NONLINEAR PULL-IN CHARACTERIZATION OF A NONLOCAL NANOBEAM WITH AN INTERMOLECULAR FORCE

Y.-G. Wang * **H.-F. Song** **W.-H. Lin**

College of Science
China Agricultural University
Beijing, China

ABSTRACT

This survey examines the geometrically nonlinear bending of a doubly clamped nanobeam that is subjected to combined actions of actuator voltage, prestress, and intermolecular force, with the pull-in instability as the primary objective. A nonlocal Euler-Bernoulli beam model, which takes the small scale effect into account, is developed making use of the principle of virtual displacement on the basis of Eringen's nonlocal theory in conjunction with von Kármán assumption. Due to complexity of the resulting equations, a shooting technique is established through taking the applied voltage as an unknown and the central deflection as a control parameter. This treatment has the capability of tackling the nonlinearities from both the large deformation and electrostatic force as well as the intermolecular force and enables the size dependent deflection response to an applied voltage of the nanobeam to be obtained conveniently. Validation is conducted in numerical examples through direct comparisons with existing solutions to confirm the proposed method. Parametric studies are undertaken addressing the impacts of the nonlocal effect, intermolecular force, residual stress, and geometry of the beam on the pull-in behaviors.

Keywords: Nanobeam, Pull-in instability, Nonlocal theory, Shooting method.

1. INTRODUCTION

By virtue of its favorable scaling property, simple configuration, lower energy consumption and better electromechanical behaviors, nano electromechanical system (NEMS) and its applications are undergoing a startling revolution in recent years [1-3]. The electrostatic NEMS is a special branch under micromechanics with a wide range of application in specific device, and is typically modeled as a conductive beam-shaped element suspending over a substrate [4,5].

An electrostatically actuated beam possesses an inherent instability that is known as the pull-in phenomenon [6,7]. Due to its importance in the design, analysis and simulation of NEMS, pull-in instability has been a subject in analysis of electrostatic beams. Typical approaches proposed for elucidating the pull-in behaviors include the lumped [8-10] and distributed parameter models. The former, which models NEMS as a spring-mass system where the elasticity of the deformable body is lumped into the stiffness of a spring, has the merit of simplicity and explicit physical meaning but inaccurate. The latter simulates the body directly from the physical continuous system, which leads the governing equations to a set of nonlinear differential equations.

At nanoscales, the decreasing size of NEMS makes surface traction due to intermolecular interaction an important force which is not important at macroscales. At such situation, the effect of intermolecular forces (dispersion forces) can be described by the vdW or Casimir force, depending on the gap between the electrodes [11,12]. Information concerning the effect of intermolecular forces on the pull-in instability of electromechanical systems is presented in Refs. [13-15] where it is highlighted that the attractive intermolecular forces decrease the pull-in voltage of nanoactuators and should be taken into account in the simulation of pull-in instability.

It is worth mentioning that surface layer characteristics often play a significant role in the pull-in performance of electromechanical nano-actuators, but limited works have been conducted for taking this effect into account. Beni *et al*. [16] studied the torsion-bending coupled pull-in instability of rotational electromechanical nano-micro mirror using a two-degree-of-freedom model with surface effect and molecular van der Waals attraction incorporated. Koochi *et al*. [17,18] investigated the influence of surface effects including residual surface stress and surface elasticity on the pull-in insta-

* Corresponding author (wangyg@cau.edu.cn)

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bility of a cantilever nano-actuators incorporating the influence of the Casimir [17] and vdW [18] attractions. Koochi *et al*. [19] made a survey on the static and dynamic pull-in instability of nano-switches with small width/height ratio. The couple stress theory combined with the surface elasticity was employed to incorporate the coupled effects of size phenomena and surface energy.

It should be noted that NEMS with electrostatic actuation is essentially nonlinear and coupled. Nonlinearity arises out of electrostatic force, dispersion force and large deformation effect [14,20]. According to the recent study, the geometric nonlinearity plays an important role in the pull-in characterization, a large discrepancy between the linear and nonlinear results has been observed [21]. When the nonlinear deformation effects due to the large displacements are accounted for, the static pull-in voltages are significantly altered by geometrical stiffening. Hence it is essential to account for large displacements appropriately [22]. In this case, it is often difficult to find an analytical solution, especially for a distributed parameter model, unless one or more simplifying assumptions are considered. In recent decades, some promising approximate analytical or semi-analytical methods such as homotopy perturbation, modified Adomian decomposition [23-26] have been proposed in modeling the pull-in instability of electrostatic nanobeams loaded by electrostatic forces and dispersion forces by Koochi *et al*. [23], Abdi *et al*. [24] and Soroush *et al*. [25,26]. Frequently, application of different numerical techniques is unavoidable in order to alleviate difficulties associated with the analysis of distributed nonlinearly coupled systems. Among many previous efforts, the reduced order model [27], shooting model [28,29], differential quadrature model [6,7], finite element model [30], and linear distributed load model [31] constitute the typical techniques in numerical procedure. A comprehensive overview of models for theoretical, numerical and experimental works has been classified by Batra *et al*. [32] according to the mechanical model used in the analysis.

