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Axial radiation reception

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Abstract. A radiating point source is abruptly switched on and, thereupon, constantly maintained while immersed within a multi-dimensional, non-dispersive, originally uniform, unbounded, propagative medium with transverse isotropy, equivalent to an intrinsic axisymmetry. The number of spatial dimensions concerned is odd. The pertinent radiation problem is posed as a partial differential equation coupled with a preactivation zero condition. This problem is, first, resolved by a Fourier integral, which is then 'frozen' onto (i.e. evaluated at points on) the axis of symmetry. Postulating appropriate hyperbolic-type as well as elliptic-type conditions, a systematic reduction of the 'frozen' integral is carried out, enabling an analytic exact solution to be constructed with reference to certain phase curves in a phase plane. Away from the source, this solution is valid throughout the symmetry axis except at certain discrete, moving singularities. These fall into three different categories. The behaviour near each typical singularity is estimated. Three significant physical features are demonstrated (as corollaries); namely, the implicit compliance with a radiation principle, the linear interception of the symmetry axis by a Petrowsky's lacuna (associated with the instantaneous point impulse), and, the eventual attainment of either a steady or quiet state after an infinite period. Interpretations are provided in terms of the group velocity. To assist in probing into these various aspects, two appendices are separately included. An application is made to consider magnetically aligned, radiation reception within a field permeated, electrically conducting gas.

1. *Introduction.* Consider the radiation problem comprising the linear differential equation

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) u = P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) H(t)\,\delta(\mathbf{x}), \tag{1·1}$$

accompanied by the (partial) zero condition

$$u = 0 \quad \text{when} \quad t < 0, \tag{1·2}$$

i.e. $u = \phi H(t)$, for some quantity ϕ, $H(t)$ being the Heaviside unit step function of the time variable t, while

$$\delta(\mathbf{x}) = \delta(x_1)\,\delta(x_2) \ldots \delta(x_n)$$

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is the Dirac delta function of the position vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ lying within the infinite n-dimensional Cartesian space R_n: $-\infty < x_v < \infty\,(v = 1, \ldots, n)$, for which

$$\nabla_1^2 = \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

is the $(n-1)$-dimensional Laplacian. For our purposes, the integer $n \geqslant 3$ and is also restricted to being odd. Furthermore,

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right), \quad Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right),$$

are each a (polynomial) operator which has real constant coefficients, is homogeneous in $\partial/\partial t$, $\partial/\partial x_1$, $\partial/\partial x_2, \ldots, \partial/\partial x_n$, and, whose respective (integral) orders are $m-l$, m. Thus, in general appearances,

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) = \sum_{\substack{\mu \geqslant 0\, v \geqslant 0 \\ (2\mu + v \leqslant m-l)}} \sum A_\mu^v \nabla_1^{2\mu} \left(\frac{\partial}{\partial x_1}\right)^v \left(\frac{\partial}{\partial t}\right)^{m-l-2\mu-v}, \tag{1·3}$$

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) = \sum_{\substack{\mu \geqslant 0\, v \geqslant 0 \\ (2\mu + v \leqslant m)}} \sum B_\mu^v \nabla_1^{2\mu} \left(\frac{\partial}{\partial x_1}\right)^v \left(\frac{\partial}{\partial t}\right)^{m-2\mu-v}, \tag{1·4}$$

where the A_μ^v's and B_μ^v's are real constants. Throughout, let us assume the integer m to be *even* and that

$$1 \leqslant l \leqslant m. \tag{1·5}$$

We shall refer to the operator $P(\ )$ as the *P-operator* and $Q(\ )$ as the *Q-operator* (of the equation (1·1)). Generally speaking, (1·1) is derived from a system of linear differential equations with homogeneous operators.

Physically, (1·1) and (1·2) are equations governing a time-dependent radiation emission starting at time $t = 0$ from a preceding state of rest throughout $t < 0$. This process is induced by the sudden activation at the instant $t = 0$, followed immediately by a constant maintenance, of the radiating source $H(t)\,\delta(\mathbf{x})$ whose distribution is, effectively, point localized at $\mathbf{x} = \mathbf{0}$ inside an anisotropic, real, homogeneous, non-dispersive, transmitting medium having transverse isotropy and occupying the entire unbounded R_n space.

The transverse isotropy in the medium is linked with a rotational symmetry about the x_1-axis, which we shall call the *axis of transverse isotropy*, and ensues from the peculiar grouping of all spatial operators, other than $\partial/\partial x_1$, in the Laplace's combination ∇_1^2. In the present state of affairs, the axis of transverse isotropy passes through the point source radiator. Transversely isotropic, non-dispersive media are encountered in the studies of magnetogasdynamics, magneto-elasticity, and the elasticity of certain crystals.

This paper is aimed, primarily, at the development and exact evaluation of certain integral representations for the solution to (1·1), (1·2) under prescribed specifications. In particular, as the title suggests, the reception point $\mathbf{x}$ is set to vary only axially, i.e. along the axis of transverse isotropy. Any measurement recorded on this axis, is, naturally, influenced by the cumulative radiation effects developing elsewhere all

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over R_n. In this respect, then, the situation being explored does not really conform to a one-dimensional wave motion in the usual accepted sense.

Results achieved depend, to a fair extent, upon the highest order of the Laplacian term ∇_1^2 present in each of the P- and Q-operators and, in particular, its relationship with the orders $m-l$, m of both $P(\,)$ and $Q(\,)$, as well as the dimension n. These results and their subsequent interpretations are also dependent on the disposition of certain *phase curves* which are implicitly determined within a phase plane by certain hyperbolic-cum-elliptic type of conditions. Among other things, one important concept is raised, viz. that the solitary radiating singularity at $\mathbf{x} = \mathbf{0}$ radiates many, what will be alluded to as, *radiated singularities*, corresponding to various fixed positions on the phase curves and falling into three principal categories. In the physical $\mathbf{x}$-space, the so-called radiated singularities are points travelling along the x_1-axis with definite characteristical axial group velocities, i.e. velocities of energy transport. Therefore, the emitted energy distribution must be heavily concentrated about these moving points. On approaching them, it is ascertained that the analytic solution, valid elsewhere, becomes mostly singular. It may be conjectured that the radiated singularities are points at which the moving wave fronts intersect the axis of transverse isotropy. Both partially infinite intervals beyond the two fastest, left and right, travelling singularities are intervals of absolute silence. This phenomenon, which crops up as a mathematical consequence, should be anticipated from a basic radiation condition that the radiative energy must flow outwards from the source, presumably with finite velocities.

The analysis of an initial value wave problem is never quite complete without some endeavour made to construct any of the existing lacunas of Petrowsky (**18**); Gårding (**8**), (**9**). These lacunas are travelling zones of absolute quiet, bounded by slow wavefronts interior to the fastest front, and, are normally associated with the Green's function, i.e. the solution corresponding to an instantaneous point impulse. The original configuration confronted by Petrowsky is severely sophisticated. For a two-dimensional medium, Bazer and Yen (**19**), (**1**) have thoroughly accomplished the construction during their determination of the Riemann matrix of symmetric hyperbolic systems, while Burridge (**4**) has beautifully succeeded in formulating sufficient existence conditions. Examples in anisotropic elasticity have been illustrated by Miller and Musgrave (**17**), Duff (**7**). As the crucial part of our investigation is focused upon the axis of transverse isotropy, we can only manage to establish, deductively, the intercept of a lacuna along this axis. Two radiated singularities coincide with the two end-points of such an intercept, within which the radiation field of the constant but abruptly triggered point source is either absolute zero or, simply, time-independent.

In passing, we note that the associated *impulse* (or *Green's*) *function*, viz. the solution to the related problem

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) v = P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) \delta(t)\,\delta(\mathbf{x}) \tag{1·6}$$

: $v = 0\,(t < 0)$, for radiation from the instantaneous point impulse $\delta(t)\,\delta(\mathbf{x})$ through the same propagating medium, is conveniently derived from the solution $u(\mathbf{x}, t)$ to

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(1·1), (1·2) via $v = \partial u/\partial t$. Also, if the point source of (1·1) is replaced by one with an arbitrary spatial distribution $\rho(\mathbf{x})$, then the solution to the appropriate governing equations

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) w = P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) H(t)\rho(\mathbf{x}) \tag{1·7}$$

: $w = 0\,(t < 0)$, is, by the usual convolution (Duhamel's principle),

$$w = \int_{R_n} u(\mathbf{x}-\mathbf{y}, t)\rho(\mathbf{y})\, d\mathbf{y}, \tag{1·8}$$

the integration being performed over R_n whose volumetric element with respect to $\mathbf{y}$ is $d\mathbf{y} = dy_1 dy_2 \ldots dy_n$.

2. *An integral formulation.* We shall now proceed to tackle the radiation problem of (1·1), (1·2). Let us introduce the n-dimensional wave vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, the two following scalar magnitudes

$$\alpha = |\boldsymbol{\alpha}| = \sqrt{(\alpha_1^2+\alpha_2^2+\ldots+\alpha_n^2)}, \quad x = |\mathbf{x}| = \sqrt{(x_1^2+x_2^2+\ldots+x_n^2)},$$

and the scalar product $\boldsymbol{\alpha}.\mathbf{x} = \alpha_1 x_1+\alpha_2 x_2+\ldots+\alpha_n x_n$. In the manner of Lighthill's ((14) Appendix B, (16)) procedure for a transient source, a Fourier integral representation of u with a Fourier transform $u_0(\omega, \alpha)$, say, is

$$u = \int_{R_n} d\boldsymbol{\alpha} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} u_0(\omega, \boldsymbol{\alpha}) \exp\left[i(\boldsymbol{\alpha}.\mathbf{x} - \omega t)\right] d\omega,$$

in which, accommodating the fact that $u = \phi H(t) \equiv 0$ whenever $t < 0$, the path $(-\infty+i\epsilon, \infty+i\epsilon)$ for the inner integration is selected so as to pass above the $\Re\,\omega$ axis as well as above all singularities of $u_0(\omega, \boldsymbol{\alpha})$ within the complex ω-plane, for any $\boldsymbol{\alpha}$ in R_n over which the outer integral ranges. Likewise,

$$H(t) = \frac{i}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \exp\left(-i\omega t\right) \frac{d\omega}{\omega},$$

and (via Lighthill(13), Jones(11)),

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{R_n} \exp\left(i\boldsymbol{\alpha}.\mathbf{x}\right) d\boldsymbol{\alpha}.$$

Whereupon (1·1) transforms into

$$(2\pi)^{n+1}\omega Q(-i\omega, i\alpha_1, \alpha_1^2-\alpha^2)\, u_0(\omega, \boldsymbol{\alpha}) = iP(-i\omega, i\alpha_1, \alpha_1^2-\alpha^2).$$

Hence

$$u = \frac{i}{(2\pi)^{n+1}} \int_{R_n} \exp\left(i\boldsymbol{\alpha}.\mathbf{x}\right) d\boldsymbol{\alpha} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{P(-i\omega, i\alpha_1, \alpha_1^2-\alpha^2)}{Q(-i\omega, i\alpha_1, \alpha_1^2-\alpha^2)} \frac{e^{-i\omega t}}{\omega} d\omega. \tag{2·1}$$

As from now, we assume that the observation point $\mathbf{x} \neq \mathbf{0}$ and is *axially aligned*, which is to say that it is never coincident with the source, and, moreover, is constrained

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to vary only along the axis of transverse isotropy, i.e. $\mathbf{x} = (x_1, 0, 0, \ldots, 0)$: $x_1 \neq 0$. Whence $\alpha_1 = \boldsymbol{\alpha}.\mathbf{x}/x_1$. So, by a law of the spherical mean (John(10)),

$$\begin{aligned}\int_{R_n} f(\boldsymbol{\alpha}.\mathbf{x})\, g(\alpha_1, \alpha^2)\, d\boldsymbol{\alpha} &= \Omega_{n-1}\int_{-1}^{1}(1-\xi^2)^{\frac{1}{2}(n-3)}d\xi\int_0^\infty f(\alpha\xi x)\, g(\alpha\xi x/x_1, \alpha^2)\,\alpha^{n-1}\,d\alpha,\\ &= \tfrac{1}{2}\Omega_{n-1}\int_{-1}^{1}(1-\xi^2)^{\frac{1}{2}(n-3)}\,d\xi\int_{-\infty}^{\infty} f(\alpha\xi x_1)\, g(\alpha\xi, \alpha^2)\,\alpha^{n-1}\,d\alpha,\end{aligned}$$

where $\Omega_n = 2\pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n)$ is the surface area of the n-dimensional unit sphere. Consequently, (2·1) simplifies into

$$u = \frac{i\Omega_{n-1}}{2(2\pi)^{n+1}}\int_{-1}^{1}(1-\xi^2)^{\frac{1}{2}(n-3)}\,d\xi\int_{-\infty}^{\infty}\alpha^{n-1}\,e^{i\alpha\xi x_1}d\alpha \times\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{P(-i\omega, i\alpha\xi, \alpha^2\xi^2-\alpha^2)}{Q(-i\omega, i\alpha\xi, \alpha^2\xi^2-\alpha^2)}\,\frac{e^{-i\omega t}}{\omega}\,d\omega. \tag{2·2}$$

Suppose, with reference to the P- and Q-operators of (1·1),

$$F(\lambda, \xi) = P(-\lambda, \xi, 1-\xi^2), \quad G(\lambda, \xi) = Q(-\lambda, \xi, 1-\xi^2), \tag{2·3}$$

which are polynomial functions (of λ and ξ). On account of the homogeneity of both the operators $P(\ )$ and $Q(\ )$ (see (1·3), (1·4)), we have, for any parameter β,

$$P(-\beta\lambda, \beta\xi, \beta^2-\beta^2\xi^2) = \beta^{m-1}F(\lambda, \xi), \quad Q(-\beta\lambda, \beta\xi, \beta^2-\beta^2\xi^2) = \beta^m G(\lambda, \xi). \tag{2·4}$$

Regarding (2·2), the ω-integral may be evaluated by means of contour integration. To do this, let us first postulate

(i) $$\deg_{(\text{in }\lambda)} \text{poly}\, G(\lambda, \xi) = m, \tag{2·5}$$

meaning that the degree of the polynomial $G(\lambda, \xi)$, when referred to λ, is m;

(ii) for any real ξ within the closed interval $[-1, 1]$, all (subsequent) m roots

$$\lambda = \lambda_j(\xi) \quad (j = 1, \ldots, m) \tag{2·6}$$

to the algebraic equation

$$G(\lambda, \xi) = 0 \tag{2·7}$$

are *real* and, moreover, if $\xi \neq 0$, are *non-vanishing* as well as *distinct*.

