim for $n = 1$ follows. For $n \geq 2$, one can repeat the same arguments using the $n$ roots of $x$, $\{y_j\}_{j \in J_{n,x}}$, since $\mu$ is invariant for the transpose of the transfer operator $\mathcal{L}_\psi^n$ for all $n \geq 2$. Thus,

$$\int_{-\infty}^{\infty} \frac{1}{\varphi(x_n) \cdots \varphi(x_1)} |f(e^{2\pi i x_n})|^2 |\tilde{\phi}(x_n)|^2\,dx = \int_{-1/2}^{1/2} |f(e^{2\pi i x})|^2 \Big( \sum_{j \in \mathbb{Z}} |\tilde{\phi}(x + j)|^2 \Big)\,d\mu(x).$$

Since $\sum_{j \in \mathbb{Z}} |\tilde{\phi}(x + j)|^2 = 1$ for all $x \in [-1/2, 1/2)$ by Theorem 13.7, $R_n$ is an isometry for all $n \geq 0$ and the conclusion follows. □

Combining the above results with the proto-multiresolution analysis, one obtains multiresolution analyses in our set-up.

THEOREM 13.12. *Assume that $\varphi : [0, 1] \to \mathbb{R}$ satisfies the hypotheses described at the beginning of the section and let $\sigma : \mathbb{T} \to \mathbb{T}$ be defined via $\sigma(e^{2\pi i x}) = e^{2\pi i \int_0^x \varphi(t)\,dt}$. Assume that $\mu$ is the Hausdorff measure on $\mathbb{T}$ that is invariant for the transfer operator $\mathcal{L}_\psi$ with potential $\psi(e^{2\pi i x}) = 1/\varphi(x)$. Fix a $\psi$-filter $\boldsymbol{m}$ and assume that the function $\mathfrak{u} \in C(X)$, $\mathfrak{u}(x) = \sqrt{\psi(x)}\boldsymbol{m}(x)$ satisfies the hypotheses of Lemma 13.4. Let $\hat{\iota} = (\mu, \mathbb{T} \times \mathbb{C}, \hat{\iota})$ be the standard fundamental representation on $\mu$ of $G(\mathbb{T}, \sigma)$ and let $\hat{\iota}_\infty = (\mu_\infty, \mathbb{T}_\infty, \hat{\iota}_\infty)$ be the induced representation of $G_\infty(\mathbb{T}, \sigma)$ defined by $D = |\mathfrak{u}|^2$. Let $(\{\mathbb{E}_n\}, U_\mathfrak{u})$ be the proto-resolution analysis in $C_c(G_\infty(\mathbb{T}, \sigma), \nu^\psi * \lambda)$ defined in the paragraph before Definition 9.2 and let $\mathfrak{U}_\mathfrak{u}$ be the unitary in $L^2(\mathbb{R})$ defined in (13-8). Set $\mathcal{H}_n = i_\infty(\mathbb{E}_n)L^2(\mathbb{T}_\infty, \mu_\infty)$ and $\mathcal{V}_n := R(\mathcal{H}_n) \subset L^2(\mathbb{R})$. Then:*

(1) $\mathcal{V}_n \subseteq \mathcal{V}_m$, *when* $n \leq m$;
(2) $\vee_{n=-\infty}^{\infty} \mathcal{V}_n = L^2(\mathbb{R})$ *and* $\wedge \mathcal{V}_n = \{0\}$;
(3) $\mathfrak{U}_\mathfrak{u}(\mathcal{V}_{n+1}) = \mathcal{V}_n$ *for all* $n \in \mathbb{Z}$;

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

(4) *$R(i_\infty \circ \Phi(\cdot))$ is reduced by $\mathcal{V}_0 \setminus \mathcal{V}_{-1}$ and the restriction is a representation of $C_c(G)$.*

*Thus, $\{\mathcal{V}_n\}_{n\in\mathbb{Z}}$ is a multiresolution analyses in $L^2(\mathbb{R})$ with possibly nonconstant scale.*

EXAMPLE 13.13.