In addition to the effect of intermolecular forces, the size dependency of material characteristics is another effect that appears at the nanostructures. It needs to be addressed that the aforementioned studies of the pull-in instability were performed within the framework of classical continuum theory, which was incapable of describing the size effects due to the lack of material length scale parameter. The nanoscale of NEMS structures may call the direct application of classical scale independent continuum models into question. Indeed, it is demonstrably approved that the pull-in instability has strong size dependency. Thus some size dependent continuum models, including couple stress elasticity [33-39], strain gradient elasticity [40,41] and nonlocal elasticity, have appealed increasing attention as alternatives to the atomistic ones for the investigation of pull-in instability due to their computational efficiency and the capability to produce accurate results. One of the most effective model among these modified continuum models is the nonlocal elasticity theory proposed by Eringen [42] in 1970s. The theory introduces the small scale effect through a spatial integral constitutive relation and allows for the stress at a point to be dependent on strains at all points in the elastic body. In what follows is a summary of selected studies in which Eringen's nonlocal elasticity model was used to deal with the size effects in investigating the pull-in instability.

The nonlocal theory has gained wide acceptance in micro/nano materials since Peddieson *et al*. [43] initiated in a nonlocal version of Euler-Bernoulli beam model. Regarding electrostatically actuated devices, the movable electrode was modeled as a mechanical beam conforming to nonlocal continuum theory. Yang *et al*. [31] studied the pull-in instability of nano-switches under electrostatic and intermolecular forces by using Eringen's nonlocal elasticity theory to take into account the small scale effect. A linear distributed load model was proposed to linearize the intermolecular and electrostatic forces and simplify the complex nonlinear differential equation to obtain a closed-form solution for cantilever and fixed-fixed nanobeams. Taghavi *et al*. [44] investigated the pull-in instability of nano-switches under electrostatic and intermolecular forces, a hybrid nonlocal theory was incorporated into the classical Euler-Bernoulli beam model to interpret the small scale effect. The nonlinear intermolecular and electrostatic interactions acting on the beam were approximated by the linear distributed load model [31]. Mousavi *et al*. [45] dealt with nano-switches subjected to electrostatic and intermolecular forces within in the framework of the nonlocal Euler-Bernoulli beam theorem. The governing equation was derived through virtual displacement principle and solved numerically via differential quadrature method. Fakhrabadi *et al*. [46] analyzed the deflection and instability of electrostatically actuated carbon nanotubes for different geometries and boundary conditions. Their results proved that application of the nonlocal elasticity theorem leads to stiffer structures with higher pull-in voltages. Miandoab *et al*. [47] modeled mathematically the mechanical behavior of an Euler-Bernoulli microbeam utilizing modified couple stress and nonlocal elasticity theories. By means of analogue equation and Gauss-Seidel iteration methods, Miandoab *et al*. [48] studied the static response of electrically actuated micro/nano beams using strain gradient and nonlocal elasticity theories, respectively. Najar *et al*. [49] modeled the nanoactuator as a nonlocal Euler-Bernoulli beam under a combined electrostatic and intermolecular forces, taking into consideration the effects of residual stress and nonlinear deformation in von Kármán sense. The established mathematical model was solved by the differential quadrature method, the influence of scale effect on the maximum stable deflection and pull-in voltage of the device was investigated. More recently, Sedighi *et al*. [50] investigated the dynamic pull-in behavior of functionally graded nano-actuators incorporating Casimir attraction applying the combination of Bubnov-Galerkin method and iteration perturbation method based on the nonlocal elasticity theory. Employing the nonlocal elasticity theory and Gurtine Murdoch model, Sedighi *et al*. [51] studied the dynamic

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stability of double-sided asymmetric functionally graded bridges. The impacts of various parameters including the finite conductivity, nonlocal parameter, surface stresses and material characteristics on the dynamic instability of the bridge are addressed.

In view of the literature survey, there are large amount of researches regarding the classical modeling of electrostatically actuated nanobeams, only a small proportion of investigations have been conducted to the size-dependent pull-in instability of nanostructures within the framework of nonlocal elasticity theory. Especially, the comprehensive model on pull-in instability of nanobeams under the combined electrostatic and intermolecular Casimir or vdW forces and axial residual stress, accounting for the nonlinearity that stems from both the forces and mid-plane stretching has received relatively little attention. Further, some conclusions from different authors can not match with each other in this direction, well-documented works are scanty. For example, a result from Najar *et al*. [49] proves that the nonlocal parameter tends to increase the pull-in voltage and decrease the maximum stable deflection at pull-in. This result is not in agreement with that given by Yang *et al*. [31] and Mousavi *et al*. [45]. This motivates us to investigate the electrostatically actuated nanobeam with large deflection and intermolecular forces by means of nonlocal elasticity theory.

Presented herein pays an effort to establish a novel mathematical model and a numerical procedure to survey the pull-in instability from the geometrically nonlinear bending of a doubly clamped Euler-Bernoulli nanobeam under electrostatic actuation, with the scale dependency, residual stress, and intermolecular forces incorporated. To this end, based on von Kármán theory, the governing equations are derived in terms of Eringen's nonlocal elasticity. The resulting coupled and strongly nonlinear boundary value problem is solved using a shooting technique with the central maximum deflection of the nanobeam as a given parameter whereas the applied voltage as an unknown. In a parametric study, the emphasis is placed on investigating the effect of veering the applied voltage, prestress, vdW and Casimir force, fringing field effect, gap-to-thickness ratio, especially nonlocal parameter on the pull-in instability behaviors. Some conclusions are reached.