Note (1). It is implicit from (ii) that, within $-1 \leqslant \xi \leqslant 1$, no member of the set (2·6) coincides identically with the line $\lambda = 0$ or with another member of the same set. In particular, then, $\partial/\partial t$ is never a factor of the Q-operator. This is, therefore, reducible to a non-trivial time independent form, viz.

$$Q\left(0, \frac{\partial}{\partial x_1}, \nabla_1^2\right) \quad (\not\equiv 0). \tag{2·8}$$

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(2) The necessary and sufficient condition that (2·5) is satisfied is that

$$Q(1, 0, 0) \neq 0,$$

$Q(1, 0, 0)$ being (via (1·4)) the constant B_0^0 = coefficient of $(\partial/\partial t)^m$ in the Q-operator.

(3) Evidently, $G(\lambda, \xi)$ is factorizable in $-1 \leqslant \xi \leqslant 1$:

$$G(\lambda, \xi) = Q(1, 0, 0) \prod_{j=1}^{m} [\lambda - \lambda_j(\xi)]. \tag{2·9}$$

(4) Suppose the requirements of 'non-vanishment' and 'distinctness' on every $\lambda_j(\xi)$ are extended to the point $\xi = 0$. Then the terms of the hypotheses (i) and (ii) become specializations, to the present case for axial alignment, of the conditions (cf. e.g. Gårding(8), Courant and Hilbert(6), Bers *et al.*(2)) that the Q-operator is *strictly hyperbolic*, and furthermore, evolves ultimately into a time-independent form (2·8) which is *elliptic* since

$$Q(0, \xi, 1 - \xi^2) = Q(1, 0, 0) \prod_{j=1}^{m} \lambda_j(\xi)$$

is now non-vanishing throughout $-1 \leqslant \xi \leqslant 1$.

(5) The reality of each $\lambda_j(\xi)$ in $-1 \leqslant \xi \leqslant 1$ is compatible with a desired stability (cf. Lighthill(14), Appendix B).

Now, generally,

$$\deg_{(\text{in } \lambda)} \text{poly } F(\lambda, \xi) \leqslant m - l, \tag{2·10}$$

so that, in view of (2·5),

$$\deg_{(\text{in } \omega)} \text{poly } \omega G(\omega\alpha^{-1}, \xi) - \deg_{(\text{in } \omega)} \text{poly } F(\omega\alpha^{-1}, \xi) \geqslant l + 1 \geqslant 2. \tag{2·11}$$

Thus, regarding (2·2), the ω-integrand factor

$$P(\ )/\omega Q(\ ) = (i\alpha)^{-l} F(\omega\alpha^{-1}, \xi)/\omega G(\omega\alpha^{-1}, \xi) \quad (-1 \leqslant \xi \leqslant 1)$$

is a meromorphic (precisely, a rational) function of ω which satisfies a uniform convergence condition of Jordan's lemma in the entire complex ω-plane, and possesses $m + 1$ real, simple poles at

$$\omega = 0, \quad \alpha\lambda_j(\xi) \quad (j = 1, \ldots, m). \tag{2·12}$$

The representation (2·2) will then satisfy $u \equiv 0$ for $t < 0$ if ϵ is real and positive. In this case, when $t > 0$ (or $t < 0$), the ω-path $(-\infty + i\epsilon, \infty + i\epsilon)$ can be closed by an infinite semicircle drawn into the half-domain $\Im\omega \leqslant \epsilon$ (or $\Im\omega \geqslant \epsilon$). Along this semicircle, the relevant integral extension vanishes under the effect (2·11), owing to Jordan's lemma. The complete contour so constructed is negatively (or positively) described and encloses every (or no) integrand singularity which goes to form the set (2·12). Whence, in accordance with the residue theory, the ω-integral of (2·2) is

$$-2\pi i H(t)\,(i\alpha)^{-l} \left\{ \frac{F(0, \xi)}{G(0, \xi)} + \sum_{j=1}^{m} \frac{F(\lambda_j(\xi), \xi)}{G_\lambda(\lambda_j(\xi), \xi)} \frac{\exp[-i\alpha\lambda_j(\xi)\, t]}{\lambda_j(\xi)} \right\}, \tag{2·13}$$

$G_\lambda(\lambda_j(\xi), \xi)$ being the λ-derivative of $G(\lambda, \xi)$ at $\lambda = \lambda_j(\xi)$. Whereupon, after some

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manipulations, we arrive at the following conclusion: the solution $u = \phi H(t)$ to (1·1), (1·2) when $\mathbf{x} = (x_1, 0, \ldots, 0)$: $x_1 \neq 0$ and $t > 0$ is determined by either

$$\phi = \phi_1(x_1, t) = \frac{\Omega_{n-1}}{(2\pi)^{n-1}} \left(\frac{\partial}{\partial x_1}\right)^{n-l-1} \frac{K(X)}{x_1} \quad (n \geqslant l+1), \tag{2·14}$$

or

$$\left(\frac{\partial}{\partial x_1}\right)^{l+1-n} \phi = \phi_2(x_1, t) = \frac{\Omega_{n-1}}{(2\pi)^{n-1}} \frac{K(X)}{x_1} \quad (n \leqslant l+1), \tag{2·15}$$

where

$$K(X) = \frac{(-1)^{\frac{1}{2}(n-1)}}{4\pi} (\operatorname{sgn} X) \int_{-1}^{1} (1-\xi^2)^{\frac{1}{2}(n-3)} \xi^{l+1-n} \, d\xi$$
$$\times \int_{-\infty}^{\infty} \left\{ \frac{F(0,\xi)}{G(0,\xi)} e^{i\alpha\xi} + \sum_{j=1}^{m} \frac{F(\lambda_j(\xi), \xi)}{G_\lambda(\lambda_j(\xi), \xi)} \frac{\exp(i\alpha[\xi - \lambda_j(\xi)/X])}{\lambda_j(\xi)} \right\} d\alpha, \tag{2·16}$$

and $X = x_1/t$. We shall call $K(X)$ the *K-element* and, $\phi_1(x_1, t)$, $\phi_2(x_1, t)$ *radiation functions* of axial reception. Under all circumstances, $X = 0$ implies $t = \infty$ (since $x_1 \neq 0$). Now, taking into consideration (2·5) and (2·10), the identity (A 9) of Appendix A assumes, at $|X| = \infty$, the form

$$\frac{F(0,\xi)}{G(0,\xi)} + \sum_{j=1}^{m} \frac{F(\lambda_j(\xi), \xi)}{\lambda_j(\xi) \, G_\lambda(\lambda_j(\xi), \xi)} \equiv 0,$$

which on application to (2·16) gives

$$\lim_{x_1 \to \pm\infty} K(X) = K(\pm\infty) \equiv 0. \tag{2·17}$$

Henceforth, as far as further development is concerned, it is enough to base all arguments upon $|X| < \infty$.

Suppose $\Phi(\mathbf{x}, t)$ determines the solution to the following problem

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) w = P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) H(t) \rho(x_1) \delta(x_2) \delta(x_3) \ldots \delta(x_n) \tag{2·18}$$

: $w = \Phi(\mathbf{x}, t) H(t)$, (with the same P- and Q-operators of (1·1)) for radiation originating from a suddenly applied source whose spatial distribution $\rho(x_1) \delta(x_2) \delta(x_3) \ldots \delta(x_n)$ is virtually restricted to an infinitesimally thin line along the axis of transverse isotropy. Then throughout this axis, the Duhamel's principle (1·8) yields

$$\Phi(x_1, 0, \ldots, 0, t) = \int_{-\infty}^{\infty} \phi_1(x_1 - y, t) \rho(y) \, dy \quad (n \geqslant l+1), \tag{2·19}$$

$$\left(\frac{\partial}{\partial x_1}\right)^{l+1-n} \Phi(x_1, 0, \ldots, 0, t) = \int_{-\infty}^{\infty} \phi_2(x_1 - y, t) \rho(y) \, dy \quad (n \leqslant l+1), \tag{2·20}$$

$\phi_1(x_1, t)$ and $\phi_2(x_1, t)$ being the radiation functions of axial reception.

3. *Phase curves.* Before an explicit evaluation of the integral for $K(X)$ can be made, certain facts must first be established. In particular, certain properties pertaining to the P- and Q-operators are required. The highest order of the ∇_1^2 term present in each of these operators will influence subsequent analysis to some extent. Generally,

$$\deg_{(\text{in } \nabla_1^2)} \operatorname{poly} P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) = \tfrac{1}{2}(m-p), \tag{3·1}$$

$$\deg_{(\text{in } \nabla_1^2)} \operatorname{poly} Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) = \tfrac{1}{2}(m-q), \tag{3·2}$$

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for *even* integers p, q satisfying $l \leqslant p \leqslant m$, $0 \leqslant q < m$.† The right inequalities $p \leqslant m$, $q < m$ ensure that both $P(\ )$ and $Q(\ )$ are, together, meaningful as operators representing, in the sense of (1·1), a truly transversely isotropic system. Regarding the left inequalities, $p = l$ if and only if $P(0, 0, 1) \neq 0$, while $q = 0$ if and only if $Q(0, 0, 1) \neq 0$. We shall refer to p and q as being the *Laplacian indices* of, respectively, the P- and Q-operators. With reference to (1·3) and (1·4), (3·1) and (3·2) implies that $A^{\nu}_{\mu} = 0$ for $\mu = \frac{1}{2}(m-p)+1, \ldots, \frac{1}{2}(m-l)$ while the $A^{\nu}_{\frac{1}{2}(m-p)}$'s are not all zeros for

$$\nu = 0, 1, \ldots, p-l, \quad \text{and} \quad B^{\nu}_{\mu} = 0 \quad \text{for} \quad \mu = \tfrac{1}{2}(m-q)+1, \ldots, \tfrac{1}{2}m$$

while the $B^{\nu}_{\frac{1}{2}(m-q)}$'s are not all zeros for $\nu = 0, 1, \ldots, q$. Whence, via (2·3), we have throughout $-\infty < X < \infty$,

$$F(\xi X, \xi) = \xi^{p-l} M(\xi, X), \quad G(\xi X, \xi) = \xi^{q} N(\xi, X), \tag{3·3}$$

where

$$M(\xi, X) = \sum_{\mu=0}^{\frac{1}{2}(m-p)} (1-\xi^2)^{\mu} \xi^{m-p-2\mu} \sum_{\nu=0}^{m-l-2\mu} A^{\nu}_{\mu} (-X)^{m-l-2\mu-\nu}, \tag{3·4}$$

$$N(\xi, X) = \sum_{\mu=0}^{\frac{1}{2}(m-q)} (1-\xi^2)^{\mu} \xi^{m-q-2\mu} \sum_{\nu=0}^{m-2\mu} B^{\nu}_{\mu} (-X)^{m-2\mu-\nu}. \tag{3·5}$$

Obviously, $M(\xi, X)$, $N(\xi, X)$ are polynomial functions of ξ and X, which we shall call the *associated polynomials* of the P- and Q-operators. We note that

$$N(0, X) = \sum_{\nu=0}^{q} B^{\nu}_{\frac{1}{2}(m-q)} (-X)^{q-\nu} \not\equiv 0 \quad \text{throughout} \quad -\infty < X < \infty. \tag{3·6}$$

A similar expression holds for $M(0, X)$. If $p = l$, $M(0, X) \equiv F(0, 0) = P(0, 0, 1)$. If $q = 0$, $N(0, X) \equiv G(0, 0) = Q(0, 0, 1)$. We also observe that

$$M(\xi, X) = M(-\xi, X), \quad N(\xi, X) = N(-\xi, X). \tag{3·7}$$

Another important aspect to consider is the statement accompanying (2·6), (2·7). This can be geometrically interpreted as follows: the characteristic equation (2·7) defines within the vertical strip $-1 \leqslant \xi \leqslant 1$ of the $\xi - \lambda$ plane, a family of m real distinct curves $\lambda = \lambda_j(\xi)$ $(j = 1, \ldots, m)$, none of which coincides identically with the segment $-1 \leqslant \xi \leqslant 1$ of the ξ-axis ($\lambda = 0$). Moreover, off the λ-axis ($\xi = 0$), these curves are non-intersecting and do not meet (or touch) the ξ-axis away from the origin. Let us call each such curve a *phase curve*. Generally, the function $\lambda_j(\xi)$ is differentiable (i.e. analytic) at $\xi = 0$.