(1) If $\varphi(t) = 2$ for all $t \in [0, 1]$, then Theorems 13.7 and 13.10 recover Mallat's theorem [23, Theorem 2].
(2) If $\varphi(t) = N \geq 2$ for all $t \in [0, 1]$, and, thus, $\sigma(z) = z^N$, our results also capture known results in the literature. Since, as described in Remark 9.4, $G_\infty(\mathbb{T}, \sigma)$ can be constructed from $G(X, \sigma)$ via a projective system, when one looks at the function spaces, $(L^2(\mathbb{T}_\infty, \mu_\infty), \tilde{U}_{\mathrm{u}})$ corresponds to the direct limit construction of [21].
(3) If $\sigma$ is the restriction of a finite Blaschke product (13-1), then under the assumption that $\sum_{i=1}^{N}(1 - |a_j|)/(1 + |a_j|) > 1$, $\sigma$ is an expansive local homeomorphism [24, Corollary on page 344]. Therefore, the Hausdorff measure $\mu$ is the unique invariant measure for the dual of the transfer operator $\mathcal{L}_\psi$. One can easily find $\psi$-filters $\boldsymbol{m}$ that satisfy the hypotheses of Lemma 13.4. Indeed, let $m : \mathbb{T} \to \mathbb{C}$ be any continuous function such that $m(1) \neq 0$ and $m(e^{2\pi i r_j}) = 0$ for all $j = 1, \ldots, N-1$, where $r_j$, $j = 0, \ldots N-1$, are the units of 0 with $r_0 = 0$ as defined earlier in the section. Assume that $m(z) \neq 0$ for all $z \notin \{e^{2\pi i r_0}, \ldots, e^{2\pi i r_{N-1}}\}$ and assume that $m$ is smooth near 0. Set $\xi(z) := \sum_{\sigma(w)=z} \psi(w)|m(w)|^2$. Then, one can easily check that the function defined via

$$\boldsymbol{m}(z) = \frac{m(z)}{\sqrt{\xi(\sigma(z))}} \quad \text{for all } z \in \mathbb{T}$$

is a $\psi$-filter that satisfies the required hypotheses.

**13.1. An example of a generalized MRA without a scaling function.** Lemma 13.4 was key to defining the scaling function $\phi$, which was the main ingredient in our identification of $L^2(\mathbb{T}_\infty, \mu_\infty)$ with $L^2(\mathbb{R})$. The next example shows that if the first hypothesis (13-3) of the lemma fails, then from the perspective of harmonic analysis, $L^2(\mathbb{T}_\infty, \mu_\infty)$ might be very different from $L^2(\mathbb{R})$. The main reason for this failure is the lack of a scaling function. The example recovers the wavelets on fractals of [10] from our set-up.

EXAMPLE 13.14. Let $X = \mathbb{T}$ and $\sigma : \mathbb{T} \to \mathbb{T}$ be defined via $\sigma(z) = z^3$. Consider the standard transfer operator $\mathcal{L}$ given by the potential $\psi(z) = 1/3$ for all $z \in \mathbb{T}$. Consider the $\psi$-filter $\boldsymbol{m} : \mathbb{T} \to \mathbb{C}$ defined via $\boldsymbol{m}(z) = 2^{-1/2}(1 + z^2)$ for all $z \in \mathbb{T}$ [3, 10] and let $\mathrm{u}(z) = \boldsymbol{m}(z)/\sqrt{3}$. Note that $|\mathrm{u}(z)| \leq \sqrt{2}/\sqrt{3}$ and, thus, it does not satisfy Hypothesis (13-3). Hence, one cannot define a scaling function in the sense of Proposition 10.1. Consider the standard fundamental representation $\hat{\iota} = (\mu, \mathbb{T} * \mathbb{C}, \hat{\iota})$, where $\mu$ is the Haar measure. Hence, the isometry $\tilde{S}_{\mathrm{u}} = \Im(S_u)$ on $L^2(\mathbb{T}, \mu)$ is given via $\tilde{S}_{\mathrm{u}}(\xi)(z) = \boldsymbol{m}(z)f(z^3)$ (see [3, Section 5]). Let $\hat{\iota}_\infty = (\mu_\infty^D, \mathbb{T}_\infty * \mathbb{C}, \hat{\iota}_\infty)$ be the induced representation

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

of $G_\infty(\mathbb{T}, \sigma)$ defined by the potential $D(z) = |\mathrm{u}(z)|^2$. Reference [3, Proposition 5.1] implies that $(L^2(\mathbb{T}_\infty, \mu_\infty^D), \tilde{U}_{\mathrm{u}})$ is unitarily equivalent to $(L^2(\mathcal{R}, \nu), \mathfrak{U}_{\mathrm{u}})$, where $\mathcal{R}$ is the 'inflated' Cantor set defined in [10, Definition 1.1], $\nu$ is the Hausdorff measure on $\mathcal{R}$, and $\mathfrak{U}_{\mathrm{u}} : L^2(\mathcal{R}, \nu) \to L^2(\mathcal{R}, \nu)$ is defined via $\mathfrak{U}_{\mathrm{u}}(\xi)(x) = 2^{-1/2}\xi(x/3)$. Thus, one recovers the MRA for the wavelets studied in [10].