2. GOVERNING EQUATIONS

The schematic view of the nanobeam under consideration is shown in Fig. 1. The beam, which is deformable and initially straight, is separated by a dielectric spacer from a fixed ground plane with an initial gap g. The beam is treated as an Euler-Bernoulli beam with electrostatic actuation force and with length l, width b, thickness h, Young's elastic modulus E, and Poisson's ratio ν. The Cartesian coordinate system (x, y, z) is used with x-axis coincident with the neutral axis of the undeformed beam, y-axis along the width, and z-axis along the thickness (the height) of the beam.

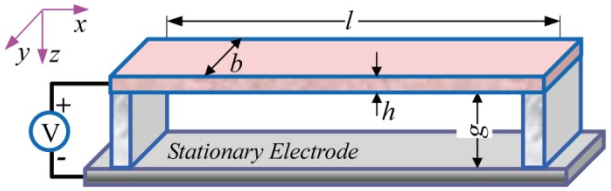


Fig. 1 Schematic view of a doubly clamped nanobeam actuated by an electrostatic force.

2.1 Strain-displacement Relations

The classical hypotheses of Euler-Bernoulli beams for displacements field $\boldsymbol{u} = (u_1, u_2, u_3)$ of an arbitrary point on the movable beam along the x-, y- and z-axes is of the form

$$u_1(x, z) = u(x) - zw_{,x}\,, \quad u_2(x, z) = 0\,, \quad u_3(x, z) = w(x) \tag{1}$$

where $x \in [0, l]$, $u(x) = u_1(x, 0)$ and $w(x) = u_3(x, 0)$ are the displacements of the neutral axis at abscissa x in x- and z-directions, respectively. Henceforward, a comma denotes the derivative with respect to the argument.

Taking into account the geometrically nonlinear deformation, the von Kármán-type nonlinear strain-displacement relationship of a beam associated with the above displacement field reads

$$\varepsilon_{xx}(x, z) = u_{,x} + \frac{1}{2}w_{,x}^2 - zw_{,xx} \tag{2}$$

2.2 Nonlocal Euler–Bernoulli Beam Theory and Stress Resultants

The essence of the nonlocal elasticity theory is that the stress field at a reference point in an elastic continuum depends not only on the strains at that point but also on strains at all other points in the domain [42]. This assumption is in accordance with the atomic theory of lattice dynamics and experimental observations on phonon dispersion.

Consider a homogeneous and isotropic elastic solid with zero body force, the nonlocal stress tensor σ at any point x in a body can be expressed as

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_{\Omega} K\left(|\,\mathbf{x}' - \mathbf{x}\,|, \tau\right)\mathbf{t}(\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,, \quad \mathbf{t}(\mathbf{x}') = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) \tag{3}$$

where $\mathbf{t}(\mathbf{x})$ is the classical local stress tensor at point $\mathbf{x}$, $\boldsymbol{\varepsilon}(\mathbf{x})$ is the strain tensor, $\mathbf{C}$ is the fourth-order elasticity tensor and ":" denotes the double dot product. The kernel function $K(|\,\mathbf{x}'\text{-}\mathbf{x}\,|,\ \boldsymbol{\tau})$ represents the nonlocal modulus which acts as an attenuation function incorporating into constitutive equations the nonlocal effects at the reference point $\mathbf{x}$ produced by local strain at the source $\mathbf{x}'$. $|\mathbf{x}'\text{-}\mathbf{x}|$ is the Euclidean distance, and τ is a material parameter that depends on internal and external characteristic lengths. In the limit when the effects of strains at points other than $\mathbf{x}$ is neglected, one obtains the classical (local) theory of elasticity.

The constitutive Eq. (3) defines the nonlocal constitutive behavior of a Hookean solid. In view of the difficulty in using the integral-partial differential con-

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stitutive relation, Eringen [52] proposed an equivalent differential model as

$$(1-\mu^2\nabla^2)\boldsymbol{\sigma}=\mathbf{t}\,,\ \ \mu=e_0\,a\,, \tag{4}$$

where e_0 is a material constant, a is an internal length scale, and ∇^2 is the Laplacian operator. The nonlocal constitutive relation for the present one dimensional nanobeam reduces to

$$\chi(\sigma_{xx})=E\varepsilon_{xx}\,,\ \ \chi(\ )=1-\mu^2(\ )_{,xx} \tag{5}$$

It is noted that the scale $\mu=e_0a$ will lead to small scale effect on the response of structures in nanoscale, moreover, the classical elasticity theory is restored if the scale μ is set to zero.

The nonlocal constitutive relation according to Eq. (5) together with the strain-displacement relation (2) can be expressed in terms of stress resultants as

$$\chi(N_{xx})=EA\left(u_{,x}+\frac{1}{2}w_{,x}^2\right) \tag{6}$$

$$\chi(M_{xx})=-EIw_{,xx} \tag{7}$$

with

$$(A,\,I)=\int_A(1,\,z^2)\,\mathrm{d}A\,,\ \ (N_{xx},\,M_{xx})=\int_A(1,\,z)\,\sigma_{xx}\mathrm{d}A$$

Here, A is the cross-sectional area, I is the moment of inertia. N_{xx} and M_{xx} being, respectively, the resultant axial force and bending moment.