The Laplacian index q has a special effect on the configuration of our phase curves. To determine this effect, we first note that (2·9) gives

$$G(\xi X, \xi) = Q(1, 0, 0) \prod_{j=1}^{m} [\xi X - \lambda_j(\xi)]. \tag{3·8}$$

This shows that a (real) zero of $G(\xi X, \xi)$ within $-1 \leqslant \xi \leqslant 1$ is the abscissa of a (real) point of intersection between a specific phase curve and the straight line $\lambda = \xi X$ passing through the origin. Now, the second identity of (3·3) coupled with the fact

† In magnetogasdynamic propagation of Alfvénic-acoustic waves (see Section 8), $q = 2$, and $m = 4$. The values p and l depend on the particular perturbation that ϕ represents.

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$N(0, X) \not\equiv 0$ (see (3·6)) implies that, throughout $|X| < \infty$, $G(\xi X, \xi)$ possesses at least q identical (simple) ξ-zeros coinciding at $\xi = 0$. The number of these zeros exceeds q at each isolated X-zero of $N(0, X)$, but is precisely q elsewhere in $|X| < \infty$. Therefore, exactly q phase curves pass simultaneously through the origin to form a *restricted subfamily*: $\lambda = \lambda_j(\xi)\,(j = 1, \ldots, q)$ satisfying

$$\lambda_j(0) = 0 \quad (j = 1, \ldots, q) \quad \text{while} \quad \lambda_j(0) \neq 0 \quad (j = q+1, \ldots, m). \tag{3·9}$$

Now, in view of (2·3) and (2·4), $G(-\lambda, -\xi) = G(\lambda, \xi)$. So, if $\lambda = \lambda_j^*(\xi)$ denotes the portion, within $0 \leqslant \xi \leqslant 1$ (or $-1 \leqslant \xi \leqslant 0$), of the phase curve $\lambda = \lambda_j(\xi)$, then

$$G(-\lambda_j^*(\xi), -\xi) = 0 \quad \text{in} \quad 0 \leqslant \xi \leqslant 1 \quad (\text{or} \quad -1 \leqslant \xi \leqslant 0).$$

Thus, equivalently,

$$G(-\lambda^*(-\xi), \xi) = 0 \quad \text{in} \quad -1 \leqslant \xi \leqslant 0 \quad (\text{or} \quad 0 \leqslant \xi \leqslant 1),$$

i.e. $\lambda = -\lambda_j^*(-\xi)$ is the portion in $-1 \leqslant \xi \leqslant 0$ (or $0 \leqslant \xi \leqslant 1$) of some phase curve $\lambda = \lambda_k(\xi)$ (which may or may not be the phase curve $\lambda = \lambda_j(\xi)$). This proves that *the entire family of phase curves is symmetric about the origin* $(\xi, \lambda) = (0, 0)$. *N.B.* For the two functions $\lambda_j(\xi)$ and $\lambda_k(\xi)$, involved in the proof, to be one and the same, it is necessary and sufficient that $q \neq 0$ and $\lambda = \lambda_j(\xi)$ is a phase curve of the restricted subfamily. Consequently, the entire family of m phase curves can generally be divided into three major subfamilies, viz.

(i) the restricted subfamily of the q phase curves

$$\lambda = \lambda_j(\xi) \quad (j = 1, \ldots, q) \tag{3·10}$$

each of which is *symmetric* about the common point $(\xi, \lambda) = (0, 0)$, i.e.

$$\lambda_j(\xi) = -\lambda_j(-\xi) \tag{3·11}$$

with $\lambda_j(0) = 0$ $(j = 1, \ldots, q)$;

(ii) a *top* subfamily of $\frac{1}{2}(m-q)$ phase curves:

$$\lambda = \lambda_j(\xi) \quad (j = q+1, \ldots, \tfrac{1}{2}m + \tfrac{1}{2}q), \tag{3·12}$$

each lying exclusively within the upper half-strip $\lambda > 0$, $-1 \leqslant \xi \leqslant 1$ (as well as above every member of the restricted subfamily);

(iii) a *bottom* subfamily of $\frac{1}{2}(m-q)$ phase curves:

$$\lambda = \lambda_{j+\frac{1}{2}(m-q)}(\xi) \quad (j = q+1, \ldots, \tfrac{1}{2}m + \tfrac{1}{2}q), \tag{3·13}$$

each lying exclusively within the lower half-strip $\lambda < 0$, $-1 \leqslant \xi \leqslant 1$ (and below every member of the restricted subfamily) and is, moreover, the image across the origin of a specific phase curve in the top subfamily, i.e.

$$\lambda_{j+\frac{1}{2}(m-q)}(\xi) = -\lambda_j(-\xi), \quad \text{say} \quad (j = q+1, \ldots, \tfrac{1}{2}m + \tfrac{1}{2}q). \tag{3·14}$$

Remarks. The restricted subfamily is null if $q = 0$, in which case all m phase curves are equally divided, numerically, between the top and bottom subfamilies. Reverting our attention to the statement associated with (2·12), we observe that for $\xi = 0$, the quantity $(i\alpha)^{-1} F(\omega\alpha^{-1}, 0)/\omega G(\omega\alpha^{-1}, 0)$ has, at $\omega = 0$, a pole which is of order $q+1$ and is, therefore, not simple unless $q = 0$. This factor does not, however, violate the residue

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evaluation, precipitating the result (2·13) on the basis of simple poles, for which, the condition that all m phase curves are distinct in $-1 \leqslant \xi \leqslant 1$ is quite sufficient. For that matter, formula (2·13) is still valid if some or all phase curves of the top subfamily (and, correspondingly, the bottom subfamily as well,) intersect each other somewhere along the λ-axis beyond the origin.

The Q-operator of (1·1) possesses two essential features. Firstly, from (2·3), (3·3), (3·8), we have

$$N(1, X) = Q(-X, 1, 0) = Q(1, 0, 0) \prod_{j=1}^{m} [X - \lambda_j(1)], \tag{3·15}$$

implying $Q(0, 1, 0) = Q(1, 0, 0)\, \lambda_1(1)\, \lambda_2(1) \ldots \lambda_m(1) \neq 0$. Hence, necessarily,

$$\deg_{(\text{in } \partial/\partial x_1)} \text{poly } Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) = m. \tag{3·16}$$

Next, using (3·3), (3·5), (3·8), (3·9) and the de l'Hôpital's rule,

$$N(0, X) = \sum_{v=0}^{q} B^v_{\frac{1}{2}(m-q)} (-X)^{q-v} = Q(1, 0, 0) \prod_{j=1}^{q} [X - \lambda_j'(0)] \prod_{j=q+1}^{m} \lambda_j(0) \quad (q > 0). \tag{3·17}$$

Evidently, $B^0_{\frac{1}{2}(m-q)} = Q(1, 0, 0)\, \lambda_{q+1}(0)\, \lambda_{q+2}(0) \ldots \lambda_m(0) \neq 0$. So, in the Q-operator of (1·1),

$$\text{coefficient of } (\partial/\partial t)^q (\nabla_1^2)^{\frac{1}{2}(m-q)} \neq 0. \tag{3·18}$$

4. *Non-singular reception.* In this section, we shall determine, under appropriate conditions, the K-element from (2·16). This becomes, on applying the symmetry relations (3·11) and (3·14) together with (2·3) and (2·4),

$$K(X) = \frac{(-1)^{\frac{1}{2}(n-1)}}{4\pi} (\operatorname{sgn} X) \left\{ \int_{-1}^{1} \frac{(1-\xi^2)^{\frac{1}{2}(n-3)} F(0, \xi)}{\xi^{n-l-1} G(0, \xi)} d\xi \int_{-\infty}^{\infty} e^{i\alpha\xi}\, d\alpha \right.$$
$$\left. + \sum_{j=1}^{m} \int_{0}^{1} \frac{(1-\xi^2)^{\frac{1}{2}(n-3)} F(\lambda_j(\xi), \xi)}{\xi^{n-l-1} \lambda_j(\xi) G_\lambda(\lambda_j(\xi), \xi)} d\xi \int_{-\infty}^{\infty} 2\cos(\alpha[\xi - \lambda_j(\xi)/X])\, d\alpha \right\}. \tag{4·1}$$

Let us now introduce the following definitions: for Laplacian index q (of the Q-operator) > 0,

$$\lambda_j'(0) \quad (j = 1, \ldots, q) \text{ is a radiated singularity of type 1 (R.S. 1);} \tag{4·2}$$

$$\lambda_j(1) \quad (j = 1, \ldots, m) \text{ is a radiated singularity of type 2 (R.S. 2);} \tag{4·3}$$

for any $j = 1, \ldots, m$ and $0 < \xi < 1$, $\lambda_j'(\xi)$ is a radiated singularity of type 3 (R.S. 3) if and only if

$$\lambda_j'(\xi) = \lambda_j(\xi)/\xi. \tag{4·4}$$

Thus, with reference to the entire family of m phase curves, the R.S. 1's are the q gradients of the restricted subfamily (3·10) at its common point, the R.S. 2's are the m ordinates at $\xi = 1$, while the R.S. 3's are the gradients of those tangents drawn from the origin to the portions of all phase curves within $0 < \xi < 1$. Algebraically, via (3·15) and (3·17), the R.S. 2's are all the (m real) X-zeros of $N(1, X)$, while the R.S. 1's are all the (q real) X-zeros of $N(0, X)$. We shall now work under the hypothesis that

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the axial radiation reception is *non-singular*, viz. the observer's velocity X is non-coincident, in $|X| < \infty$, with any radiated singularity, i.e. R.S. 1, R.S. 2, or R.S. 3. The infinite (velocity) range $-\infty < X < \infty$ is composed of a finite number of intervals in which there is non-singular axial reception. These intervals are separated from each other by the (travelling) radiated singularities, which are, essentially, isolated. Since (via Lighthill(13), Jones(11))

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha\xi}\,d\alpha = \delta(\xi) \quad \text{(the delta function)}, \tag{4·5}$$

so, in $0 \leqslant \xi \leqslant 1$,

$$\frac{1}{2\pi|X|}\int_{-\infty}^{\infty} \cos\left(\alpha[\xi - \lambda_j(\xi)/X]\right) d\alpha = \delta(\xi X - \lambda_j(\xi)), \tag{4·6}$$

$$= \begin{cases} \dfrac{\delta(\xi)}{|X-\lambda_j'(0)|} + \displaystyle\sum_{\{\xi_v\}_j} \frac{\delta(\xi-\xi_v)}{|X-\lambda_j'(\xi_v)|} & (j = 1, \ldots, q), \qquad (4·7) \\ \displaystyle\sum_{\{\xi_v\}_j} \frac{\delta(\xi-\xi_v)}{|X-\lambda_j'(\xi_v)|} & (j = q+1, \ldots, m), \qquad (4·8) \end{cases}$$

where each summation is performed over $\{\xi_v\}_j$, the set of ξ-roots: $\xi_v = \xi_v(X)$ to

$$\xi X = \lambda_j(\xi) \quad \text{in} \quad 0 < \xi < 1 \quad \text{(all necessarily) satisfying } X \neq \lambda_j'(\xi), \tag{4·9}$$

i.e. the set of abscissae of the (non-tangential) intersections, within $0 < \xi < 1$, between the straight line $\lambda = \xi X$ and the phase curve $\lambda = \lambda_j(\xi)$. Non-singular reception ensures that all denominators occurring in (4·7), (4·8) are non-vanishing, and that $\delta(\xi X - \lambda_j(\xi))$ receives no contribution from $\xi = 1$. The first term in (4·7) exists only if $q > 0$, is contributed by the permanent simple root at $\xi = 0$ to $\xi X = \lambda_j(\xi)$ for $j = 1, \ldots, q$ (cf. (3·9)) and is, in view of the de l'Hôpital's rule, $\delta(\xi)/|X - \lambda_j(\xi)/\xi|$.