## Acknowledgment

The views expressed in this paper are those of the authors and do not reflect the official policy or position of the U.S. Naval Academy, Department of the Navy, the Department of Defense, or the U.S. Government. The authors are very grateful to the referee who made suggestions that led to a substantial improvement of our original manuscript.

## References

[1] C. Anantharaman-Delaroche and J. Renault, *Amenable Groupoids*, Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique], 36 (L'Enseignement Mathématique, Geneva, 2000); with a foreword by G. Skandalis and Appendix B by E. Germain.

[2] L. W. Baggett, N. S. Larsen, K. D. Merrill, J. A. Packer and I. Raeburn, 'Generalized multiresolution analyses with given multiplicity functions', *J. Fourier Anal. Appl.* **15**(5) (2009), 616–633.

[3] L. W. Baggett, N. S. Larsen, J. A. Packer, I. Raeburn and A. Ramsay, 'Direct limits, multiresolution analyses, and wavelets', *J. Funct. Anal.* **258**(8) (2010), 2714–2738.

[4] L. W. Baggett, H. A. Medina and K. D. Merrill, 'Generalized multi-resolution analyses and a construction procedure for all wavelet sets in $\mathbb{R}^n$', *J. Fourier Anal. Appl.* **5**(6) (1999), 563–574.

[5] V. Baladi, *Positive Transfer Operators and Decay of Correlations*, Advanced Series in Nonlinear Dynamics, 16 (World Scientific Publishing Co., Inc., River Edge, NJ, 2000).

[6] O. Bratteli and P. E. T. Jorgensen, 'Isometries, shifts, Cuntz algebras and multiresolution wavelet analysis of scale $N$', *Integral Equations Operator Theory* **28**(4) (1997), 382–443.

[7] O. Bratteli and P. E. T. Jorgensen, *Wavelets Through a Looking Glass: The World of the Spectrum*, Applied and Numerical Harmonic Analysis (Birkhäuser, Boston, MA, 2002).

[8] D. Courtney, P. S. Muhly and S. W. Schmidt, 'Composition operators and endomorphisms', *Complex Anal. Oper. Theory* **6** (2012), 163–188.

[9] V. Deaconu, 'Groupoids associated with endomorphisms', *Trans. Amer. Math. Soc.* **347**(5) (1995), 1779–1786.

[10] D. E. Dutkay and P. E. T. Jorgensen, 'Wavelets on fractals', *Rev. Mat. Iberoam.* **22**(1) (2006), 131–180.

[11] D. E. Dutkay and P. E. T. Jorgensen, 'Disintegration of projective measures', *Proc. Amer. Math. Soc.* **135**(1) (2007), 169–179.

[12] R. Exel, 'A new look at the crossed-product of a $C^*$-algebra by an endomorphism', *Ergodic Theory Dynam. Systems* **23**(6) (2003), 1733–1750.

[13] S. R. Garcia, J. Mashreghi and W. T. Ross, *Finite Blaschke Products and their Connections* (Springer, Cham, 2018).

[14] R. D. Holkar, 'Topological construction of $C^*$-correspondences for groupoid $C^*$-algebras', *J. Operator Theory* **77**(1) (2017), 217–241.

[15] R. D. Holkar, 'Topological construction of $C^*$-correspondences for groupoid $C^*$-algebras', PhD Thesis, der Georg-August-Universität Göttingen, 2014.

[16] M. Ionescu and A. Kumjian, 'Hausdorff measures and KMS states', *Indiana Univ. Math. J.* **62**(2) (2013), 443–463.