2.3 Nonlinear Equilibrium Equations

The governing equation of the nanobeam in bending and all boundary conditions are identified using the principle of virtual displacement that is expressed by

$$\delta\pi=(\delta U_m+\delta U_\sigma)-\delta W=0 \tag{8}$$

The virtual strain energy δU_m due to bending and the change of stretch with respect to the initial configuration is

$$\begin{aligned}\delta U_m&=\int_0^l\int_A\sigma_{xx}\delta\varepsilon_{xx}\mathrm{d}A\mathrm{d}x\\&=\int_0^l\left[N_{xx}(\delta u_{,x}+w_{,x}\delta w_{,x})-M_{xx}\delta w_{,xx}\right]\mathrm{d}x\end{aligned} \tag{9}$$

The increase in the stored energy with respect to the initial configuration due to the existence of initially axial loads takes the form

$$U_\sigma=\int_0^l\frac{N_0}{A}\int_A\left(u_{,x}-zw_{,xx}+\frac{1}{2}w_{,x}^2\right)\mathrm{d}A\mathrm{d}x \tag{10}$$

its first variation reads

$$\delta U_\sigma=(N_0\delta u+N_0w_{,x}\,\delta w)\Big|_0^l-\int_0^l\left[N_{0,x}\delta u+(N_0w_{,x})_{,x}\delta w\right]\mathrm{d}x \tag{11}$$

Here, $N_0=\sigma A$ is the initial axial load. For a wide beam ($b\geq 5h$), $\sigma=\sigma_0(1-\nu)$, otherwise, σ simply becomes σ_0 [39], σ_0 is the biaxial residual stress.

Designating $f(x)$ and $q(x)$ being the distributed axial load and transverse load, respectively, then the virtual work done by the external forces is

$$\delta W=\int_0^l(f\delta u+q\delta w)\,\mathrm{d}x \tag{12}$$

The distributed load $q(x)$ in Eq. (12) for the case is the sum of the electrostatic force q_e and an intermolecular force q_n (n = 3,4), both nonlinearly dependent on the gap between the deformed beam and the electrode. With the first order fringing field correction included, the electrostatic force per unit length of the beam is approximated by the parallel capacitor formula as

$$q_e=\frac{1}{2}\frac{\varepsilon_0bV^2}{(g-w)^2}\left(1+\frac{2}{\pi}\frac{g-w}{b}\right) \tag{13}$$

in which ε_0 is the permittivity in vacuum, V is the applied voltage, respectively.

The intermolecular force appears in two forms, vdW (q_3) and Casimir (q_4) forces. The vdW force per unit length of the beam is expressed as [15]

$$q_3=q_{vdW}=\frac{1}{6\pi}\frac{\mathscr{H}b}{(g-w)^3} \tag{14}$$

while the Casimir force per unit length of the beam can be calculated by adopting the proximity force approximation as [11]

$$q_4=q_{Casimir}=\frac{\pi^2}{240}\frac{\hbar cb}{(g-w)^4} \tag{15}$$

where $\mathscr{H}$ is the Hamaker constant, $\hbar$ is Plank's constant divided by 2π and c the speed of light in vacuum.

The principle of virtual work requires Eq. 8 be satisfied, which yields the equilibrium equations

$$(N_{xx}+N_0)_{,x}+f=0 \tag{16}$$

$$M_{xx,xx}+\left[(N_{xx}+N_0)w_{,x}\right]_{,x}+q=0 \tag{17}$$

and the corresponding boundary conditions at the beam ends

$$(N_{xx}+N_0)\big|_{x=0,l}=0\ \text{ or }\ \delta u\big|_{x=0,l}=0 \tag{18}$$

$$V_x\big|_{x=0,l}=0\ \text{ or }\ \delta w\big|_{x=0,l}=0 \tag{19}$$

$$M_{xx}\big|_{x=0,l}=0\ \text{ or }\ \delta w_{,x}\big|_{x=0,l}=0 \tag{20}$$

with

$$V_x=(N_{xx}+N_0)w_{,x}+M_{xx,x} \tag{21}$$

Expressions in Eqs. (16) and (17) for N_{xx} and M_{xx} can be substituted into Eqs. (6) and (7) to obtain the equilibrium equations in terms of the nonlocal displace-

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ments u and w

$$EA\left(u_{,x}+\frac{1}{2}w_{,x}^{2}\right)_{,x}+\chi(f+N_{0,x})=0 \tag{22}$$

$$EIw_{,xxxx}-\chi\left[(N_{xx}w_{,x})_{,x}+N_{0}w_{,xx}+q\right]=0 \tag{23}$$

In a similar vein, the nonlocal stress resultants be expressed in terms of the displacements as

$$N_{xx}=EA\left(u_{,x}+\frac{1}{2}w_{,x}^{2}\right)-\mu^{2}(f_{,x}+N_{0,xx}) \tag{24}$$

$$M_{xx}=-EIw_{,xx}-\mu^{2}\left[(N_{xx}+N_{0})w_{,x}\right]_{,x}-\mu^{2}q \tag{25}$$

$$V_{x}=-EIw_{,xxx}+\chi\left[(N_{xx}+N_{0})w_{,x}\right]-\mu^{2}q_{,x} \tag{26}$$

with no distributed axial load and with constant biaxial residual stress, the axial equilibrium leads to a simplified formulation. To facilitate theoretical computation, the following coordinate transformation and nondimensional notations are introduced