Since no phase curve cuts the ξ-axis at $\xi = 1$ or touches it (tangentially) in $0 < \xi < 1$, there is neither a vanishing R.S. 2 nor a vanishing R.S. 3. Hence, because the reception at $X = 0$ (i.e. when $t = \infty$) is non-singular by hypothesis, then, necessarily,

$$\lambda_j'(0) \neq 0 \quad \text{for every} \quad j = 1, \ldots, q. \tag{4·10}$$

In particular, if $q = 0$, no R.S. 1 exists so that the reception is definitely non-singular at $X = 0$. Whereupon, in terms of the associated polynomials defined by (3·3)–(3·5), the rational function

$$\xi^{l+q-p} F(0, \xi)/G(0, \xi) = M(\xi, 0)/N(\xi, 0) \tag{4·11}$$

is differentiable at $\xi = 0$ since, via (3·17),

$$N(0, 0) = Q(1, 0, 0) \prod_{j=1}^{q} \lambda_j'(0) \prod_{j=q+1}^{m} \lambda_j(0) \neq 0. \tag{4·12}$$

With regards to the Q-operator of (1·1), this is equivalent to

$$B^q_{\frac{1}{2}(m-q)} = \text{coefficient of } (\partial/\partial x_1)^q (\nabla_1^2)^{\frac{1}{2}(m-q)} \neq 0,$$

a companion to the requirement (3·18). Also, from (3·3),

$$\frac{\xi^{l+q-p} F(\lambda_j(\xi), \xi)}{\lambda_j(\xi)\, G_\lambda(\lambda_j(\xi), \xi)} = \frac{M(\xi, \lambda_j(\xi)/\xi)}{N_X(\xi, \lambda_j(\xi)/\xi)\, \lambda_j(\xi)/\xi}, \tag{4·13}$$

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$N_X(\xi, \lambda_j(\xi)/\xi)$ being the X-derivative of $N(\xi, X)$ at $X = \lambda_j(\xi)/\xi$. Let us assume that all R.S. 1's are distinct, i.e.

$$\lambda_j'(0) \neq \lambda_k'(0) \quad \text{if} \quad j \neq k \quad \text{for} \quad j, k = 1, \ldots, q. \tag{4·14}$$

Then, for $j = 1, \ldots, q$, the expression (4·13) is differentiable at $\xi = 0$ because, via (3·8),

$$\lim_{\xi \to 0} N_X(\xi, \lambda_j(\xi)/\xi)\, \lambda_j(\xi)/\xi = Q(1, 0, 0)\, \lambda_j'(0) \prod_{k=1:k\neq j}^{q} \lambda_j'(0) - \lambda_k'(0) \prod_{k=q+1}^{m} \lambda_k(0),$$

which is non-zero. Consequently, substituting (4·5)–(4·8), (4·11), (4·13) into (4·1), accounting for the fact (see Lighthill (13), Jones(11))

$$\frac{\delta(\xi)}{\xi^{n+q-p-1}} = \begin{cases} \left(\dfrac{d}{d\xi}\right)^{n+q-p-1} \delta(\xi)/(n+q-p-1)! & (n+q \geqslant p+1), \quad (4·15) \\ 0 & (n+q < p+1), \quad (4·16) \end{cases}$$

and invoking the usual convolution principle pertaining to the delta function and its derivatives, we eventually arrive at the following results for non-singular reception in $|X| < \infty$:

$$K(X) = K_{n+q-p}(X) \quad \text{for} \quad n+q < p+1, \tag{4·17}$$

but,

$$\begin{aligned} K(X) = K_{n+q-p}(X) &+ \frac{(-1)^{\frac{1}{2}(n-1)} \operatorname{sgn} X}{2(n+q-p-1)!} \left[\left(\frac{d}{d\xi}\right)^{n+q-p-1} (1-\xi^2)^{\frac{1}{2}(n-3)} \frac{M(\xi, 0)}{N(\xi, 0)}\right]_{\xi=0} \\ &+ \frac{(-1)^{\frac{1}{2}(n-1)} X}{2(n+q-p-1)!} \sum_{j=1}^{q} \operatorname{sgn}(X - \lambda_j'(0)) \\ &\times \left[\left(\frac{\partial}{\partial \xi}\right)^{n+q-p-1} \frac{(1-\xi^2)^{\frac{1}{2}(n-3)} M(\xi, \lambda_j(\xi)/\xi)}{(X - \lambda_j(\xi)/\xi) N_X(\xi, \lambda_j(\xi)/\xi)\, \lambda_j(\xi)/\xi}\right]_{\xi=0} \end{aligned} \tag{4·18}$$

for $n+q \geqslant p+1$; where

$$K_{n+q-p}(X) = (-1)^{\frac{1}{2}(n-1)} \sum_{j=1}^{m} \sum_{\{\xi_v\}_j} \frac{(1-\xi_v^2)^{\frac{1}{2}(n-3)} M(\xi_v, X)}{\xi_v^{n+q-p-1} |X - \lambda_j'(\xi_v)| N_X(\xi_v, X)}, \tag{4·19}$$

with the double summation ranging over all (non-tangential) intersections in

$$0 < \xi < 1$$

between the line $\lambda = \xi X$ and the entire family of m phase curves. By virtue of (3·3) and (3·8), the abscissae of these intersections constitute the set of all (real, distinct) ξ-zeros of the associated polynomial $N(\xi, X)$ in $0 < \xi < 1$. Now if $\xi_v = \xi_v(X) \in \{\xi_v\}_j$, then (4·9) yields

$$(X - \lambda_j'(\xi_v))\, \xi_v'(X) = -\xi_v. \tag{4·20}$$

So,

$$\xi_v^{-1}(X - \lambda_j'(\xi_v))\, N_X(\xi_v, X) = -N_X(\xi_v, X)/\xi_v'(X) = N_\xi(\xi_v, X). \tag{4·21}$$

Whereupon, (4·19) is alternatively expressible as

$$K_{n+q-p}(X) = (-1)^{\frac{1}{2}(n+1)} \sum_{v} (\operatorname{sgn} \xi_v'(X)) \frac{(1-\xi_v^2)^{\frac{1}{2}(n-3)} M(\xi_v, X)}{\xi_v^{n+q-p} N_\xi(\xi_v, X)}, \tag{4·22}$$

where the single summation runs over every (distinct) ξ-zero $\xi_v(X)$ of $N(\xi, X)$ in $0 < \xi < 1$.

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Note. (i) As $N(\xi, X)$ is an even polynomial function of ξ with a degree which $\leqslant m-q$ (cf. (3·5), (3·7)), the number of all such zeros cannot exceed $\frac{1}{2}(m-q)$. For $X > 0$ (or $X < 0$), they are determined by existing intersections in $0 < \xi < 1$ between $\lambda = \xi X$ and the top subfamily (3·12) (or bottom subfamily (3·13)) of phase curves as well as those members of the restricted subfamily (3·10) which do pass through the first quadrant $0 < \xi < 1$, $\lambda > 0$ (or fourth quadrant $0 < \xi < 1$, $\lambda < 0$).

(ii) In (4·18), the third term involving the sum $\sum_{j=1}^{q}$ is absent if $q = 0$.

(iii) Via (4·4) and (4·21), we deduce that, provided $0 < |\xi| < 1$, the real X-roots to the two simultaneous equations $N(\xi, X) = 0$ and $N_\xi(\xi, X) = 0$ constitute the R.S. 3's, each obviously determined as a function of just the constant coefficients of $N(\xi, X)$. Corresponding to any R.S. 3, then, $N(\xi, X)$ possesses at least two coincident zeros in $0 < \xi < 1$. Whence, a necessary (but not sufficient) condition that R.S. 3's exist is that $m-q \geqslant 4$.

Taking into consideration the inequalities accompanying the formulae (2·14), (2·15) for the radiation functions ϕ_1 and ϕ_2, we assert: the *relevant* radiation function (obtainable to problem (1·1), (1·2)) is ϕ_1 if either

$$l+q-p \leqslant n+q-p-1 < 0 \quad \text{(with } K(X) \text{ given by (4·17))}, \tag{4·23}$$

or

$$l+q \leqslant p \quad \text{and} \quad n+q \geqslant p+1, \quad \text{or} \quad n+q-p-1 \geqslant l+q-p \geqslant 0, \tag{4·24}$$

(with $K(X)$ given by (4·18)). But the *relevant* radiation function is ϕ_2 if either

$$n+q-p-1 \leqslant l+q-p < 0, \quad \text{or} \quad n+q < p+1 \quad \text{and} \quad l+q \geqslant p, \tag{4·25}$$

(with $K(X)$ given by (4·17)), or

$$l+q-p \geqslant n+q-p-1 \geqslant 0 \quad \text{(with } K(X) \text{ given by (4·18))}. \tag{4·26}$$

All possible independent values of n, l, p, q are covered in (4·23)–(4·26). Regarding the differentiation derivation of ϕ_1 (see (2·14)) under (4·23), (4·24), and via (4·17)–(4·19), we note that all x_1-derivatives of $\operatorname{sgn} X \equiv \operatorname{sgn} x_1$, $\operatorname{sgn}(X - \lambda_j'(0))$ and $\operatorname{sgn}(X - \lambda_j'(\xi_v))$ vanish during non-singular reception within $0 < |x_1| < \infty$ and $0 < t \leqslant \infty$.

5. *Intervals of steady state and silence.* Suppose the finite, observer's velocity X varies in an interval within which

$$X \neq \lambda_j(\xi)/\xi \quad \text{in} \quad 0 < \xi \leqslant 1 \quad \text{for every} \quad j = 1, \ldots, m, \tag{5·1}$$

and

$$X \neq \lambda_j'(0) \quad j = 1, \ldots, q; \tag{5·2}$$

i.e. the associated polynomial $N(\xi, X)$ never vanishes for $0 \leqslant \xi \leqslant 1$. In such an X-interval, radiation reception is obviously non-singular, and the results (4·17)–(4·19) are applicable: $K_{n+q-p}(X) \equiv 0$. Thus if $n+q < p+1$, $K(X) \equiv 0$ and so both *relevant* radiation functions ϕ_1, ϕ_2 (corresponding, respectively, to (4·23) and (4·25)) vanish identically. In this respect, our X-interval is one of *absolute silence* within which the axially travelling observer never experiences any radiation. On the other hand, if $n+q \geqslant p+1$, $K(X)$ is just the sum of the remaining two terms in (4·18), and is generally time-dependent when $q > 0$. However, if $q = 0$, (the provision (5·2) must be deleted, and)

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the third term involving $\sum_{j=1}^{q}$ is absent, in which case, the radiation reception is non-trivial but *steady*, i.e. *time-independent.*

That at least one interval of the type (5·1), (5·2) always exists can be easily proved as follows. Since no phase curve ever encounters, tangentially or otherwise, the ξ-axis away from the origin (see section 3) and since, if $q > 0$, no phase curve of the restricted subfamily (3·10) is, owing to (4·10), tangential to the ξ-axis at $\xi = 0$, there is a wedge about the ξ-axis with its vertex at the origin into which no phase curve ever penetrates in $0 < \xi \leqslant 1$. Wherefore, we infer that there exist real finite values $X_+(>0)$ and $X_-(<0)$, precisely, $X_+ = \min$ [set of all positive radiated singularities] and $X_- = \max$ [set of all negative radiated singularities], such that

$$X_- < X < X_+ \tag{5·3}$$

is an innermost interval (about the radiating point source at $\mathbf{x} = \mathbf{0}$) consistent with (5·1), (5·2). Any other such interval is normally bounded by two radiated singularities. Obviously, $X = 0$ (i.e. $t = \infty$) is contained within the interval (5·3). So the results associated with (5·1), (5·2) are applicable in the limit as $t \to \infty$. Thus we conclude that at time $t = \infty$, the radiation reception at every axial position $\mathbf{x} = (x_1, 0, \ldots, 0) \neq \mathbf{0}$ either (i) culminates in complete silence with absolute zero reception: $K(0) = 0$ whenever $n+q < p+1$, or, (ii) attains a steady state:

$$K(0) = \frac{(-1)^{\frac{1}{2}(n-1)} \operatorname{sgn} x_1}{2(n+q-p-1)!} \left[\left(\frac{d}{d\xi} \right)^{n+q-p-1} (1-\xi^2)^{\frac{1}{2}(n-3)} \frac{M(\xi, 0)}{N(\xi, 0)} \right]_{\xi=0} \tag{5·4}$$

whenever $n+q \geqslant p+1$. Under the stipulations of this paper, the radiation process governed by (2·18), for a line source of, effectively, finite length $b-a(>0)$: $\rho(x_1) \equiv 0$ in $x_1 > b$ and $x_1 < a$ is, therefore, determined (via (2·19) and (2·20)) at any axial point and during time $t = \infty$ by $\Phi = 0$ under (4·23), $(\partial/\partial x_1)^{l+1-n}\Phi = 0$ under either condition of (4·25),

$$\Phi = \frac{(-1)^{\frac{1}{2}(n-1)} \Omega_{n-1}}{2(2\pi)^{n-1}} \left[\left(\frac{d}{d\xi} \right)^{n+q-p-1} \frac{(1-\xi^2)^{\frac{1}{2}(n-3)} M(\xi, 0)}{(n+q-p-1)! N(\xi, 0)} \right]_{\xi=0} \left(\frac{\partial}{\partial x_1} \right)^{n-l-1} \int_a^b \frac{\rho(y)}{|x_1 - y|} dy$$

under either condition of (4·24), and,

$$\left(\frac{\partial}{\partial x_1} \right)^{l+1-n} \Phi = \frac{(-1)^{\frac{1}{2}(n-1)} \Omega_{n-1}}{2(2\pi)^{n-1}} \left[\left(\frac{d}{d\xi} \right)^{n+q-p-1} \times \frac{(1-\xi^2)^{\frac{1}{2}(n-3)} M(\xi, 0)}{(n+q-p-1)! N(\xi, 0)} \right]_{\xi=0} \int_a^b \frac{\rho(y)}{|x_1 - y|} dy$$

under (4·26). The physical phenomenon evidently resembles that corresponding to the point source.