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press

[17] M. Ionescu and P. S. Muhly, 'Groupoid methods in wavelet analysis', in: *Group Representations, Ergodic Theory, and Mathematical Physics: A Tribute to George W. Mackey*, Contemporary Mathematics, 449 (eds. R. S. Doran, C. C. Moore and R. J. Zimmer) (American Mathematical Society, Providence, RI, 2008), 193–208.
[18] A. Kumjian, P. S. Muhly, J. N. Renault and D. P. Williams, 'The Brauer group of a locally compact groupoid', *Amer. J. Math.* **120**(5) (1998), 901–954.
[19] A. Kumjian and J. Renault, 'KMS states on $C^*$-algebras associated to expansive maps', *Proc. Amer. Math. Soc.* **134**(7) (2006), 2067–2078.
[20] E. C. Lance, *Hilbert $C^*$-Modules: A Toolkit for Operator Algebraists*, London Mathematical Society Lecture Note Series, 210 (Cambridge University Press, Cambridge, 1995).
[21] N. S. Larsen and I. Raeburn, 'From filters to wavelets via direct limits', in: *Operator Theory, Operator Algebras, and Applications*, Contemporary Mathematics, 414 (eds. D. Han, P. E. T. Jorgensen and D. R. Larson) (American Mathematical Society, Providence, RI, 2006), 35–40.
[22] N. S. Larsen and I. Raeburn, 'Projective multi-resolution analyses arising from direct limits of Hilbert modules', *Math. Scand.* **100**(2) (2007), 317–360.
[23] S. G. Mallat, 'Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$', *Trans. Amer. Math. Soc.* **315**(1) (1989), 69–87.
[24] N. F. G. Martin, 'On finite Blaschke products whose restrictions to the unit circle are exact endomorphisms', *Bull. Lond. Math. Soc.* **15** (1983), 343–348.
[25] P. S. Muhly, J. N. Renault and D. P. Williams, 'Equivalence and isomorphism for groupoid $C^*$-algebras', *J. Operator Theory* **17**(1) (1987), 3–22.
[26] J. Renault, *A Groupoid Approach to $C^*$-Algebras*, Lecture Notes in Mathematics, 793 (Springer-Verlag, New York, 1980).
[27] J. Renault, 'The ideal structure of groupoid crossed product $C^*$-algebras', *J. Operator Theory* **25** (1991), 3–36.
[28] J. Renault, 'Cuntz-like algebras', in: *Operator Theoretical Methods (Timişoara, 1998)* (eds. A. Gheondea, R. N. Gologan and D. Timotin) (Theta Foundation, Bucharest, 2000), 371–386.
[29] J. Renault, 'AF equivalence relations and their cocycles', in: *Operator Algebras and Mathematical Physics (Constanţa, 2001)* (eds. J.-M. Combes, J. Cuntz, G. A. Elliott, G. Nenciu, H. Siedentop and S. Stratila) (Theta Foundation, Bucharest, 2003), 365–377.
[30] J. Renault, 'The Radon–Nikodým problem for appoximately proper equivalence relations', *Ergodic Theory Dynam. Systems* **25**(5) (2005), 1643–1672.
[31] J. Renault, *$C^*$-Algebras and Dynamical Systems* (Publicações Matemáticas do IMPA. [IMPA Mathematical Publications], Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2009).
[32] J. Renault, 'Induced representations and hypergroupoids', *SIGMA Symmetry Integrability Geom. Methods Appl.* **10** (2014), Article no. 057, 18.
[33] M. A. Rieffel, 'Induced representations of $C^*$-algebras', *Adv. Math.* **13** (1974), 176–257.
[34] M. Rosenblatt, *Markov Processes. Structure and Asymptotic Behavior*, Die Grundlehren der Mathematischen Wissenschaften, 184 (Springer-Verlag, New York–Heidelberg, 1971).
[35] D. Ruelle, *Themodynamic Formalism*, Encyclopedia of Mathematics and its Applications, 5 (Addison-Wesley, Reading, MA, 1978).
[36] D. P. Williams, *A Tool Kit for Groupoid $C^*$-Algebras*, Mathematical Surveys and Monographs, 241 (American Mathematical Society, Providence, RI, 2019).

MARIUS IONESCU, United States Naval Academy, Annapolis, MD 21402-5002, USA
e-mail: ionescu@usna.edu

PAUL MUHLY, University of Iowa, Iowa City, Iowa 52242, USA
e-mail: paul-muhly@uiowa.edu

https://doi.org/10.1017/S1446788725101377 Published online by Cambridge University Press