$$(X,\ \eta)=\frac{1}{l}(x,\ \mu),\quad W=\frac{w}{g},\quad U=\frac{lu}{g^{2}},$$

$$\zeta_{1}=\frac{g}{l},\quad \zeta_{2}=\frac{2}{\pi}\frac{g}{b},\quad \kappa=l\sqrt{\frac{A}{I}}$$

$$(\tilde{N},\ \tilde{N}_{0})=\frac{l^{2}}{EI}(N_{xx},\ N_{0}),$$

$$(P_{V},\ \tilde{M})=\frac{l^{2}}{EIg}(lV_{x},\ M_{xx}),\quad Q=\frac{l^{4}q}{EIg}$$

$$\alpha=\frac{bl^{4}\varepsilon_{0}}{2EIg^{3}}V^{2},\quad \beta_{3}=\frac{1}{6\pi}\frac{\hbar bl^{4}}{EIg^{4}},\quad \beta_{4}=\frac{\pi^{2}}{240}\frac{\hbar cbl^{4}}{EIg^{5}},$$

$$\tilde{\chi}(\ )=1-\eta^{2}(\ )_{,XX},\quad P_{H}=\tilde{N}+\tilde{N}_{0}$$

Utilizing these quantities, the dimensionless expressions of nonlinear governing equation can be given by

$$U_{,XX}=-W_{,X}W_{,XX} \tag{27}$$

$$W_{,XXXX}=\tilde{\chi}\left[(P_{H}W_{,X})_{,X}+Q\right] \tag{28}$$

with

$$P_{H}=\kappa^{2}\left[\zeta_{1}^{2}\left(U_{,X}+\frac{1}{2}W_{,X}^{2}\right)+\frac{\sigma}{E}\right] \tag{29}$$

$$P_{V}=-W_{,XXX}+\tilde{\chi}(P_{H}W_{,X})-\eta^{2}Q_{,X} \tag{30}$$

$$\tilde{M}=-W_{,XX}-\eta^{2}\left[(P_{H}W_{,X})_{,X}+Q\right] \tag{31}$$

$$Q=\frac{\alpha\zeta_{2}}{1-W}+\frac{\alpha}{(1-W)^{2}}+\frac{\beta_{n}}{(1-W)^{n}},\quad n=3,4 \tag{32}$$

For a doubly clamped nanobeam, the deformation will be symmetrical about the center of the beam. Therefore, only half-span of the beam is to be considered in the present investigation and the boundary conditions must be defined and specified for the end constraints as

$$U(0)=0,\ \ W(0)=0,\ \ W_{,X}(0)=0 \tag{33}$$

$$P_{V}(1/2)=0,\ \ U(1/2)=0,\ \ W_{,X}(1/2)=0,\ \ W(1/2)=\varepsilon \tag{34}$$

It may be mentioned that κ expresses the slenderness of the nanobeam, ζ_1 is the ratio of gap to length, α is the voltage parameter, β_3 and β_4 are the vdW and Casimir parameter, respectively. It is to be further noted according to the definition of the non-dimensional variable that physically meaningful solutions exist only in the region of $0<\varepsilon<1$, where ε is the dimensionless central deflection of the beam.

3. BOUNDARY VALUE PROBLEM AND SOLUTIONS

Accurate determination of pull-in voltage is very challenging in virtue of the electromechanical coupling and the nonlinearity. The simultaneous effects of the electrostatic force, dispersion force, fringing field and the residual stress further complicate the modeling and make the analytical methods fail or become too cumbersome. Accordingly, a primacy must be given to develop an algorithm able to capture the pull-in parameter. Imposing boundary conditions (33) ~ (34) on the governing Eqs. (27) ~ (28) turns them into a two-point boundary value problem, which is to be solved using the methodology based on shooting technique, Runge-Kutta integration scheme and Newton-Raphson iteration method [53,54].

To apply the numerical method, the field equations together with the boundary conditions are summarized as seven coupled set of first order, nonlinear ordinary differential equations, such that

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}X}=\mathbf{H}(X,\mathbf{Y};\beta_{n},\eta,\varepsilon),\quad X\in(0,1/2) \tag{35}$$

$$\mathbf{B}_{1}\mathbf{Y}(0)=\{\mathrm{v}\}^{\mathrm{T}},\ \ \mathbf{B}_{2}\mathbf{Y}(1/2)=\{\varepsilon,0,0,0\}^{\mathrm{T}} \tag{36}$$

with the denotations of the forms

$$\mathbf{Y}=\{Y_{1},Y_{2},Y_{3},Y_{4},Y_{5},Y_{6},Y_{7}\}^{\mathrm{T}}$$
$$=\{W,W_{,X},W_{,XX},W_{,XXX},U,U_{,X},\alpha\}^{\mathrm{T}}$$

$$\mathbf{H}=\{Y_{2},Y_{3},Y_{4},\varphi,Y_{6},-Y_{2}Y_{3},0\}^{\mathrm{T}}$$

in which φ represents the right hand of Eq. (28), $\mathbf{B}_1$ and $\mathbf{B}_2$ are two boundary related matrixes. Here, the applied voltage parameter α is selected as a resolving function while the central deflection of the nanobeam acts as a control parameter to facilitate the solution procedure.