We shall next demonstrate, among other things, that our non-singular solution does substantiate a basic *radiation condition* that net energy flux propagates away from the source and so, with finite characteristic speeds, a sufficiently 'fast' outgoing observer can escape the advancing emission field. In view of the fact (2·17), escape is certainly achieved at the extreme ends $x_1 = \pm\infty$. Consider the wedge

$$\max_{j=q+1, \ldots, \frac{1}{2}(m+q)} \lambda_j(1) < \lambda/\xi < \infty$$

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contained inside the first quadrant of the $\xi-\lambda$ plane, $\lambda_j(1)$ $(j = q+1, \ldots, \frac{1}{2}m+\frac{1}{2}q)$ being the ordinates of the top subfamily (3·12) at $\xi = 1$. Any straight line $\lambda = \xi X$ $(X > 0)$, say, passing into this wedge through its vertex at the origin $(\xi, \lambda) = (0, 0)$, clearly intersects, within $0 < \xi < 1$, all $\frac{1}{2}(m-q)$ phase curves of the top subfamily (3·12), each intersection corresponding to a zero of $N(\xi, X)$ in $0 < \xi < 1$. Since, according to note (i) accompanying (4·22), these intersections cannot exceed $\frac{1}{2}(m-q)$ in number, the specified line therefore intersects each phase curve of the top subfamily exactly once and non-tangentially in $0 < \xi < 1$. Moreover,

$$X > \lambda_j'(\xi) \quad (j = q+1, \ldots, \tfrac{1}{2}m+\tfrac{1}{2}q)$$

at every such intersection. Furthermore, if $q > 0$, the same line neither touches any member of the restricted subfamily (3·10), nor intersects it away from the origin. Evidently, then, $X > \lambda_j'(0)$ $(j = 1, \ldots, q)$. Likewise, any line $\lambda = \xi X (X < 0)$, drawn from the origin into the wedge

$$-\infty < \lambda/\xi < \min_{j=q+1, \ldots, \frac{1}{2}(m+q)} \lambda_{j+\frac{1}{2}(m-q)}(1)$$

contained in the fourth quadrant, intersects non-tangentially, every phase curve of the bottom subfamily (3·13) exactly once within $0 < \xi < 1$, such that

$$X < \lambda'_{j+\frac{1}{2}(m-q)}(\xi) \quad (j = q+1, \ldots, \tfrac{1}{2}m+\tfrac{1}{2}q)$$

at the particular intersection. Again, there is no tangential contact, or intersection off the origin, between the given line and any member of the restricted subfamily (3·10), and $X < \lambda_j'(0)$ $(j = 1, \ldots, q)$. Hence, throughout each of the two partially infinite (outer) intervals

$$-\infty < X < \min_{j=1, \ldots, m} \lambda_j(1), \qquad \max_{j=1, \ldots, m} \lambda_j(1) < X < \infty, \tag{5·5}$$

the associated polynomial $N(\xi, X)$ attains its highest possible degree (viz. $m-q$; cf. (3·5)) in ξ, vanishing at exactly $\frac{1}{2}(m-q)$ distinct zeros $\xi = \xi_\nu$ within $0 < \xi < 1$. Furthermore, the radiation reception is permanently non-singular. We can therefore apply the results (4·17) and (4·18), using (4·22) with $\sum_\nu$ taken over all these $\frac{1}{2}(m-q)$ ξ_ν-zeros of $N(\xi, X)$ and with $\operatorname{sgn} X = -\operatorname{sgn} \xi_\nu'(X) = \operatorname{sgn}(X - \lambda_j'(0))$ $(j = 1, \ldots, q$ if $q > 0)$. Whereupon, employing the results (A 1)–(A 4) of Appendix A, we arrive at the conclusion that *the particular radiation function* (viz. ϕ_1 or ϕ_2), *whichever is 'relevant' in the context of* (4·23)–(4·26), *vanishes identically throughout both the ranges defined by* (5·5). These are therefore outermost ranges of absolute silence. Any observer who is axially travelling right (or left) from the radiating source with a speed exceeding $\max_{j=1, \ldots, m} \lambda_j(1)$ (or $|\min_{j=1, \ldots, m} \lambda_j(1)|$) will never receive any radiation. This phenomenon is indeed fully compatible with the radiation condition. *Note.* The above arguments also prove that to the right (or left) of the source, the 'fastest (axially) departing' radiated singularity has the 'velocity' $\max_{j=1, \ldots, m} \lambda_j(1)$ (or $\min_{j=1, \ldots, m} \lambda_j(1)$). This singularity is of type 2 and is non-coincident with any other radiated singularity of type 1, 2 or 3.

Whenever a state of silence or of steady reception develops within some axial interval, the corresponding solution to the related radiation problem (1·6) for the instantaneous point impulse is identically zero in this interval. Except for the two outer

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regions of (5·5), which evidently lie just outside the fastest n-dimensional wavefront generated by the source, any other such interval, interior to both 'fastest' singularities, must in fact be the line intercept of a time-varying lacuna of Petrowsky ((18); see also Burridge(4) for the two-dimensional case) along the axis of transverse isotropy. In particular, any interval consistent with (5·1) and (5·2), e.g. the interval (5·3), is one such 'lacunary' intercept for $n+q < p+1$, and, provided $q = 0$, for $n \geqslant p+1$ as well.

6. *Near-singular reception.* We shall now attempt to estimate the behaviour of the K-element for reception which is non-singular but significantly close to some radiated singularity. Let us start with any R.S. 1, say, $\lambda_k'(0)$ for some $k = 1, \ldots, q$ ($q > 0$). Since the particular $\lambda_k(\xi)$ is an odd function of ξ (cf. (3·11)), then

$$\xi X - \lambda_k(\xi) = \xi(X - \lambda_k'(0)) - \xi^3 \lambda_k'''(0)/6 + \ldots .$$

This has a permanent zero at $\xi = 0$ which, provided $n+q \geqslant p+1$, yields a finite contribution to $K(X)$ accommodated within the series $\sum_{j=1}^{q}$ of (4·18). For $|X - \lambda_k'(0)| \ll |\lambda_k'''(0)|$, two additional distinct ξ-zeros appear near $\xi = 0$ at the approximate positions $\xi_0, -\xi_0$:

$$\xi_0 = \sqrt{[6(X - \lambda_k'(0))/\lambda_k'''(0)]}. \tag{6·1}$$

As $X \to \lambda_k'(0)$, both $\xi_0, -\xi_0 \to 0$. Here $X - \lambda_k'(\xi_0) \simeq -2(X - \lambda_k'(0))$. If $(X - \lambda_k'(0))/\lambda_k'''(0) > 0$, ξ_0 is real and provides $K_{n+q-p}(X)$ (see (4·19)) with a contribution that is approximately

$$\frac{(-1)^{\frac{1}{2}(n-1)}}{2}\left(\frac{\lambda_k'''(0)}{6}\right)^{\frac{1}{2}(n+q-p-1)} \frac{M(0, X)\,\mathrm{sgn}\,(X - \lambda_k'(0))}{N_X(0, X)\,(X - \lambda_k'(0))^{\frac{1}{2}(n+q-p+1)}}, \tag{6·2}$$

assuming $M(0, X) \neq 0$. Also assuming $X - \lambda_k'(0) = o(\lambda_j'(0) - \lambda_k'(0))$ ($j = 1, \ldots, m : j \neq k$), $N_X(0, X)$, which is non-zero, is not small. So if $n+q < p-1$, the amount (6·2) is negligible. It is finite at $n+q = p-1$, but maintains a modest singularity associated with the factor $\mathrm{sgn}\,(X - \lambda_k'(0))$. For the remaining case $n+q \geqslant p+1$, (6·2) represents a large singular quantity dominating $K_{n+q-p}(X)$ and tending to infinity as $X \to \lambda_k'(0)$. On the other hand, when $(X - \lambda_k'(0))/\lambda_k'''(0) < 0$, the zero ξ_0 is imaginary and cannot contribute to $K_{n+q-p}(X)$. Let us now concentrate solely on the important singular case $n+q \geqslant p+1$. Whatever the sign of $(X - \lambda_k'(0))/\lambda_k'''(0)$, there always exists one singular contribution to the $\sum_{j=1}^{q}$ series in (4·18), viz.

$$\frac{(-1)^{\frac{1}{2}(n-1)} X}{2(n+q-p-1)!}\,\mathrm{sgn}\,(X - \lambda_k'(0)) \times \left[\left(\frac{\partial}{\partial \xi}\right)^{n+q-p-1} \frac{(1-\xi^2)^{\frac{1}{2}(n-3)}\, M(\xi, \lambda_k(\xi)/\xi)}{(X - \lambda_k(\xi)/\xi)\, N_X(\xi, \lambda_k(\xi)/\xi)\, \lambda_k(\xi)/\xi}\right]_{\xi=0}, \tag{6·3}$$

which, as it stands, is not sufficiently informative. By direct differentiation, the highest order term involving $(X - \lambda_k'(0))^{-(n+q-p)}$ vanishes, as well as several subsequent terms. It is, generally, difficult to determine from (6·3), the 'strongest' non-vanishing singular term. This can be accomplished, instead, by resorting to certain results established in Appendix A. We assume that for the particular range of X near $\lambda_k'(0)$, $Q(-X, 1, -1) \neq 0$, so that the condition (A 5) is satisfied. In particular, then, $N(\xi, X)$ possesses $m-q$ zeros. Among these, not necessarily all are confined to the real slit

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$-1 < \Re\,\xi < 1$, $\Im\,\xi = 0$. Those which are stay simple under non-singular reception. The remaining zeros distributed outside this slit may be simple or multiple. Accordingly, the identity (A 7) remains valid if its left side is replaced by

$$\sum_{0<\xi_\mu<1} \frac{(1-\xi_\mu^2)^{\frac{1}{2}(n-3)}\,M(\xi_\mu, X)}{\xi_\mu^{n+q-p}\,N_\xi(\xi_\mu, X)} + \frac{1}{2}\sum\nolimits^{(\text{relevant residue at each } \xi\text{-zero of } N(\xi, X) \text{ outside} -1\leqslant \Re\,\xi\leqslant 1, \Im\,\xi=0),} \tag{6·4}$$

the ξ_μ's being those simple zeros of $N(\xi, X)$ along $0 < \Re\,\xi < 1$, $\Im\,\xi = 0$. By the same sequence of arguments carrying through to (A 11), we deduce that on substitution of the expression (6·4) for the leading series $\sum_{\nu=1}^{\frac{1}{2}(m-q)}$ occurring in the expression (A 2), the latter retains either its zero identity for $n+q-p-1 \leqslant l+q-p$ (cf (A 3)), or, for

$$n+q-p-1 \geqslant l+q-p+1,$$

the form (A 11), this being finite when $X \simeq \lambda_k'(0)$. If $(X-\lambda_k'(0))/\lambda_k'''(0) > 0$, the expression (6·4) is dominated by the residue term associated with ξ_0 (given by (6·1)) and participating in the series $\sum_{0<\xi_\mu<1}$. If $(X-\lambda_k'(0))/\lambda_k'''(0) < 0$, the dominant quantity is jointly contributed by both the simple zeros $\xi_0, -\xi_0$, which are now imaginary, to the other residue series $\frac{1}{2}\Sigma(\ )$. In either case, the contribution dominating (6·4) is

$$\frac{(1-\xi_0^2)^{\frac{1}{2}(n-3)}\,M(\xi_0, X)}{\xi_0^{n+q-p}N_\xi(\xi_0, X)} \simeq -\frac{1}{2}\left(\frac{\lambda_k'''(0)}{6}\right)^{\frac{1}{2}(n+q-p-1)} \frac{M(0, X)}{N_X(0, X)\,(X-\lambda_k'(0))^{\frac{1}{2}(n+q-p+1)}},$$

which is comparable with just one other singular term present in (A 2), viz. the term corresponding to $j = k$ of the series $\sum_{j=1}^{q}$. Whereupon, in view of the above remarks pertaining to the modified (A 2), we then see that the functions given by (6·2) and (6·3) are approximately equal. Besides these, $K(X)$ acquires no other singular contributions. Thus we conclude that for $n+q \geqslant p+1$, $K(X)$ is dominated by either (i) the expression (6·2) if $(X-\lambda_k'(0))/\lambda_k'''(0) \lesssim 0$, or (ii) twice the expression (6·2) if

$$(X-\lambda_k'(0))/\lambda_k'''(0) \gtrsim 0.$$

Let us next consider reception near the R.S. 2 at $\lambda_k(1)$ ($k = 1, \ldots, m$). Now,

$$\xi X - \lambda_k(\xi) = X - \lambda_k(1) + (\xi-1)\,(X-\lambda_k'(1)) + \ldots$$

which, assuming $|X-\lambda_k(1)| \ll |X-\lambda_k'(1)|$, has one simple ξ-zero near $\xi = 1$ at approximately $1-(X-\lambda_k(1))/(X-\lambda_k'(1))$. This particular zero contributes (or does not contribute) to $K(X)$, depending on $(X-\lambda_k(1))/(X-\lambda_k'(1)) > 0$ (or < 0). The appropriate contribution is therefore, via (4·19), approximately

$$(-1)^{\frac{1}{2}(n-1)}\,2^{\frac{1}{2}(n-3)}\frac{|X-\lambda_k(1)|^{\frac{1}{2}(n-3)}\,M(1, X)}{|X-\lambda_k'(1)|^{\frac{1}{2}(n-1)}N_X(1, X)}\,H\left(\frac{X-\lambda_k(1)}{X-\lambda_k'(1)}\right). \tag{6·5}$$

Here, $N_X(1, X) \neq 0$, and is, moreover, not small if

$$X-\lambda_k(1) = o(\lambda_j(1)-\lambda_k(1)) \quad (j = 1, \ldots, m: j \neq k).$$

In the non-trivial case, this contribution is, normally, relatively weak, tending to zero as X approaches $\lambda_k(1)$, unless the number of dimensions $n = 3$, in which event it is characterized by a finite 'jump' discontinuity, disappearing suddenly across

$$X = \lambda_k(1).$$

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Suppose there is a R.S. 3 at $\lambda_k'(\xi_1) = \lambda_k(\xi_1)/\xi_1$, with $0 < \xi_1 < 1$. Then

$$\xi X - \lambda_k(\xi) = \xi(X - \lambda_k'(\xi_1)) - (\xi - \xi_1)^2 \lambda_k''(\xi_1)/2 + \dots$$

which, for $|X - \lambda_k'(\xi_1)| \ll |\lambda_k''(\xi_1)|$, vanishes at two distinct ξ-zeros near ξ_1 at approximately

$$\xi_1 + (X - \lambda_k'(\xi_1))/\lambda_k''(\xi_1) \pm \sqrt{[2\xi_1(X - \lambda_k'(\xi_1))/\lambda_k''(\xi_1)]}.$$

Provided $(X - \lambda_k'(\xi_1))\,\lambda_k''(\xi_1) > 0$ (or < 0), these zeros are real (or complex) and so do (or do not) contribute to $K(X)$ (see (4·19)). Their net approximate contribution is

$$2(-1)^{\frac{1}{2}(n-1)} \frac{(1-\xi_1^2)^{\frac{1}{2}(n-3)} M(\xi_1, X)\, H\{(X - \lambda_k'(\xi_1))\,\lambda_k''(\xi_1)\}}{\xi_1^{n+q-p-\frac{1}{2}} N_X(\xi_1, X) \sqrt{[2(X - \lambda_k'(\xi_1))\,\lambda_k''(\xi_1)]}}. \tag{6·6}$$

This contribution is singular, being infinitely and abruptly discontinuous across $X = \lambda_k'(\xi_1)$. It dominates $K(X)$ for $(X - \lambda_k'(\xi_1))\,\lambda_k''(\xi_1) \gtrsim 0$, but disappears altogether when $(X - \lambda_k'(\xi_1))\,\lambda_k''(\xi_1) \lesssim 0$.

7. *A group velocity interpretation.* The results (B 16)–(B 18) of Appendix B disclose that the Q-operator of (1·1) is associated with an n-dimensional travelling wave system, propagating m real discrete modes of transmission in which the jth mode ($j = 1, \dots, m$) is characterized by the phase velocity

$$\boldsymbol{\lambda}_j(\boldsymbol{\eta}) = \boldsymbol{\eta}\lambda_j(\xi), \tag{7·1}$$

and the group velocity

$$\mathbf{U}_j(\boldsymbol{\eta}) = \boldsymbol{\eta}[\lambda_j(\xi) - \xi\lambda_j'(\xi)] + (\lambda_j'(\xi), 0, 0, \dots, 0). \tag{7·2}$$

Here, $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_n)$ denotes the unit wave vector: $\eta_1 = \xi$ $(-1 \leqslant \xi \leqslant 1)$, and $\lambda_j = \lambda_j(\xi)$ is a phase curve. From the classification of phase curves via (3·10)–(3·14), we observe, in passing, that

$$\boldsymbol{\lambda}_j(\boldsymbol{\eta}) = -\boldsymbol{\eta}\lambda_j(-\xi) = \boldsymbol{\lambda}_j(-\boldsymbol{\eta}) \quad (j = 1, \dots, q), \tag{7·3}$$

$$\boldsymbol{\lambda}_{j+\frac{1}{2}(m-q)}(\boldsymbol{\eta}) = -\boldsymbol{\eta}\lambda_j(-\xi) = \boldsymbol{\lambda}_j(-\boldsymbol{\eta}) \quad (j = q+1, \dots, \tfrac{1}{2}m + \tfrac{1}{2}q), \tag{7·4}$$

$$\mathbf{U}_j(\boldsymbol{\eta}) = \mathbf{U}_j(-\boldsymbol{\eta}) \quad (j = 1, \dots, q), \tag{7·5}$$

$$\mathbf{U}_{j+\frac{1}{2}(m-q)}(\boldsymbol{\eta}) = \mathbf{U}_j(-\boldsymbol{\eta}) \quad (j = q+1, \dots, \tfrac{1}{2}m + \tfrac{1}{2}q); \tag{7·6}$$

i.e. the $\{\mathbf{U}_j\}$ surface family (of sheets), as well as the $\{\boldsymbol{\lambda}_j\}$ surface family, are each symmetric about its origin of reference.

Now, when $\boldsymbol{\eta} = (0, 0, \dots, 1, \dots, 0)$, i.e. in a direction perpendicular to $(1, 0, \dots, 0)$,

$$\boldsymbol{\lambda}_j(0, 0, \dots, 1, \dots, 0) = \begin{cases} \mathbf{0} & (j = 1, \dots, q) \quad (7·7) \\ (0, 0, \dots, \lambda_j(0), \dots, 0) & (j = q+1, \dots, m); \quad (7·8) \end{cases}$$

$$\mathbf{U}_j(0, 0, \dots, 1, \dots, 0) = \begin{cases} (\lambda_j'(0), 0, 0, \dots, 0) & (j = 1, \dots, q), \quad (7·9) \\ (\lambda_j'(0), 0, \dots, \lambda_j(0), \dots, 0) & (j = q+1, \dots, m), \quad (7·10) \end{cases}$$

(cf. (B 22), (B 23) Appendix B). Also, comparable with (B 20), for $\boldsymbol{\eta} = (1, 0, 0, \dots, 0)$,

$$\mathbf{U}_j(1, 0, 0, \dots, 0) = (\lambda_j(1), 0, 0, \dots, 0) = \boldsymbol{\lambda}_j(1, 0, 0, \dots, 0) \quad (j = 1, \dots, m). \tag{7·11}$$

Thus, *the (axially moving) R.S. 1's and R.S.2's occur in* $\mathbf{x}/t$*-space at, respectively,*

$$\mathbf{U}_j(0, 0, \dots, 1, \dots, 0) \quad (j = 1, \dots, q) \quad \textit{and} \quad \mathbf{U}_k(1, 0, 0, \dots, 0) \quad (k = 1, \dots, m).$$

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Finally, it follows from (7·2) that, for $0 < \xi < 1$, $\lambda_j'(\xi) = \lambda_j(\xi)/\xi$ if and only if

$$\mathbf{U}_j(\boldsymbol{\eta}) = (\lambda_j'(\xi), 0, 0, \ldots, 0).$$

In this case, $\mathbf{U}_j(\boldsymbol{\eta})$ *coincides with an (axial) R.S.* 3.

8. *An application in magnetogasdynamics.* The propagation of magnetogasdynamic waves (see e.g. Kulikovskiy and Lyubimov(12)) in a perfect fluid is non-dispersive, and the symmetry of the associated Friedrichs wavefronts reveals a transverse isotropy about a magnetic line of force. These two phenomena satisfy the basic postulates of this paper. To illustrate its magnetogasdynamic application, let us consider an unbounded, undisturbed region of inviscid, infinitely conducting, adiabatic gas having density ρ_0, magnetic permeability μ, speed of sound c, and traversed by a magnetic field $\mathbf{H}_0$. For a homogeneous medium, $\rho_0, \mu, c, \mathbf{H}_0$. are all constants. Suppose within this configuration, small disturbances are generated by a point effect which is suddenly triggered and constantly maintained from time $t = 0$, exerting a force in the magnetic field direction taken to be the positive x_1-direction relative to a 3-dimensional Cartesian frame: $\mathbf{x} = (x_1, x_2, x_3)$. Let $p(\mathbf{x}, t), \mathbf{v}(\mathbf{x}, t), \mathbf{h}(\mathbf{x}, t)$ denote, respectively, the fluid pressure, velocity and magnetic field perturbations. The linearized equations governing non-relativistic motion are then

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p + \frac{\mu \mathbf{H}_0}{4\pi} \times (\nabla \times \mathbf{h}) = (\rho_0 H(t)\, \delta(\mathbf{x}), 0, 0),$$

$$\frac{\partial p}{\partial t} + c^2 \rho_0 \nabla . \mathbf{v} = 0, \quad \frac{\partial \mathbf{h}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{H}_0), \quad \nabla . \mathbf{h} = 0.$$

Here, ∇ is the gradient operator $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. The solution to these equations, accompanied by the Cauchy zero initial conditions

$$p(\mathbf{x}, 0) = 0, \quad \mathbf{v}(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{h}(\mathbf{x}, 0) = \mathbf{0}, \tag{8·1}$$

for the x_1-component of $\mathbf{v}$ is $v_1 = \partial\phi/\partial t\,(t > 0)$ provided

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) \phi H(t) = P_1\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) H(t)\, \delta(\mathbf{x}), \tag{8·2}$$

where

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) = \frac{\partial^2}{\partial t^2} + c^2 \frac{\partial^2}{\partial x_1^2} - (a^2 + c^2)\, \nabla^2, \tag{8·3}$$

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right) = \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2} - (a^2 + c^2)\, \nabla^2\right) + a^2 c^2 \nabla^2 \frac{\partial^2}{\partial x_1^2}, \tag{8·4}$$

with $\nabla^2 = \partial^2/\partial x_1^2 + \nabla_1^2$ (the 3-dimensional Laplacian), and $a = \sqrt{(\mu \mathbf{H}_0^2/4\pi\rho_0)}$ (the Alfvén speed).

Equation (8·2) is of the type (1·1), while the operators $P(\ )$, $Q(\ )$ of (8·3) and (8·4) conform to the general, transversely isotropic, homogeneous forms of (1·3), (1·4) with $m = 4$, $l = 2$. The axis of transverse isotropy (i.e. the x_1-axis) coincides with the equilibrium magnetic line of force passing through the point source. Regarding (8·4), $Q(1, 0, 0) = 1$, and $Q(-\lambda, \xi, 1 - \xi^2)$ has four λ-zeros determined by

$$\lambda^2 = \lambda_\pm^2(\xi) \equiv \tfrac{1}{2}(a^2 + c^2 \pm \sqrt{\{(a^2 + c^2)^2 - 4a^2c^2|\xi|^2\}}), \tag{8·5}$$

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all four zeros being obviously real in $|\xi| \leqslant 1$, and distinct as well as non-vanishing if $\xi \neq 0$. Thus, the sufficiency conditions of (2·5)–(2·7) are clearly satisfied. The principles of this paper are therefore applicable for *magnetically aligned reception*: $\mathbf{x}/t = (X, 0, 0)$. The Laplacian indices of the above P- and Q-operators are, respectively, $p = 2$, $q = 2$ (cf. (3·1), (3·2)). From (8·5), $\lambda_-(0) = 0$. Whereupon, the entire family of four phase curves is composed of (cf. (3·10)–(3·14))

(i) the restricted subfamily of two phase curves $\lambda = \lambda_j(\xi)$ $(j = 1, 2)$:

$$\lambda_1(\xi) \equiv -\lambda_-(\xi) \quad (-1 \leqslant \xi < 0), \quad \lambda_-(\xi) \quad (0 < \xi \leqslant 1), \quad \lambda_1(0) = 0;$$
$$\lambda_2(\xi) \equiv \lambda_-(\xi) \quad (-1 \leqslant \xi < 0), \quad -\lambda_-(\xi) \quad (0 < \xi \leqslant 1), \quad \lambda_2(0) = 0;$$

(ii) the top subfamily: $\lambda = \lambda_3(\xi) \equiv \lambda_+(\xi)$ $(-1 \leqslant \xi \leqslant 1)$;

(iii) the bottom subfamily: $\lambda = \lambda_4(\xi) \equiv -\lambda_+(\xi)$ $(-1 \leqslant \xi \leqslant 1)$.

The associated polynomial of the P-operator of (8·3) is

$$M(\xi, X) = (X^2 + c^2)\,\xi^2 - (a^2 + c^2), \tag{8·6}$$

while that of the Q-operator of (8·4) is

$$N(\xi, X) = \xi^2 X^4 - (a^2 + c^2)\,X^2 + a^2 c^2. \tag{8·7}$$

The latter vanishes at $\xi = \xi_v, -\xi_v$:

$$\xi_v = \sqrt{(X^2(a^2 + c^2) - a^2 c^2)/X^2}. \tag{8·8}$$

For $|X| \neq ac(a^2 + c^2)^{-\frac{1}{2}}$, both these zeros are non-vanishing and simple. There are two R.S. 1's at

$$\lambda_1'(0) = ac(a^2 + c^2)^{-\frac{1}{2}}, \quad \lambda_2'(0) = -ac(a^2 + c^2)^{-\frac{1}{2}} = -\lambda_1'(0); \tag{8·9}$$

and four R.S. 2's at

$$\lambda_1(1) = \lambda_-(1) = \min(a, c), \quad \lambda_2(1) = -\lambda_-(1) = -\min(a, c), \tag{8·10}$$

$$\lambda_3(1) = \lambda_+(1) = \max(a, c), \quad \lambda_4(1) = -\lambda_+(1) = -\max(a, c). \tag{8·11}$$

Evidently, there are no R.S. 3's. Also, $\lambda_+(1) > \lambda_-(1) > \lambda_1'(0)$.