In view of the shooting method, the boundary value

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problem is first reduced to an initial one

$$\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}X} = \mathbf{H}(X, \mathbf{Z}; \beta_n, \eta, \varepsilon), \quad X > 0 \tag{37}$$

$$\mathbf{Z}(0) = \mathbf{I}(0, \mathbf{D}) \tag{38}$$

where $\mathbf{Z} = \{Z_1, Z_2, \ldots, Z_7\}^{\mathrm{T}}$ and $\mathbf{D} = \{d_1, d_2, d_3, d_4\}^{\mathrm{T}}$ is an unknown vector related to the missing initial values of $\mathbf{Y}$ at $X = 0$. For a given $\mathbf{D}$, there exists

$$\mathbf{Z} = \mathbf{Z}(X; \beta_n, \eta, \varepsilon, \mathbf{D}) = \mathbf{I}(0, \mathbf{D}) + \int_0^X \mathbf{H}(\xi, \mathbf{Z}; \beta_n, \eta, \varepsilon)\, \mathrm{d}\xi \tag{39}$$

Proceeding integration for Eq. (39) from the starting point at $X = 0$ utilizing the assigned initial value $\mathbf{Z}(0)$ to its finial location at $X = 1/2$. The fourth order Runge-Kutta method with variable steps may be used with this purpose. Now, one defines a discrepancy vector $\mathbf{F}$ at $X = 1/2$ as

$$\mathbf{F} = \mathbf{B}_2\mathbf{Z}(1/2; \beta_n, \eta, \varepsilon, \mathbf{D}) - \{\varepsilon, 0, 0, 0\}^{\mathrm{T}} \tag{40}$$

For prescribed parameter $\{\beta_n, \eta, \varepsilon\}$, one is seeking for a vector value of $\mathbf{D}$ that zeros the vector value of $\mathbf{F}$. This can be implemented by applying Newton-Raphson iteration method, and the initial guess is sought and updated until convergence is achieved. If there exists $\mathbf{D} = \mathbf{D}^*$ making Eq. (40) zero, then the solution for two-point boundary value problem is obtained as

$$\mathbf{Y}(X; \beta_n, \eta, \varepsilon) = \mathbf{Z}(X; \beta_n, \eta, \varepsilon, \mathbf{D}^*) \tag{41}$$

and the variation of applied voltage with central deflection reads

$$\alpha = Y_7(\beta_n, \eta, \varepsilon) \tag{42}$$

from which the pull-in parameters be extracted.

4. NUMERICAL RESULTS AND DISCUSSIONS

For the problem at hand, the solution obtained allows to investigate a nonlinear bending configuration and pull-in instability caused by electrostatic pressure and intermolecular force arising in a nonlocal nanobeam.

Unless otherwise specified, the following set of material and geometric parameters for an electrostatically actuated nanobeam with rectangular cross section is considered for illustration purpose [49]

$$l = 200\text{nm}, \ b = 30\text{nm}, \ h = 3.5\text{nm},$$
$$g = 30\text{nm}, \ E = 166\text{GPa}, \ \varepsilon_0 = 8.854\text{PF/m} \tag{43}$$

4.1 Comparison with Published Results

For comparison with the results available from the literature, the influence of intermolecular force on the pull-in voltage for different values of nonlocal parameter η is studied and abstracted in Table 1, in which, a classical local nanobeam result is rendered when η vanishes. Herein, R. D. denotes the relative difference percentage with the result tabulated in [49] defined as (present solution-published one)/published solution. It can be seen that the results of the present work agree with the ones obtained by Najar *et al*. [49], particularly for lower nonlocal parameter values. The difference between the two solutions increases with the increase in the nonlocal parameter. The reason for this deserves further study.

The influence of residual stress and nonlocal effect on the pull-in voltage is listed in Table 2. A good agreement among the results can be witnessed. As has been concluded by Najar *et al*. [49], augmenting the nonlocal parameter results in an increase in the pull-in voltage. It is inferred from the comparative studies that the methodology proposed herein can yield reliable numerical results for an electrostatically actuated nanobeam.

Table 1 also shows that the pull-in voltage with vdW force is always higher slightly than that with Casimir force. According to the previous study [15], the vdW force operatives at separations below 20nm and the Casimir interaction presents at separations above 20nm. The effect of dispersion (van der Waals and Casimir) forces is taken into consideration in what follows.

4.2 Nonlocal Characteristic Relation of Applied Voltage Versus Central Deflection

Figure 2(a) is plotted for the η-dependent characteristic curves of dimensionless central deflection versus applied voltage including the Casimir force. Here, the solid diamond denotes a critical state, at which, the slope of voltage-central deflection curve approaches infinity, and the pull-in instability occurs. The corresponding voltage is referred to as pull-in voltage which has been extracted in Table 1 in its dimensional form. For a given applied voltage less than its pull-in value, two solutions exist, the upper branch is unstable whereas the lower one corresponds to the stable equilibrium configuration [20,28]. It is observed that the nonlocal parameter has a significant effect in predicting the pull-in voltage, while it affects pull-in deflection slightly. The small scale causes the stiffening effect of the nonlocal beam and delays the pull-in instability, leading the beam to collapse at higher value of applied voltage. Such an observation has been mentioned in Ref. [49]. The trend is similar to what is observed in Fig. 2(b) for a nanobeam under vdW attraction.