Under zero initial conditions, the solution to (8·2) at any $\mathbf{x} = (x_1, 0, 0) \neq \mathbf{0}$ is, via (2·14) or (2·15), expressible as $\phi = K(X)/2\pi x_1$, where the K-element $K(X)$ can be evaluated from (4·18), (4·22) with $n + q - p - 1 = 2$. But first, let us divert our attention to Appendix A. The condition (A 5) is obviously satisfied for $X \neq 0$ (i.e. $t \neq \infty$) since $Q(-X, 1, -1) = X^4 \neq 0$. We can then apply a slightly modified (A 2) coupled with (A 3). For values of X for which $N(\xi, X)$ does vanish, in the general problem, outside the real slit $-1 \leqslant \xi \leqslant 1$ (as well as possibly inside it), the leading sum $\sum_{v=1}^{\frac{1}{2}(m-q)}$ occurring in (A 2) (and (A 1)) must be replaced by

$$\Sigma\, \tfrac{1}{2}\,\text{residue}\left[\frac{(1-\xi^2)^{\frac{1}{2}(n-3)} M(\xi, X)}{\xi^{n+q-p} N(\xi, X)}\right] \quad \text{at each } \xi\text{-zero (real or complex) of } N(\xi, X).$$

Consequently, with reference to (8·6)–(8·8),

$$\frac{M(\xi_v, X)}{\xi_v^3 N_\xi(\xi_v, X)} + \frac{1}{4}\left[\frac{d^2}{d\xi^2}\frac{M(\xi, 0)}{N(\xi, 0)}\right]_{\xi=0} + \frac{X}{4}\sum_{j=1}^{2}\left[\frac{\partial^2}{\partial\xi^2}\frac{M(\xi, \lambda_j(\xi)/\xi)}{(X - \lambda_j(\xi)/\xi)\, N_X(\xi, \lambda_j(\xi)/\xi)\, \lambda_j(\xi)/\xi}\right]_{\xi=0} \equiv 0, \tag{8·12}$$

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if $|X| \neq 0, \lambda_1'(0)$. We shall use this identity to simplify the form for $K(X)$. There are, altogether, four different cases to examine. These correspond to seven consecutive ranges of the observer's velocity, separated from each other by the radiated singularities (8·9)–(8·11).

Case: $\lambda_+(1) < |X| < \infty$. From (8·8), ξ_v is real and, moreover, $0 < \xi_v < 1$. Also, since

$$X^5 \xi_v \frac{d\xi_v}{dX} = a^2(c^2 - X^2) + c^2(a^2 - X^2), \tag{8·13}$$

then $-\operatorname{sgn}(d\xi_v/dX) = \operatorname{sgn} X = \operatorname{sgn}(X - \lambda_j'(0))$ $(j = 1, 2)$. Whence, combining (4·18), (4·22) with (8·12), we see that

$$K(X) \equiv 0. \tag{8·14}$$

So, $-\infty < X < -\lambda_+(1)$ and $\lambda_+(1) < X < \infty$ are simply the (external) intervals of absolute silence (cf. (5·5)).

Case: $\lambda_-(1) < |X| < \lambda_+(1)$. Here, ξ_v is real but $\xi_v > 1$. Again,

$$\operatorname{sgn} X = \operatorname{sgn}(X - \lambda_j'(0)) \quad (j = 1, 2),$$

and, utilizing (4·18), (8·12), we arrive at

$$K(X) = (\operatorname{sgn} X)\, M(\xi_v, X)/\xi_v^3 N_\xi(\xi_v, X) \equiv \tfrac{1}{2}c^4(\operatorname{sgn} X)(X^2 - a^2)[X^2(a^2 + c^2) - a^2c^2]^{-2}. \tag{8·15}$$

Case: $\lambda_1'(0) < |X| < \lambda_-(1)$. We have $0 < \xi_v < 1$, and via (8·13),

$$\operatorname{sgn}(d\xi_v/dX) = \operatorname{sgn} X = \operatorname{sgn}(X - \lambda_j'(0)) \quad (j = 1, 2).$$

Thus, (4·18), (4·22), (8·12) lead to

$$K(X) = c^4(\operatorname{sgn} X)(X^2 - a^2)[X^2(a^2 + c^2) - a^2c^2]^{-2}, \tag{8·16}$$

which is twice the value expressed by (8·15).

Case: $|X| < \lambda_1'(0)$. Both ξ-zeros of $N(\xi, X)$ are imaginary. Whereupon, (4·18) yields

$$K(X) = \frac{X}{4}\left[\frac{\partial^2}{\partial\xi^2}\frac{M(\xi, \lambda_1(\xi)/\xi)}{(X - \lambda_1(\xi)/\xi)\, N_X(\xi, \lambda_1(\xi)/\xi)\, \lambda_1(\xi)/\xi}\right]_{\xi=0} - \frac{X}{4}\left[\frac{\partial^2}{\partial\xi^2}\frac{M(\xi, \lambda_2(\xi)/\xi)}{(X - \lambda_2(\xi)/\xi)\, N_X(\xi, \lambda_2(\xi)/\xi)\, \lambda_2(\xi)/\xi}\right]_{\xi=0} - \frac{1}{2a^2}\operatorname{sgn} X. \tag{8·17}$$

APPENDIX A. AN EXTENSION OF A LAGRANGE'S IDENTITY

Let $M(\xi, X)$, $N(\xi, X)$ and p, q denote (as usual), the associated polynomials and Laplacian indices of the respective P- and Q-operators of equation (1·1), n being odd and $\geqslant 3$ while $1 \leqslant l \leqslant m$ (see (1·3)–(1·5)). Let us assume the hypothesis associated with (2·5)–(2·7), as well as the provisions (4·10), (4·14) if $q > 0$. Suppose $X = x_1/t$ is in an interval in which (i) the axial radiation reception is *non-singular*, and that (ii) $N(\xi, X)$ vanishes within the real slit $0 < \xi < 1$ at exactly $\frac{1}{2}(m - q)$ points $\xi = \xi_v$ $(v = 1, \ldots, \frac{1}{2}m - \frac{1}{2}q)$. It can then be proved that

$$\sum_{v=1}^{\frac{1}{2}(m-q)} \frac{(1 - \xi_v^2)^{\frac{1}{2}(n-3)} M(\xi_v, X)}{\xi_v^{n+q-p} N_\xi(\xi_v, X)} = 0 \quad \text{for} \quad n + q < p + 1; \tag{A 1}$$

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while for $n+q \geqslant p+1$, the quantity

$$\sum_{v=1}^{\frac{1}{2}(m-q)} \frac{(1-\xi_v^2)^{\frac{1}{2}(n-3)} M(\xi_v, X)}{\xi_v^{n+q-p} N_\xi(\xi_v, X)} + \frac{1}{2(n+q-p-1)!}\left[\left(\frac{d}{d\xi}\right)^{n+q-p-1}(1-\xi^2)^{\frac{1}{2}(n-3)}\frac{M(\xi,0)}{N(\xi,0)}\right]_{\xi=0}$$
$$+\frac{X}{2(n+q-p-1)!}\sum_{j=1}^{q}\left[\left(\frac{\partial}{\partial\xi}\right)^{n+q-p-1}\frac{(1-\xi^2)^{\frac{1}{2}(n-3)}M(\xi,\lambda_j(\xi)/\xi)}{(X-\lambda_j(\xi)/\xi)\,N_X(\xi,\lambda_j(\xi)/\xi)\,\lambda_j(\xi)/\xi}\right]_{\xi=0} \quad (A\,2)$$

I is identically zero if $l+q-p \geqslant n+q-p-1 \geqslant 0$, (A 3)

II vanishes identically when operated on by

$$\left(\frac{\partial}{\partial x_1}\right)^{n-l-1}\frac{1}{x_1} \quad \text{if either} \quad l+q \leqslant p \quad (n+q \geqslant p+1), \quad \text{or} \quad n+q-p-1 \geqslant l+q-p \geqslant 0, \quad (A\,4)$$

with $\lambda = \lambda_j(\xi)$ $(j = 1, \ldots, q)$ constituting the restricted subfamily (3·10).

Note. The condition (4·10), which need be imposed only if $q > 0$, secures the validity of the second term in (A 2). The third term involving $\sum_{j=1}^{q}$ is absent if $q = 0$, but exists for $q > 0$ under (4·10), (4·14) and the condition (i). From the condition (ii), it is implicit that in the particular X-interval, $N(\xi, X)$ attains its highest possible degree in ξ, viz.

$$\deg_{(\text{in }\xi)} \text{poly } N(\xi, X) = m-q, \quad (A\,5)$$

or equivalently, via (1·4), (3·2) and (3·5), $Q(-X, 1, -1) \neq 0$. Furthermore, all $m-q$ zeros of $N(\xi, X)$ $(= N(-\xi, X)$, see (3·7)) are confined to the real slit $-1 < \xi < 1$ in symmetric pairs at $\xi = \xi_v, -\xi_v$ $(v = 1, \ldots, \frac{1}{2}m-\frac{1}{2}q)$. By virtue of the condition (i), these zeros are all distinct, and none appears at $\xi = 0, +1, -1$. In particular, regarding (A 1) and (A 2), the denominator factors $N_\xi(\xi_v, X) \neq 0$.

The results (A 1)–(A 4) can be considered as representing an extension of the Lagrange's identity (Carrier, Krook and Pearson(5)) for the partial fraction decomposition of a rational function. To prove them, we first observe from (3·4) that, generally, $\deg_{(\text{in }\xi)} \text{poly } M(\xi, X) \leqslant m-p$. Incorporating (A 5), we get,

$$\deg_{(\text{in }\xi)} \text{poly}\,[N(\xi, X)] - \deg_{(\text{in }\xi)} \text{poly}[\xi^{p-q-n}(1-\xi^2)^{\frac{1}{2}(n-3)}M(\xi, X)] \geqslant 3$$

in the specified X-interval. So, if C is an infinite, closed, circular contour in the complex ξ-plane, then

$$\int_C \xi^{p-q-n}(1-\xi^2)^{\frac{1}{2}(n-3)}\frac{M(\xi, X)}{N(\xi, X)}d\xi \equiv 0. \quad (A\,6)$$

Since the integrand here is, clearly, a rational function, its only singularities are poles, all of which are enclosed inside C. If $n+q < p+1$, these poles are simple, being, precisely, the simple zeros $\xi = \xi_v, -\xi_v$ $(v = 1, \ldots, \frac{1}{2}m-\frac{1}{2}q)$ of $N(\xi, X)$. Whence, applying the residue theorem to (A 6), accounting for the fact (via (3·7)) that the sum of all relevant residues is simply twice the sum of the residues at just those $\frac{1}{2}(m-q)$ ξ_v's inside $0 < \xi < 1$, we arrive at the result (A 1). If $n+q \geqslant p+1$, we can follow similar arguments, but must allow for one exception, viz. in addition to the simple poles at the ξ_v's,

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$-\xi_v$'s, there is an extra residue-contributing pole of order $n+q-p$ at $\xi = 0$. Thus, (A 6) leads to

$$\sum_{v=1}^{\frac{1}{2}(m-q)} \frac{(1-\xi_v^2)^{\frac{1}{2}(n-3)} M(\xi_v, X)}{\xi_v^{n+q-p} N_\xi(\xi_v, X)} = -\frac{1}{2}\left[\left(\frac{\partial}{\partial \xi}\right)^{n+q-p-1} \frac{(1-\xi^2)^{\frac{1}{2}(n-3)} M(\xi, X)}{(n+q-p-1)!\, N(\xi, X)}\right]_{\xi=0} \quad (A\,7)$$

for $n+q \geqslant p+1$. Regarding the right side, $N(0, X) \neq 0$ during non-singular reception. We shall now proceed to (A 2)–(A 4). To do this, we first require the fact

$$\int_C \frac{F(\lambda, \xi)\, d\lambda}{\lambda(\xi X - \lambda)\, G(\lambda, \xi)} \equiv 0, \quad (A\,8)$$

where C again denotes the infinite closed circular contour, but described in the complex λ-plane, and $F(\lambda, \xi)$, $G(\lambda, \xi)$ are the polynomials defined by (2·3). The integral vanishes because the integrand is a rational function of λ such that, via (2·5) and (2·10),

$$\deg_{(\text{in } \lambda)} \text{poly}\,[\lambda(\xi X - \lambda)\, G(\lambda, \xi)] - \deg_{(\text{in } \lambda)} \text{poly}\,[F(\lambda, \xi)] \geqslant l+2 \geqslant 3.$$

The only integrand singularities lying inside C are simple poles at $\lambda = 0$, $\lambda = \xi X$, as well as, in view of (2·6) and (2·7), simple poles at

$$\lambda = \lambda_j(\xi) \quad (j = 1, \ldots, m) \quad \text{if} \quad -1 \leqslant \xi \leqslant 1.$$

So, by the residue theorem, (A 8) yields for $-1 \leqslant \xi \leqslant 1$,

$$\frac{F(\xi X, \xi)}{G(\xi X, \xi)} = \frac{F(0, \xi)}{G(0, \xi)} + \sum_{j=1}^{m} \frac{F(\lambda_j(\xi), \xi)}{G_\lambda(\lambda_j(\xi), \xi)} \frac{\xi X}{\lambda_j(\xi)\,(\xi X - \lambda_j(\xi))}. \quad (A\,9)$$

Hence, using (3·3) and (4·13),

$$\frac{M(\xi, X)}{N(\xi, X)} = \frac{M(\xi, 0)}{N(\xi, 0)} + X \sum_{j=1}^{q} \frac{M(\xi, \lambda_j(\xi)/\xi)}{(X - \lambda_j(\xi)/\xi)\, N_X(\xi, \lambda_j(\xi)/\xi)\, \lambda_j(\xi)/\xi}$$
$$+ X \sum_{j=q+1}^{m} \frac{F(\lambda_j(\xi), \xi)\, \xi^{l+q-p+1}}{G_\lambda(\lambda_j(\xi), \xi)\, \lambda_j(\xi)\,(\xi X - \lambda_j(\xi))}. \quad (A\,10)$$

This is now multiplied by

$$(1-\xi^2)^{\frac{1}{2}(n-3)} \left(\frac{d}{d\xi}\right)^{n+q-p-1} \delta(\xi)/(n+q-p-1)!$$

and then integrated with respect to ξ from $\xi = -1$ to $\xi = +1$, accommodating the rule (cf. Lighthill (13), Jones (11))

$$\xi^{l+q-p+1}(d/d\xi)^{n+q-p-1}\, \delta(\xi)/(n+q-p-1)! \quad (n+q \geqslant p+1)$$
$$= \begin{cases} (-1)^{l+1} \left(\dfrac{d}{d\xi}\right)^{n-l-2} \delta(\xi)/(n-l-2)! & (n+q-p-1 \geqslant l+q-p+1), \\ 0 & (n+q-p-1 \leqslant l+q-p), \end{cases}$$

and appealing to the usual convolution principle involving derivatives of the delta function $\delta(\xi)$. Substitution of the subsequent result (thus derived from (A10)) into (A 7) reveals that the function defined by (A 2) is indeed consistent with the phrase (A 3) but assumes, whenever $n+q-p-1 \geqslant l+q-p+1$, the form

$$\frac{X}{2(n-l-2)!} \sum_{j=q+1}^{m} \left[\left(\frac{\partial}{\partial \xi}\right)^{n-l-2} \frac{(1-\xi^2)^{\frac{1}{2}(n-3)} F(\lambda_j(\xi), \xi)}{G_\lambda(\lambda_j(\xi), \xi)\, \lambda_j(\xi)\,(\lambda_j(\xi) - \xi X)}\right]_{\xi=0}. \quad (A\,11)$$

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In the case of equalities, viz. $l+q-p = 0 = n+q-p-1$ or

$$n+q-p-1 = l+q-p \geqslant 0,$$

(A 4) overlaps with (A 3). However, for its remaining cases represented by

$$n+q-p-1 \geqslant 0 \geqslant l+q-p+1 \quad \text{or} \quad n+q-p-1 \geqslant l+q-p+1 > 0,$$

the phrase (A 4) follows from the fact that the expression (A 11), when divided by x_1, remains a polynomial in x_1 but whose degree $\leqslant n-l-2$ and so vanishes identically when further operated on by $(\partial/\partial x_1)^{n-l-1}$. The proof is now complete.

APPENDIX B. THE GROUP VELOCITY

Suppose the 'free' perturbation of a non-dispersive, anisotropic, real, homogeneous medium conforms to the linear differential equation

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) u = 0 \tag{B 1}$$

in which $L(\ )$ is a homogeneous polynomial function of $\partial/\partial t, \partial/\partial x_1, \ldots, \partial/x_n$, and possesses real constant coefficients. In this appendix, the dimension n can be either odd or even and $\geqslant 2$. Let us introduce the unit wave vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$, satisfying

$$\boldsymbol{\eta} = \frac{\boldsymbol{\alpha}}{\alpha} \equiv \frac{(\alpha_1, \alpha_2, \ldots, \alpha_n)}{\sqrt{(\alpha_1^2+\alpha_2^2+\ldots+\alpha_n^2)}}, \tag{B 2}$$

and associate (cf. Lighthill(14), (15); Bretherton and Garrett (3)) with the plane wave

$$u_0 \exp[i(\boldsymbol{\alpha}.\mathbf{x}-\omega t)] \tag{B 3}$$

travelling in $\mathbf{x}$-space, the phase velocity (velocity of a wave crest)

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda\boldsymbol{\eta} \tag{B 4}$$

having a magnitude λ given by

$$\lambda = \frac{\omega}{\alpha}, \tag{B 5}$$

as well as the group velocity (velocity of energy propagation)

$$\mathbf{U} = (U_1, U_2, \ldots, U_n) \tag{B 6}$$

defined by

$$U_\nu = \frac{\partial\omega}{\partial\alpha_\nu} \quad (\nu = 1, 2, \ldots, n). \tag{B 7}$$

Then (B 3) constitutes a non-trivial, constant amplitude plane wave solution to (B 1) provided the dispersion relation

$$L(-\lambda, \eta_1, \ldots, \eta_n) = 0 \tag{B 8}$$

determines $\lambda = \lambda(\eta_1, \ldots, \eta_n)$ implicitly. Whence we obtain

$$U_\nu = \lambda\eta_\nu + \sum_{\mu=1}^{n} \alpha \frac{\partial\eta_\mu}{\partial\alpha_\nu}\frac{\partial\lambda}{\partial\eta_\mu}.$$

But

$$\alpha\frac{\partial\eta_\nu}{\partial\alpha_\nu} = 1-\eta_\nu^2, \quad \alpha\frac{\partial\eta_\mu}{\partial\alpha_\nu} = -\eta_\mu\eta_\nu \quad (\mu \neq \nu).$$

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Consequently, denoting the scalar product between $\boldsymbol{\eta}$ and the gradient operator

$$\nabla_\eta = \left(\frac{\partial}{\partial\eta_1}, \frac{\partial}{\partial\eta_2}, \ldots, \frac{\partial}{\partial\eta_n}\right)$$

by

$$\boldsymbol{\eta}.\nabla_\eta = \eta_1\frac{\partial}{\partial\eta_1} + \eta_2\frac{\partial}{\partial\eta_2} + \ldots + \eta_n\frac{\partial}{\partial\eta_n},$$

we arrive at

$$\mathbf{U} = \boldsymbol{\eta}(\lambda - \boldsymbol{\eta}.\nabla_\eta\lambda) + \nabla_\eta\lambda. \tag{B 9}$$

If the anisotropic medium possesses transverse isotropy about the x_1-axis, then (B 1) specializes into

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \nabla_1^2\right)u = 0, \tag{B 10}$$

where $Q(\ )$ is an operator of the type (1·4), having an arbitrary order m. We let

$$\kappa = \eta_2^2 + \eta_3^2 + \ldots + \eta_n^2. \tag{B 11}$$

In this case, the dispersion relation (B 8) reduces to

$$Q(-\lambda, \eta_1, \kappa) = 0. \tag{B 12}$$

This implicitly establishes $\lambda = \lambda(\eta_1, \kappa)$, leading to

$$\frac{\partial\lambda}{\partial\eta_1}(\eta_1, \ldots, \eta_n) = \frac{\partial\lambda}{\partial\eta_1}(\eta_1, \kappa), \quad \frac{\partial\lambda}{\partial\eta_v}(\eta_1, \ldots, \eta_n) = 2\eta_v\frac{\partial\lambda}{\partial\kappa}(\eta_1, \kappa) \quad (v = 2, \ldots, n).$$

Whereupon, (B 9) yields

$$U_1 = \eta_1\lambda + \kappa\left(\frac{\partial\lambda}{\partial\eta_1} - 2\eta_1\frac{\partial\lambda}{\partial\kappa}\right), \tag{B 13}$$

$$U_v = \eta_v\left[\lambda - \eta_1\left(\frac{\partial\lambda}{\partial\eta_1} - 2\eta_1\frac{\partial\lambda}{\partial\kappa}\right)\right] \quad (v = 2, \ldots, n). \tag{B 14}$$

Now,

$$\kappa = 1 - \xi^2 \quad \text{if} \quad \eta_1 = \xi \quad (-1 \leqslant \xi \leqslant 1). \tag{B 15}$$

So, if $\lambda = \lambda(\xi)$ is implicitly given by

$$G(\lambda, \xi) \equiv Q(-\lambda, \xi, 1 - \xi^2) = 0 \quad (-1 \leqslant \xi \leqslant 1), \tag{B 16}$$

then, as $\lambda'(\xi) = \partial\lambda/\partial\eta_1 - 2\eta_1(\partial\lambda/\partial\kappa)$, we get (B 13) and (B 14) combining into

$$\mathbf{U} \equiv \mathbf{U}(\boldsymbol{\eta}) = \boldsymbol{\eta}[\lambda(\xi) - \xi\lambda'(\xi)] + (\lambda'(\xi), 0, 0, \ldots, 0) \quad (-1 \leqslant \xi \leqslant 1). \tag{B 17}$$

Note that

$$\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}(\boldsymbol{\eta}) = \boldsymbol{\eta}\lambda(\xi). \tag{B 18}$$

Remarks (1) Since, by (B 17) and (B 18),

$$U_1(\xi) - \lambda_1(\xi) = (1 - \xi^2)\lambda'(\xi) \quad (-1 \leqslant \xi \leqslant 1), \tag{B 19}$$

we deduce that, anywhere in the open interval $-1 < \xi < 1$,

$$\mathbf{U}(\boldsymbol{\eta}) = \boldsymbol{\lambda}(\boldsymbol{\eta})$$

if and only if $U_1(\xi) = \lambda_1(\xi)$. Thus *the group and phase velocities coincide whenever their respective components along the axis of transverse isotropy coincide* (and vice versa) within the open interval $-1 < \xi < 1$.

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(2) At $\boldsymbol{\eta} = (1, 0, 0, \ldots, 0)$ and $\boldsymbol{\eta} = (-1, 0, 0, \ldots, 0)$,

$$\mathbf{U}(1, 0, 0, \ldots, 0) = (\lambda(1), 0, 0, \ldots, 0) = \boldsymbol{\lambda}(1, 0, 0, \ldots, 0), \tag{B 20}$$

$$\mathbf{U}(-1, 0, 0, \ldots, 0) = (-\lambda(-1), 0, 0, \ldots, 0) = \boldsymbol{\lambda}(-1, 0, 0, \ldots, 0). \tag{B 21}$$

In particular, these results provide a sort of extension of the above result regarding the group and phase velocities, to the right and left limits $\xi = 1, -1$. Also, (B 20) and (B 21) imply that *corresponding sheets of the group velocity* $\mathbf{U}$*-surface and the phase velocity* $\boldsymbol{\lambda}$*-surface* (determined by (B 12)) *always intersect simultaneously, the axis of transverse isotropy at points defined by*

$$\mathbf{x}/t = (\lambda(1), 0, 0, \ldots, 0) \quad \textit{and} \quad \mathbf{x}/t = (-\lambda(-1), 0, 0, \ldots, 0).$$

(3) When $\boldsymbol{\eta} = (0, 0, \ldots, 1, \ldots, 0)$, the unit vector in the x_μ-direction, say, then

$$\mathbf{U}(0, 0, \ldots, 1, \ldots, 0) = (\lambda'(0), 0, \ldots, \lambda(0), \ldots, 0), \tag{B 22}$$

whose only non-trivial components are $U_1 = \lambda'(0)$ and $U_\mu = \lambda(0)$. If $\lambda(\xi)$ is an odd function of ξ (so that $\lambda(0) = 0$), then

$$\mathbf{U}(0, 0, \ldots, 1, \ldots, 0) = (\lambda'(0), 0, 0, \ldots, 0), \quad \boldsymbol{\lambda}(0, 0, \ldots, 1, \ldots, 0) = \mathbf{0}, \tag{B 23}$$

in which case, *corresponding sheets of the* $\mathbf{U}$*-surface and* $\boldsymbol{\lambda}$*-surface intersect the axis of transverse isotropy at points given, respectively, by* $\mathbf{x}/t = (\lambda'(0), 0, 0, \ldots, 0)$ *and* $\mathbf{x}/t = \mathbf{0}$, *whenever the vector parameter* $\boldsymbol{\eta}$ *points in any direction perpendicular to* $(1, 0, 0, \ldots, 0)$. On the other hand, if $\lambda(\xi)$ is an even function of ξ (so that $\lambda'(0) = 0$), then

$$\mathbf{U}(0, 0, \ldots, 1, \ldots, 0) = (0, 0, \ldots, \lambda(0), \ldots, 0) = \boldsymbol{\lambda}(0, 0, \ldots, 1, \ldots, 0), \tag{B 24}$$

i.e. *corresponding sheets of the* $\mathbf{U}$*-surface and* $\boldsymbol{\lambda}$*-surface intersect, simultaneously, a plane normal to the axis of transverse isotropy, in a circle (n-dimensional) centred at* $\mathbf{x}/t = \mathbf{0}$ *and having the radius* $|\mathbf{x}/t| = |\lambda(0)|$.

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