Figure 3 demonstrates the relationship between the applied voltage and central deflection for a nanobeam under some specific values of initial residual stress σ_0. Examining the results reveals that either an increment in the tensile residual stress or a decrement in the compressive one causes an augmentation in the pull-in voltage. This is due to the stiffening and softening behavior of a nanobeam related to the residual stress, respectively. In addition, the beam presents the same behavior in the presence of Casimir attraction as that of vdW attraction.

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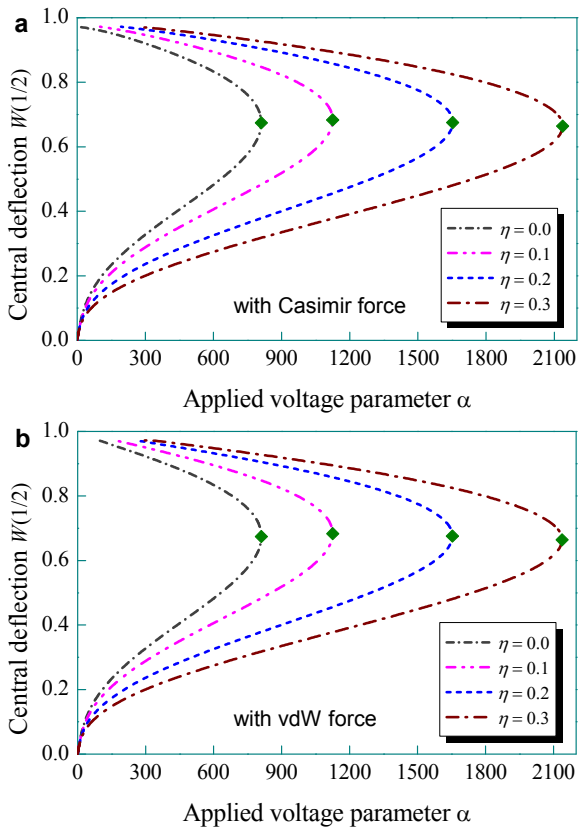


Fig. 2 Characteristic curve of the applied voltage versus central deflection for a nanobeam with nonlocal effect considering the dispersion force effect.

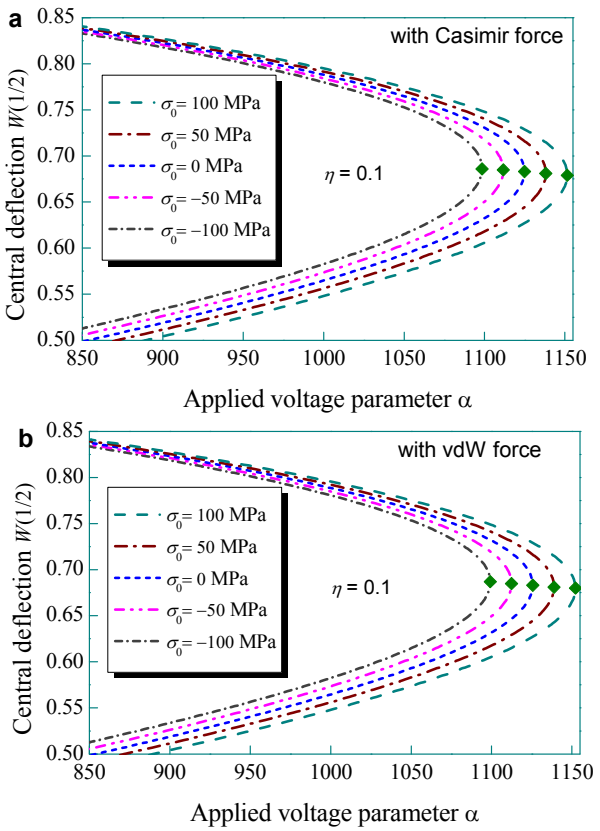


Fig. 3 Characteristic curve of the applied voltage versus central deflection for a nanobeam with residual stress considering the nonlocal effect and dispersion force effect.

Table 1 Comparison of the pull-in voltage (in V) for various nonlocal parameters of a nanobeam when the residual stress is vanished.

η	without dispersion force			with vdW	with Casimir		
	Present	Ref. [49]	R.D. [%]		Present	Ref. [49]	R.D. [%]
0.0	42.8233	42.8	0.054	42.8205	42.8022	42.778	0.057
0.1	50.4472	49.58	1.749	50.4448	50.4285	49.56	1.752
0.2	61.1435	58.11	5.220	61.1414	61.1281	58.09	5.230
0.3	69.5192	64.51	7.765	69.5174	69.5060	64.5	7.761

Table 2 Comparison of the pull-in voltage (in V) for various nonlocal parameters of a nanobeam in consideration of the residual stress and Casimir effect.

σ_0 (MPa)	$\eta = 0.0$			$\eta = 0.1$			$\eta = 0.2$			$\eta = 0.3$		
	Present	Ref. [49]	R.D. [%]	Present	Ref. [49]	R.D. [%]	Present	Ref. [49]	R.D. [%]	Present	Ref. [49]	R.D. [%]
−10	42.7533	42.66	0.218	50.3691	49.287	2.196	61.0539	57.636	5.930	69.4188	63.909	8.621
−50	42.5577	42.47	0.206	50.1314	49.035	2.236	60.7567	57.292	6.047	69.0701	63.48	8.806
−100	42.3128	42.215	0.232	49.8340	48.72	2.287	60.3852	56.8615	6.197	68.6345	62.944	9.040

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Figure 4 investigates the fringing field effect on the pull-in voltage for a nanobeam with zero residual stress and dispersion force based on the framework of classical and nonclassical theories, respectively. It is clear that the fringing field effect leads to a reduction in the pull-in voltage. On the other hand, since the pull-in voltage from the nonlocal theory is larger compared with the one from classical beam theory, the nonlocal effect causes the beam to become stiffen.

Figure 5 depicts the variation of the pull-in voltage with the gap to thickness ratio for some assigned nonlocal parameters with dispersion force and residual stress in consideration. Here, the gap to thickness ratio is varied by keeping the thickness constant. It is manifest that augmenting the gap to thickness ratio leads to an increase in the pull-in voltage of the nanobeam. This matter is attributed to weakening the effects of electrostatic and dispersion force with gap increment. Thus, in order to reach the pull-in, higher voltage should be applied. Similar to the previous cases, the nonlocal parameter has a tendency to increase the pull-in voltage.

5. CONCLUSIONS

In the present work, an attempt has been made to study the size-dependent pull-in instability exhibited by an electrostatically actuated nanobeam in the context of Eringen's nonlocal elasticity theory with von Kármán nonlinearity and intermolecular attractive forces incorporated. The formulation has been derived using the principle of virtual displacement, and a computational scheme has been constructed to establish the bending equilibrium configuration, from which, the pull-in parameters have been extracted. The influences of nonlocal parameter, prestress, vdW and Casimir force, fringing field effect, and the gap-to-thickness ratio on the pull-in performances are examined. Some general inferences are mentioned as below:

- The nonlocal parameter increases the pull-in voltage, the nonlocal beam tends to become stiffer compared to its local counterpart. The pull-in voltages obtained from local and nonlocal theories have more significant differences for high value of the nonlocal parameter.
- Presence of an intermolecular force reduces the pull-in voltage.
- A tensile stress prior to deformation increases the pull-in voltage. It is in contrast to a compressive stress which results in a lower pull-in voltage.
- The fringing field effect makes a reduction in pull-in voltage in both cases of without or with nonlocal effect.
- Increase in the gap-to-thickness ratio induces an increased pull-in voltage.

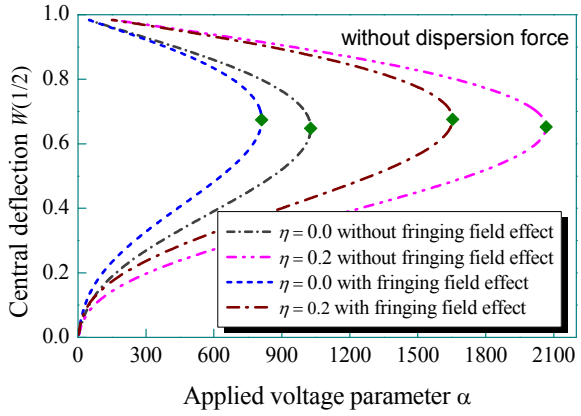


Fig. 4 Characteristic curve of the applied voltage versus central deflection for a nanobeam without residual stress and dispersion force effect considering the fringing field effect and nonlocal effect.

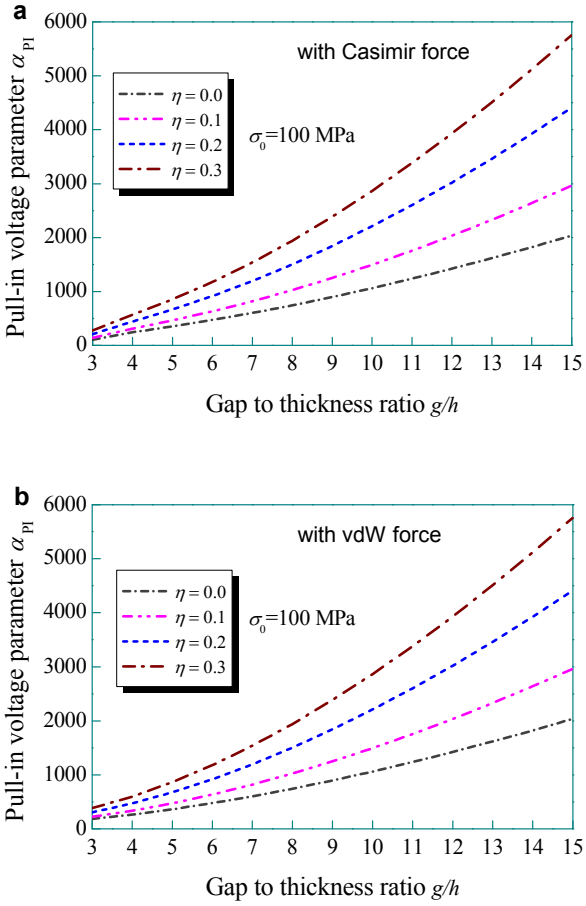


Fig. 5 Effect of the gap to thickness ratio on the pull-in voltage for a nanobeam with residual stress considering the nonlocal effect and dispersion force effect.

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- The problem concerned is inherently nonlinear and coupled, the nonlinearity arises out of the electrostatic force, intermolecular force, and large deformation effect of the deformable nanobeam. The numerical treatment proved to be useful and efficient one for this kind of problem.
- For vanishing nonlocal parameter, the electrostatically actuated nanobeam model furnished by the classical elasticity theory is recovered.

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